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On the computing time of the continued fractions method

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ABSTRACT

The maximum computing time of the continued fractions method for polynomial real root isolation is at least quintic in the degree of the input polynomial. This computing time is realized for an infinite sequence of polynomials of increasing degrees, each having the same coefficients. The recursion trees for those polynomials do not depend on the use of root bounds in the continued fractions method. The trees are completely described. The height of each tree is more than half the degree. When the degree exceeds one hundred, more than one third of the nodes along the longest path are associated with primitive polynomials whose low-order and high-order coefficients are large negative integers. The length of the forty-five percent highest order coefficients and of the ten percent lowest order coefficients is at least linear in the degree of the input polynomial multiplied by the level of the node. Hence the time required to compute one node from the previous node using classical methods is at least proportional to the cube of the degree of the input polynomial multiplied by the level of the node. The intervals that the continued fractions method returns are characterized using a matrix factorization algorithm.

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2 As the result of a tragic accident Werner Krandick passed away on March 26, 2012, after this article was accepted for publication. He was a major contributor to real root isolation of univariate polynomials with rational coefficients. He worked on both theoretical complexity and practically efficient algorithms and implementations. This article, reporting joint work with George Collins, provides a lower bound, conclusively resolving some long discusssions on comparisons between the continued fractions method and bisection-based methods. The article exemplifies Werner's work: precise, dedicated and cogent, giving generous credit to related results. He will be greatly missed by the JSC community.

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1. Motivation

Algorithms for polynomial real root isolation are an important part of computer algebra but few lower bounds are known for their maximum computing time functions. One exception is the continued fractions method (CF-method) due to Vincent (1836) and recommended by Uspensky (1948). Collins and Akritas (1976) proved that the maximum computing time of the CF-method is at least exponential in the length of the coefficients of the input polynomial. That lower bound motivated two algorithmic innovations, the bisection method by Collins and Akritas (1976) and the CF-method with root bounds. The computing time of the bisection method has a polynomial upper bound. The CF-method with root bounds was proposed by Akritas (1978, 1980). Sharma (2007, 2008) modified Akritas's method by employing different root bounds in order to obtain a polynomial upper bound for the computing time. To this day no non-trivial lower bounds are known for the maximum computing time functions of the bisection method and the CF-method with root bounds.

We show that, when classical computation is used, the maximum computing time of the CF-method with root bounds dominates n^5 where n is the degree of the input polynomial. Our result applies to Akritas's original method (1978, 1980) and to variants that use other root bounds. Such variants were recently considered by Akritas et al. (2007), by Tsigaridas and Emiris (2008), and by Sharma (2008).

There is no conjecture in the literature that the CF-method with root bounds can require computing times as large as n^5 when classical computation is used. The computing times reported by Akritas (1978, 1980) for Chebyshev polynomials and for polynomials with random roots seem to be dominated by n^4. Also the computing times reported by Tsigaridas and Emiris (2008, Table 1) seem to be dominated by n^4 or perhaps by $n^4 \log n$; Tsigaridas and Emiris consider Laguerre polynomials, Chebyshev polynomials of the first and second kind, Wilkinson polynomials, Mignotte polynomials and also the reducible polynomials

$$(x^n - 2(101x - 1)^2)(x^n - 2((101 + 1/101)x - 1)^2), \tag{1.1}$$

whose computing times seem to be dominated even by n^3. Tsigaridas and Emiris then assert that larger computing times can be obtained using the reducible polynomials

$$(x^n - 2(ax - 1)^2)(x^n - (ax - 1)^2) \tag{1.2}$$

which were introduced by Eigenwillig et al. (2006) in a paper on the bisection method. The reducible polynomials in lines (1.1) and (1.2) could be called double-Mignotte polynomials since their two factors were introduced by Mignotte (1981, 1982, 1995). Sharma (2007, Section 5) conjectures that, for the polynomials in line (1.2), the number of nodes in the recursion trees of the CF-method with root bounds dominates $n \log a$.

In the present paper we consider input polynomials for which the CF-method with root bounds operates identically to the CF-method. We prove for the polynomials

$$A_n(x) = x^n - 2(x^2 - 3x + 1)^2 \tag{1.3}$$

that the tree height of the CF-method is $\lfloor n/2 \rfloor + 2$ if n is different from 6 and 10. Unlike the polynomials in line (1.2), the polynomials A_n are irreducible. We completely describe the recursion trees for those polynomials. We then investigate the coefficients of the polynomials that are associated with the nodes of the recursion trees. This allows us to show that the computing time of the CF-method with input A_n dominates n^5.

Johnson (1991, Theorem 57) claimed to have proved in just a few lines that the maximum computing time of the bisection method by Collins and Akritas (1976) also dominates n^5. But Johnson's original result and also a slightly revised version (1998, Theorem 17) are seriously flawed. Johnson considers the polynomials

$$S_n(x) = x^n - 2(ax - 1)^2 \tag{1.4}$$

where $a \geq 3$ and $n \geq 3$. Johnson argues that either the polynomials $S_n(x)$ or the polynomials $S_n(-x + 1)$ require a computing time that dominates n^5. The polynomials $S_n(x)$, introduced by

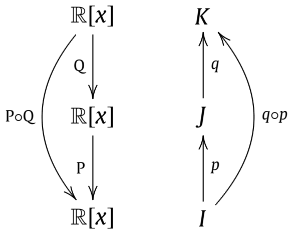

Fig. 2.1. Composition of root matchings.

Mignotte (1981, 1982), have a pair of close real roots, one on either side of $1/a$. The distance between either root and $1/a$ is less than $a^{-n/2-1}$. Johnson concludes that the number of bisections required to separate the two roots dominates n, a fallacy in case $1/a$ is a bisection point. The polynomials $S_n(-x+1)$ have a pair of close real roots, one on either side of $1-1/a$. This solves a perceived number-theoretical problem: If the binary expansion of $1/a$ does not contain sufficiently many digits 1, then the expansion of $1-1/a$ will. Johnson asserts without proof that the bisection method will call itself for polynomials that are dense and have long coefficients. How dense? How many of the coefficients are long? Do their signs matter? There is no proof here, just speculation. Some statements do not make any sense, for instance the assertion that the length of an arbitrary level number in the recursion tree of the bisection method is codominant with the height of the tree. A few more errors can be discerned but much of the presentation is unclear.

Eigenwillig et al. (2006) show that the height of the recursion tree of the bisection method for the double-Mignotte polynomials in line (1.2) dominates $n \log a$. This is the only lower bound those authors prove for the maximum computing time of the bisection method. So it is still an open question whether the maximum computing time of the bisection method dominates n^5. Experimental evidence (Johnson, 1991, 1998; Rouillier and Zimmermann, 2004; Akritas et al., 2006) suggests that Mignotte polynomials with a suitable choice of a perhaps require a computing time that dominates n^5.

In Section 2 we present the CF-method and characterize the intervals that can appear in its output. We also define the universal CF-tree, a notion we use in Section 3, "A Road Map", to construct difficult input polynomials. That section also provides an overview of the remainder of the paper.

2. The CF-method

Our statement of the CF-method uses only the reciprocal transformation and translation by one. But our results carry over to versions of the method that use root bounds and translations by integers greater than one.

2.1. The algorithm

Definition 1. The *translation transformation* transforms a polynomial A into the polynomial $\mathrm{T}(A)(x) = A(x+1)$. The *translation mapping* $t : \mathbb{C} \longrightarrow \mathbb{C}$ is defined by $t(z) = z+1$. The *reciprocal transformation* transforms a polynomial A into the polynomial $\mathrm{R}(A)(x) = x^{\deg(A)}A(1/x)$. The *reciprocal mapping* $r : \mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}$ is defined by $r(z) = 1/z$. Let M be a polynomial transformation and I and J bounded or unbounded intervals and $m : I \to J$ a bijective mapping. We call the pair (M, m) a *root matching* for (I, J) if, for all real polynomials A and all elements $a \in I$, a is a root of $\mathrm{M}(A)$ if and only if $m(a)$ is a root of A.

Note that the pair (T, t) is a root matching for $((0, \infty), (1, \infty))$ and that (R, r) is a root matching for $((1, \infty), (0, 1))$. Root matchings can be composed as shown in Fig. 2.1: If I, J and K are intervals and (P, p) is a root matching for (I, J) and (Q, q) is a root matching for (J, K) then $(\mathrm{P} \circ \mathrm{Q},\ q \circ p)$ is a root matching for (I, K). In particular, $(\mathrm{T} \circ \mathrm{R},\ r \circ t)$ is a root matching for $((0, \infty), (0, 1))$. Thus, if, for any polynomial A and any interval I, $\mathcal{Z}(A, I)$ designates the set of roots of A in I then

$$\mathcal{Z}(A, (0,\infty)) = t(\mathcal{Z}(\mathrm{T}(A), (0,\infty))) \cup \mathcal{Z}(A, \{1\}) \cup (r \circ t)(\mathcal{Z}((\mathrm{T}\circ\mathrm{R})(A), (0,\infty))). \tag{2.1}$$

Eq. (2.1) allows us to isolate the roots of A in $(0,\infty)$ by isolating the roots of $A_1 = \mathrm{T}(A)$ and $A_2 = (\mathrm{T}\circ\mathrm{R})(A)$ in $(0,\infty)$. We will now state the Descartes rule of signs which will serve as a terminating condition.

Definition 2. Let $a = (a_0, \ldots, a_n)$ be a finite sequence of real numbers. The *number of sign variations* in a, $\mathrm{var}(a)$, is the number of pairs (i, j) with $0 \le i < j \le n$ and $a_i a_j < 0$ and $a_{i+1} = \cdots = a_{j-1} = 0$. Let A be the polynomial $a_0 + a_1 x + \cdots + a_n x^n$. The *number of coefficient sign variations* in A, $\mathrm{var}(A)$, is $\mathrm{var}(a)$.

Theorem 3 (*Descartes Rule of Signs*)**.** *For any nonzero real polynomial the number of coefficient sign variations exceeds the number of positive real roots – counting multiplicities – by a nonnegative, even integer.*

The Descartes rule of signs is well known; Krandick and Mehlhorn (2006) provide a proof and historical remarks. Only two special cases are needed for the CF-method: if a polynomial has no coefficient sign variations then it has no positive roots, and if it has exactly one coefficient sign variation then it has exactly one positive root. Otherwise, Eq. (2.1) can be applied where the roots of A_1 and A_2 are isolated recursively as shown in Algorithm 1. Line 17 of the algorithm maps the isolating intervals for A_1 and A_2 back to isolating intervals for A.

Algorithm 1 Continued fractions algorithm (CF-method). The expressions $m(\mathrm{CF}(B))$ in line 17 stand for $\{m(I) \mid I \in \mathrm{CF}(B)\}$.

1: **procedure** CF(A)
2: **Input:** A, a squarefree integral polynomial.
3: **Output:** L, a set of isolating open or one-point positive intervals for the positive roots of A.
4: $v \leftarrow \mathrm{var}(A)$
5: **if** $v = 0$ **then**
6: $L \leftarrow \emptyset$
7: **else if** $v = 1$ **then**
8: $L \leftarrow \{(0,\infty)\}$
9: **else**
10: $A_1 \leftarrow \mathrm{T}(A)$
11: $A_2 \leftarrow (\mathrm{T}\circ\mathrm{R})(A)$
12: **if** $A(1) = 0$ **then**
13: $K \leftarrow \{\{1\}\}$
14: **else**
15: $K \leftarrow \emptyset$
16: **end if**
17: $L \leftarrow t(\mathrm{CF}(A_1)) \cup (r \circ t)(\mathrm{CF}(A_2)) \cup K$
18: **end if**
19: **return**(L)
20: **end procedure**

Definition 4. We call the recursion tree that Algorithm 1 associates with an input polynomial A the *CF-tree* of A. The *root* of the tree is the original invocation of the algorithm for the input polynomial A. The *left child* of each internal node is the call $\mathrm{CF}(A_1)$, the *right child* is the call $\mathrm{CF}(A_2)$. We represent each node of the tree by a string of 1's and 2's. The root of the tree is represented by the empty string ϵ. If a parent is represented by the string s then the left child is represented by $s1$, and the right child by $s2$. In contexts where a string might be misconstrued as an integer we will enclose the string in quotation marks; for example, we may say that node "2" is the parent of node "21". The *level* of node

s is the length of s; the *height* of the tree is the maximum level of any node. We associate each node s with a root matching (M_s, m_s) as follows.

$$\begin{aligned}(\mathrm{M}_\epsilon, m_\epsilon) &= (\mathrm{Id}, \mathrm{id})\\ (\mathrm{M}_{s1}, m_{s1}) &= (\mathrm{T}\circ \mathrm{M}_s, m_s \circ t)\\ (\mathrm{M}_{s2}, m_{s2}) &= ((\mathrm{T}\circ \mathrm{R})\circ \mathrm{M}_s, m_s \circ (r\circ t)).\end{aligned}$$

Furthermore we define

$$A_s = \mathrm{M}_s(A) \quad \text{and} \quad I_s = m_s((0,\infty)).$$

As Algorithm 1 descends from the root of the tree to a node s, it transforms the input polynomial A into the polynomial A_s by successively applying transformations T and T $\circ$ R. The composition of these transformations is the transformation M_s. When the recursive calls return, the interval $(0, \infty)$ is mapped onto the interval I_s by successive application of mappings t and $r \circ t$. The composition of these mappings is the mapping m_s.

Whenever Algorithm 1 terminates, the results are correct; this can be shown by induction on the height of the CF-tree. It is not obvious that Algorithm 1 will always terminate, but Vincent (1836) proved that it will. Alesina and Galuzzi (1998, 1999) provide an insightful discussion of Vincent's result.

