Asymptotic Expansions of a Class of Hypergeometric Polynomials with Respect to the Order, II

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INTRODUCTION

In a previous paper [8], asymptotic properties of the hypergeometric polynomials

\[ F_n(z) = \binom{-n, n + \lambda, \alpha_p}{1 + \rho_q} \]

for large order were discussed. However, only the case \( q = p + 1 \) (Case I) was treated extensively. In this paper, the above polynomials for \( q < p \) and \( q > p + 2 \), designated as Cases II and III, respectively, are treated more fully. By confluence, analogous results for lower order hypergeometric polynomials and functions are deduced. In particular, a useful form of the asymptotic expansion for Bessel functions is given.

III. Case II, \( q \leq p \)

As before \( F_q(z) \) obeys a differential equation (see (1.8)) of order \( M = p + 2 \). By direct computation there are \( p + 2 \) descending series solutions of (1.8) of the following form:

\[
L_{p+2,q}(z) = \frac{(\alpha_p - \alpha_i)(z)^{-\alpha_i}}{(1 + \rho_q)_{-\alpha_i}} \\
\times \frac{\Gamma(\alpha_t, \alpha_t - \rho_q)}{\Gamma(1 + \alpha_t + n, 1 + \alpha_t - n - \lambda, 1 + \alpha_t - \alpha_p)} \left( \frac{(-1)^{q-p+1}}{z} \right),
\]

\( t = 1, \ldots, p \).

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1 Numbers in square brackets pertain to references at end of paper. References 1–7 are given at the end of [8] which we assume is handy to the reader.

\( \ast \) Sections I and II are contained in [8].
\[ \mathcal{H}^{(1)}_2(z) = \frac{(\alpha_p)_n}{(1 + \rho_q)_n} (-z)^n \binom{\cdots n - n}{1 - 2n - \lambda, 1 - n - \alpha_p} \left( \frac{(-1)^{q - p + 1}}{z} \right), \]

\[ \mathcal{H}^{(2)}_2(z) = \frac{(\alpha_p)_n - \lambda}{(1 + \rho_q)_n} (-z)^{-n - \lambda} \times q + 1 F_{p + 1} \left( \begin{array}{c} n + \lambda, n + \lambda - \rho_q \\ 1 + 2n + \lambda, 1 + n + \lambda - \alpha_p \end{array} \right) \left( \frac{(-1)^{q - p + 1}}{z} \right), \]

where the notation should be interpreted as in (2.1) and (2.2). Under the restriction that no \( \alpha_i - \alpha_j, i \neq j \), is equal to an integer or zero, the solutions in (3.1) and (3.2) are linearly independent. This restriction can be relaxed by taking limiting forms. In general, \( \mathcal{F}_n(z) \) is equal to a linear combination of (3.1) and (3.2). In fact, under any particular set of restrictions on the parameters \( \alpha_i \) and \( \lambda \), \( \mathcal{F}_n(z) \) equals one and only one of these \( p + 2 \) solutions; e.g.,

\[ \mathcal{F}_n(z) = \frac{(n + \lambda)_m}{(n + 1)_m} \mathcal{L}^{(\alpha_m)}_{p + 2, q}(z), \quad (3.3) \]

if some one \( \alpha_m \) is a negative integer, \(-\alpha_m \leq n, \lambda \) is not a negative integer,

\[ \mathcal{F}_n(z) = \frac{1}{(n + 1)_{n + \lambda}} \mathcal{H}^{(2)}_2(z), \quad (3.4) \]

if no \( \alpha_i \) is a negative integer, \( \lambda \) is a negative integer, \( 2n > -\lambda \), etc. We remark that although the \( \mathcal{L}^{(\alpha_m)}_{p + 2, q}(z) \), \( \mathcal{H}^{(1)}_2(z) \), and \( \mathcal{H}^{(2)}_2(z) \) are descending series in \( z \), the way \( n \) appears makes them suitable for computation for large \( n \). For the purposes of asymptotic equivalence for large \( n \), one permits all solutions to appear, and writes for \( q < p \),

