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Commuting involution graphs for symmetric groups

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1. Introduction

Suppose G is a finite group and X is a subset of G. The commuting graph on the set X, which we denote by $\mathcal{C}(G,X)$, has X as its vertex set with $x, y \in X$ joined by an edge whenever $xy = yx$. If X consists entirely of involutions, then we call $\mathcal{C}(G,X)$ a commuting involution graph. Many authors have studied $\mathcal{C}(G,X)$ for different choices of G and X, and from a number of different perspectives. For example, in the seminal paper of Brauer and Fowler [2] this graph is studied in the case when G has even order and $X = G\backslash\{1\}$. Typical of a number of results obtained is that in a group with more than one conjugacy class of involutions, any two involutions are distance at most 3 apart in $\mathcal{C}(G,X)$. More recently, Segev and Seitz [10] in resolving the Margulis–Platonov conjecture for inner forms of type A_n needed to look at the diameter of $\mathcal{C}(G,X)$ for G a non-abelian simple group and $X = G\backslash\{1\}$. Rapinchuk, Segev, and Seitz [8] in their work on finite quotients of the multiplicative group of a finite-dimensional division algebra are also forced to examine certain configurations in this graph. And, in related work, Segev [9] proves that the diameter of $\mathcal{C}(G,X)$ (where $X = G\backslash\{1\}$) is always greater than or equal to 3 when G is a minimal nonsoluble group (meaning that G is not soluble but any proper quotient of G is). Further investigations along these lines are to be found in the thesis of Moshe [6]. Of an entirely different flavour we have the contributions of Marchionna Tibiletti [5] and Pyber [7]. Commuting involution graphs arose in the work of Fischer [4] during his investigation of the so-called 3-transposition groups (see also [1]), one outcome of which was the discovery of three new sporadic simple groups. There X was the conjugacy class of involutions which are 3-transpositions.

In this paper we analyse the commuting involution graph $\mathcal{C}(G,X)$ where X is an involution conjugacy class of G and G is $\mathrm{Sym}(n)$, the symmetric group of degree n. From now on G denotes $\mathrm{Sym}(n)$, for some n, and X is a conjugacy class of involutions

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of G. Clearly G, acting by conjugation, induces graph automorphisms of $\mathcal{C}(G, X)$ and is transitive on its vertices. For $x \in X$ and $i \in \mathbb{N}$, $\Delta_i(x)$ denotes the set of vertices of $\mathcal{C}(G, X)$ which are distance i from x, using the usual distance function for graphs. This distance function will be denoted by d(,). We use G_x $(= C_G(x))$ to denote the stabilizer in G of x. Evidently $\Delta_i(x)$ will be a union of certain G_x-orbits.

Now let a stand for a fixed element of X (so $X = a^G$) and we suppose, without loss of generality, that

$$a = (12)(34)\cdots(2m-1\ 2m).$$

Set $r = n - 2m$. Thus a has cycle type $1^r 2^m$ and

$$G_a \cong \bigl(2^m : \mathrm{Sym}(m)\bigr) \times \mathrm{Sym}(r).$$

The properties of $\mathcal{C}(G, X)$ we shall primarily focus upon are the structure and sizes of the set $\Delta_i(a)$, the ith disc of a, and, when $\mathcal{C}(G, X)$ is connected, the diameter of $\mathcal{C}(G, X)$. Our first theorem shows that in the majority of cases $\mathcal{C}(G, X)$ is connected.

Theorem 1.1. *$\mathcal{C}(G, X)$ is disconnected if and only if $n = 2m + 1$ or $n = 4$ and $m = 1$.*

We remark that when $n = 4$ and $m = 1$, $\mathcal{C}(G, X)$ consists of three connected components each of size 2 while, when $n = 2m + 1$, $\mathcal{C}(G, X)$ has n connected components each of which is isomorphic to $\mathcal{C}(H, Y)$ where $H \cong \mathrm{Sym}(2m)$, $Y = b^H$, and $b = (12)(34)\cdots(2m-1\ 2m)$.

Using various results concerning $\Delta_i(a)$ we can pin down $\mathrm{Diam}\,\mathcal{C}(G, X)$, the diameter of $\mathcal{C}(G, X)$.

Theorem 1.2. *Suppose that $\mathcal{C}(G, X)$ is connected. Then one of the following holds*:

(i) $\mathrm{Diam}\,\mathcal{C}(G, X) \leqslant 3$, *or*
(ii) $2m + 2 = n \in \{6, 8, 10\}$ *and* $\mathrm{Diam}\,\mathcal{C}(G, X) = 4$.

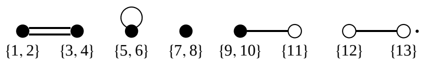
We observe that there are many such graphs of diameter 3—for more on such matters we refer the reader to Proposition 3.6 and Theorem 3.7.

This paper is organised as follows. We begin Section 2 by defining x-graphs. These parameterize the G_a-orbits of X and as a consequence frequently play an important role in our arguments—using these graphs we give a formula for the order of ax for any involution $x \in X$. We also start looking at the disc $\Delta_1(a)$ in Section 2.2 and in the following subsection we prove Theorem 1.1. Section 3 is primarily concerned with the diameter of $\mathcal{C}(G, X)$. Our first result there, Lemma 3.1, is the lynchpin of many of our later arguments. A further noteworthy result is Proposition 3.6, which gives an algorithm for deciding whether $\mathrm{d}(a, x) \leqslant 2$ or $\mathrm{d}(a, x) \geqslant 3$ for a vertex x of $\mathcal{C}(G, X)$. This result has a number of consequences which are recorded in Theorem 3.7. Finally, Section 4 contains detailed descriptions of the three exceptional diameter 4 graphs which arise in part (ii) of Theorem 1.2.

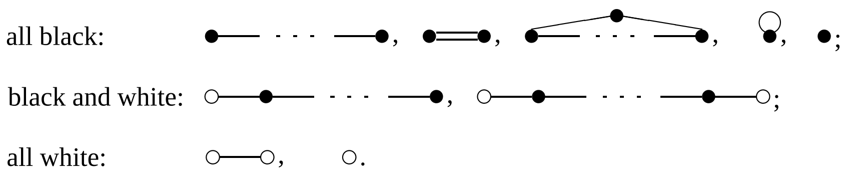
2. Preliminary results

2.1. *x*-graphs

We assume $G = \mathrm{Sym}(n)$ acts on the set $\Omega = \{1, 2, \ldots, n\}$ in the usual manner. Put $\mathcal{V} = \{\{1,2\}, \{3,4\}, \ldots, \{2m-1, 2m\}, \{2m+1\}, \ldots, \{n\}\}$; so $\mathcal{V}$ is the set of orbits of a on Ω. For $x \in X$ we define a graph, denoted $\mathcal{G}_x$, whose vertex set is $\mathcal{V}$ and $v_1, v_2 \in \mathcal{V}$ are joined by an edge whenever there exist α in v_1 and β in v_2 with $\beta \neq \alpha$ such that x interchanges α and β. We shall refer to $\mathcal{G}_x$ as the x-graph. The vertices corresponding to the 2-cycles of a will be coloured black ($\bullet$) and the other vertices white ($\circ$). So the number of black vertices is m and the number of white vertices is r. As an example, suppose $n = 13$, $a = (12)(34)(56)(78)(9\,10)$, and $x = (13)(24)(56)(10\,11)(12\,13)$. Then $\mathcal{G}_x$ is



We note that the number of edges in an x-graph must equal the number of black vertices and they both equal m. Further, a black vertex has valency at most two, and a white vertex has valency at most one. Hence the possible connected components of $\mathcal{G}_x$ are:



If we say, for $x, y \in X$, the graphs $\mathcal{G}_x$ and $\mathcal{G}_y$ are isomorphic it will be understood that the graph isomorphism preserves the black and white vertices. Our interest in x-graphs is sparked by the following lemma.