In our computing time analysis we will, for given input polynomials A_ϵ, track the real and nonreal roots of the polynomials A_s on the basis of the following theorem.

Theorem 5. *Let A be a nonzero polynomial that does not have any rational root, and let* (M, m) *be a composition of root matchings* (T, t) *and* (R, r). *Then* $\mathrm{M}(A)$ *is a nonzero polynomial that does not have any rational root, and the linear fractional mapping m bijectively maps the roots of* $\mathrm{M}(A)$ *onto the roots of A, preserving the multiplicity of each root, the relation of complex conjugacy, the property of being real and the property of being nonreal.*

Proof. It suffices to consider compositions of length 1. Write A as the product of its linear factors over $\mathbb{C}$, and verify the assertions for each of the two root matchings. □

2.2. The universal CF-tree

We now define a tree that does not depend on any particular input polynomial. The tree will contain all the nodes and all the intervals that might arise in the CF-tree of a given polynomial. Each node will have exactly two children, so the tree will be infinite.

Definition 6. The *universal CF-tree* is defined as follows. The set of nodes is the infinite set $\{1, 2\}^*$ of the strings of 1's and 2's. The empty string, ϵ, is the root of the tree. Each node s has a left child, the node $s1$, and a right child, the node $s2$. The *level* of node s is $|s|$, the *length* of s. Each node s is associated with a root matching (M_s, m_s) using the recursive formulas of Definition 4. Moreover, each node s is associated with the interval $I_s = m_s((0, \infty))$.

One can show that, in the universal CF-tree, every finite, full subtree that contains the root arises as the CF-tree of some input polynomial.

Let s be a node of a CF-tree or of the universal CF-tree. Then the mapping m_s is a composition of translations $t(x)$ and reciprocal mappings $r(x)$. Each of these mappings can be written as a *linear fractional mapping*,

$$x \longmapsto \frac{ax+b}{cx+d}, \quad \text{and be represented by the matrix} \quad \begin{bmatrix} a & b\\ c & d\end{bmatrix}.$$

In the literature, linear fractional mappings are sometimes called *Möbius transformations*, but here we reserve the word "transformations" for operations on polynomials. It is well known that the coefficients of a composition of linear fractional mappings can be obtained by multiplying the corresponding matrices (Knopp, 1952, for example). We use this fact to define a representation of the mapping m_s by a matrix $\mathbf{m}_s$ with coefficients a_s, b_s, c_s, d_s.

Definition 7. Let

$$\mathbf{m}_\epsilon = \begin{bmatrix} a_\epsilon & b_\epsilon \\ c_\epsilon & d_\epsilon \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and, for any $s \in \{1, 2\}^*$,

$$\mathbf{m}_{s1} = \begin{bmatrix} a_{s1} & b_{s1} \\ c_{s1} & d_{s1} \end{bmatrix} = \begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_s & a_s + b_s \\ c_s & c_s + d_s \end{bmatrix}$$

and

$$\mathbf{m}_{s2} = \begin{bmatrix} a_{s2} & b_{s2} \\ c_{s2} & d_{s2} \end{bmatrix} = \begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} b_s & a_s + b_s \\ d_s & c_s + d_s \end{bmatrix}.$$

Theorem 8. *The matrix*

$$\mathbf{m} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a non-empty product of matrices

$$\mathbf{m}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{m}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

if and only if a, b, c, d *are integers such that* $0 \le a \le b$ *and* $0 \le c \le d$ *and* $|ad - bc| = 1$*; in that case, the factorization of* $\mathbf{m}$ *into matrices* $\mathbf{m}_1$ *and* $\mathbf{m}_2$ *is unique, and* $ad - bc = (-1)^h$ *where* h *is the number of times the matrix* $\mathbf{m}_2$ *occurs as a factor in the product.*

Proof. The "only if"–part and the assertion concerning h can be shown using the multiplicativity of the determinant function and complete induction on the length of the total number of matrices in the product. Uniqueness: Let $S = (\mathbf{s}_1, \dots, \mathbf{s}_k)$ and $T = (\mathbf{t}_1, \dots, \mathbf{t}_l)$ be sequences of matrices $\mathbf{m}_1$ and $\mathbf{m}_2$ such that $S \neq T$. Each sequence defines a path from the root of the universal CF-tree. Let s and t be the respective end nodes. Then $\mathbf{m}_s = \mathbf{s}_1 \cdots \mathbf{s}_k$, $\mathbf{m}_t = \mathbf{t}_1 \cdots \mathbf{t}_l$, and $s \neq t$. It is easy to see that $I_s \neq I_t$, but that means $m_s((0, \infty)) \neq m_t((0, \infty))$ and, in particular, $m_s \neq m_t$. But then $\mathbf{m}_s \neq \mathbf{m}_t$.

To show the "if"-part, let

$$a, b, c, d \text{ integers and } 0 \le a \le b \text{ and } 0 \le c \le d \text{ and } |ad - bc| = 1. \tag{2.2}$$

We will show that $\mathbf{m}$ can be factored into a product of matrices $\mathbf{m}_1$ and $\mathbf{m}_2$. It suffices to show that $\mathbf{m}$ can be factored into a product of matrices

$$\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & f \end{bmatrix} \tag{2.3}$$

where f takes positive integer values. Indeed, the first matrix equals $\mathbf{m}_1^f$ and the second equals $\mathbf{m}_2 \cdot \mathbf{m}_1^{f-1}$. We will show that Algorithm 2 performs the latter factorization. We start by showing that, whenever the loop condition in line 6 of the algorithm is tested,

$$\text{line (2.2) holds and } \mathbf{m} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \mathbf{s}_1 \cdots \mathbf{s}_k \text{ where } S = (\mathbf{s}_1, \dots, \mathbf{s}_k). \tag{2.4}$$

Line (2.4) clearly holds when the loop condition in line 6 of the algorithm is evaluated for the first time since, at that point,

$$\mathbf{m} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad S = ().$$

We now assume that line (2.4) holds, and that $a \neq 0$ and $c \neq 0$. Then $1 \le a \le b$ and $1 \le c \le d$, the quotients in the assignment $e \leftarrow \min(\lfloor b/a \rfloor, \lfloor d/c \rfloor)$ on line 7 of the algorithm are well-defined,

Algorithm 2 The algorithm serves to characterize the intervals in the universal CF-tree. The proof of Theorem 8 contains a correctness proof.

1: **procedure** FACTOR($\mathbf{m}$)

2: **Input:** $\mathbf{m} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are integers, $0 \leq a \leq b$ and $0 \leq c \leq d$ and $|ad - bc| = 1$.

3: **Output:** $S = (\mathbf{s}_1, \ldots, \mathbf{s}_k)$, a list of matrices $\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & f \end{bmatrix}$ where f takes positive integer values, such that $\mathbf{m} = \mathbf{s}_1 \cdots \mathbf{s}_k$.

4: $S \leftarrow ()$

5: $(a, b, c, d) \leftarrow$ the coefficients of $\mathbf{m}$

6: **while** $a \neq 0$ and $c \neq 0$ **do**

7: $e \leftarrow \min(\lfloor b/a \rfloor, \lfloor d/c \rfloor)$

8: $b' \leftarrow b - ae$

9: $d' \leftarrow d - ce$

10: insert $\begin{bmatrix} 0 & 1 \\ 1 & e \end{bmatrix}$ at the head of S

11: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \leftarrow \begin{bmatrix} b' & a \\ d' & c \end{bmatrix}$

12: **end while**

13: **if** $c = 0$ **then**

14: insert $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ at the head of S

15: **else**

16: insert $\begin{bmatrix} 0 & 1 \\ 1 & d \end{bmatrix}$ at the head of S

17: **end if**

18: **return**(S)

19: **end procedure**

and $e \geq 1$. Moreover, the integers $b' = b - ae$ and $d' = d - ce$ in lines 8 and 9 of the algorithm are nonnegative. Also, $ad' - b'c = a(d - ce) - (b - ae)c = ad - bc$ so that

$$|ad' - b'c| = 1. \tag{2.5}$$

By the definition of e we have $b' < a$ or $d' < c$. In case $b' < a$ we have $0 \leq b' \leq a - 1$, so, using Eq. (2.5),

$$d' \leq \frac{1 + b'c}{a} \leq \frac{1 + (a-1)c}{a} = c + \frac{1 - c}{a} \leq c.$$

The case $d' < c$ is analogous; we have $0 \leq d' \leq c - 1$ and hence, again using Eq. (2.5),

$$b' \leq \frac{1 + ad'}{c} \leq \frac{1 + a(c-1)}{c} = a + \frac{1 - a}{c} \leq a.$$

Combining the two cases, we have both $0 \leq b' < a$ and $0 \leq d' \leq c$ or both $0 \leq d' < c$ and $0 \leq b' \leq a$. Furthermore,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b - ae & a \\ d - ce & c \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & e \end{bmatrix} = \begin{bmatrix} b' & a \\ d' & c \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & e \end{bmatrix}$$

so that line (2.4) holds when line 6 of the algorithm is executed again. This proves that line (2.4) is a loop invariant.

The loop condition will eventually become false. Indeed, we have seen that the loop body computes new values for a and c that are each at most as large as the corresponding old value, and such that at least one of the new values is strictly less than the corresponding old value. But since a and c are nonnegative, one of a and c will eventually become 0. Then line 13 of the algorithm will be executed. At that point, $a = 0$ or $c = 0$, and the loop invariant, line (2.4), holds.

If $c = 0$ then $|ad| = 1$ so $a = d = 1$. Therefore, if $S = (\mathbf{s}_1, \ldots, \mathbf{s}_k)$ then

$$\mathbf{m} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \mathbf{s}_1 \cdots \mathbf{s}_k = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \cdot \mathbf{s}_1 \cdots \mathbf{s}_k.$$

Note that $b \geq 1$ since $1 = a \leq b$. If, however, $c \neq 0$ when line 13 is reached then $a = 0$, so $|bc| = 1$ and hence $b = c = 1$. Therefore, if $S = (\mathbf{s}_1, \ldots, \mathbf{s}_k)$ then

$$\mathbf{m} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \mathbf{s}_1 \cdots \mathbf{s}_k = \begin{bmatrix} 0 & 1 \\ 1 & d \end{bmatrix} \cdot \mathbf{s}_1 \cdots \mathbf{s}_k.$$

Note that $d \geq 1$ since $1 = c \leq d$. So, when line 18 of the algorithm is executed and $S = (\mathbf{s}_1, \ldots, \mathbf{s}_k)$ then $\mathbf{m} = \mathbf{s}_1 \cdots \mathbf{s}_k$. Since all the elements that were inserted into S are of the form described in line (2.3), the proof is complete. □

We can now characterize the intervals in the universal CF-tree.

Theorem 9. *An interval I occurs in the universal CF-tree if and only if $I = (b, \infty)$ where b is a nonnegative integer or $I = (a/c, b/d)$ where a, b, c, d are integers such that $ad - bc = -1$ and either both $0 \leq a \leq b$ and $1 \leq c \leq d$ or both $a \geq b \geq 0$ and $c \geq d \geq 1$.*

Proof. An interval I occurs in the universal CF-tree if and only if $I = m((0, \infty))$ where $m(x) = (ax + b)/(cx + d)$ is a mapping represented by a matrix

$$\mathbf{m} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

that occurs in the tree. By Theorem 8, the matrix $\mathbf{m}$ occurs in the tree if and only if $\mathbf{m}$ is the identity matrix or a, b, c, d are integers such that $0 \leq a \leq b$ and $0 \leq c \leq d$ and $|ad - bc| = 1$. In case $c = 0$ both a and d must be equal to 1 so that $m(x) = x + b$ and, hence, $I = m((0, \infty)) = (b, \infty)$. In case $c > 0$ and $ad - bc = -1$ we have $d > 0$ and $a/c - b/d = -1/(cd) < 0$ so that $I = (a/c, b/d)$. In case $c > 0$ and $ad - bc = 1$ we have $d > 0$ and $a/c - b/d = 1/(cd) > 0$ so that $I = (b/d, a/c)$. Letting $(a', b', c', d') = (b, a, d, c)$ we see that $I = (a'/c', b'/d')$ and $a'd' - b'c' = bc - ad = -1$ and $a' \geq b' \geq 0$ and $c' \geq d' \geq 1$. □

Remark 10. Theorem 9 can be used to show that every nonnegative rational number occurs as an interval endpoint in the universal CF-tree. One can further construct a level-preserving permutation σ on the set $\{1, 2\}^*$ of nodes s such that the function $s \mapsto m_{\sigma(s)}(1)$ defines the Stern–Brocot tree described by Graham et al. (1994).

Lemma 11. *The matrix coefficients a_s, b_s, c_s, d_s have the following properties.*

1. *If $s \in \{1\}^*$ then*

$$\mathbf{m}_s = \begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix} = \begin{bmatrix} 1 & |s| \\ 0 & 1 \end{bmatrix}.$$

2. *If $s \notin \{1\}^*$ then*

$$\mathbf{m}_s = \begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix}$$

and a_s, b_s, c_s, d_s are integers and $0 \leq a_s \leq b_s$ and $1 \leq c_s \leq d_s$ and $a_s d_s - b_s c_s = (-1)^h$ where h is the number of times 2 occurs in s.