\[ p + 2 F_q \left( \begin{array}{c} -n, n + \lambda, \alpha_p \\ 1 + \rho_q \end{array} \right) \sim \sum_{i=1}^p \frac{(n + \lambda)_{-\alpha_i}}{(n + 1)_{\alpha_i}} \mathcal{L}^{(\alpha_i)}_{p + 2, q}(z) \]

\[ + \frac{1}{(n + 1)_{n + \lambda}} \mathcal{H}^{(2)}_2(z) \]

where the connecting constants of the various solutions are those values which hold in the particular situation when \( \mathcal{F}_n(z) \) equals exactly that solution. Then the dominant term of (3.5) under any set of conditions is that solution to which \( \mathcal{F}_n(z) \) is equal exactly.

If \( p > q \), (3.5) is suitable for computational purposes for large \( n \). However,
if \( p = q \), the (in general) dominant term \( \mathcal{H}^{(1)}_{2}(z) \) converges very slowly and we note that in this case,

\[
\mathcal{H}^{(1)}_{2}(z) = \frac{(\alpha_p)_{\lambda}}{(1 + \rho_p)n} (-z)^n \exp \left( - \frac{n(n + \rho_p)}{z(2n + \lambda - 1)(n - 1 + \alpha_p)} \right) - (1 + \rho_{\lambda + \rho_{p + 1}}) (2n + \lambda - 1)(n - 1 + \alpha_p) z \]

as can be shown by elementary series manipulation. As before, the notation \((n + \rho_p)\) should be interpreted as \(\Pi_{j=1}^{p} (n + \rho_j)\), and similar remarks hold for \((n - 1 + \alpha_p)\), \((\alpha_p)_{n}\), and \((1 + \rho_p)_{n}\).

Equation (3.5) holds for all \( z \) except at the singular points zero and infinity, as \( n \to \infty \). For fixed \( n \) and variable \( z \), we require \( |z| \geq O(1) \) to insure that the correction terms remain small.

Finally, we note that by confluence,

\[
\lim_{\sigma \to \infty} \text{P}_q\text{F}_p \left( \frac{-n, n + \lambda, \alpha_p, \sigma}{\sigma} \right) = \text{P}_q\text{F}_p \left( \frac{-n, n + \lambda, \alpha_p}{1 + \rho_{\lambda + \rho_{p + 1}}} \right) \]

and that confluences on the \( \mathcal{L}_{p+2,q}^{(\alpha_p)}(z) \) terms in (3.5) can be carried out in a similar manner. By \( p + 1 - q \) such confluences, we drop from Case II to Case I, and by comparison of the \( \mathcal{L}_{p+2,q+1}(z) \) terms, we see the consistency of our choice of the connecting constants \( A_t, t = 1, 2, \ldots, p \), in (2.5).

IV. Case III, \( q \geq p + 2, N^\beta = n(n + \lambda) \)

Setting \( \beta = q - p + 1 \), our development for \( \beta \geq 3 \) is similar to Case I (\( \beta = 2 \)). \( F_n(z) \) obeys a differential equation, (1.8), of order \( M = p + \beta \). The \( \mathcal{L}_{p+2,q}^{(\alpha_p)}(z) \) functions defined in (3.1) are \( p \) formal, descending series in \( z \) solutions of (1.8). Although in general they are divergent, they serve as asymptotic expansions for large \( z \) to valid solutions of (1.8), and as mentioned before, are suitable for computation if \( n \) is large. The confluence argument at the end of Section III indicates that the \( \mathcal{L}_{p+2,q}^{(\alpha_p)}(z) \) solutions and their connecting constants for \( F_n(z) \) are independent of the case number. The lead terms of the exponential asymptotic expansions of the remaining \( M - p = \beta \) solutions around infinity are computed by the formal procedure given in
section 1, and are denoted by \( \mathcal{X}^{(j)}_{\beta}(z) \), \( j = 1, \cdots, \beta \) — see [4] and the references given there. Thus

\[
F_n(z) \sim \sum_{i=1}^{p} \frac{(n + \lambda) - \alpha_i}{(n + 1)\alpha_i} \mathcal{L}_{\rho_{z+2,q}}(z) + \sum_{j=1}^{\beta} A_{\rho_1,} \mathcal{X}^{(j)}_{\beta}(z) \tag{4.1}
\]

under suitable restrictions on \( z \).