Lemma 2.1. (i) *Every graph with m black vertices of valency at most two, r white vertices of valency at most one and exactly m edges is the x-graph for some $x \in X$.*

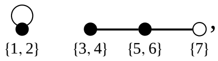
(ii) *Let $x, y \in X$. Then x and y are in the same G_a-orbit if and only if $\mathcal{G}_x$ and $\mathcal{G}_y$ are isomorphic graphs.*

Proof. Part (i) is immediate from the definition of x-graphs and part (ii) follows from the fact that G_a is m-transitive on $\{\{1,2\}, \{3,4\}, \ldots, \{2m-1, 2m\}\}$ and r-transitive on $\{\{2m+1\}, \ldots, \{n\}\}$. $\square$

Our next result is concerned with the possible orders of ax for $x \in X$. For each connected component C_i of the x-graph, let x_i and a_i be the corresponding parts of x and a. For example, let $n = 7$, $a = (12)(34)(56)$, and $x = (12)(45)(67)$. Then the x-graph is

Table 1

C_i	x_i	$a_i x_i$
	$(1s)(23)(45)\cdots(s-2\ s-1)$	$(135\cdots s-1)(2s\cdots 4)$
	$(23)(45)\cdots(s-2\ s-1)$	$(135\cdots s-1\ s\cdots 642)$
	$(\alpha\beta)$ some $\alpha,\beta>2m$	$(\alpha\ \beta)$
	$(\alpha\ s)(23)(45)\cdots(s-2\ s-1)$	$(135\cdots s-1\ \alpha\ s\cdots 642)$
	$(1\ \alpha)(\beta\ s)(23)(45)\cdots(s-2\ s-1)$	$(135\cdots s-1\ \beta\ s\cdots 642\ \alpha)$



where $C_1=\{\{1,2\}\}$, $a_1=x_1=(12)$, $C_2=\{\{3,4\},\{5,6\},\{7\}\}$, $a_2=(34)(56)$, and $x_2=(45)(67)$.

Proposition 2.2. *Suppose that $x\in X$ and that $C_1,\dots,C_k$ are the connected components of $\mathcal{G}_x$. Let m_i, r_i, and c_i be, respectively, the number of black vertices, white vertices, and cycles in C_i. Then*

(i) *the order of ax is the least common multiple of the orders of a_ix_i $(i=1,\dots,k)$; and*
(ii) *for $i=1,\dots,k$, the order of a_ix_i is $(2m_i+r_i)/(c_i+1)$.*

Proof. Part (i) follows immediately from the observation that if $i\neq j$, then both a_i and x_i commute with both of a_j and x_j. Note that for $g\in G_a$, the orders of ax and ax^g are the same, so we may choose the most suitable x from each orbit of G_a. Assume then that $a_i=(12)(34)\cdots(s-1\ s)$ for some $s\leqslant 2m$. Then there are five possibilities for x_i, presented in Table 1 along with C_i and a_ix_i. A simple check shows that the order of a_ix_i is, in each case, given by $(2m_i+r_i)/(c_i+1)$. (Note that permutations are applied from left to right, so they operate on the right.) □

2.2. The disc $\Delta_1(a)$

We begin with the following elementary observation.

Lemma 2.3. *Let $x\in X$. Then $x\in\Delta_1(a)\cup\{a\}$ if and only if each connected component of $\mathcal{G}_x$ is one of*  ,  , , , *and* .

We look more closely at the G_a-orbits in $\Delta_1(a)$.

Lemma 2.4. *Let $b\in\Delta_1(a)$ and suppose that $\mathcal{G}_b$ has k double edges, l loops, and s edges between two white vertices. Then the number of elements in the G_a-orbit of b is*

$$\frac{m!\,r!}{2^s k!\,l!\,(r-2s)!\,(s!)^2}.$$

Proof. There are $\binom{m}{2k}$ ways of choosing the $2k$ vertices of $\mathcal{G}_b$ for the k double edges and then $(2k-1)(2k-3)\cdots 1$ ways of pairing these vertices up. Since two permutations give rise to the same double edge, there are

$$2^k\binom{m}{2k}(2k-1)(2k-3)\cdots 1=\frac{m!}{(m-2k)!\,k!}$$

possible permutations associated with the double edges. Clearly, there are $\binom{m-2k}{l}$ ways of choosing the l loops and $\binom{r}{2s}$ ways of choosing white vertices that will be joined by an edge. For the latter there are $(2s-1)(2s-3)\cdots 1$ ways of pairing up the resulting white vertices and so

$$\binom{r}{2s}(2s-1)(2s-3)\cdots 1=\frac{r!}{2^s(r-2s)!\,s!}$$

ways of choosing s edges joining two white vertices. Therefore, the number of elements in the G_a-orbit of b is

$$\frac{m!}{(m-2k)!\,k!}\binom{m-2k}{l}\frac{r!}{2^s(r-2s)!\,s!}=\frac{m!\,r!}{2^s k!\,l!\,(r-2s)!\,(s!)^2}$$

because m is the number of edges in $\mathcal{G}_b$, so $m-2k-l=s$. $\square$

Lemma 2.5. *Let $\mu=\min\{m,[r/2]\}$ and $\nu_i=[(m-i)/2]$. Then*

$$\left|\Delta_1(a)\right|=\left(\sum_{i=0}^{\mu}\sum_{j=0}^{\nu_i}\frac{m!\,r!}{2^i j!\,(r-2i)!\,(m-i-2j)!\,(i!)^2}\right)-1.$$

Proof. Let $x\in\Delta_1(a)\cup\{a\}$ and suppose $\mathcal{G}_x$ has exactly i components of the form $\bullet$. Since m is both the number of black vertices and the number of edges in $\mathcal{G}_x$, i is the number of edges in $\mathcal{G}_x$ between two white vertices. In particular, $0\leqslant i\leqslant\mu$. Also let j be the number of double edges (between black vertices) in $\mathcal{G}_x$; j can take any value between 0 and ν_i. Note also that the number of loops in $\mathcal{G}_x$ is $m-i-2j$. Using Lemma 2.4 and the fact that $i=0=j$ implies that $x=a$, gives the result. $\square$

We next obtain a closed formula for the number of G_a-orbits in $\Delta_1(a)$.

Proposition 2.6. *Let $\mu=\min\{m,[r/2]\}$. Then the number $N_1(a)$ of orbits in $\Delta_1(a)$ is*

$$N_1(a)=\frac{1}{4}\left\{(2m+3-\mu)(\mu+1)+\frac{1}{2}\left((-1)^m+(-1)^{m-\mu}\right)\right\}-1.$$

Proof. For $x\in\Delta_1(a)\cup\{a\}$, the possible connected components of $\mathcal{G}_x$ are $\bullet$, , $\circ\!-\!\circ$, $\circ$, and $\bullet\!=\!\bullet$ by Lemma 2.3. Since the number of components $\circ\!-\!\circ$ and $\circ$ are determined by the components consisting only of black vertices, we only need count the latter. As observed earlier the number of • components is between 0 and μ. If there are i of these, then the number of possible x-graphs is clearly the number of nonisomorphic partitions of $m-i$ into parts of size at most 2, which is $[(m-i+2)/2]$. Therefore we have

$$N_1(a)=\sum_{i=0}^{\mu}\left[\frac{m-i+2}{2}\right]-1=\left[\frac{m+2}{2}\right]+\cdots+\left[\frac{m-\mu+2}{2}\right]-1$$

(the -1 being there to remove $\{a\}$ from the count). Notice that

$$N_1(a)=\sum_{i=0}^{\mu}\frac{m-i+2}{2}-\frac{1}{2}K-1,$$

where $K=|\{i \mid m-i+2 \text{ is odd}\}|$. Hence,

$$K=\begin{cases}\dfrac{\mu+1}{2} & \text{if the parities of } m \text{ and } m-\mu \text{ are distinct,}\\ \dfrac{\mu+2}{2} & \text{if } m \text{ and } m-\mu \text{ are odd,}\\ \dfrac{\mu}{2} & \text{if } m \text{ and } m-\mu \text{ are even.}\end{cases}$$

Now $\sum_0^{\mu}(m-i+2)/2=((2m+4-\mu)(\mu+1))/4$, so the lemma follows. □

2.3. *Connectedness of $\mathcal{C}(G,X)$*

As promised we give the

Proof of Theorem 1.1. When $n=4$ and $m=1$ a trivial calculation shows that $\mathcal{C}(G,X)$ has three connected components consisting of pairs of vertices. If $n=2m+1$, then a fixes only the point $2m+1$ and so any conjugate of a which commutes with a also fixes $2m+1$. By induction every conjugate of a in the connected component of a fixes $2m+1$. So $\mathcal{C}(G,X)$ is not connected.