Proof. Assertion (1): Use induction on $|s|$, the length of s, starting with $|s| = 0$. Assertion (2): By Theorem 8, $0 \leq a_s \leq b_s$ and $0 \leq c_s \leq d_s$ and $a_s d_s - b_s c_s = (-1)^h$ for all $s \in \{1, 2\}^*$, so it suffices to show that $c_s > 0$ for all $s \notin \{1\}^*$. Note first that $d_s > 0$ for any $s \in \{1, 2\}^*$. Indeed, if $d_s = 0$ then $c_s = 0$ and hence $|a_s d_s - c_s b_s| = 0$, a contradiction. So, for any $s \in \{1, 2\}^*$, $c_{s2} = d_s > 0$. Also note that, for all $s \in \{1, 2\}^*$ and all $v \in \{1\}^*$, $c_{sv} = c_s$ by induction on $|v|$. Now let $s \notin \{1\}^*$. Then $s = u2v$ for some $u \in \{1, 2\}^*$ and $v \in \{1\}^*$, so $c_s = c_{u2v} = c_{u2} > 0$. □

Theorem 12. *The intervals in the universal CF-tree have the following properties.*

1. *If* $s \in \{1\}^*$ *then* I_s *is an unbounded interval,*

$$I_s = (|s|, \infty).$$

2. *If* $s \notin \{1\}^*$ *then* I_s *is a bounded interval with nonnegative rational endpoints. If the number of occurrences of* 2 *in* s *is even then*

$$I_s = \left(\frac{b_s}{d_s}, \frac{a_s}{c_s}\right).$$

If the number of occurrences of 2 *in* s *is odd then*

$$I_s = \left(\frac{a_s}{c_s}, \frac{b_s}{d_s}\right).$$

In either case, the width of I_s *is* $1/(c_s d_s)$, *and* $0 \le a_s \le b_s$, $1 \le c_s \le d_s$, $\gcd(a_s, c_s) = 1$, $\gcd(b_s, d_s) = 1$.

Proof. Use Lemma 11 and that $I_s = m_s((0, \infty))$ for all $s \in \{1, 2\}^*$. Note in particular that $|a_s d_s - c_s b_s| = 1$ implies $\gcd(a_s, c_s) = 1$ and $\gcd(b_s, d_s) = 1$. □

3. A road map

We explain our construction of difficult input polynomials by comparing it to related work in the literature, and we provide an outline for the remainder of the paper.

3.1. Constructing difficult polynomials

The universal CF-tree can be used to construct difficult input polynomials. Indeed, one can show, using Definition 7 and Theorem 12, that the two widest intervals < 1 at level h, $h > 0$, are the intervals $(0, 1/h)$ and $(1 - 1/h, 1)$. Let a be an integer, $a > 2$. Then the latter interval contains the roots $1 - 1/a$ and $1 - 1/(a+1)$ of the quadratic polynomial $(ax - (a-1))((a+1)x - a)$ if and only if $h < a$. Thus, the height of the CF-tree of that polynomial is at least a. In particular, the height is not dominated by any polynomial function of the maximum coefficient length. This observation is due to Collins and Akritas (1976) who also assert that one can construct a similar example for every degree ≥ 2. Note that, for $h \geq 2$, the intervals $(1 - 1/h, 1)$ appear in the universal CF-tree along the leftmost path that starts at node 22; indeed, $(1 - 1/h, 1) = I_s$ where $|s| = h$ and $s = 221\ldots1$.

Akritas (1978) proposes to jump over leftmost paths in CF-trees by computing root bounds. Let s be a node in the CF-tree of some input polynomial A_ϵ, and let the integer h be a nonnegative lower bound for the positive roots of the polynomial A_s. Then the roots of A_s in the interval $(0, \infty)$ are the roots of A_s in the interval (h, ∞), and those roots are the roots of $\mathrm{T}^h(A_s)$ in the interval $(0, \infty)$, translated by h. In the CF-tree, the polynomial $\mathrm{T}^h(A_s)$ appears on level $|s| + h$, at node $s1\ldots1$, and its computation from A_s requires h translations by 1. Akritas proposes to perform a single translation of A_s by h instead to obtain the polynomial $A_{s1\ldots1} = \mathrm{T}^h(A_s)$, effectively jumping from node s to node $s1\ldots1$.

We will construct input polynomials whose CF-trees each contain a long rightmost path; this will preclude the application of Akritas's idea. Each rightmost path will be an initial segment of the infinite rightmost path S that starts at node 21. Let $s \in S$ be a node on that path. Then $s = 21\tau$ for some string $\tau \in \{2\}^*$, and

$$I_s = ((r \circ t) \circ t \circ (r \circ t)^{|\tau|})((0, \infty)).$$

The intersection of the intervals I_s, $s \in S$, consists of the point $\alpha = ((r \circ t) \circ t)(\xi)$ where ξ is the positive fixed point of the linear fractional mapping $r \circ t$. We have $\xi = (\sqrt{5} - 1)/2$ and, hence, $\alpha = (3 - \sqrt{5})/2$. One can show, using Definition 7 and Theorem 12, that the intervals containing α or ξ are the intervals of smallest width among the intervals on the same level of the universal CF-tree. If an input polynomial has two real roots close to α, the CF-tree will contain an initial segment of the

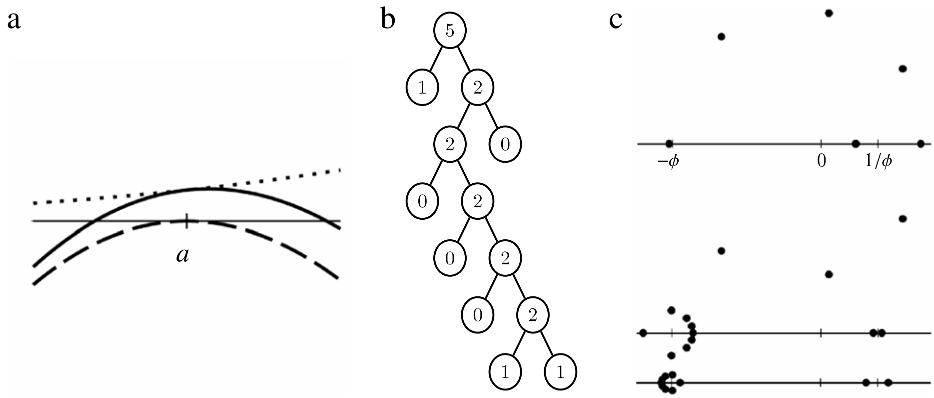

Fig. 3.1. (a) Applying Mignotte's construction. The diagram shows the graphs of x^5 (dotted), $-2(x^2 - 3x + 1)^2$ (dashed) and $A_5(x) = x^5 - 2(x^2 - 3x + 1)^2$ (solid) in the interval $(a - 0.04, a + 0.04) \subset (0, 1)$ where $a = (3 - \sqrt{5})/2$; the horizontal line represents the x-axis. (b) The CF-tree of the polynomials A_8, A_9 and A_{10}; for each node, the number of sign variations is shown. (c) The roots of – from top to bottom – $B_0 = A_{10}$, B_3 and B_4. The horizontal lines represent the real axis, ϕ is the golden ratio. The two close real roots, indistinguishable for B_0, are repelled from $1/\phi$ in the diagrams for B_3 and B_4 while the other roots are attracted to $-\phi$.

path S. We obtain such input polynomials by exploiting an idea of Mignotte (1981, 1982) as shown in Fig. 3.1(a).

Mignotte subtracts from x^n, $n \geq 3$, the square of a linear polynomial that has a root in the interval $(0, 1)$. Let the linear polynomial be $bx-1$ where b is an integer, $b \geq 2$. Then one obtains the polynomial $x^n - (bx - 1)^2$ which has two close real roots, one on either side of $1/b$. As n increases, the distance between the roots approaches 0. Finally, Mignotte inserts the factor 2, yielding $x^n - 2(bx - 1)^2$, so that Eisenstein's irreducibility criterion becomes applicable. But these polynomials are not difficult for the CF-method. Indeed, by the first paragraph of this section, the two close roots are elements of disjoint intervals on level b of the CF-tree, regardless of the value of n. Moreover, the node with the isolating interval $(0, 1/b)$ can be reached from node 2 using a leftmost path of length $b - 1$.

We modify Mignotte's construction by subtracting from x^n, $n \geq 5$, the square of a quadratic polynomial with the root $a \in (0, 1)$. The quadratic polynomial is easily determined by noting that $a = (r \circ t)(\phi)$ where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. Since ϕ is a root of the polynomial $x^2 - x - 1$, the number a is a root of the polynomial $(\mathrm{T} \circ \mathrm{R})^{-1}(x^2 - x - 1) = x^2 - 3x + 1$. So we will consider the polynomials $x^n - 2(x^2 - 3x + 1)^2$ for $n \geq 5$.

3.2. *Proof strategy*

We now outline our plan for proving Theorem 87. We start with some notation. Let $B(x) = x^2 - 3x + 1$ and $A_n(x) = x^n - 2B(x)^2$ with $n \geq 5$. Let $\phi = (1 + \sqrt{5})/2$ and $a = 1/\phi^2 = (3 - \sqrt{5})/2$. Then a is a root of $B(x)$. We will often use approximate values of ϕ and a, which are $\phi = 1.6180339887\ldots$ and $a = 0.3819660113\ldots$. We will also frequently use the equation $\phi^2 = \phi + 1$. Furthermore we will use the Fibonacci numbers F_k, which are inductively defined for $k \geq 0$ by $F_0 = 0$, $F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$. On occasion we will extend the definition to all $k < 0$ by the same equation, used in the form $F_k = F_{k+2} - F_{k+1}$. We then have $F_{-1} = 1$ and, for all integers k, $F_{-k} = (-1)^{k+1} F_k$. Table 1 lists more symbols that we will use.

The first half of the table lists symbols used in Sections 4 and 5 where we determine the CF-trees of the polynomials A_n.

Section 4 We show that A_n has two real roots that are very close to a.

Section 5.1 We describe the structure of the CF-tree of A_n in terms of the tree height. This involves an explicit determination of $\mathrm{var}(A_s)$ for the nodes $s = 1$ and $s = 2$; in each case, $\mathrm{var}(A_s)$ does not depend on n. For the other nodes s of the tree we use a property called "subadditivity" to

Table 1
Symbols in order of their appearance.

$B(x) = x^2 - 3x + 1$.
$A_n(x) = x^n - 2B(x)^2 = x^n - 2x^4 + 12x^3 - 22x^2 + 12x - 2, \quad n \geq 5$.
$\phi = (1 + \sqrt{5})/2 = 1.6180339887\ldots$, the golden ratio, $\phi^2 = \phi + 1$.
$a = 1/\phi^2 = (3 - \sqrt{5})/2 = 0.3819660113\ldots$, $B(a) = 0$.
a_1, a_2 the roots of A_n in $(0, a)$ and $(a, 1)$, respectively.
(M_k, m_k) root matching at node $s = 21\tau$ where $\tau \in \{2\}^*$, $\|s\| = k$ and $k \geq 2$.
$M_k = (\mathrm{T} \circ \mathrm{R})^{k-2} \circ \mathrm{T} \circ (\mathrm{T} \circ \mathrm{R})$, $B_k = M_k(A_n)$.
$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ for any integer n, the Fibonacci numbers.
$m_k(x) = \frac{F_{k-2}x + F_{k-1}}{F_k x + F_{k+1}}$, $m_k^{-1}(x) = \frac{F_{k+1}x - F_{k-1}}{-F_k x + F_{k-2}}$, $m_{k+1}^{-1} = (t^{-1} \circ r) \circ m_k^{-1}$ for $k \geq 2$.
$\mathcal{C}_a = \left\{z \in \mathbb{C} : \|z - 1/\phi^2\| < 1/\phi^2\right\}$ contains a_1, a_2 and no other root of A_n.
$P_k(x) = \prod(x - m_k^{-1}(\alpha))$ where α traverses the roots of A_n except a_1 and a_2
$P_k(x) = x^{n-2} + p_{k,n-3}x^{n-3} + \cdots + p_{k,0}$.
$a_1 = (1 + \delta_1)a$, so $\delta_1 < 0$. $a_2 = (1 + \delta_2)a$, so $\delta_2 > 0$.
$b_{k,1} = m_k^{-1}(a_1)$ and $b_{k,2} = m_k^{-1}(a_2)$ (Sections 10 and 11).
$Q_k(x) = (x - b_{k,1})(x - b_{k,2}) = x^2 - c_k x + d_k$.
$R_k(x) = P_k(x)Q_k(x) = x^n + r_{k,n-1}x^{n-1} + \cdots + r_{k,0}$ the monic associate of $B_k(x)$.
$B_k(x) = b_{k,n}x^n + \cdots + b_{k,0}$ (Sections 12 and 13).

reduce the determination of $\mathrm{var}(A_s)$ to determining the number of roots of A_n in the interval I_s. That number depends on the signs of A_n on the endpoints of I_s. Those endpoints are of the form F_{k-2}/F_k. The height of the tree turns out to be one less than the least positive integer k such that $A_n(F_{k-2}/F_k) > 0$.

Section 5.2 We approximate $A_n(F_{k-2}/F_k)$ using a closed formula for the Fibonacci numbers. Since the sign of $F_k - \phi^k/\sqrt{5}$ alternates as k traverses the integers, we obtain one lower bound for even heights of the CF-tree of A_n, and a different lower bound for odd heights. The minimum of those bounds is a lower bound for heights of any parity; that bound dominates n.

Section 5.3 We approximate $A_n(F_{k-2}/F_k)$ more precisely and determine the height exactly. This completes the description of the CF-tree of A_n. The CF-trees for A_8, A_9 and A_{10} are identical; the tree is shown in Fig. 3.1(b).