The constants \( A_{\rho_1,} j = 1, \cdots, \beta \) are determined by the exploitation of the fact

\[
\lim_{n \to \infty} F_n \left( \frac{z}{n(n + \lambda)} \right) = \lim_{n \to \infty} {}_{\rho_{z+2}}F_{\rho_{z}} \left( \frac{-n, n + \lambda, \alpha_p}{1 + \rho_q} \left| \frac{z}{n(n + \lambda)} \right. \right)
\]

\[
= {}_{\rho_{z}}F_{\rho_{z}} \left( \frac{\alpha_p}{1 + \rho_q} \left| -z \right. \right) \tag{4.2}
\]

and that for \( 0 \leq \arg z \leq \pi - \epsilon, \epsilon > 0 \), the representation (4.1) coalesces with the asymptotic representation of \( {}_{\rho_{z}}F_{\rho_{z}}\left( \frac{\alpha_p}{1 + \rho_q} \left| -z \right. \right) \) for large \( z \), see [6]. Thus we write, through the \( \tau_3 \) terms (see (1.20)),

\[
{}_{\rho_{z+2}}F_{\rho_{z}} \left( \frac{-n, n + \lambda, \alpha_p}{1 + \rho_q} \left| z \right. \right) \sim \sum_{i=1}^{p} \frac{(n + \lambda) - \alpha_i}{(n + 1)\alpha_i} \mathcal{L}_{\rho_{z+2,q}}(z) \\
+ \frac{2(2\pi)^{1-\beta/2} \Gamma(1 + \rho_q)}{(\beta)^{1/2} \Gamma(\alpha_p)} (N^2 \lambda \rho_{z})^{\nu} \exp \{N^{2+1/2} \beta \cos(\pi/\beta) + az/3 \} \tag{4.3}
\]

\[
- (N^{2+1/2} \lambda \rho_{z}) \cos(\pi/\beta) + O(N^{-2}) \exp \{N^{2+1/2} \beta \sin(\pi/\beta) + \pi \gamma \}
\]

\[
+ (\beta - 2) \text{ exponentially lower order terms,}
\]

where

\[
\begin{align*}
N^\beta &= n(n + \lambda) \\
\beta &= q - p + 1 \\
D_1 &= \sum_{i=1}^{p} \alpha_i \\
E_1 &= \sum_{i=1}^{q} (1 + \rho_i) \\
D_2 &= \sum_{i=1}^{p} \sum_{t=1}^{(\alpha_i - 1)} \alpha_i \\
E_2 &= \sum_{i=2}^{q} \sum_{t=1}^{(1 - \rho_i)} (1 + \rho_i) (1 + \rho_i) \\
\gamma &= (2\beta - 1)^{-1}((\beta - 1) + 2D_1 - 2E_1) \\
\Omega(z) &= (2\beta - 1)^{-1} \lambda_1 z^2 + (\beta - 1)^{-1} \lambda_2 z - \lambda_3 
\end{align*}
\tag{4.4}
\]
section 1, and are denoted by \( \mathcal{X}(z) \), \( j = 1, \cdots, \beta \) — see [4] and the references given there. Thus

\[
F_n(z) \sim \sum_{i=1}^{\beta} \frac{(n + \lambda - a_i)}{(n + 1)a_i} \mathcal{L}_{n+2,a_i}(z) + \sum_{j=1}^{\beta} A_{\beta+1} \mathcal{X}(z) \tag{4.1}
\]

under suitable restrictions on \( z \).