We must show that in all other cases $\mathcal{C}(G,X)$ is connected. Let $x\in X\backslash\{a\}$. Since $\mathrm{Sym}(n)$ is generated by transpositions, a can be transformed into x by a series of conjugations by transpositions. Hence it is sufficient to show that a and a^t are connected for any transposition t. Suppose that $t=(\alpha\ \beta)$ where $\alpha,\beta\in\Omega$. If $a^t=a$, then there is nothing to prove. So we may assume that a does not interchange α and β nor fix both α and β. Suppose that a fixes neither of α and β. Then a must contain the 2-cycles $(\alpha\ \gamma)$ and $(\beta\ \delta)$ for $\gamma,\delta\in\Omega\backslash\{\alpha,\beta\}$, $\gamma\neq\delta$. Since t conjugates these 2-cycles to $(\beta\ \gamma)$ and $(\alpha\ \delta)$ and leaves all other 2-cycles and 1-cycles of a unchanged, a and a^t commute as required. Thus we may suppose that a fixes β but not α. So $n\neq 2m$ and hence $n\geqslant 2m+2$. By conjugation in $G_a=C_G(a)$ we may assume $t=(1\ 2m+1)$ and so $a^t=(2m+1\ 2)(34)\cdots(2m-1\ 2m)$. If $n\geqslant 2m+3$, then $b=(34)(56)\cdots(2m-1\ 2m)(2m+2\ 2m+3)\in X$ and b commutes with both a and a^t, so we are done. Therefore, it remains to consider the case $n = 2m + 2 \geqslant 6$. Let $b = (12)(56) \cdots (2m-1\ 2m)(2m+1\ 2m+2)$, $c = (1\ 2m+1)(2\ 2m+2)(56) \cdots (2m-1\ 2m)$, and $d = (2\ 2m+2)(34)(56) \cdots (2m-1\ 2m)$. Then $b, c, d \in X$, $ab = ba$, $bc = cb$, and $cd = dc$. Since $d^t = d$, there is a path from a^t to d and hence a path from a to a^t completing the proof of Theorem 1.1. $\square$

3. Diameter of $\mathcal{C}(G, X)$

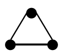
Our first result is important in the proof of Theorem 3.4.

Lemma 3.1. *Let $x \in X$. If $\mathcal{G}_x$ has no white vertices connected to black vertices, then* $\mathrm{d}(a, x) \leqslant 2$.

Proof. Let $C_1, C_2, \ldots, C_k$ be the connected components of $\mathcal{G}_x$ and let a_i and x_i be the corresponding parts of a and x, respectively. By assumption, for each C_i the vertices are either all black or all white. Suppose C_i is a cycle consisting of black vertices and, for ease of notation, assume that $a_i = (12)(34) \cdots (2s-1\ 2s)$ and $x_i = (23)(45) \cdots (1\ 2s)$. Set $b_i = (12)(3\ 2s)(4\ 2s-1) \cdots (s+1\ s+2)$, and note that a_i and b_i have the same number of 2-cycles. Further, we see that b_i commutes with both of a_i and x_i. Now suppose that C_i is a chain consisting of (one or more) black vertices and again for clarity we assume that $a_i = (12)(34) \cdots (2s-1\ 2s)$ and $x_i = (23)(45) \cdots (2s-2\ 2s-1)$. This time we define $b_i = (1\ 2s)(2\ 2s-1) \cdots (s\ s+1)$ and again check that b_i commutes with both a_i and x_i and that a_i and b_i have the same number of 2-cycles. Now let b be the product of all the b_i's defined above (and note that the fixed points of b equal the fixed points of a). By design, b has the same number of 2-cycles as a and so $b \in X$. Since b commutes with x and a, $\mathrm{d}(a, x) \leqslant 2$, so proving the lemma. $\square$

As a straightforward consequence of Lemmas 2.3 and 3.1 we have

Corollary 3.2. *If $n = 2m \geqslant 6$, then* $\mathrm{Diam}\,\mathcal{C}(G, X) = 2$.

Proof. Suppose $n = 2m \geqslant 6$. Then there are no white vertices in any x-graph, so, by Lemma 3.1, $\mathrm{Diam}\,\mathcal{C}(G, X) \leqslant 2$. However, since $n \geqslant 6$, there exists $x \in X$ such that $\mathcal{G}_x$ contains . By Lemma 2.3 $x \notin \Delta_1 \cup \{a\}$ and hence $\mathrm{Diam}\,\mathcal{C}(G, X) = 2$. $\square$

Theorem 3.3. *If $n > 2m + 2$, then* $\mathrm{Diam}\,\mathcal{C}(G, X) \leqslant 3$.

Proof. Let $x \in X$. Then we may write x as

$$\prod_{i=1}^{r_1}(\alpha_i \alpha_i') \prod_{i=1}^{r_2}(\beta_i \beta_i') \prod_{i=1}^{r_3}(\gamma_i \gamma_i') \prod_{i=1}^{r_4}(\delta_i) \prod_{i=1}^{r_5}(\varepsilon_i),$$

where α_i, α_i', β_i, and δ_i are in $\{1, \ldots, 2m\}$ and β_i', γ_i, γ_i', and ε_i are in $\{2m+1, \ldots, n\}$. Since $n > 2m + 2$ we have $r_4 + r_5 \geqslant 3$.

Suppose that r_2 is even. Let

$$c = \left(\prod_{i=1}^{r_1} \bigl(\alpha_i \alpha_i'\bigr) \prod_{i=1}^{r_3} \bigl(\gamma_i \gamma_i'\bigr) \right) (\beta_1 \beta_2) \cdots (\beta_{r_2-1} \beta_{r_2}) \bigl(\beta_1' \beta_2'\bigr) \cdots \bigl(\beta_{r_2-1}' \beta_{r_2}'\bigr).$$

It is easily checked that $xc = cx$ and that $c \in X$. Thus, since $\mathcal{G}_c$ has no white vertices connected to black vertices, Lemma 3.1 implies that $\mathrm{d}(a, x) \leqslant 3$. Next we consider the case when r_2 is odd. Let

$$d = \left(\prod_{i=1}^{r_1} \bigl(\alpha_i \alpha_i'\bigr) \prod_{i=1}^{r_3} \bigl(\gamma_i \gamma_i'\bigr) \right) (\beta_1 \beta_2) \cdots (\beta_{r_2-2} \beta_{r_2-1}) \bigl(\beta_1' \beta_2'\bigr) \cdots \bigl(\beta_{r_2-2}' \beta_{r_2-1}'\bigr) (\alpha\beta),$$

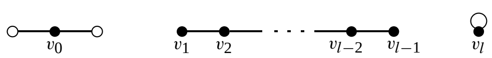
where $\alpha = \delta_1$, $\beta = \delta_2$ if $r_4 > 1$, otherwise $\alpha = \varepsilon_1$, $\beta = \varepsilon_2$. We have that $d \in X$, $dx = xd$ and using Lemma 3.1 again gives $\mathrm{d}(a, x) \leqslant 3$. Therefore, we have proved that $\mathrm{Diam}\,\mathcal{C}(G, X) \leqslant 3$. $\square$

Theorem 3.4. *Suppose that* $n = 2m + 2 \geqslant 12$. *Then* $\mathrm{Diam}\,\mathcal{C}(G, X) \leqslant 3$.

Proof. Let $x \in X$. If neither of the two white vertices is joined to a black vertex in $\mathcal{G}_x$, then $\mathrm{d}(a, x) \leqslant 2$ by Lemma 3.1. When both white vertices are joined to a black vertex, then (see the r_2 even case in Theorem 3.3) x commutes with an involution c with $\mathcal{G}_c$ having no white vertices joined to black vertices and so $\mathrm{d}(a, x) \leqslant 3$.

It remains to consider the case when $\mathcal{G}_x$ has exactly one white vertex joined to a black vertex. So the other white vertex is isolated and $\mathcal{G}_x$ consists of some cycles on the black vertices together with one chain which has a white vertex at the end.

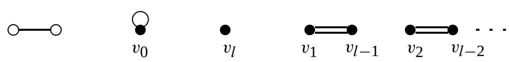
(3.4.1) If there exists $y \in X$ such that $xy = yx$ and $\mathcal{G}_y$ is of the form



$+ \{\text{cycles}\}$,

then $\mathrm{d}(a, x) \leqslant 3$.

Now y commutes with an involution $b \in X$ with $\mathcal{G}_b$ being of the form



$+ \{\text{cycles of length 1 or 2}\}$

and b commuting with a. Hence (3.4.1) holds.