The symbols in the second half of Table 1 arise in the computing time analysis. Let S be the infinite rightmost path defined in Section 3.1, but augmented by the nodes ϵ and 2 so that S starts at the root of the universal CF-tree. By Section 5.3, any node $s \in S$ is a node of the CF-tree of A_n, $n > 10$, if and only if $|s| \leq \lfloor n/2 \rfloor + 2$. For any nonnegative integer k, let (M_k, m_k) denote the root matching associated with the unique node $s \in S$ on level k. Table 1 gives a recursion formula for the polynomial transformations M_k. For any nonnegative integer k, let $B_k = \mathrm{M}_k(A_n)$.

The CF-method computes

$$B_{k+1} = (\mathrm{T} \circ \mathrm{R})(B_k)$$

for all k, $2 \leq k \leq \lfloor n/2 \rfloor + 1$. If the transformation T is computed by a classical algorithm, we can obtain a lower bound for the cost of the transformation by showing that the high-order coefficients of $\mathrm{R}(B_k)$ are large negative integers. We do that by showing that the low-order coefficients of B_k are large negative integers. We keep track of the coefficients of B_k indirectly, by tracking all the complex roots of the polynomials B_k. Separately, we track the leading coefficient of B_k. We then reconstruct the coefficients of B_k as sums of products of the roots, multiplied by the leading coefficient.

The roots of B_k are images of the roots of A_n under the mapping m_k^{-1}. Indeed, by Section 4, A_n does not have any rational roots. Hence, by Theorem 5, the linear fractional mapping m_k^{-1} bijectively maps the roots of A_n onto the roots of B_k, preserving the multiplicity of each root, the relation of complex conjugacy, the property of being real and the property of being nonreal. We note in particular that $\deg(B_k) = n$.

For $k \geq 2$ we have $m_{k+1}^{-1} = (t^{-1} \circ r) \circ m_k^{-1}$ so that

$$m_k^{-1} = (t^{-1} \circ r)^{k-2} \circ m_2^{-1}.$$

The linear fractional mapping $t^{-1} \circ r$ has the fixed points $1/\phi$ and $-\phi$, and the normal form (Knopp, 1952, for example)

$$\frac{(t^{-1} \circ r)(z) - 1/\phi}{(t^{-1} \circ r)(z) - (-\phi)} = (-\phi^2)\frac{z - 1/\phi}{z - (-\phi)} \quad \text{where } -\phi^2 = e^{2\ln(\phi)+\pi i},$$

so that the mapping is loxodromic. The fixed point $1/\phi$ is repulsive with the local repulsion rate $\phi^2 = 2.618\ldots$ and counterclockwise rotation by π, and the fixed point $-\phi$ is attractive with the local attraction rate $1/\phi^2 = 0.381\ldots$ and clockwise rotation by π. As k increases, the images of the roots of A_n under m_k^{-1} will move away from $1/\phi$ and towards $-\phi$, see Fig. 3.1(c). We will track the images of the two close real roots with respect to $1/\phi$, and the images of the other roots with respect to $-\phi$. It will be convenient to describe the mappings m_k^{-1} using Fibonacci numbers as shown in Table 1. The formulas can be verified using the matrix calculus of Section 2.2.

Section 6 We use Rouché's theorem to show that the two close real roots of A_n are the only roots of A_n inside the circle $\mathcal{C}_a$ with center and radius a.

Section 7 Since the non-close roots of A_n are outside of $\mathcal{C}_a$, their images under $m_2^{-1} = t^{-2} \circ r$ are in the left half-plane. But then their images under any mapping m_k^{-1}, $k \geq 2$, are in the left half-plane. In fact, those images converge to $-\phi$.

Section 8 Let $P_k(x) = p_{k,n-2}x^{n-2} + \cdots + p_{k,0}$ be the monic polynomial whose roots are $m_k^{-1}(\alpha)$ where α is a root of A_n that is different from the two close real roots. Since the $m_k^{-1}(\alpha)$ are close to $-\phi$, we expect $P_k(x)$ to be approximately equal to $(x+\phi)^{n-2}$. We note that A_n is real, so the nonreal roots of A_n occur in complex conjugate pairs. But m_k^{-1} preserves complex conjugacy, so the nonreal roots of P_k occur in complex conjugate pairs as well, hence P_k is real, too. On the other hand, the coefficients of P_k are sums of possibly nonreal products of the $m_k^{-1}(\alpha)$. We bound those sums from below by bounding the real part of each product from below. We do that by considering both, the absolute value and the argument of each $m_k^{-1}(\alpha)$. In order for those arguments to be small, we require k to be larger than a logarithmic function of n. As expected, the coefficients $p_{k,n-2-i}$ turn out to be close to $\binom{n-2}{i}\phi^i$.

Sections 9–11 Let a_1 and a_2 be the two close real roots of A_n, let $b_{k,1}$ and $b_{k,2}$ be their respective images under m_k^{-1}, and let $Q_k(x) = x^2 - c_k x + d_k$ be the monic quadratic polynomial that has $b_{k,1}$ and $b_{k,2}$ as roots. Then $c_k = b_{k,1} + b_{k,2}$ and $d_k = b_{k,1}b_{k,2}$. We determine lower and upper bounds for c_k and d_k. We know that a_1 and a_2 are close to $a = 1/\phi^2$. So $m_2^{-1}(a_1)$ and $m_2^{-1}(a_2)$ are close to $m_2^{-1}(a) = (t^{-2} \circ r)(1/\phi^2) = 1/\phi$. But $1/\phi$ is a fixed point of $t^{-1} \circ r$, so $m_k^{-1}(a_1) = (t^{-1} \circ r)^{k-2}(m_2^{-1}(a_1))$ and $m_k^{-1}(a_2)$ remain close to $1/\phi$. Hence c_k is close to $2/\phi$, and d_k is close to $1/\phi^2$; the respective distances increase as k increases.

Section 12 Let $R_k(x) = P_k(x)Q_k(x)$, and let $b_{k,n}$ be the leading coefficient of $B_k(x)$. Then $B_k(x) = b_{k,n} \cdot R_k(x)$. So $B_k(x)$ is approximately $b_{k,n} \cdot (x+\phi)^{n-2} \cdot (x^2 - (2/\phi)x + 1/\phi^2)$. We analyze various coefficients of $R_k(x)$ in three separate theorems. For example, letting $R_k(x) = r_{k,n}x^n + r_{k,n-1}x^{n-1} + \cdots + r_{k,0}$, we show that the coefficients $r_{k,i}$, $0 \leq i \leq n/10$, are greater than 1. Concerning $b_{k,n}$ we prove at first only that $b_{k,n}$ is negative if $2 \leq k \leq n/2 - 2$. Then, letting $B_k(x) = b_{k,n}x^n + \cdots + b_{k,0}$, we show, by induction on k, that the coefficients $b_{k,i}$ are large negative integers for all i, $0 \leq i \leq n/10$. We also show that $B_k(x)$ is primitive.

Section 13 We show that the time to compute the polynomial B_{k+1} from the polynomial $\mathrm{R}(B_k)$ using classical translation by 1 dominates $n^3(k - k_1)$ where k_1 is an integer that depends logarithmically on n. The sum, over k, of those computing times yields the lower bound n^5 for the computing time of the CF-method for the polynomials A_n.

4. A set of input polynomials

Let $a = (3 - \sqrt{5})/2$ be one of the roots of the polynomial $B(x) = x^2 - 3x + 1$. For all n, $n \geq 5$, let $A_n(x) = x^n - 2B(x)^2$, see Table 1. We show that A_n has two real roots that are very close to a.

Theorem 13. *All roots of A_n are simple, no root is rational.*

Proof. By the Eisenstein irreducibility criterion (van der Waerden, 1949), $A_n(x)$ is irreducible. In particular, all roots of A_n are simple. Since A_n is irreducible over the integers, A_n is irreducible over the rationals by Gauss's Lemma (van der Waerden, 1949, Section 23). In particular, A_n does not have any rational roots. □

Theorem 14. *$A_n(x)$ has exactly three positive real roots, namely one in each of the three intervals $(0, a)$, $(a, 1)$ and $(1, 2)$. If n is even, A_n has exactly one negative real root; if n is odd, A_n has no negative real root.*

Proof. $A_n(0) = -2$, $A_n(a) = a^n$, $A_n(1) = -1$ and $A_n(2) > 0$. So A_n has a least one root in each of the three specified intervals. The proof will be completed by showing that A_n cannot have more than three positive real roots. Consider the third derivative $A_n^{(3)}(x) = n(n-1)(n-2)x^{n-3} - 48x + 72$. If $0 < x < 1$ then $A_n^{(3)}(x) > -48x + 72 > -48 + 72 > 0$. If $x \geq 1$ then $A_n^{(3)}(x) \geq 60x^{n-3} - 48x + 72 \geq 60x^2 - 48x + 72$, which is positive by the quadratic formula. So $A_n^{(3)}$ has no positive real roots, $A_n^{(2)}$ has at most one, $A_n^{(1)}$ has at most two, and A_n has at most three. If n is even, $A_n(-x) = x^n - 2(x^2 + 3x + 1)^2$ has one sign variation; if n is odd it has no sign variations. □

We will denote the root of A_n in $(0, a)$ by a_1 and the root in $(a, 1)$ by a_2.

Lemma 15. *Let $h > 0$. Then $B(a - h)^2 > B(a + h)^2$.*

Proof. $B(a + h) = (a + h)^2 - 3(a + h) + 1 = h^2 + (2a - 3)h + (a^2 - 3a + 1) = h^2 + (2a - 3)h$ since a is a root of $B(x)$. Therefore $B(a + h)^2 = h^4 + 2(2a - 3)h^3 + (2a - 3)^2h^2$ and so $B(a - h)^2 = h^4 - 2(2a - 3)h^3 + (2a - 3)^2h^2$. Therefore $B(a - h)^2 - B(a + h)^2 = -4(2a - 3)h^3 > 0$ since $2a - 3 < 0$. □

Lemma 16. *If $0 < h < a$ then $A_n(a - h) < A_n(a + h)$.*

Proof. $0 < a - h < a + h$ so $(a - h)^n < (a + h)^n$ and, by Lemma 15, $-2B(a - h)^2 < -2B(a + h)^2$. Adding these two inequalities completes the proof. □

Theorem 17. *Let $h = a^{n/2+1}$. Then $A_n(x)$ has a root in each of the intervals $(a - h, a)$ and $(a, a + h)$ if $n \geq 6$. The polynomial A_5 has a root in $(a - h, a)$ and a root in $(a + h, a + h + 0.002)$.*

Proof. Assume first that $n \geq 7$. Since $A_n(a) > 0$ it suffices to prove that $A_n(a - h)$ and $A_n(a + h)$ are negative. Then, by Lemma 16, it suffices to prove that $A_n(a + h) < 0$. We will prove that $(a+h)^n < 2B(a+h)^2$. First note that $(a+h)^n = (a+a^{n/2+1})^n = a^n(1+a^{n/2})^n$. That $1+a^{n/2} < 1+1/(3n)$ for $n \geq 7$ is easily proved by induction on n. So $(a + h)^n < a^n(1 + 1/(3n))^n < e^{1/3}a^n = e^{1/3}h^2/a^2 < 1.3957h^2/0.1458 < 9.573h^2$. Since $a^2-3a+1 = 0$ we have $B(a+h) = h^2+(2a-3)h = (h+2a-3)h$ and so $B(a + h)^2 = (3 - 2a - h)^2h^2$. But $2a < 0.7640$ and $h = a^{n/2+1} \leq \phi^{-9} < 0.0132$ so $(3 - 2a - h) > 2.2228$ and $2B(a + h)^2 > 2(4.940h^2) = 9.880h^2 > 9.573h^2 > (a + h)^n$ so $A_n(a + h) < 0$.

Now assume that $n = 6$. Then $h < 0.021287$, $(a + h)^6 < 0.004300$ and $2B(a + h)^2 > 0.004445$, so $A_n(a + h) < 0$.

Now let $n = 5$. Then $h > 0.034441$, $(a + h)^5 > 0.012519$ and $2B(a + h)^2 < 0.011500$ so that $A_5(a+h) > 0$. Therefore A_5 does not have a root in $(a, a+h)$. But $h < 0.034442$, $(a-h)^5 < 0.005070$ and $2B(a - h)^2 > 0.012230$, so $A_n(a - h) < 0$ and, hence, A_5 has a root in $(a - h, a)$. Furthermore, $a + h + 0.002 < 0.418408$, $(a + h + 0.002)^5 < 0.012824$ and $2B(a + h + 0.002)^2 > 0.012850$, so $A_5(a + h + 0.002) < 0$ and, hence, A_5 has a root in $(a + h, a + h + 0.002)$. □

5. CF-trees

We now determine the CF-tree of A_n for all n, $n \geq 5$.

5.1. Tree structure

Table 1 shows that $\mathrm{var}(A_n) = 5$. Hence the root, node ϵ, is an internal node. We now compute the number of sign variations at the children, nodes "1" and "2".