The constants \( A_{\beta+1} \), \( j = 1, \cdots, \beta \) are determined by the exploitation of the fact

\[
\text{Lim}_{n \to \infty} F_n \left( \frac{z}{n(n + \lambda)} \right) = \text{Lim}_{n \to \infty} \mathcal{F}_{-2} \left( 1 + \rho_q \left| \frac{z}{n(n + \lambda)} \right) \right.
\]

and that for \( 0 \ll |z| \ll n(n + \lambda) \), the representation (4.1) coalesces with the asymptotic representation of \( \mathcal{F}_{-2}(z) \) for large \( z \), see [6]. Thus we write, through the \( \tau_3 \) terms (see (1.20)),

\[
\mathcal{F}_{-2} \left( 1 + \rho_q \left| \frac{z}{n(n + \lambda)} \right) \right.
\]

\[
+ \frac{2(2\pi)^{1-\beta/2}}{(\beta)^{\gamma/2} \Gamma(\alpha_p)} (N^\beta z^\gamma \exp \{Nz^{1/\beta} \cos (\pi/\beta) + az/3 \} \tag{4.3}
\]

\[
- (Nz^{1/\beta})^2 \cos (\pi/\beta) + O(N^{-2}) \cos (Nz^{1/\beta} \sin (\pi/\beta) + \pi \gamma
\]

\[
+ (Nz^{1/\beta})^2 \sin (\pi/\beta) + O(N^{-2})
\]

\[
+ (\beta - 2) \text{ exponentially lower order terms},
\]

where

\[
N^\beta = n(n + \lambda) \quad \beta = q - p + 1
\]

\[
D_1 = \sum_{i=1}^{\beta} \alpha_i \quad E_1 = \sum_{i=1}^{q} (1 + \rho_i)
\]

\[
D_2 = \sum_{i=2}^{\beta} \sum_{l=1}^{\beta-1} (a_i)(\alpha_i) \quad E_2 = \sum_{p=2}^{q} \sum_{l=1}^{p-1} (1 + \rho_i)(1 + \rho_i)
\]

\[
\gamma = (2\beta - 1)(\beta - 1 + 2D_1 - 2E_1)
\]

\[
\Omega(z) = (2\beta - 1)^{-1}\lambda_1 z^2 + (\beta - 1)^{-1}\lambda_2 z - \lambda_3 \tag{4.4}
\]
Since cosine terms of large argument are related to Bessel functions of large argument by

\[ J_\mu(z) \sim \left( \frac{2}{\pi z} \right)^{1/2} \cos \left[ z - \frac{\mu \pi}{4} - \frac{\pi}{4} \right] \left\{ 1 + O(z^{-2}) \right\} \]

\[ - \sin \left[ z - \frac{\mu \pi}{4} - \frac{\pi}{4} \right] \left[ \frac{\mu^2 - \frac{1}{4}}{2z} + O(z^{-3}) \right], \quad (5.2) \]

\[ | \arg z | \leq \pi - \epsilon, \quad \epsilon > 0, \]

for large \( z \), see [9], the zeros of \( F_n(z) \) for Cases I and III can be related to the zeros of Bessel functions.

Let

\[ \mu = -(2\gamma + \frac{1}{2}), \quad (5.3) \]

and denote by \( l_j \) the \( j \)th positive zero of \( J_\mu(z) \), then (5.1) can be rewritten

\[ n(n + \lambda) A_{j,n} \sim \{ l_j \left[ \beta \sin (\pi/\beta) \right] \}^\beta, \quad (5.4) \]

\[ \beta = q - p + 1 \geq 2, \]

for fixed \( j \) and \( n \rightarrow \infty \).

Similarly, let \( z_{r,n} \) be the value of \( z \) at which the \( n \)th order polynomial takes on its \( r \)th extremal value, i.e., maximum or minimum, counted in the positive direction. The \( r \)th extremal value of \( \varphi^{+n}_{+n} F_q(-n+\lambda+1,1|z) \) occurs at the \( r \)th zero of \( \varphi^{+n}_{+n} F_{q}(n+\lambda+1,1+1+1|z) \). Therefore, if \( t_r \) is the \( r \)th positive zero of \( J_r(z) \),

\[ v = \mu + \frac{2(\beta - 1)}{\beta} = -(2\gamma + \frac{1}{2}) + \frac{2(\beta - 1)}{\beta}, \quad (5.5) \]

we can write

\[ (n - 1)(n + \lambda - 1) z_{r,n} \sim \left[ \frac{t_r}{\beta \sin (\pi/\beta)} \right]^\beta, \quad (5.6) \]

\[ \beta = q - p + 1 \geq 2, \]

for fixed \( r \) and \( n \rightarrow \infty \). The combination of (5.6) with (2.5) and (4.3) generalize results given in [10]-[15].