Let k be the length of the chain in $\mathcal{G}_x$. We relabel the points of Ω so that the chain corresponds to

$$a_1 = (n-2\ n-3)(n-4\ n-5)\cdots(n-2k\ n-2k-1) \quad \text{and}$$
$$x_1 = (n-1\ n-2)(n-3\ n-4)\cdots(n-2k+1\ n-2k),$$

with the fixed points of a being n, $n-1$, while those of x are n, $n-2k-1$. Suppose for the moment that $k \geqslant 5$. Now the involution

$$(n\ n-2k-1)(n-1\ n-2k)(n-2\ n-2k+1)(n-4\ n-5)(n-3\ n-6)$$

multiplied, when $k \geqslant 6$, by the involution

$$(n-7\ n-8)\cdots(n-2k+5\ n-2k+4),$$

and having fixed points $n-2k+3$, $n-2k+2$, commutes with x_1. Taking this involution and multiplying it by the remaining cycles of x gives us an involution $y \in X$ with $xy = yx$ and $\mathcal{G}_y$ as in (3.4.1) (with v_0 corresponding to $\{n-2k, n-2k-1\}$ and v_l corresponding to $\{n-4, n-5\}$). Notice that for this choice of y,

$$b = (n\ n-1)(n-2k\ n-2k-1)\cdots,$$

and, of course, the fixed points of b are $n-4$, $n-5$. Hence $\mathrm{d}(a,x) \leqslant 3$, by (3.4.1).

Next we consider the cases $k=3$ and $k=4$. Then x_1 commutes with

$$(n\ n-2k-1)(n-1\ n-2k)(n-2\ n-2k+1) \quad (\text{if } k=3)$$

and

$$(n\ n-2k-1)(n-1\ n-2k)(n-2\ n-2k+1)(n-3\ n-4) \quad (\text{if } k=4).$$

The graphs corresponding to these involutions contain $\circ\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\circ$ (with $\{n-2k-1, n-2k\}$ being the black vertex) and a chain of length 1 or 2. In order to make an involution y as in (3.4.1) it suffices to show that a cycle in $\mathcal{G}_x$ commutes with an involution whose graph contains a loop. Because $n \geqslant 12$ there are at least five black vertices and so there exist cycles in $\mathcal{G}_x$. By relabelling points of Ω we may assume, without loss of generality, that a_2 and x_2 correspond to

$$a_2 = (12)(34)\cdots(2l-1\ 2l) \quad \text{and} \quad x_2 = (23)(45)\cdots(2l-2\ 2l-1)(1\ 2l).$$

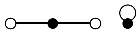
Since x_2 commutes with

$$(12)(3\ 2l)(4\ 2l-1)\cdots(l+1\ l+2),$$

which has a loop on $\{1,2\}$, we may form an involution y of the form in (3.4.1) commuting with x (notice that the fixed points of y are $n-3$, $n-4$, when $k=3$ and $n-5$, $n-6$ when $k=4$). Thus $\mathrm{d}(a,x) \leqslant 3$ by (3.4.1).

Now we assume that $k=2$. The involution

$$(n\ n-5)(n-1\ n-4)(n-2\ n-3)$$

commutes with x_1 and has corresponding graph . Forming an involution y by multiplying the above involution with all except one of the transpositions from the cycles of $\mathcal{G}_x$, yields an involution as in (3.4.1) which commutes with x. So $\mathrm{d}(a,x) \leqslant 3$.

Finally, we examine the case $k = 1$. This time we seek an involution y which is the product of $(n\ n-3)(n-1\ n-2)$ with transpositions which commute with the elements corresponding to cycles of $\mathcal{G}_x$, with one less edge on the corresponding vertices, and containing a loop. If $\mathcal{G}_x$ has two or more cycles, then this can be achieved by removing an edge from one cycle and forming a loop from another cycle as was done in the cases $k = 3$ and 4. So it remains to consider the situation when $\mathcal{G}_x$ has only one cycle. Since $n \geqslant 12$, this cycle must have length $l = (n-4)/2 \geqslant 4$. Again, by relabelling points of Ω we may assume that this cycle corresponds to

$$a_2 = (12)(34)\cdots(2l-1\ 2l) \quad \text{and} \quad x_2 = (23)(45)\cdots(2l-2\ 2l-1)(1\ 2l).$$

Since x_2 commutes with

$$(12)(3\ 2l)(45)\cdots(2l-4\ 2l-3)$$

which has one edge less and contains a loop in its corresponding graph, we may find a y satisfying the conditions of (3.4.1), whence $\mathrm{d}(a,x) \leqslant 3$. Thus in all cases $\mathrm{d}(a,x) \leqslant 3$, so proving Theorem 3.4. □

We are now able to prove Theorem 1.2. If $n > 2m+2$, then by Theorem 3.3 we have $\mathrm{Diam}\,\mathcal{C}(G,X) \leqslant 3$ and if $n = 2m \geqslant 6$, then $\mathrm{Diam}\,\mathcal{C}(G,X) = 2$ by Corollary 3.2. Since $\mathcal{C}(G,X)$ is assumed to be connected, this by Theorem 1.1, only leaves the case $2m+2 = n$. Then Theorem 3.4 and the information gathered in Section 4 ahead complete the proof of Theorem 1.2. As a consequence of Theorem 1.2 we have

Corollary 3.5. *Suppose $H \cong \mathrm{Alt}(n)$, the alternating group of degree n, and let X be an H-conjugacy class of involutions. If $\mathcal{C}(H,X)$ is connected, then either* $\mathrm{Diam}\,\mathcal{C}(H,X) \leqslant 3$ *or* $2m+2 = n \in \{6,10\}$ *and* $\mathrm{Diam}\,\mathcal{C}(H,X) = 4$.

We now give an algorithm for deciding from $\mathcal{G}_x$ whether $\mathrm{d}(a,x) \leqslant 2$ or $\mathrm{d}(a,x) \geqslant 3$. Given $x \in X$ we begin by constructing various sets whose elements are, apart from one, connected components of $\mathcal{G}_x$. So let $\mathcal{C}_x$ denote the set of connected components of $\mathcal{G}_x$ and $\mathcal{C}_x^o$ those connected components with one white vertex and at least one black vertex. Note that the connected components in $\mathcal{C}_x^o$ are chains. Now pair up, arbitrarily, each chain in $\mathcal{C}_x^o$ (if possible) with another chain in $\mathcal{C}_x^o$ of the same length. Of course, there may be some chains in $\mathcal{C}_x^o$ which have not been paired up, and we denote the set of such chains by $U(x)$. The subgraph of $\mathcal{G}_x$ given by the union of two paired connected components of $\mathcal{C}_x^o$ will be called a *double chain*. Let $P(x)$ denote the set of double chains. Also let

$$N(x) = \{C \in \mathcal{C}_x \mid C \text{ is a chain with all vertices black and at least one edge}\};$$

$$R(x) = \bigl\{C \in \mathcal{C}_x \bigm| C \notin \mathcal{C}_x^o \cup N(x),\ C \text{ has at least one edge}\bigr\}; \quad \text{and}$$

$$F(x) = \{C \in \mathcal{C}_x \mid C \text{ has no edges}\}.$$

So $C \in F(x)$ consists of one vertex of $\mathcal{G}_x$ (and we shall sometimes think of $F(x)$ as a set of vertices). Finally, set $b(x)$, respectively $w(x)$, to be the number of black, respectively white, vertices in $F(x)$.

We now use the sets above to define an integer $l(x)$. This is done using the following procedure which cancels edges and components or double chains from the sets above and concludes by counting the edges remaining after cancellation.

Step 1. Cancel an edge from a chain in $U(x)$ and cancel a component or double chain with a maximal number of edges in $P(x) \cup N(x)$.

Step 2. Repeat Step 1 until either there are no edges left in $U(x)$ or there are no components or double chains left in $P(x) \cup N(x)$.

Step 3. If $P(x) \cup N(x)$ is now empty, then let $l(x)$ be the total number of edges remaining in $U(x)$ and stop the procedure here. Otherwise continue to Step 4.

Step 4. Now cancel an edge from a component or a double chain from which an edge had already been cancelled, if such an edge exists. If not, cancel an edge from a component or a double chain with a minimal number of edges in $P(x) \cup N(x) \cup R(x)$. In either case we then also cancel a component or double chain with a maximal number of edges in $P(x) \cup N(x)$, providing the cancelled edge is not in that component or double chain. If there is a choice of elements with a minimal number of edges in $P(x) \cup N(x) \cup R(x)$, edges from elements of $R(x)$ should always be removed in preference to those from elements of $P(x) \cup N(x)$.