Theorem 18. *The polynomial* $\mathrm{T}(A_n)$ *has exactly one sign variation.*

Proof. Let $n \geq 5$. Then $\mathrm{T}(A_n) = \mathrm{T}(x^n) + \mathrm{T}(-2B(x)^2)$ where

$$\mathrm{T}(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i$$

and $\mathrm{T}(-2B(x)^2) = \mathrm{T}(-2x^4 + 12x^3 - 22x^2 + 12x - 2) = -2x^4 + 4x^3 + 2x^2 - 4x - 2$. So, if $\mathrm{T}(A_n) = \bar{a}_n x^n + \cdots + \bar{a}_0$ then $\bar{a}_i > 0$ for $i > 4$ and also for $i = 3$ and for $i = 2$. Since $n \geq 5$, $\binom{n}{4} \geq 5$ and therefore $\bar{a}_4 > 0$. Since $\binom{n}{1} \geq 5$, $\bar{a}_1 > 0$. And since $\binom{n}{0} = 1$, $\bar{a}_0 = -1$. We have proved more than necessary: All coefficients of $\mathrm{T}(A_n)$ are positive except for the last, which is -1. □

Theorem 19. *The polynomial* $\mathrm{T}\,\mathrm{R}(A_n)$ *has exactly two sign variations.*

Proof. Let $n \geq 5$. By Theorem 14, A_n has exactly two positive roots < 1. Hence $\mathrm{R}(A_n)$ has exactly two roots > 1. Thus, $\mathrm{T}\,\mathrm{R}(A_n)$ has exactly two positive roots, and hence $\mathrm{var}(\mathrm{T}\,\mathrm{R}(A_n)) \geq 2$. To show equality we will use induction on n. The assertion clearly holds for $n = 5$ since

$$\mathrm{T}\,\mathrm{R}(A_5) = -2x^5 + 2x^4 + 6x^3 - 2x^2 - 6x - 1.$$

The induction step requires some preparation. For any $n \geq 5$,

$$A_n = x^n - 2x^4 + 12x^3 - 22x^2 + 12x - 2$$

and, hence,

$$\mathrm{R}(A_n) = -2x^n + 12x^{n-1} - 22x^{n-2} + 12x^{n-3} - 2x^{n-4} + 1.$$

Thus,

$$\mathrm{R}(A_{n+1}) = (\mathrm{R}(A_n) - 1)x + 1,$$

and hence

$$\mathrm{T}\,\mathrm{R}(A_{n+1}) = (\mathrm{T}\,\mathrm{R}(A_n) - 1)(x + 1) + 1.$$

So, letting $\mathrm{T}\,\mathrm{R}(A_n) = b_n x^n + \cdots + b_0$ and $\mathrm{T}\,\mathrm{R}(A_{n+1}) = c_{n+1}x^{n+1} + \cdots + c_0$ we have

$$\begin{aligned} c_{n+1} &= b_n, \\ c_k &= b_k + b_{k-1} \quad \text{for } 2 \leq k \leq n, \\ c_1 &= b_1 + b_0 - 1, \\ c_0 &= b_0. \end{aligned}$$

In particular, all polynomials $\mathrm{T}\,\mathrm{R}(A_n)$, $n \geq 5$, have the same constant term, $b_0 = -1$, and the same leading coefficient, $b_n = -2$.

As an induction hypothesis assume now that $\mathrm{var}(\mathrm{T}\,\mathrm{R}(A_n)) = 2$ for some $n \geq 5$. Let (p, q) and (r, s), $0 \leq p < q \leq r < s \leq n$, be the index pairs that contribute to $\mathrm{var}(\mathrm{T}\,\mathrm{R}(A_n))$. Then the coefficient signs of $\mathrm{T}\,\mathrm{R}(A_n)$ are as follows.

	$k = 0$	$0 < k < q$	$k = q$	$q < k < r$	$k = r$	$r < k < n$	$k = n$
b_k	< 0	≤ 0	> 0	≥ 0	> 0	≤ 0	< 0

Hence, in case $q = r$, the coefficient signs of $\mathrm{T}\,\mathrm{R}(A_{n+1})$ are

	$k = 0$	$0 < k < q$	$k = q$	$k = q + 1 = r + 1$	$r + 1 < k < n + 1$	$n + 1$
c_k	< 0	≤ 0	?	?	≤ 0	< 0

where the question marks stand for undetermined signs. But regardless of the signs of c_q and c_{q+1} we have $\mathrm{var}(\mathrm{T}\,\mathrm{R}(A_{n+1})) \leq 2$.

In case $q < r$ the coefficient signs of $\mathrm{T\,R}(A_{n+1})$ are as follows.

	$k = 0$	$0 < k < q$	$k = q$	$q < k < r + 1$	$k = r + 1$	$r + 1 < k < n + 1$	$k = n + 1$
c_k	< 0	≤ 0	?	≥ 0	?	≤ 0	< 0

So we have $\mathrm{var}(\mathrm{T\,R}(A_{n+1})) \leq 2$ in this case as well.

Since, by the first paragraph of the proof, $\mathrm{var}(\mathrm{T\,R}(A_{n+1})) \geq 2$, we obtain $\mathrm{var}(\mathrm{T\,R}(A_{n+1})) = 2$. This completes the proof by induction. □

As the CF-method descends the CF-tree, the number of sign variations will decrease or stay the same by the next theorem.

Definition 20. The polynomial A is *subadditive* in case

$$\mathrm{var}(\mathrm{T}(A)) + \mathrm{var}((\mathrm{T} \circ \mathrm{R})(A)) \leq \mathrm{var}(A).$$

Theorem 21. *All polynomials are subadditive.*

Proof. Schoenberg's proof (1934) uses Schoenberg's theorem (1930) that states that the linear transformations given by totally positive matrices are variation-diminishing. □

By Theorem 18, node "1" is a leaf node with 1 sign variation. By Theorem 19, node "2" is an internal node with 2 sign variations; its associated interval, $(0, 1)$, contains exactly two roots of A_n by Theorem 14. Moreover, by Theorem 13, A_n does not have any rational roots. Hence the following theorem applies with $A = A_n$ and $s = 2$.

Theorem 22. *Let A be a polynomial that does not have any rational root. Let $s \in \{1, 2\}^*$ represent an internal node of the CF-tree of A. Let $\mathrm{var}(A_s) = 2$, and let I_s contain exactly two roots of A. Then, for any descendant t of s, $\mathrm{var}(A_t)$ equals the number of roots of A in the interval I_t.*

Proof. Due to subadditivity, Theorem 21, $\mathrm{var}(A_t) \leq 2$ for any descendant t of s. This can be shown by induction on the length of the path from s to t. The proof that $\mathrm{var}(A_t)$ equals the number of roots of A in I_t is similar. Indeed, if $t = s$ then, trivially, $\mathrm{var}(A_t)$ equals the number of roots of A in the interval I_t. Now assume that the assertion holds for some descendant $\bar{s}$ of s, and that t is a child of $\bar{s}$. Then $\mathrm{var}(A_{\bar{s}})$ equals the number of roots of A in $I_{\bar{s}}$ and, since $\bar{s}$ is an internal node, $\mathrm{var}(A_{\bar{s}}) = 2$; moreover, $t = \bar{s}1$ or $t = \bar{s}2$. The bisection points in the CF-method are rational numbers and A does not have any rational root. Hence each root of A in $I_{\bar{s}}$ is either in $I_{\bar{s}1}$ or in $I_{\bar{s}2}$. So, letting r_i be the number of roots of A in the interval $I_{\bar{s}i}$, $i = 1, 2$, we have

$$r_1 + r_2 = 2. \tag{5.1}$$

Let $v_i = \mathrm{var}(A_{\bar{s}i})$, $i = 1, 2$. Then, again by subadditivity,

$$v_1 + v_2 \leq 2. \tag{5.2}$$

Also, by the Descartes rule of signs,

$$r_1 \leq v_1 \quad \text{and} \quad r_2 \leq v_2. \tag{5.3}$$

In case $r_1 = 0$ Eq. (5.1) yields $r_2 = 2$, lines (5.3) and (5.2) imply $v_2 = 2$ and Inequality (5.2) yields $v_1 = 0$. In case $r_1 = 1$ we similarly obtain $r_2 = 1$ and $v_1 = v_2 = 1$. The remaining case is $r_1 = 2$; we obtain $r_2 = 0$, $v_1 = 2$, and $v_2 = 0$. In all three cases, $r_1 = v_1$ and $r_2 = v_2$, that is, $\mathrm{var}(A_t)$ equals the number of roots of A in the interval I_t. □

Theorem 23. *Let s be a descendant of node "2" in the universal CF-tree. Then s is an internal node in the CF-tree of A_n if and only if $a \in I_s$ and A_n is negative at the endpoints of I_s.*

Proof. First note that $0 < a_1 < a < a_2 < 1$ and that A is positive on (a_1, a_2), zero at a_1 and at a_2, and negative on the remaining parts of $(0, 1)$. If s is an internal node in the CF-tree of A_n then, by Theorem 22, I_s contains more than one root of A_n. But $I_s \subset (0, 1)$, and a_1 and a_2 are the only roots of A_n in $(0, 1)$, so I_s contains both a_1 and a_2. Hence I_s contains a and A_n is negative at the endpoints of I_s. Conversely, if s is such that $a \in I_s$ and A_n is negative at the endpoints of I_s then I_s contains both a_1 and a_2. So, again by Theorem 22, s is an internal node in the CF-tree of A_n. □

We now describe the intervals in the universal CF-tree that contain a. As a preparation we prove a Fibonacci identity that is well known for positive odd indices (Ledin, 1970; Carlitz, 1971) and has recently been stated for all positive indices (Bouhamida, 2009).

Lemma 24. *For all integers k,*

$$F_{k+2}^2 - 3F_{k+2}F_k + F_k^2 = (-1)^k.$$

Proof. Let $D(x, y) = x^2 - 3xy + y^2$. Then $D(2a - b, a - b) = -D(a, b)$ for all a, b. In particular, $D(F_{k+3}, F_{k+1}) = D(2F_{k+2} - F_k, F_{k+2} - F_k) = -D(F_{k+2}, F_k)$ for all integers k. Since $D(F_2, F_0) = D(1, 0) = 1$, we obtain by induction proofs on k that $D(F_{k+2}, F_k) = (-1)^k$ for all integers k. This proves the assertion. □

Lemma 25. *Let k be a nonzero integer. Then* $B(F_{k-2}/F_k) = (-1)^k/F_k^2$.

Proof. Since $k \neq 0$, $F_k \neq 0$. Then, using Lemma 24, $B(F_{k-2}/F_k) = (F_{k-2}^2 - 3F_{k-2}F_k + F_k^2)/F_k^2 = (-1)^k/F_k^2$. □

Theorem 26. *Let s be a node in the universal CF-tree. Then* $a \in I_s$ *if and only if* $s = \epsilon$ *or* $s = 2$ *or* $s = 21\tau$ *where* $\tau \in \{2\}^*$. *Moreover,* $I_\epsilon = (0, \infty)$, $I_2 = (0, 1)$ *and, for* $s = 21\tau$ *and* $\tau \in \{2\}^*$, *the interval* I_s *has the endpoints* $F_{|\tau|}/F_{|\tau|+2}$ *and* $F_{|\tau|+1}/F_{|\tau|+3}$ *and width* $1/(F_{|s|}F_{|s|+1})$.

Proof. Definition 7 associates each node of the universal CF-tree with a matrix. The matrices of nodes ϵ and "2" are immediate from the definition; the matrices of nodes 21τ are obtained using an easy induction on $|\tau|$. We have

$$\mathbf{m}_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{m}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{m}_{21\tau} = \begin{bmatrix} F_{|\tau|} & F_{|\tau|+1} \\ F_{|\tau|+2} & F_{|\tau|+3} \end{bmatrix}.$$

The intervals $I_\epsilon, I_2, I_{21\tau}$ are obtained as the images of the interval $(0, \infty)$ under the corresponding linear fractional mappings

$$m_\epsilon(x) = x, \qquad m_2(x) = \frac{1}{x+1}, \qquad m_{21\tau}(x) = \frac{F_{|\tau|}x + F_{|\tau|+1}}{F_{|\tau|+2}x + F_{|\tau|+3}}.$$

The number a is contained in the interval $I_\epsilon = (0, \infty)$ on level 0 and in the interval $I_2 = (0, 1)$ on level 1. The endpoints of the interval $I_{21\tau}$ on level $|\tau| + 2$ are $F_{|\tau|}/F_{|\tau|+2}$ and $F_{|\tau|+1}/F_{|\tau|+3}$. By Lemma 25, B is positive on one endpoint and negative on the other. Hence, $a \in I_{21\tau}$. By Theorem 12, the width of $I_{21\tau}$ is $1/(F_{|\tau|+2}F_{|\tau|+3})$. Our list of intervals containing a is complete because it includes one interval from each level of the tree. □

Corollary 27. *The polynomial* A_n *is positive on* F_{k-2}/F_k *for almost all positive integers k.*

Proof. Let $k \geq 2$. By Theorem 26, F_{k-2}/F_k is among the endpoints of the intervals $I_{21\tau}$, $\tau \in \{2\}^*$, and those endpoints converge to a. But A_n is a continuous function, and $A_n(a) = a^n > 0$. Hence A_n is positive on almost all F_{k-2}/F_k. □

Theorem 28. *Let* $s = 21\tau$, $\tau \in \{2\}^*$, *and let k be the least positive integer such that* $A_n(F_{k-2}/F_k) > 0$. *Then* $k \geq 5$. *Moreover,* A_n *is negative at both endpoints of* I_s *if and only if* $|\tau| \leq k - 4$.