VI. CONFLUENT POLYNOMIALS

By confluence

\[ \lim_{\lambda \rightarrow \infty} \varphi^{+n}_{+n} F_q \left( \frac{-n, n + \lambda, \alpha_p}{1 + \rho_q} \bigg| \frac{\pi}{n + \lambda} \right) = \varphi^{+n}_{+n} F_q \left( \frac{-n, \alpha_p}{1 + \rho_q} \bigg| z \right). \quad (6.1) \]
Thus the asymptotic representation for the polynomials $p+1F_p^{(-n, \alpha_p} \mid z)$ may be derived from the corresponding results for $p+2F_p^{(-n, \alpha_p} \mid z)$ except in the case when $p = q$ which can be treated directly by the methods used in [8].

We now treat the case $q = p + 1$ explicitly. Replacing $z$ by $z/(n + \lambda)$ corresponds to replacing $\theta$ by \{z/(n + \lambda)\} in (2.5). Since the lead constants of (2.5) are only asymptotic in nature for terms $O(N^{-2})$ and higher, one considers \(\lim_{\lambda \to \infty} \frac{\tau_m \left( \frac{t}{n + \lambda} \right)}{\lambda^{(m+1)/2}} \) instead of \(\lim_{\lambda \to \infty} N^{-m-1} \frac{t^{1/2}}{(n + \lambda)^{1/2}} \) in the notation of (2.5) and (1.20). Thus

\[

p+1F_{p+1}^{(-n, \alpha_p} \mid z) \sim \frac{(\alpha_p - \alpha_t - \alpha_t - \alpha_t - \frac{1}{2})}{(n + 1)_{\alpha_t}} \\
\times p+2F_p^{(\alpha_t, \alpha_t - \rho_p + 1} \mid z) \\
+ \frac{\Gamma(1 + \rho_{p+1})}{\Gamma(n+1)^{1/2}} (n\pi)^{1/2} \exp \left\{ \frac{1}{2} + (n\pi)^{-1/2} \psi_1(z) + \mathcal{O}(n^{-5/2}) \right\} \\
\times \cos \left( 2(n\pi)^{1/2} + \gamma \right) - (n\pi)^{-1/2} \psi_1(z) - (n\pi)^{-3/2} \psi_3(z) + \mathcal{O}(n^{-5/2}),
\]

where

\[
B_1 = \sum_{t=1}^{p} \alpha_t \\
B_2 = \sum_{s=2}^{p} \sum_{t=1}^{s-1} (\alpha_s)(\alpha_t) \\
B_3 = \sum_{r=3}^{p} \sum_{s=2}^{r-1} \sum_{t=1}^{s-1} (\alpha_r)(\alpha_s)(\alpha_t) \\
C_1 = \sum_{t=1}^{p+1} (1 + \rho_t) \\
C_2 = \sum_{s=2}^{p+1} \sum_{t=1}^{s-1} (1 + \rho_s)(1 + \rho_t) \\
C_3 = \sum_{r=3}^{p+1} \sum_{s=2}^{r-1} \sum_{t=1}^{s-1} (1 + \rho_r)(1 + \rho_s)(1 + \rho_t)
\]

etc.

\[
\gamma = (4)^{-1}(1 + 2B_1 - 2C_1) \\
\psi_1(z) = (12)^{-1}z^2 + (2)^{-1}(B_1 - C_1)z + \omega_1 \\
\psi_2(z) = (16)^{-1}z^2 + \omega_2 \\
\psi_3(z) = (320)^{-1}z^4 + (48)^{-1}(B_1 - C_1)z^3 + \omega_3z^2 + \omega_4z + \omega_5
\]
\[ \omega_1 = (4)^{-1}(B_1 - C_1)(3B_1 + C_1 - 2) + C_2 - B_2 - 3/16 \]

\[ \omega_2 = (16)^{-1}(C_1 - B_1)(8B_2 - 8B_1^2 + 11B_1 + C_1 - 2) \]
\[ + (4)^{-1}(C_2 - B_2)(2B_1 - 3) - (2)^{-1}(C_2 - B_2) + 3/64 \]