Step 5. Repeat Step 4 until either:

(a) $P(x) \cup N(x)$ contains no edges or
(b) $P(x) \cup N(x)$ has just one component or double chain C left with edges (and this has fewer edges than every component in $R(x)$).

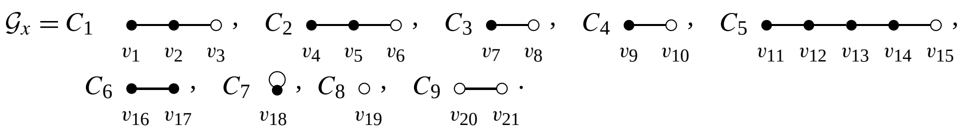
For possibility (b) we distinguish two cases:

(i) edges were cancelled from C or
(ii) edges were *not* cancelled from C.

In case (a) and (b)(i) let $l(x)$ be the number of edges left in the last component or double chain from which edges had been cancelled. In case (b)(ii) let $l(x)$ be the number of edges in C.

We give two examples to illustrate the calculation of $l(x)$.

Example 1.



Pairing C_1 with C_2 and C_3 with C_4 (the only possible choice here) gives $P(x) = \{C_1 \cup C_2, C_3 \cup C_4\}$ and $U(x) = \{C_5\}$. Also $N(x) = \{C_6\}$, $R(x) = \{C_7, C_9\}$, and $F(x) = \{C_8\}$.

So $b(x) = 0$ and $w(x) = 1$. Since $P(x) \cup N(x)$ contains two double chains and one chain, applying Step 1 three times leaves $P(x) \cup N(x) = \emptyset$ and

$$U(x) = \bigl\{\bigl\{ \underset{v_{11}}{\bullet} \quad \underset{v_{12}}{\bullet} \quad \underset{v_{13}}{\bullet} \quad \underset{v_{14}}{\bullet}\!\!-\!\!\!-\!\!\underset{v_{15}}{\circ} \bigr\}\bigr\}.$$

Therefore, by Step 3, $l(x) = 1$. Consequently, as $b(x) + w(x)/2 = 1/2 < 1 = l(x)$, $\mathrm{d}(a, x) \geqslant 3$ by Proposition 3.6 below.

Example 2.

$$\begin{aligned} \mathcal{G}_x = C_1 \; & \underset{v_1}{\bullet}\!\!-\!\!\!-\!\!\underset{v_2}{\bullet}\!\!-\!\!\!-\!\!\underset{v_3}{\circ}, \quad C_2 \; \underset{v_4}{\bullet}\!\!-\!\!\!-\!\!\underset{v_5}{\bullet}\!\!-\!\!\!-\!\!\underset{v_6}{\circ}, \quad C_3 \; \underset{v_7}{\bullet}\!\!-\!\!\!-\!\!\underset{v_8}{\circ}, \quad C_4 \; \underset{v_9}{\bullet}\!\!-\!\!\!-\!\!\underset{v_{10}}{\circ}, \quad C_5 \; \underset{v_{11}}{\bullet}\!\!-\!\!\!-\!\!\underset{v_{12}}{\circ}, \\ & C_6 \; \underset{v_{13}}{\bullet}\!\!-\!\!\!-\!\!\underset{v_{14}}{\bullet}, \quad C_7 \underset{v_{15}}{\bullet}, \quad C_8 \; \underset{v_{16}}{\circ}\!\!-\!\!\!-\!\!\underset{v_{17}}{\circ}. \end{aligned}$$

We have $P(x) = \{C_1 \cup C_2, C_3 \cup C_4\}$, $U(x) = \{C_5\}$, $N(x) = \{C_6\}$, $R(x) = \{C_7, C_8\}$, and $F(x) = \emptyset$. One pass through Step 1 yields

$$U(x) = \bigl\{\bigl\{ \underset{v_{11}}{\bullet} \quad \underset{v_{12}}{\circ} \bigr\}\bigr\} \quad \text{and} \quad P(x) \cup N(x) = \{C_3 \cup C_4, C_6\}.$$

As there are no further edges in $U(x)$ and $P(x) \cup N(x) \neq \emptyset$, we go to Step 4. Looking at $P(x) \cup N(x) \cup R(x) = \{C_3 \cup C_4, C_6, C_7, C_8\}$ we can (using Step 4) cancel $C_3 \cup C_4$ with the edge in C_8 and then cancel C_6 with the edge in C_7. Then

$$P(x) \cup N(x) \cup R(x) = \bigl\{\bigl\{ \underset{v_{15}}{\bullet} \bigr\}, \bigl\{ \underset{v_{16}}{\circ} \quad \underset{v_{17}}{\circ} \bigr\}\bigr\}.$$

The last edge was removed from C_7, an element of $R(x)$, and hence, by Step 5, $l(x) = 0$. As $b(x) = w(x) = 0$, Proposition 3.6 below implies $\mathrm{d}(a, x) \leqslant 2$.

Proposition 3.6. *Suppose $\mathcal{C}(G, X)$ is connected and let $x \in X$. Then $\mathrm{d}(a, x) \leqslant 2$ if and only if $l(x) \leqslant b(x) + w(x)/2$.*

Proof. Suppose first that $l(x) \leqslant b(x) + w(x)/2$. We will use the connected components of $\mathcal{G}_x$ to construct $\mathcal{G}_y$ for some $y \in X$ which commutes with x and a. If a and y are to commute, then by Lemma 2.3 the only possible connected components for $\mathcal{G}_y$ are $\circ\!\!-\!\!\!-\!\!\circ$, $\bullet\!=\!=\!\bullet$, $\bullet$, $\bullet$, and $\circ$. The following diagrams show, for various possible connected components of $\mathcal{G}_x$ on the left, some arrangement of edges for $\mathcal{G}_y$ on the right so that y will commute with x and a. Note that in the first four cases the number of edges in the graphs on the left and right are the same but in the last three cases the right-hand graph has one more edge than that on the left.

(1)


(2)


(3)


(4)


(5)


(6)


(7)


For each connected component in $R(x)$ we may use (1)–(4) to construct the corresponding part of $\mathcal{G}_y$ so that y commutes with x and a. It may be necessary to change this construction if edges have to be taken from one of these components in the argument which follows. Now consider the chains in $N(x)$ and the double chains in $P(x)$. These can be dealt with using the graphs in (5)–(7) but in each case we must cancel an edge from some other part of $\mathcal{G}_x$ so as to make $y \in X$. The algorithm for calculating $l(x)$ tells us the order in which to do this. We first use edges from chains in $U(x)$ and, if these are exhausted, then we use edges from an element of $P(x) \cup N(x) \cup R(x)$ with a minimal number of edges (taking edges from $R(x)$ if there is a choice). For each component of $\mathcal{G}_x$ from which edges have been cancelled let all the corresponding vertices in $\mathcal{G}_y$ have valency 0. Furthermore, if case (b)(ii) in Step 5 occurs, let also all vertices corresponding to the component C have valency 0. By the definition of $l(x)$, at the end of this process we have $l(x)$ edges left amongst components of $\mathcal{G}_x$ which have not been accounted for in $\mathcal{G}_y$. Since $l(x) \leqslant b(x) + w(x)/2$ there are enough vertices in $F(x)$ to accommodate these edges using loops in $\mathcal{G}_y$ on the black vertices of $F(x)$ and using edges between pairs of white vertices in $F(x)$. This completes the construction of $\mathcal{G}_y$ and hence of y, so $\mathrm{d}(a, x) \leqslant 2$.

To prove the converse, suppose that $\mathrm{d}(a, x) \leqslant 2$ and let $y \in \Delta_1(a) \cap \Delta_1(x)$. We need to prove that $\mathcal{G}_y$ has at least $l(x)$ edges between the vertices of $F(x)$. We first prove

(3.6.1) Let C be a connected component of $\mathcal{G}_x$. Then one of the following holds:

(i) the vertices of C are vertices of $F(y)$;
(ii) there exists a connected component C' of $\mathcal{G}_x$ and an isomorphism $\psi : C \to C'$ (preserving black and white vertices) such that v is connected to $\psi(v)$ in $\mathcal{G}_y$, for every vertex $v \in C$. Note that in the case when $C' = C$, the vertices of C are connected in $\mathcal{G}_y$ as shown in the right-hand side of (1)–(6).