Proof. By Lemma 25, $A_n(F_{j-2}/F_j) = (F_{j-2}/F_j)^n - 2/F_j^4$ for all nonzero integers j, so we have $A_n(F_{j-2}/F_j) < 0$ for $j = 1, 2, 3, 4$ since $n \geq 5$. Hence $k \geq 5$. By Theorem 26, the endpoints of I_s are $F_{|\tau|}/F_{|\tau|+2}$ and $F_{|\tau|+1}/F_{|\tau|+3}$. If $|\tau| \leq k - 4$, then $|\tau| + 3 < k$, so, by the hypothesis on k, A_n is nonpositive at both endpoints. But A_n is nonzero at the endpoints by Theorem 13, so A_n is negative at the endpoints. Conversely, assume $|\tau| > k - 4$. Then $|\tau| \geq k - 3$, so $I_{21\tau}$ is a subinterval of the interval $I_{21\sigma}$, $\sigma \in \{2\}^*$ and $|\sigma| = k - 3$. By Theorem 26, the interval $I_{21\sigma}$ has the endpoints F_{k-3}/F_{k-1} and F_{k-2}/F_k and both intervals contain a. So one of the endpoints of $I_{21\tau}$ is between F_{k-2}/F_k and a. But A_n is positive at a and, by hypothesis, at F_{k-2}/F_k. Moreover, by Theorems 13 and 14, the roots of A_n in $(0, 1)$ are simple and lie on opposite sides of a. Thus, A_n is positive at all points between F_{k-2}/F_k and a, and, in particular, at one endpoint of $I_{21\tau}$. □

Theorem 29. *Let k be the least positive integer such that $A_n(F_{k-2}/F_k) > 0$; recall that, by Theorem 28, $k \geq 5$. Then the CF-tree of A_n has height $k-1$.*

1. *The internal nodes of the tree are the nodes ϵ, "2" and all nodes 21τ where $\tau \in \{2\}^*$ and $|\tau| \leq k-4$. The number of sign variations is 5 at ϵ and 2 at each of the other internal nodes.*
2. *The leaves with one sign variation are the nodes 1, $21\tau 1$ and $21\tau 2$ where $\tau \in \{2\}^*$ and $|\tau| = k-4$.*
3. *The leaves with no sign variation are the node 22 and the nodes $21\tau 1$ where $\tau \in \{2\}^*$ and $0 \leq |\tau| < k-4$.*

Proof. We already remarked that $\mathrm{var}(A_n) = 5$ by Table 1. Hence node ϵ is an internal node. The assertions regarding nodes "1" and "2" follow from Theorems 18 and 19, respectively; in particular, node "2" has two sign variations.

Let s be a proper descendant of "2". Then, by Theorem 23, s is an internal node in the CF-tree of A_n if and only if $a \in I_s$ and A_n is negative at the endpoints of I_s. By Theorems 26 and 28, this is equivalent to $s = 21\tau$ where $\tau \in \{2\}^*$, $|\tau| \leq k-4$.

From $|\tau| = 0$ we obtain that "21" is an internal node. Hence, by Theorem 22, "21" has two sign variations and "22" none. If $|\tau| < k-4$ then $s2$ is an internal node. Hence, by Theorem 22, $s2$ has two sign variations and $s1$ none. From $|\tau| = k-4$ we obtain that $s1$ and $s2$ are leaves. Hence, again by Theorem 22, those nodes have one sign variation each. Those nodes are at level $|s1| = |21\tau 1| = 2 + (k-4) + 1 = k-1$. □

5.2. A lower bound for the height

Theorem 29 describes the CF-tree of A_n in terms of the height, and it reduces the determination of the height to identifying the least positive integer k such that $A_n(F_{k-2}/F_k) > 0$.

Lemma 30. *Let k be an integer, $k \neq 0$ and $k \neq 2$. Then*

$$A_n\left(\frac{F_{k-2}}{F_k}\right) > 0 \quad \text{if and only if} \quad 4 \ln F_k > \ln 2 + n \ln \frac{F_k}{F_{k-2}}.$$

Proof. Using Lemma 25,

$$\begin{aligned} A_n\left(\frac{F_{k-2}}{F_k}\right) &= \left(\frac{F_{k-2}}{F_k}\right)^n - 2B\left(\frac{F_{k-2}}{F_k}\right)^2 \\ &= \left(\frac{F_{k-2}}{F_k}\right)^n - \frac{2}{F_k^4}. \end{aligned}$$

Hence, $A_n(F_{k-2}/F_k) > 0$ if and only if

$$\left(\frac{F_{k-2}}{F_k}\right)^n > \frac{2}{F_k^4}.$$

Taking reciprocals and taking logarithms yields the assertion. □

In the proof of the next result – and also later – we will use the formula

$$F_k = (\phi^k - \hat{\phi}^k)/\sqrt{5} \tag{5.4}$$

where $\phi = (1+\sqrt{5})/2$ and $\hat{\phi} = (1-\sqrt{5})/2$. The formula holds for all integers k. Knuth (1968) derives it for $k \geq 0$ using a generating function; Hardy and Wright (1938, Section 10.14) use the theory of continued fractions; Benjamin and Quinn (2003, Identity 240) give a combinatorial proof. Note that $\hat{\phi} = 1 - \phi = -1/\phi < 0$.

Theorem 31. *Let k be an even positive integer such that $A_n(F_{k-2}/F_k) > 0$. Then $k > n/2 + 2$.*

Proof. By Theorem 28, $k \geq 5$. By Lemma 30,

$$4 \ln F_k > \ln 2 + n \ln \frac{F_k}{F_{k-2}}.$$

Since k is even, the formula for the Fibonacci numbers, Eq. (5.4), implies $F_k < \phi^k/\sqrt{5}$, hence

$$4k \ln \phi - 4 \ln \sqrt{5} > 4 \ln F_k > \ln 2 + n \ln \frac{F_k}{F_{k-2}}.$$

Since $\hat{\phi} = -1/\phi$ and k is even,

$$\begin{aligned}\frac{F_k}{F_{k-2}} &= \frac{\phi^k - \hat{\phi}^k}{\phi^{k-2} - \hat{\phi}^{k-2}} = \frac{\phi^k - 1/\phi^k}{\phi^{k-2} - 1/\phi^{k-2}} = \frac{(\phi^{2k} - 1)/\phi^k}{(\phi^{2k-4} - 1)/\phi^{k-2}} \\ &= \frac{\phi^{2k} - 1}{\phi^{2k-4} - 1} \cdot \frac{\phi^{k-2}}{\phi^k} > \frac{\phi^{2k} - \phi^4}{\phi^{2k-4} - 1} \cdot \frac{1}{\phi^2} = \phi^2.\end{aligned}$$

Thus, $4k \ln \phi - 4 \ln \sqrt{5} > \ln 2 + 2n \ln \phi$ and, hence,

$$k > \frac{\ln \sqrt{5}}{\ln \phi} + \frac{\ln 2}{4 \ln \phi} + \frac{n}{2}.$$

But $\ln \sqrt{5}/ \ln \phi > 1.672$ and $\ln 2/(4 \ln \phi) > 0.360$, so $k > 2.032 + n/2$. □

Lemma 32. *If $k \geq 2$ then*

$$\phi^{k-2} \leq F_k < \phi^{k-3/2}.$$

Proof. The assertion clearly holds for $k = 2$ and $k = 3$. Let j be an integer, $j \geq 2$, and assume that the assertion holds for $k = j$ and $k = j + 1$. Then

$$\phi^{j-2} + \phi^{j-1} \leq F_j + F_{j+1} < \phi^{j-3/2} + \phi^{j-1/2}.$$

Since $1 + \phi = \phi^2$, the left-hand sum equals ϕ^j, and the right-hand sum equals $\phi^{j+1/2}$. The sum in the middle equals F_{j+2}. Hence the assertion holds for $k = j + 2$. □

Theorem 33. *Let k be an odd positive integer such that $A_n(F_{k-2}/F_k) > 0$. Then $k > (3/8)n + 2$.*

Proof. By Theorem 28, $k \geq 5$. By Lemma 30,

$$-n \ln \frac{F_k}{F_{k-2}} + 4 \ln F_k - \ln 2 > 0. \tag{5.5}$$

Since k is odd, the formula for the Fibonacci numbers, Eq. (5.4), implies $F_k = (\phi^k + \phi^{-k})/\sqrt{5} = \phi^k(1+\phi^{-2k})/\sqrt{5}$. Hence, $4 \ln F_k = 4k \ln \phi + 4 \ln(1+\phi^{-2k}) - 2 \ln 5$. Substituting into Inequality (5.5),

$$-n \ln \frac{F_k}{F_{k-2}} + 4k \ln \phi + 4 \ln(1 + \phi^{-2k}) - 2 \ln 5 - \ln 2 > 0. \tag{5.6}$$

By Lemma 32,

$$\frac{F_k}{F_{k-2}} > \frac{\phi^{k-2}}{\phi^{k-2-3/2}} = \phi^{3/2}.$$

So, by Inequality (5.6),

$$-\frac{3}{2} n \ln \phi + 4k \ln \phi + 4 \ln(1 + \phi^{-2k}) - 2 \ln 5 - \ln 2 > 0.$$

Dividing by $4 \ln \phi$ yields

$$k > \frac{3}{8} n - \frac{\ln(1 + \phi^{-2k})}{\ln \phi} + \frac{\ln 5}{2 \ln \phi} + \frac{\ln 2}{4 \ln \phi}. \tag{5.7}$$

Since $k \geq 5$, $\ln(1 + \phi^{-2k}) \leq \ln(1 + \phi^{-10})$. Hence,

$$-\frac{\ln(1 + \phi^{-2k})}{\ln \phi} \geq -\frac{\ln(1 + \phi^{-10})}{\ln \phi} > -0.017.$$

Also, $\ln 5/(2 \ln \phi) > 1.672$ and $\ln 2/(4 \ln \phi) > 0.360$. So we obtain from Inequality (5.7)

$$k > \frac{3}{8} n - 0.017 + 1.672 + 0.360 = \frac{3}{8} n + 2.015. \quad \square$$

By Theorems 29, 31 and 33, the height of the CF-tree of A_n is greater than $(3/8)n + 1$.

5.3. The height

We now determine the tree height precisely.

Lemma 34. *For all x, $x > 1$,*

$$\frac{x-1}{x} < \ln(x) < \frac{x-1}{x}\Big(1 + \frac{x-1}{2}\Big).$$

Proof. For all x, $x \geq 1/2$, we have the well-known equation (Abramowitz and Stegun, 1965, (4.1.25))

$$\ln(x) = \frac{x-1}{x} + \frac{1}{2}\Big(\frac{x-1}{x}\Big)^2 + \frac{1}{3}\Big(\frac{x-1}{x}\Big)^3 + \cdots .$$

In our case, $x > 1$, so all summands are positive, hence the first inequality is clearly true. Moreover, $|(x-1)/x| < 1$, so

$$\begin{aligned}
&\frac{x-1}{x} + \frac{1}{2}\Big(\frac{x-1}{x}\Big)^2 + \frac{1}{3}\Big(\frac{x-1}{x}\Big)^3 + \cdots \\
&\quad < \frac{x-1}{x} + \frac{1}{2}\Big(\frac{x-1}{x}\Big)^2 + \frac{1}{2}\Big(\frac{x-1}{x}\Big)^3 + \cdots \\
&\quad = \frac{x-1}{x} + \frac{1}{2}\Big(\frac{x-1}{x}\Big)^2 \left(1 + \frac{x-1}{x} + \Big(\frac{x-1}{x}\Big)^2 + \cdots\right) \\
&\quad = \frac{x-1}{x} + \frac{1}{2}\Big(\frac{x-1}{x}\Big)^2 \left(\frac{1}{1-\frac{x-1}{x}}\right) \\
&\quad = \frac{x-1}{x} + \frac{1}{2}\Big(\frac{x-1}{x}\Big)^2 x \\
&\quad = \frac{x-1}{x}\Big(1 + \frac{x-1}{2}\Big)
\end{aligned}$$

which yields the second inequality. □

Lemma 35. Let n and k be integers such that $n > 0$, $k \geq 8$ and k is even. Then

$$2n \ln \phi < n \ln \frac{F_k}{F_{k-2}}.$$

If, in addition, $n < 2k - 4$ then

$$n \ln \frac{F_k}{F_{k-2}} < 2n \ln \phi + 0.032.$$

Proof. For $k \geq 4$ let

$$u(k) = \frac{\phi^{-2k+4}(1-\phi^{-4})}{1-\phi^{-2k+4}}$$

and

$$v(k) = (2k-4)\ln(1+u(k)).$$

We start by proving the following assertions.

1. $\ln(F_k/F_{k-2}) = 2\ln\phi + \ln(1+u(k))$ for all even k, $k \geq 4$.
2. The function $v(k)$ is a positive, decreasing function of k for $k \geq 4$.
3. If $k \geq 8$ then $0 < v(k) < 0.032$.