\[ \omega_3 = (32)^{-1}(B_1 - C_1)(2B_1 - 3C_1) + (8)^{-1}(B_2 - C_2) - 1/128 \]

\[ \omega_4 = (16)^{-1}(B_1 - C_1)[8B_2 - 5B_1^2 - C_1^2 - 2B_1C_1 + 6B_1 + 2C_1 - 3/4] \]
\[ + (4)^{-1}(B_2 - C_2)(B_1 + C_1 - 2) - (2)^{-1}(B_2 - C_2) \]

\[ \omega_5 = (128)^{-1}(B_1 - C_1)[C_1^3 + 5B_1C_1^2 + 35B_2C_1^2 - 105B_1^2 + 236B_1^3 \]
\[ + 160B_1B_2 - 24B_2C_1 - 8C_1C_2 - 40B_1C_1 - 4C_1^2 - 192B_2 \]
\[ - 64B_3 - (291/2) B_1 - (1/2) C_1 + 9] + (24)^{-1}(B_2 - C_2)[2C_2 - 10B_2 \]
\[ + 6C_1 - 30B_1 - 7B_1C_1 + 15B_2^2 + 73/4] \]
\[ + (6)^{-1}(B_2 - C_2)(C_1 - 3B_1 + 6) \]
\[ + (3)^{-1}(B_4 - C_4) + 21/1024 \]

and the notation in (6.2) is interpreted to make sense, i.e., \( \Gamma(\alpha_p) \) stands for \( \Pi_{j=1}^p \Gamma(\alpha_j) \) and the denominator parameters of the hypergeometric functions on the right-hand side of (6.2) are \((1 + \alpha_t + n)\) and \((1 + \alpha_t - \alpha_j), (j = 1, \ldots, p, j \neq t)\).

As usual (6.2) holds for all fixed \( z \) as \( n \to \infty \) except at the singular points zero and infinity, and along the negative real axis. If \( n \) is fixed and \( z \) is allowed to vary, additional restrictions must be put on \( z \) to insure that the correction terms remain small.

Just as in Section V, the values of \( z \) at which \( \psi_{p+1}F_{p+1}(\frac{-n+\alpha_p}{1+\rho_{p+1}}|z) \) takes on its zeros and relative extrema can be related to the zeros of certain Bessel functions, and this in conjunction with (6.2) generalizes known results, see [16] and [17].

Treating (2.7) in a similar fashion, and using the same notations as in (6.2), we have

\[
\psi_{p+1}F_{p+1}\left(\begin{array}{c}
-n, \alpha_p \\
1 + \rho_{p+1}
\end{array}\mid -z\right) \sim \sum_{t=1}^{\infty} \frac{(\alpha_p)_{-\alpha_t}(e^{\pi i z})^{-\alpha_t}}{(n + 1)_{\alpha_t}(1 + \rho_{p+1})^{-\alpha_t}} \]
\[ \times \frac{\Gamma(1 + \rho_{p+1})(nz)^{\nu}}{\Gamma(\alpha_p) \pi^{1/2}} \exp\{-z/2 - (nz)^{-1}\psi_{p+1}(z) + O(n^{-2})\} \]
\[ \times \cosh\{(2nz)^{1/2} + (nz)^{-1/2}\psi_{p+1}(-z) - (nz)^{-3/2}\psi_{p+1}(-z) + O(n^{-5/2})\}, \]
\[ |\arg z| \leq \pi - \epsilon, \epsilon > 0, \delta = + (-) \text{ if } \arg z \leq (> ) 0. \]
VII. Confluent Functions