Given $C \in \mathcal{C}_x$, we define the following subset of Ω

$$\Omega_C = \bigcup_{v \in C} v.$$

Notice that Ω_C is an orbit of $\langle a, x\rangle$, and that $\langle a, x\rangle$ commute with y. Hence if y fixes some element of Ω_C, then Ω_C is fixed by y, and case (i) holds.

Suppose that no element of Ω_C is fixed by y. For $z \in X$, and two vertices $v, v' \in \mathcal{G}_z$ we will say that v, v' are adjacent in $\mathcal{G}_z$ if there exists at least one edge between v and v' in $\mathcal{G}_z$ (notice that $v = v'$ is allowed only in case of a loop). Let $B_z(v)$ be the set of vertices $v' \in \mathcal{G}_z$ adjacent to v. It is easy to verify that if two vertices v, w are adjacent in $\mathcal{G}_y$, then, since x and y commute, each $v' \in B_x(v)$ is adjacent in $\mathcal{G}_y$ to a unique $w' \in B_x(w)$. Now since y fixes no element of Ω_C, and since y commutes with a, every vertex in $v \in C$ is adjacent in $\mathcal{G}_y$ to a unique vertex $\psi(v)$ (Lemma 2.3). The above argument shows that $C' := \{\psi(v) \mid v \in C\}$ is a connected component of $\mathcal{G}_x$ and that ψ is the required isomorphism. This completes the proof of (3.6.1).

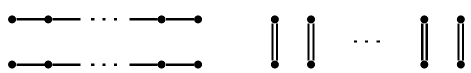
Let now $y \in X$ satisfy (i) and (ii) of (3.6.1). Set

$$\mathfrak{A}_y = \bigl\{C \in \mathcal{C}_x \mid C \notin F(x) \text{ and } \Omega_C \subseteq \operatorname{Fix}(y)\bigr\},$$

where $\operatorname{Fix}(y) \subset \Omega$ are the fixed points of y. Let

$$\mathfrak{B}_y := \mathcal{C}_x - \bigl(\mathfrak{A}_y \cup F(x)\bigr).$$

Let $C \in \mathfrak{B}_y$ and let C' and ψ be as in (ii) of (3.6.1). If $C = C'$, then the subgraph of $\mathcal{G}_y$ on the vertices of C is as shown in the right-hand side of (1)–(6). If $C \neq C'$, then the subgraph of $\mathcal{G}_y$ on the vertices of $C \cup C'$ is obtained by drawing two edges between $v \in C$ and $\psi(v)$ if v is a black vertex and drawing a single edge between $v \in C$ and $\psi(v)$ if v is a white edge. Suppose $C \in R(x)$. Since the components of $R(x)$ already have the maximal possible number of edges, it follows that the number of edges on the vertices of $C \cup C'$ (both in the case when $C = C'$ and in the case when $C \neq C'$) is the same in $\mathcal{G}_x$ and in $\mathcal{G}_y$. If $C \in N(x)$ and $C' = C$, then the number of edges on the vertices of C is exactly one more in $\mathcal{G}_y$ than in $\mathcal{G}_x$ (see (5) and (6)), while if $C \neq C'$, then the number of edges on the vertices of $C \cup C'$ is exactly two more in $\mathcal{G}_y$ than in $\mathcal{G}_x$ (see (7) and (8) below).

(8) 

Finally, if $C \in \mathcal{C}_x^o$, then necessarily $C' \neq C$ and then the number of edges on the vertices of $C \cup C'$ is exactly one more in $\mathcal{G}_y$ than in $\mathcal{G}_x$ (see (7)).

Hence the number of edges in $\mathcal{G}_y$ between vertices that are not in $F(x)$ is exactly

$$\sum_{C \in \mathfrak{B}_y \cap R(x)} |C| + \sum_{C \in \mathfrak{B}_y \cap N(x)} \big(|C| + 1\big) + \sum_{C \in \mathfrak{B}_y \cap \mathcal{C}_x^o} \big(|C| + 1/2\big),$$

where $|C|$ is the total number of edges in the component C. Since the number of edges in $\mathcal{G}_y$ is the same as that in $\mathcal{G}_x$, it follows that the number of edges between vertices of $F(x)$ in $\mathcal{G}_y$ is

$$\mu_y := \sum_{C \in \mathfrak{A}_y} |C| - \big|\mathfrak{B}_y \cap N(x)\big| - \frac{1}{2}\big|\mathfrak{B}_y \cap \mathcal{C}_x^o\big|.$$

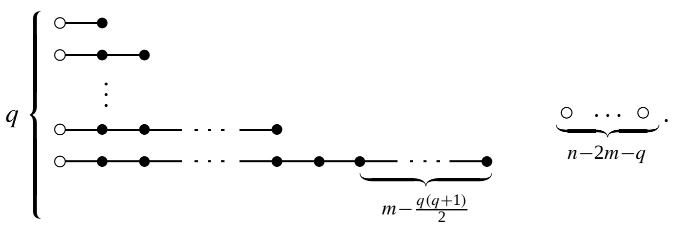
We claim that $l(x) \leqslant \mu_y$.

Notice now that the algorithm used to obtain $l(x)$ may be thought of as a way to construct $y \in X$ satisfying (i) and (ii) of (3.6.1) and such that the sum $\sum_{C \in \mathfrak{A}_y} |C|$ is minimized while the number of elements of $P(x) \cup N(x)$ "thrown out" is maximized, i.e., the sum $|\mathfrak{B}_y \cap N(x)| + (1/2)|\mathfrak{B}_y \cap \mathcal{C}_x^o|$ is maximized, and hence $l(x) \leqslant \mu_y$, for any $y \in X$ satisfying (i) and (ii) of (3.6.1). For example, notice that (3.6.1) shows that after a suitable pairing of the components in $\mathcal{C}_x^o$ all the vertices of components in $U(x)$ must be in $F(y)$. Thus if the total number of edges of $U(x)$ is larger than the number of elements in $P(x) \cup N(x)$, then it is clear that $l(x) \leqslant \mu_y$. This completes the proof of Proposition 3.6. □

We close this section with an application of Proposition 3.6.

Theorem 3.7. *Let* $m \geqslant 3$. *If* $n \geqslant 4m + [(-1 + \sqrt{1 + 8m}\,)/2]$, *then* $\operatorname{Diam}\mathcal{C}(G, X) = 2$ *and if* $2m + 3 \leqslant n \leqslant 4m + [(-1 + \sqrt{1 + 8m}\,)/2] - 1$, *then* $\operatorname{Diam}\mathcal{C}(G, X) = 3$.

Proof. Fix m and let $x \in X$ be such that $\mathcal{G}_x$ has the following form, where q is the largest integer such that $q(q+1)/2 \leqslant m$,



An easy calculation shows that

$$q = \left[\frac{-1 + \sqrt{1 + 8m}}{2}\right].$$

Since all the chains are in $U(x)$ we see that $l(x) = m$ and $b(x) = 0$. So by Proposition 3.6, if $\mathrm{d}(a, x) = 2$, then $2m \leqslant w(x) = n - 2m - q$. So $n \geqslant 4m + q$. This implies that for $n \leqslant 4m + q - 1$ there exist $x \in X$ for which $\mathrm{d}(a, x) \geqslant 3$ and so $\mathrm{Diam}\,\mathcal{C}(G, X) \geqslant 3$. So if $4m + q - 1 \geqslant n \geqslant 2m + 3$, then $\mathrm{Diam}\,\mathcal{C}(G, X) = 3$ by Theorem 3.3.