We obtain Assertion (1) from the exact formula for F_k, Eq. (5.4). Indeed, we have for even k, $k \geq 4$,

$$\begin{aligned}\frac{F_k}{F_{k-2}} &= \frac{\phi^k - \phi^{-k}}{\phi^{k-2} - \phi^{-k+2}} \\ &= \phi^2 + \frac{\phi^{-k+4} - \phi^{-k}}{\phi^{k-2} - \phi^{-k+2}} \\ &= \phi^2 \left(1 + \frac{\phi^{-k+2} - \phi^{-k-2}}{\phi^{k-2} - \phi^{-k+2}}\right) \\ &= \phi^2 \left(1 + \phi^{-2k+4} \frac{1 - \phi^{-4}}{1 - \phi^{-2k+4}}\right) \\ &= \phi^2(1 + u(k)).\end{aligned}$$

Taking logarithms proves Assertion (1). Since $u(k)$ is positive, $\ln(1 + u(k))$ is positive. So, multiplying the assertion by n proves the first assertion of the lemma. Also, again because $\ln(1 + u(k))$ is positive, the function $v(k)$ is positive. To complete the proof of Assertion (2) we show next that $v(k+1)/v(k) < 1$ for all k, $k \geq 4$.

$$\frac{v(k+1)}{v(k)} = \frac{(2k-2)\ln(1+u(k+1))}{(2k-4)\ln(1+u(k))} \leq \frac{6}{4} \cdot \frac{\ln(1+u(k+1))}{\ln(1+u(k))}.$$

We remove the logarithms in the quotient on the right using Lemma 34 as follows. We let $x = 1 + u(k+1)$ and apply the second inequality of the lemma to the numerator, and we let $x = 1 + u(k)$ and apply the first inequality of the lemma to the denominator. We thus obtain

$$\begin{aligned}\frac{6}{4} \cdot \frac{\ln(1+u(k+1))}{\ln(1+u(k))} &< \frac{6}{4} \cdot \frac{u(k+1)}{1+u(k+1)} \cdot \left(1 + \frac{u(k+1)}{2}\right) \Big/ \frac{u(k)}{1+u(k)} \\ &= \frac{6}{4} \cdot \frac{u(k+1)}{1+u(k+1)} \cdot \frac{1+u(k)}{u(k)} \cdot \left(1 + \frac{u(k+1)}{2}\right) \\ &= \frac{6}{4} \cdot \frac{u(k+1)}{u(k)} \cdot \frac{1+u(k)}{1+u(k+1)} \cdot \left(1 + \frac{u(k+1)}{2}\right) \\ &< 1.5 \cdot 0.382 \cdot 1.184 \cdot 1.026 \\ &< 1\end{aligned}$$

where the constants in the fourth line are obtained as follows.

$$\begin{aligned}\frac{u(k+1)}{u(k)} &= \frac{1}{\phi^2} \cdot \frac{1 - \phi^{-2k+4}}{1 - \phi^{-2k+2}} \\ &< \frac{1}{\phi^2} \\ &< 0.382, \\ \frac{1+u(k)}{1+u(k+1)} &= \frac{1 - \phi^{-2k}}{1 - \phi^{-2k+4}} \cdot \frac{1 - \phi^{-2k+2}}{1 - \phi^{-2k-2}} \\ &= \frac{1 - \phi^{-2k} - \phi^{-2k+2} + \phi^{-4k+2}}{1 - \phi^{-2k+4} - \phi^{-2k-2} + \phi^{-4k+2}} \\ &< \frac{1 + \phi^{-4k+2}}{1 - \phi^{-2k+4} - \phi^{-2k-2}} \\ &\leq \frac{1 + \phi^{-14}}{1 - \phi^{-4} - \phi^{-10}} \\ &< 1.184,\end{aligned}$$

$$\begin{aligned} 1+\frac{u(k+1)}{2} &= 1+\frac{1}{2}\cdot\frac{\phi^{-2k+2}(1-\phi^{-4})}{1-\phi^{-2k+2}} \\ &\le 1+\frac{1}{2}\cdot\frac{\phi^{-6}(1-\phi^{-4})}{1-\phi^{-6}} \\ &< 1.026. \end{aligned}$$

This completes the proof of Assertion (2). Assertion (3) follows from Assertion (2) by evaluation of $v(k)$ at $k = 8$.

To prove the second assertion of the lemma we multiply the equation in Assertion (1) by n, obtaining

$$n \ln(F_k/F_{k-2}) = 2n \ln \phi + n \ln(1 + u(k)),$$

and we assume $n < 2k - 4$ so that

$$n \ln(1 + u(k)) < (2k - 4) \ln(1 + u(k)).$$

The right-hand side equals $v(k)$ which, by Assertion (3), is less than 0.032. So,

$$n \ln(F_k/F_{k-2}) < 2n \ln \phi + 0.032,$$

and the proof is complete. □

Lemma 36. *For all* x, $0 < x < 1$,

$$x < -\ln(1-x) < x\left(1+\frac{1}{2}\cdot x\cdot\frac{1}{1-x}\right).$$

Proof. For all x, $-1 < x < 1$, we have the well-known equation (Abramowitz and Stegun, 1965, (4.1.24))

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

In our case, $0 < x < 1$, so all summands are positive, hence the first inequality is clearly true. Moreover, $|x| < 1$, so

$$\begin{aligned} x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots &< x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{2}+\cdots \\ &= x+\frac{1}{2}x^2(1+x+x^2+\cdots) \end{aligned}$$

yields the second inequality. □

Lemma 37. Let n and k be integers such that $n > 0$, $k \ge 8$ and k is odd. Then

$$n \ln \frac{F_k}{F_{k-2}} < 2n \ln \phi.$$

If, in addition, $n < 3k - 5$ then

$$2n \ln \phi - 0.051 < n \ln \frac{F_k}{F_{k-2}}.$$

Proof. The proof is similar to the proof of Lemma 35. For $k \ge 4$ let

$$u(k) = \frac{\phi^{-2k+4}(1-\phi^{-4})}{1+\phi^{-2k+4}}$$

and

$$v(k) = (3k - 5) \ln(1 - u(k)).$$

We start by proving the following assertions.

1. $\ln(F_k/F_{k-2}) = 2 \ln \phi + \ln(1 - u(k))$ for all odd k, $k \ge 4$.

2. The function $v(k)$ is a negative, increasing function of k for $k \geq 4$.
3. If $k \geq 8$ then $-0.051 < v(k) < 0$.

We obtain Assertion (1) from the exact formula for F_k, Eq. (5.4). Indeed, we have for odd k, $k \geq 4$,

$$\begin{aligned}\frac{F_k}{F_{k-2}} &= \frac{\phi^k + \phi^{-k}}{\phi^{k-2} + \phi^{-k+2}} \\ &= \phi^2 - \frac{\phi^{-k+4} - \phi^{-k}}{\phi^{k-2} + \phi^{-k+2}} \\ &= \phi^2 \left(1 - \frac{\phi^{-k+2} - \phi^{-k-2}}{\phi^{k-2} + \phi^{-k+2}}\right) \\ &= \phi^2 \left(1 - \phi^{-2k+4} \frac{1 - \phi^{-4}}{1 + \phi^{-2k+4}}\right) \\ &= \phi^2 (1 - u(k)).\end{aligned}$$

Taking logarithms proves Assertion (1). Since $u(k)$ is positive, $\ln(1 - u(k))$ is negative. So, multiplying the assertion by n proves the first assertion of the lemma. Also, again because $\ln(1 - u(k))$ is negative, the function $v(k)$ is negative. To complete the proof of Assertion (2) we show next that $v(k+1)/v(k) < 1$ for all k, $k \geq 4$.

$$\frac{v(k+1)}{v(k)} = \frac{(3k-2)\ln(1 - u(k+1))}{(3k-5)\ln(1 - u(k))} < \frac{3}{2} \cdot \frac{\ln(1 - u(k+1))}{\ln(1 - u(k))}.$$

We remove the logarithms in the quotient on the right using Lemma 36 as follows. We let $x = u(k+1)$ and apply the second inequality of the lemma to the negated numerator, and we let $x = u(k)$ and apply the first inequality of the lemma to the negated denominator. We thus obtain

$$\begin{aligned}\frac{3}{2} \cdot \frac{\ln(1 - u(k+1))}{\ln(1 - u(k))} &= \frac{3}{2} \cdot \frac{-\ln(1 - u(k+1))}{-\ln(1 - u(k))} \\ &< \frac{3}{2} \cdot u(k+1) \left(1 + \frac{u(k+1)}{2} \cdot \frac{1}{1 - u(k+1)}\right) \Big/ u(k) \\ &= \frac{3}{2} \cdot \frac{u(k+1)}{u(k)} \cdot \left(1 + \frac{1}{2} \cdot \frac{u(k+1)}{1 - u(k+1)}\right) \\ &< 1.5 \cdot 0.415 \cdot (1 + 0.5 \cdot 0.048) \\ &< 1\end{aligned}$$

where the constants in the fourth line are obtained as follows.

For the constant 0.415 consider that

$$\frac{u(k+1)}{u(k)} = \frac{1}{\phi^2} \cdot \frac{1 + \phi^{-2k+4}}{1 + \phi^{-2k+2}}$$

is a decreasing function of k. Indeed, let $w(k) = (1 + \phi^{-2k+4})/(1 + \phi^{-2k+2})$. Then $w(k+1) = (1 + \phi^{-2k+2})/(1 + \phi^{-2k})$ and

$$\begin{aligned}\frac{w(k+1)}{w(k)} &= \frac{(1 + \phi^{-2k+2})^2}{(1 + \phi^{-2k})(1 + \phi^{-2k+4})} \\ &= \frac{1 + 2\phi^{-2k+2} + \phi^{-4k+4}}{1 + \phi^{-2k} + \phi^{-2k+4} + \phi^{-4k+4}}.\end{aligned}$$

Therefore $w(k+1) < w(k)$ is equivalent to $2\phi^{-2k+2} < \phi^{-2k} + \phi^{-2k+4}$ or, dividing by ϕ^{-2k}, to $2\phi^2 < 1 + \phi^4$, and therefore to $(\phi^2 - 1)^2 > 0$. Hence, for $k \geq 4$,

$$\begin{aligned}\frac{u(k+1)}{u(k)} &\leq \frac{1}{\phi^2} \cdot \frac{1+\phi^{-4}}{1+\phi^{-6}} \\ &< 0.415.\end{aligned}$$

For the constant 0.048 consider

$$\begin{aligned}\frac{u(k+1)}{1-u(k+1)} &= \frac{\phi^{-2k+2}(1-\phi^{-4})}{1+\phi^{-2k+2}} \Big/ \left(1 - \frac{\phi^{-2k+2}(1-\phi^{-4})}{1+\phi^{-2k+2}}\right) \\ &= \frac{\phi^{-2k+2}(1-\phi^{-4})}{1+\phi^{-2k+2} - \phi^{-2k+2}(1-\phi^{-4})} \\ &= \frac{1-\phi^{-4}}{\phi^{2k-2}+\phi^{-4}} \\ &\leq \frac{1-\phi^{-4}}{\phi^{6}+\phi^{-4}} \\ &< 0.048.\end{aligned}$$

This completes the proof of Assertion (2). Assertion (3) follows from Assertion (2) by evaluation of $v(k)$ at $k = 8$.

To prove the second assertion of the lemma we multiply the equation in Assertion (1) by n, obtaining

$$n \ln(F_k/F_{k-2}) = 2n \ln \phi + n \ln(1 - u(k)),$$

and we assume $n < 3k - 5$ so that

$$n \ln(1 - u(k)) > (3k - 5) \ln(1 - u(k)).$$

The right-hand side equals $v(k)$ which, by Assertion (3), is greater than -0.051. So,

$$n \ln(F_k/F_{k-2}) > 2n \ln \phi - 0.051,$$

and the proof is complete. □

Lemma 38. *Let n and k be integers, $n \geq 5$, $k \geq 8$. Then the following implications hold.*

1. *If k is even and $k - n/2 < 2.032$ then $A_n(F_{k-2}/F_k) < 0$.*
2. *If k is even and $k - n/2 > 2.052$ then $A_n(F_{k-2}/F_k) > 0$.*
3. *If k is odd and $k - n/2 < 2.004$ and $n < 3k - 5$ then $A_n(F_{k-2}/F_k) < 0$.*
4. *If k is odd and $k - n/2 > 2.034$ then $A_n(F_{k-2}/F_k) > 0$.*

Proof. For any $k \geq 8$ we have, by Lemma 30,

$$A(F_{k-2}/F_k) > 0 \quad \text{if and only if} \quad 4 \ln F_k - \ln 2 - n \ln \frac{F_k}{F_{k-2}} > 0. \tag{5.8}$$

We first consider the case that k is even. Then $F_k = (\phi^k - \phi^{-k})/\sqrt{5} = \phi^k(1 - \phi^{-2k})/\sqrt{5}$ and, hence,

$$4 \ln F_k = 4k \ln \phi + 4 \ln(1 - \phi^{-2k}) - 2 \ln 5.$$

Moreover, by Lemma 35,

$$n \ln(F_k/F_{k-2}) = 2n \ln \phi + e_1$$

where $e_1 > 0$ and, in case $n < 2k - 4$, $e_1 < 0.032$. Substituting into Equivalence (5.8) we obtain that $A(F_{k-2}/F_k) > 0$ if and only if

$$4k \ln \phi + 4 \ln(1 - \phi^{-2k}) - 2 \ln 5 - \ln 2 - 2n \ln \phi - e_1 > 0.$$

Dividing by $4\ln\phi$, $A(F_{k-2}/F_k) > 0$ if and only if $f(k,n) > 0$ where

$$f(k,n) = k - \frac{n}{2} + \frac{\ln(1-\phi^{-2k})}{\ln\phi} - \frac{\ln 5}{2\ln\phi} - \frac{\ln 2}{4\ln\phi} - \frac{e_1}{4\ln\phi}. \tag{5.9}$$

The numerical quantities can be bounded as follows; the first line uses $k \geq 8$.

$$\begin{array}{rcl} -0.001 < & \ln(1-\phi^{-2k})/\ln\phi & < 0, \\ -1.673 < & -\ln 5/(2\ln\phi) & < -1.672, \\ -0.361 < & -\ln 2/(4\ln\phi) & < -0.360, \\ -0.017 < & -0.032/(4\ln\phi) & < 0. \end{array}$$

So, given line (5.9), we have, on the one hand,

$$f(n,k) < k - n/2 - 1.672 - 0.360 = k - n/2 - 2.032,$$

proving Assertion (1). On the other hand, in case $n < 2(k - 2.052) < 2k - 4$,

$$f(n,k) > k - n/2 - 0.001 - 1.673 - 0.361 - 0.017 = k - n/2 - 2.052,$$

proving Assertion (2).