Since by confluence,

$$\lim_{n \to \infty} pF_{q} \left( \begin{array}{c} -n, n + \lambda, \alpha_v \\ 1 + \rho_q \end{array} \right) \frac{z}{n(n + \lambda)} = \lim_{n \to \infty} p+1F_{q} \left( \begin{array}{c} -n, \alpha_p \\ 1 + \rho_q \end{array} \right) \frac{z}{n}$$

$$= \sum F_{p} \left( \begin{array}{c} \alpha_p \\ 1 + \rho_q \end{array} \right) \frac{z}{n} \quad (7.1)$$

asymptotic representations of $pF_{q} \left( \begin{array}{c} \alpha_p \\ 1 + \rho_q \end{array} \right) \frac{z}{n}$, $p < q$, for large $z$ can be deduced from (2.9), (2.7), (4.3), (4.5) and (6.2). In particular for $q = p + 1$, one obtains from (6.2),

$$pF_{p+1} \left( \begin{array}{c} \alpha_p \\ 1 + \rho_{p+1} \end{array} \right) \sim \sum_{i=1}^{p} (z)^{-\alpha_i} \frac{(\alpha_p - \alpha_i)}{(1 + \rho_{p+1} - \alpha_i)}$$

$$\times 2F_{p+1} \left( \begin{array}{c} \alpha_t, \alpha_t - \rho_{p+1} \\ 1 + \alpha_t - \alpha_p \end{array} \right) \frac{1}{z} + \frac{\Gamma(1 + \rho_{p+1})}{\Gamma(\alpha_p) \pi^{1/2}} (z)^{\nu} \exp \left( \omega_2 z^{-1} + O(z^{-2}) \right)$$

$$\times \cos \left( 2\pi z^{1/2} + \pi \gamma - \omega_2 z^{-1/2} - \omega_2 z^{-3/2} + O(z^{-5/2}) \right), \quad (7.2)$$

$$\arg z \leq \pi - \epsilon, \epsilon > 0,$$

for large $z$, and where the notation is the same as that in (6.2). Since the case $p = 0$ in (7.2) is of particular interest in connection with Bessel functions, we develop that case further by recourse to the original methods of [8].

Assume that the differential equation satisfied by

$$I(\nu + 1) \left( \frac{2}{z} \right) J_{\nu}(z) = pF_{1}(1 + \nu | -z^2/4) \quad (7.3)$$

has a solution of the form

$$K \exp \left[ c_0 z + 2c_1 \log z - \sum_{m=2}^{\infty} \frac{c_m}{m-1} z^{1-m} \right] \quad (7.4)$$

where $K$ and $c_m$ are constants. Put (7.4) into this differential equation. Equating the coefficients of powers of $z^{-1}$ to zero leads to the recursion formulas

$$c_0^2 + 1 = 0 \quad (7.5)$$

$$(2\nu + 1 - m) c_{m+1} + \sum_{l=0}^{m+1} c_l c_{m+1-l} = 0.$$
combination of the two possible solutions of the form (7.4) with the known
asymptotic expansion of \( J_\nu(x) \) for large \( x \) then yields the formula

\[
J_\nu(x) \sim \left( \frac{2}{\pi x} \right)^{1/2} \exp \{ A_\nu(x) \} \cos \left( B_\nu(x) - \frac{\pi \nu}{2} - \frac{\pi}{4} \right),
\]

(7.6)

for large \( x \) where

\[
A_\nu(x) = - \sum_{m=1}^{\infty} \left( \frac{\mu - 1}{2m - 1} \right) \frac{4^m}{m} = \frac{(\mu - 1)}{16z^2} \left( \frac{\mu - 13}{8z^2} + \frac{(\mu^2 - 53\mu + 412)}{48z^4} + O(x^{-6}) \right),
\]

\[
B_\nu(x) = \frac{1}{2} \left\{ 1 + \frac{(\mu - 25)}{48z^2} + \frac{(\mu^2 - 114\mu + 1073)}{640z^4} + \frac{(5\mu^3 - 1535\mu^2 + 54703\mu - 375733)}{128z^6} + O(x^{-8}) \right\},
\]

\( \mu = 4\nu^2 \).