Now let $n \geqslant 4m + q$, with q as defined above, and suppose there exists $z \in X$ such that $\mathrm{d}(a, z) \geqslant 3$. Let $t(z)$ be the number of connected components of $\mathcal{G}_z$ with two white vertices. In addition, let $p(z)$ be the number of double chains in $P(z)$ which have been cancelled (as whole double chains) when the algorithm for calculating $l(z)$ terminates. Let $S(z)$ be one of $U(z)$, $R(z)$, $P(z)$, and $P(z) \cup N(z)$. Then $|S(z)|$ will denote the total number of edges in $S(z)$. Clearly, $p(z) \leqslant |P(z)|$. We will show that $l(z) \leqslant m - t(z) - p(z)$. Since each connected component of $\mathcal{G}_z$ with two white vertices contains at least one edge, $t(z)$ cannot exceed the total number of edges occurring in such components. Now $R(z)$ contains all components of $\mathcal{G}_z$ with two white vertices, along with all cycles of black vertices in $\mathcal{G}_z$. So clearly $t(z) \leqslant |R(z)|$. Consider the algorithm for finding $l(z)$. If this stops at Step 3, then $l(z) \leqslant |U(z)| \leqslant m - |R(z)| - |P(z)| \leqslant m - t(z) - p(z)$. Now suppose the algorithm continues to Steps 4 and 5. If case (b) of Step 5 occurs then $l(z)$ is simply the number of edges remaining in $P(z) \cup N(z)$ when the algorithm terminates, so $l(z) \leqslant |P(z) \cup N(z)| - p(z)$. If case (a) occurs then either $l(z) = 0$ or, if an edge of an element C of $R(z)$ was removed in the last step, $l(z)$ is the number of edges left in C. But for this to happen, C must have had a minimal number of edges in the remaining members of $P(z) \cup N(z) \cup R(z)$, and at the same time a component or double chain C' with a maximal number of edges would have been cancelled from $P(z) \cup N(z)$. Now $P(z)$ contains at least $p(z)$ double chains and thus $(P(z) \cup N(z)) \setminus \{C'\}$ contains at least $p(z) - 1$ components or double chains (each of which must contain at least one edge). Therefore,

$$\bigl|P(z) \cup N(z)\bigr| \geqslant p(z) - 1 + |C'|, \quad \text{hence} \quad |C'| - 1 \leqslant \bigl|P(z) \cup N(z)\bigr| - p(z),$$

where $|C'|$ denotes the number of edges in C'. We have that $l(z)$ is the number of edges left in C after at least one edge has been removed. But C had a minimal number of edges in the remaining members of $P(z) \cup N(z) \cup R(z)$, and so $l(z) \leqslant |C'| - 1 \leqslant |P(z) \cup N(z)| - p(z)$. Therefore, when the algorithm stops at Step 5, in either case, we have

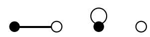
$$l(z) \leqslant \bigl|P(z) \cup N(z)\bigr| - p(z) \leqslant \bigl(m - \bigl|R(z)\bigr|\bigr) - p(z) \leqslant m - t(z) - p(z).$$

So in every case $l(z) \leqslant m - t(z) - p(z)$. Rewriting this we get

$$l(z) + t(z) + p(z) \leqslant m.$$

By Proposition 3.6, $l(z) > b(z) + w(z)/2$. Therefore $w(z) + 2(t(z) + p(z)) < 2(l(z) + t(z) + p(z)) \leqslant 2m$. Since $n \geqslant 4m + q$ there are at least $2m + q$ white vertices in $\mathcal{G}_z$. Now $w(z)$ is the number of white vertices with valency 0 in $\mathcal{G}_z$, $2t(z)$ is the number of white vertices which occur in components containing 2 white vertices, and $2p(z)$ is the number of white vertices occurring in the double chains which have been cancelled. Given that $w(z) + 2t(z) + 2p(z) < 2m$ and that $n \geqslant 4m + q$, at least $q + 1$ white vertices are in chains of $\mathcal{C}_z^o$. Furthermore, these chains are not contained in double chains of $P(z)$ which have been cancelled. Let $V(z)$ denote the collection of chains in $\mathcal{C}_z^o$ which are not contained in double chains of $P(z)$ that have been cancelled. If distinct chains in $V(z)$ have distinct length than the total number of black vertices involved in chains of $V(z)$ would be at least $(q+1)(q+2)/2 > m$, a contradiction. By the algorithm defining $l(z)$, there is at most one double chain which has not been cancelled. Thus, there is a subset $V'(z) \subset V(z)$ of size $q-1$ such that distinct chains in $V'(z)$ have distinct length and two additional chains $C_1, C_2 \in V(z) - V'(z)$ having the same length. Note that $V'(z) \subseteq U(z)$ and hence the total number of edges (and black vertices) occurring in components in $U(z)$ is at least $q(q-1)/2$. Furthermore, since the algorithm defining $l(z)$ ended with a double chain that had not been cancelled, all the edges in chains of $U(z)$ were cancelled. It follows that

Table 2
The G_a-orbits in the case $n=6$, $m=2$

	x-graph	Size		x-graph	Size
$\Delta_1^1(a)$		2	$\Delta_1^2(a)$		2
$\Delta_2^1(a)$		4	$\Delta_2^2(a)$		4
$\Delta_3^1(a)$		8	$\Delta_3^2(a)$		8
$\Delta_4^1(a)$		16			

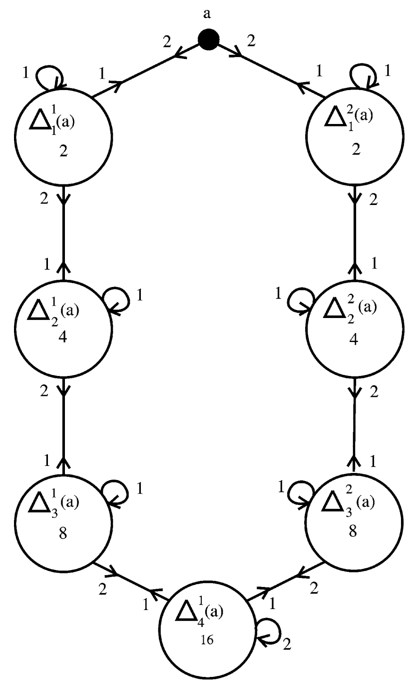


Fig. 1. $n=6$, $m=2$.

there were at least $q(q-1)/2$ components or double chain in $P(z) \cup N(z)$ that had been cancelled. Since each such component or double chain contains at least two black vertices we have at least $q(q-1)$ black vertices in components or double chains in $P(z) \cup N(z)$ that had been cancelled. Also in $C_1 \cup C_2$ there are at least 2 black vertices. Finally, as we

Table 3
The G_a-orbits in the case $n=8$, $m=3$

	x-graph	Size		x-graph	Size		x-graph	Size
$\Delta_1^1(a)$		3	$\Delta_1^2(a)$		6	$\Delta_1^3(a)$		6
$\Delta_2^1(a)$		12	$\Delta_2^2(a)$		24	$\Delta_2^3(a)$		24
$\Delta_2^4(a)$		12	$\Delta_2^5(a)$		24	$\Delta_2^6(a)$		8
$\Delta_3^1(a)$		12	$\Delta_3^2(a)$		48	$\Delta_3^3(a)$		96
$\Delta_3^4(a)$		24	$\Delta_3^5(a)$		96	$\Delta_4^1(a)$		24

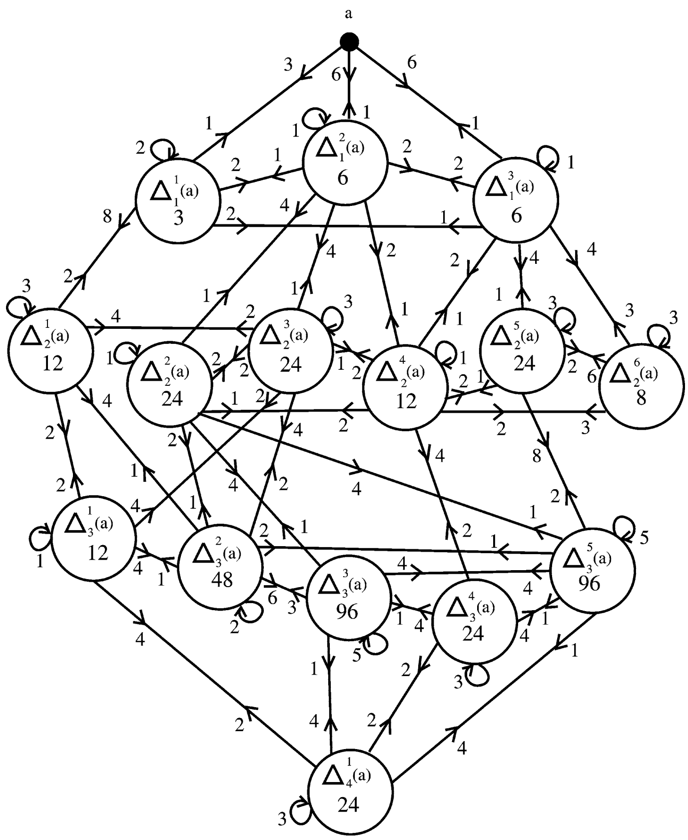


Fig. 2. $n=8$, $m=3$.

mentioned above, the total number of black vertices occurring in a components in $U(z)$ is at least $q(q-1)/2$. It follows that the total number of black vertices is at least

$$q(q-1)+2+\frac{q(q-1)}{2}=\frac{3q(q-1)}{2}+2.$$

However, by the definition of q we must have

$$\frac{3q(q-1)}{2}+2<\frac{(q+1)(q+2)}{2},$$

Table 4
The G_a-orbits in the case $n=10$, $m=4$

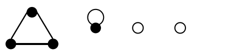
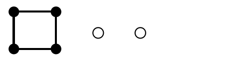
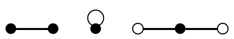
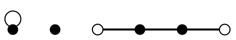
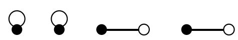
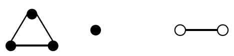
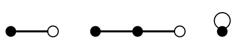
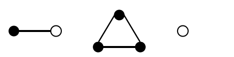
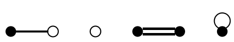
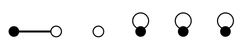
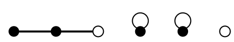
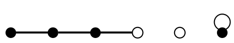
	x-graph	Size		x-graph	Size
$\Delta_1^1(a)$		12	$\Delta_1^2(a)$		12
$\Delta_1^3(a)$		4	$\Delta_1^4(a)$		24
$\Delta_2^1(a)$		32	$\Delta_2^2(a)$		48
$\Delta_2^3(a)$		48	$\Delta_2^4(a)$		24
$\Delta_2^5(a)$		96	$\Delta_2^6(a)$		192
$\Delta_2^7(a)$		96	$\Delta_2^8(a)$		48
$\Delta_2^9(a)$		96	$\Delta_2^{10}(a)$		24
$\Delta_2^{11}(a)$		192	$\Delta_2^{12}(a)$		48
$\Delta_2^{13}(a)$		32	$\Delta_3^1(a)$		768
$\Delta_3^2(a)$		384	$\Delta_3^3(a)$		96
$\Delta_3^4(a)$		384	$\Delta_3^5(a)$		128
$\Delta_3^6(a)$		96	$\Delta_3^7(a)$		16
$\Delta_3^8(a)$		192	$\Delta_3^9(a)$		96
$\Delta_3^{10}(a)$		384	$\Delta_3^{11}(a)$		192
$\Delta_3^{12}(a)$		192	$\Delta_4^1(a)$		768

Table 5
$n = 10$, $m = 4$ (For $n = 10$, a ‘picture’ would not be easy on the eye, so we give the data in matrix form)

	$\{a\}$	Δ_1^1	Δ_1^2	Δ_1^3	Δ_1^4	Δ_2^1	Δ_2^2	Δ_2^3	Δ_2^4	Δ_2^5	Δ_2^6	Δ_2^7	Δ_2^8	Δ_2^9	Δ_2^{10}	Δ_2^{11}	Δ_2^{12}	Δ_2^{13}	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_3^6	Δ_3^7	Δ_3^8	Δ_3^9	Δ_3^{10}	Δ_3^{11}	Δ_3^{12}	Δ_4^1
$\{a\}$	0	12	12	4	24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	1	3	4	2	4	8	8	4	2	8	0	0	0	0	0	0	0	8	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^2	1	4	7	0	0	0	12	8	4	0	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^3	1	6	0	3	12	0	0	0	0	0	0	0	12	0	18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^4	1	2	0	2	3	4	0	0	2	4	0	4	4	8	2	8	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	3	0	0	3	9	12	0	3	6	12	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	2	3	0	0	8	11	4	0	8	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	1	2	0	0	0	4	5	2	0	8	0	0	0	0	0	2	0	0	8	8	0	0	0	0	0	0	0	0	12	0
Δ_2^4	0	1	2	0	2	4	0	4	1	4	0	8	4	4	0	0	2	4	0	0	4	0	0	0	0	0	0	0	8	0	0
Δ_2^5	0	1	0	0	1	2	4	0	1	3	8	2	0	0	0	4	0	2	8	0	0	8	0	0	0	0	0	0	8	0	0
Δ_2^6	0	0	1	0	0	2	4	2	0	4	9	0	0	0	0	0	0	0	16	8	0	4	0	0	0	0	0	0	0	2	0
Δ_2^7	0	0	0	0	1	0	0	0	2	2	0	3	2	0	0	4	0	0	8	0	0	4	4	2	0	4	0	4	4	8	0
Δ_2^8	0	0	0	1	2	0	0	0	2	0	0	4	7	0	2	12	0	0	0	0	0	8	0	4	2	0	0	8	0	0	0
Δ_2^9	0	0	0	0	2	0	0	0	1	0	0	0	0	11	4	12	2	2	0	0	0	4	0	0	2	0	4	8	0	0	0
Δ_2^{10}	0	0	0	3	2	0	0	0	0	0	0	0	4	16	5	8	0	0	0	0	0	0	0	4	2	0	8	0	0	0	0
Δ_2^{11}	0	0	0	0	1	0	0	0	0	2	0	2	3	6	1	11	2	2	0	0	0	8	0	2	0	0	4	8	0	0	0
Δ_2^{12}	0	0	0	0	2	0	0	2	1	0	0	0	0	4	0	8	1	0	0	8	4	8	0	0	0	4	2	8	0	0	0
Δ_2^{13}	0	3	0	0	3	4	0	0	3	6	0	0	0	6	0	12	0	3	0	0	0	12	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	1	4	1	0	0	0	0	0	0	14	5	2	5	1	0	0	2	0	3	3	3	8
Δ_3^2	0	0	0	0	0	0	0	1	0	0	4	0	0	0	0	0	1	0	10	9	1	6	2	0	0	2	0	2	4	2	8
Δ_3^3	0	0	0	0	0	0	0	4	1	0	0	0	0	0	0	0	2	0	16	4	3	0	0	0	0	4	2	0	4	4	8
Δ_3^4	0	0	0	0	0	0	0	0	0	2	2	1	1	1	0	4	1	1	10	6	0	7	0	1	0	0	1	4	0	2	8
Δ_3^5	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	6	6	0	0	7	6	0	0	0	6	6	0	12
Δ_3^6	0	0	0	0	0	0	0	0	0	0	0	2	2	0	1	4	0	0	0	0	0	4	8	3	2	4	2	4	4	4	8
Δ_3^7	0	0	0	0	0	0	0	0	0	0	0	0	6	12	3	0	0	0	0	0	0	0	0	12	1	12	6	0	0	0	0
Δ_3^8	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	1	0	8	4	2	0	0	2	1	9	3	0	0	4	16
Δ_3^9	0	0	0	0	0	0	0	0	0	0	0	0	0	4	2	8	1	0	0	0	2	4	0	2	1	6	2	12	0	0	8
Δ_3^{10}	0	0	0	0	0	0	0	0	0	0	0	1	1	2	0	4	1	0	6	2	0	4	2	1	0	0	3	9	0	0	16
Δ_3^{11}	0	0	0	0	0	0	0	0	1	4	0	2	0	0	0	0	0	0	12	8	2	0	4	2	0	0	0	0	9	0	8
Δ_3^{12}	0	0	0	0	0	0	0	3	0	0	2	4	0	0	0	0	0	0	12	4	2	4	0	2	0	4	0	0	0	7	8
Δ_4^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	4	1	4	2	1	0	4	1	8	2	2	15

and an easy calculation shows that $q = 1$. Since $q = [(-1 + \sqrt{1 + 8m})/2]$, $m \leqslant 2$, which contradicts the hypothesis. So if $n \geqslant 4m + q$, then $\mathrm{d}(a, z) \leqslant 2$ for all $z \in X$. Hence $\mathrm{Diam}\,\mathcal{C}(G, X) = 2$. $\square$

4. The diameter 4 graphs

Here we display the collapsed adjacency graphs for $\mathcal{C}(G, X)$ when $2m + 2 = n \in \{6, 8, 10\}$. We use $\Delta_i^j(a)$ to denote a G_a orbit contained in the ith disc $\Delta_i(a)$. The specific definitions of the $\Delta_i^j(a)$, in terms of x-graphs, are given in Tables 2–4. The required calculations were carried out by hand, except in Table 5, for which we used the computer algebra package MAGMA [3].

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