We remark that (7.6) is exact for \( \nu = \pm \frac{1}{2} \) since

\[
J_{1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \sin x,
\]

(7.8)

\[
J_{-1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \cos x.
\]

For future reference, we note

\[
A_\nu(-x) = A_\nu(x), \quad A_{-\nu}(x) = A_\nu(x),
\]

\[
B_\nu(-x) = -B_\nu(x), \quad B_{-\nu}(x) = B_\nu(x).
\]

(7.9)

Also, in terms of (7.7), the asymptotic expansions of other Bessel functions
can be written by altering the connecting constants, e.g.,

\[
Y_\nu(x) \sim \left( \frac{2}{\pi x} \right)^{1/2} \exp \{ A_\nu(x) \} \sin \left( B_\nu(x) - \frac{\pi \nu}{2} - \frac{\pi}{4} \right),
\]

\[
H^{(1)}_\nu(x) \sim \left( \frac{2}{\pi x} \right)^{1/2} \exp \{ A_\nu(x) + i \left[ B_\nu(x) - \frac{\pi \nu}{2} - \frac{\pi}{4} \right] \},
\]

(7.10)

\[
\Phi_\nu(x) \sim \left( \frac{2}{\pi x} \right)^{1/2} \exp \{ A_\nu(x) \} \cos \left( B_\nu(x) - \frac{\pi \nu}{2} - \frac{\pi}{4} + \alpha \right),
\]

\( | \arg x | \leq \pi - \epsilon, \epsilon > 0 \),
for large $z$, where $Y_r(z)$ is the Bessel function of the second kind, $H_r^{(1)}(z)$ is the Hankel function of the first kind, and $\mathcal{G}_r(z)$ is the cylinder function defined by

$$\mathcal{G}_r(z) = J_r(z) \cos \alpha - Y_r(z) \sin \alpha. \quad (7.11)$$

We remark that (7.6) and (7.10) hold for all $\nu$ as $z \to \infty$, $|\arg z| \leq \pi - \epsilon$, $\epsilon > 0$. However, if $z$ is held fixed one needs the additional restriction $|\nu| < |z|$ on $z$ for the correction terms of (7.6) and (7.10) to remain small.

Inverting the series $B_r(z) = w$, one obtains

$$z = \Psi_r(w) = w^\mu + \left(1 + \frac{(7\mu - 31)}{48\pi^2} + \frac{(83\mu^2 - 982\mu + 3779)}{1920\pi^4} + O(w^{-8})\right),$$

$$\mu = 4\nu^2. \quad (7.12)$$

Thus, the $j$th positive zero of $\mathcal{G}_r(z)$ occurs at $\Psi_r((\pi/4) [4j + 2\nu - 1] - \alpha)$. This result corresponds to the McMahon expansions given in [9] and [18].

For an application of the above results, consider expressions of the form $J_{\nu}(z) + J_{\nu}(z)$. Using (7.6), (7.9) and elementary identities, we can write

$$J_{-\nu}(z) + J_{\nu}(z) \sim 2 \cos (\pi \nu/2)(2/\pi z)^{1/2} \exp \{A_r(\xi)\} \cos \{B_r(\xi) - (\pi/4)\},$$

$$J_{-\nu}(z) - J_{\nu}(z) \sim -2 \sin (\pi \nu/2)(2/\pi z)^{1/2} \exp \{A_r(\xi)\} \sin \{B_r(\xi) - (\pi/4)\},$$

$$|\arg \xi| \leq \pi - \epsilon, \epsilon > 0, \quad (7.13)$$

as $\xi \to \infty$. Thus the $j$th positive zeros of $J_{-\nu}(z) + J_{\nu}(z)$ and $J_{-\nu}(z) - J_{\nu}(z)$ occur at $\Psi_r((\pi/4) [4j + 2\nu - 1])$ and $\Psi_r((\pi/4) [4j + 1])$, respectively. If $\nu = 1/3$, (7.13) gives information on the Airy functions $Ai(-z)$ and $Bi(-z)$, since

$$Ai(-z) = (3)^{-1/2} \left[J_{-1/3}(\xi) + J_{1/3}(\xi)\right],$$

$$Bi(-z) = (z/3)^{1/3} \left[J_{-1/3}(\xi) - J_{1/3}(\xi)\right], \quad (7.14)$$

$$\zeta = (2/3) \; z^{3/2}.$$

**References**


12. Vacca, Maria Teresa, Determinazione asintotica per $n \to \infty$ degli estremi relativi dell' $n$esimo polinomio di Jacobi. *Boll. Unione Mat. Ital.* (3) 8, 277-280 (1953).