We now consider Equivalence (5.8) in the case that k is odd. Then $F_k = (\phi^k + \phi^{-k})/\sqrt{5} = \phi^k(1+\phi^{-2k})/\sqrt{5}$ and, hence,

$$4\ln F_k = 4k\ln\phi + 4\ln(1+\phi^{-2k}) - 2\ln 5.$$

By Lemma 37,

$$n\ln(F_k/F_{k-2}) = 2n\ln\phi - e_2$$

where $e_2 > 0$ and, in case $n < 3k - 5$, $e_2 < 0.051$. Substituting into Equivalence (5.8) we obtain that $A(F_{k-2}/F_k) > 0$ if and only if

$$4k\ln\phi + 4\ln(1+\phi^{-2k}) - 2\ln 5 - \ln 2 - 2n\ln\phi + e_2 > 0.$$

Dividing by $4\ln\phi$, $A(F_{k-2}/F_k) > 0$ if and only if $g(k,n) > 0$ where

$$g(k,n) = k - n/2 + \frac{\ln(1+\phi^{-2k})}{\ln\phi} - \frac{\ln 5}{2\ln\phi} - \frac{\ln 2}{4\ln\phi} + \frac{e_2}{4\ln\phi}. \tag{5.10}$$

The numerical quantities can be bounded as follows; the first line uses $k \geq 8$.

$$\begin{array}{rcl} 0 < & \ln(1+\phi^{-2k})/\ln\phi & < 0.001, \\ -1.673 < & -\ln 5/(2\ln\phi) & < -1.672, \\ -0.361 < & -\ln 2/(4\ln\phi) & < -0.360, \\ 0 < & 0.051/(4\ln\phi) & < 0.027. \end{array}$$

So, given line (5.10) we have, on the one hand, in case $n < 3k - 5$,

$$g(k,n) < k - n/2 + 0.001 - 1.672 - 0.360 + 0.027 = k - n/2 - 2.004,$$

proving Assertion (3). On the other hand,

$$g(k,n) > k - n/2 - 1.673 - 0.361 = k - n/2 - 2.034,$$

proving Assertion (4). □

Theorem 39. *Let* $n \geq 16$, *and let* k *be the least positive integer that satisfies* $A_n(F_{k-2}/F_k) > 0$. *Then* $k = \lfloor n/2 \rfloor + 3$.

Proof. Since $n \geq 16$ and $A_n(F_{k-2}/F_k) > 0$, we have $k > 8$ by Theorems 31 and 33; hence Lemma 38 applies.

We will first show that if $k \leq \lfloor n/2 \rfloor + 2$ then $A_n(F_{k-2}/F_k) \leq 0$, contradicting the definition of k. So assume that $k \leq \lfloor n/2 \rfloor + 2$. Then $k - n/2 \leq k - \lfloor n/2 \rfloor \leq 2$. Therefore, if k is even then $A_n(F_{k-2}/F_k) < 0$ by Lemma 38(1); if k is odd then, by Lemma 38(3), $A_n(F_{k-2}/F_k) < 0$ provided that $n < 3k - 5$. If, however, $n \geq 3k - 5$ then $k \leq n/3 + 5/3 < (3/8)n + 2$ so $A_n(F_{k-2}/F_k) \leq 0$ by the contrapositive of Theorem 33. Thus $k \leq \lfloor n/2 \rfloor + 2$ implies $A_n(F_{k-2}/F_k) \leq 0$, contradicting the definition of k. As a result, $k \geq \lfloor n/2 \rfloor + 3$.

To complete the proof we show that $k \leq \lfloor n/2 \rfloor + 3$. We do that by showing that $A_n(F_{K-2}/F_K) > 0$ for $K = \lfloor n/2 \rfloor + 3$. Note that $K > 8$ and $K - n/2 \geq K - \lfloor n/2 \rfloor - 1/2 = 2.5$. Therefore, if K is even then $A_n(F_{K-2}/F_K) > 0$ by Lemma 38(2); if K is odd then $A_n(F_{K-2}/F_K) > 0$ by Lemma 38(4). So, $A_n(F_{K-2}/F_K) > 0$ regardless of the parity of K. But then the minimality of k implies $k \leq K = \lfloor n/2 \rfloor + 3$.

Since $k \geq \lfloor n/2 \rfloor + 3$ and $k \leq \lfloor n/2 \rfloor + 3$ we conclude $k = \lfloor n/2 \rfloor + 3$. □

Theorem 40. *The height of the CF-tree of A_n is 4 if $n = 6$, 6 if $n = 10$, and otherwise $\lfloor n/2 \rfloor + 2$.*

Proof. Let k_n be the least of the positive integers k that satisfy $A_n(F_{k-2}/F_k) > 0$. By Theorem 29, $k_n - 1$ is the height of the CF-tree of A_n. For $n \leq 15$, the assertion can be verified by calculating k_n explicitly. For all other values of n, the assertion holds by Theorem 39. □

Taken together, Theorems 29 and 40 completely describe the CF-trees of the polynomials A_n.

6. The non-close roots

Following the plan of Section 3.2, we now consider the roots of A_n that are different from a_1 and a_2. We show that those roots are outside of or on a circle $\mathcal{C}_a$ with center and radius $a = 1/\phi^2$.

Lemma 41. *The minimum absolute value of the polynomial B on $\mathcal{C}_a$ is $3\sqrt{8a-3}$.*

Proof. Let $z = x + iy$ where x and y are real. Then

$$B(z) = z^2 - 3z + 1 = (x^2 - y^2 - 3x + 1) + i(2xy - 3y).$$

If z lies on $\mathcal{C}_a$ then $(x-a)^2 + y^2 = a^2$. In particular, $y^2 = 2ax - x^2$ and $0 \leq x \leq 2a$. Making the substitution for y^2 we obtain

$$B(z) = (2x^2 - 2ax - 3x + 1) + i(2x - 3)y.$$

Squaring the real part and the imaginary part yields

$$|B(z)|^2 = (4x^4 - 8ax^3 - 12x^3 + 4a^2x^2 + 12ax^2 + 13x^2 - 4ax - 6x + 1) + (4x^2 - 12x + 9)y^2.$$

Substituting again for y^2 and combining terms,

$$|B(z)|^2 = 4a^2x^2 - 12ax^2 + 4x^2 + 14ax - 6x + 1 = 14ax - 6x + 1,$$

since $a^2 - 3a + 1 = 0$. Thus the square of the norm on $\mathcal{C}_a$ is a linear function of x, $x \in [0, 2a]$, that achieves its minimum at $2a$, where the square of the norm is $28a^2 - 12a + 1$, which simplifies to $72a - 27 = 9(8a - 3)$, again using $a^2 = 3a - 1$. □

The following theorem is a specialization of Rouché's theorem which is proven, for example, in Marden's book (1949, Theorem 1.3).

Theorem 42. *If P and Q are polynomials, $\mathcal{C}$ is a circle, and $|P(z)| < |Q(z)|$ on $\mathcal{C}$, then, counting multiplicities, the polynomial $F = P + Q$ has the same number of zeros interior to $\mathcal{C}$ as does Q.*

Theorem 43. *The only roots of A_n inside $\mathcal{C}_a$ are a_1 and a_2.*

Proof. By Lemma 41, the minimum absolute value of the polynomial $-2B(z)^2$ on $\mathcal{C}_a$ is $18(8a - 3)$, which is greater than 1. The maximum absolute value of the polynomial z^n on $\mathcal{C}_a$ is $(2a)^n < 1$. Thus, by Rouché's theorem, Theorem 42, the polynomials $-2B(z)^2$ and $z^n - 2B(z)^2 = A_n(z)$ have the same number of roots inside $\mathcal{C}_a$. But the only root of $-2B(z)^2$ inside $\mathcal{C}_a$ is the double root a, so the polynomial A_n has exactly two roots inside $\mathcal{C}_a$. By Theorem 17, those roots must be a_1 and a_2. □

7. Linear fractional mappings of the non-close roots

We now track the images of the roots of A_n that are different from a_1 and a_2 under the linear fractional mappings m_k^{-1} as k increases. Those images turn out to be in the left half-plane and to converge to $-\phi$, the negative fixed point of $t^{-1} \circ r$.

Theorem 44. *Let α be any root of A_n other than a_1 and a_2, and let $k \geq 2$. Then $m_k^{-1}(\alpha)$ is in the left half-plane.*

Proof. Let α be a root of A_n, $\alpha \neq a_1$ and $\alpha \neq a_2$; note that $\alpha \neq 0$. We prove the theorem by induction on k. We first show that $m_2^{-1}(\alpha) = (F_3\alpha - F_1)/(-F_2\alpha + F_0) = \alpha^{-1} - 2$ is in the left half-plane. Let r_1 and r_2 be real numbers such that $\alpha = r_1 + ir_2$. By Theorem 43, α is not in the interior of circle $\mathcal{C}_a$ with center a and radius a. So, $(r_1 - a)^2 + r_2^2 \geq a^2$ which simplifies to

$$r_1^2 + r_2^2 \geq 2ar_1. \tag{7.1}$$

Now let r_3 and r_4 be real numbers such that $\alpha^{-1} = r_3 + ir_4$. We need to show that $r_3 < 2$. In case $|\alpha| > 1/2$ we clearly have $|\alpha^{-1}| < 2$ and hence $r_3 < 2$. In case $r_1 \leq 0$ we use $r_3 = r_1/(r_1^2 + r_2^2)$ to conclude $r_3 < 2$. In all other cases, $|\alpha| \leq 1/2$ and $r_1 > 0$. Then Inequality (7.1) yields $r_3 = r_1/(r_1^2 + r_2^2) \leq r_1/(2ar_1) = 1/(2a) < 4/3 < 2$. So, $r_3 < 2$ in every case, and hence $m_2^{-1}(\alpha)$ is in the left half-plane.

Now assume that, for some k, $k \geq 2$, $m_k^{-1}(\alpha)$ is in the left half-plane. Since the mappings r and t^{-1} map the left half-plane to itself, also $(t^{-1} \circ r \circ m_k^{-1})(\alpha) = m_{k+1}^{-1}(\alpha)$ is in the left half-plane. Thus, $m_k^{-1}(\alpha)$ is in the left half-plane for all $k \geq 2$. □

The following two lemmas are special cases of the well-known equality

$$F_{n+h}F_{n+k} - F_nF_{n+h+k} = (-1)^n F_h F_k$$

which is proved in a book by Vajda (1989, (20a)). For Lemma 45, let $h = 1$, $k = -1$ and then replace n by k; for Lemma 46, let $h = 1$, $k = 2$ and then replace n by $k - 2$.

Lemma 45. *For any integer k,*

$$F_{k+1}F_{k-1} - F_k^2 = (-1)^k.$$

Proof. Graham et al. (1994) call the equation "Cassini's identity" and give a short proof by induction on k. □

Lemma 46. *For any integer k,*

$$F_kF_{k-1} - F_{k+1}F_{k-2} = (-1)^k.$$

Proof. The left-hand side equals $F_kF_{k-1} - F_{k+1}(F_k - F_{k-1}) = F_kF_{k-1} - F_{k+1}F_k + F_{k+1}F_{k-1} = F_kF_{k-1} - (F_{k-1} + F_k)F_k + F_{k+1}F_{k-1} = -F_k^2 + F_{k+1}F_{k-1}$. Now apply Lemma 45. □

Lemma 47. *Let k be an integer, $k \geq 2$. Then*

$$m_k^{-1}(x) + \frac{F_{k+1}}{F_k} = \frac{(-1)^k}{F_k(F_kx - F_{k-2})}.$$

Proof. To $m_k^{-1}(x) = (-F_{k+1}x + F_{k-1})/(F_kx - F_{k-2})$ add F_{k+1}/F_k, and express the result with a common denominator,

$$\frac{F_k(-F_{k+1}x + F_{k-1}) + F_{k+1}(F_kx - F_{k-2})}{F_k(F_kx - F_{k-2})} = \frac{F_kF_{k-1} - F_{k+1}F_{k-2}}{F_k(F_kx - F_{k-2})}.$$

Apply Lemma 46 to the numerator to complete the proof. □

Lemma 48. *Let $k \geq 2$. Then a is between F_{k-2}/F_k and F_{k-1}/F_{k+1}, and ϕ is between F_k/F_{k-1} and F_{k+1}/F_k.*

Proof. The first assertion was shown in Theorem 26. The second assertion is a well-known fact in the theory of continued fractions (Hardy and Wright, 1938, Sections 10.13, 10.14). Indeed, F_k/F_{k-1} and F_{k+1}/F_k are successive convergents to ϕ. □

Theorem 49. *Let α be any root of A_n other than a_1 or a_2. For $k \geq 3$ let $\beta = m_k^{-1}(\alpha)$. Then $|\beta + F_{k+1}/F_k| < 5/F_k^2$. If $k \geq 4$ then $|\beta + F_{k+1}/F_k| < 4/F_k^2$. If $k \geq 5$ then $|\beta + F_{k+1}/F_k| < 3/F_k^2$.*

Proof. By Lemma 47,

$$\left|\beta + \frac{F_{k+1}}{F_k}\right| = \left|\frac{1}{F_k(F_k\alpha - F_{k-2})}\right| = \left|\frac{1}{F_k^2(\alpha - F_{k-2}/F_k)}\right|. \tag{7.2}$$

By the triangle inequality,

$$|\alpha - F_{k-2}/F_k| \geq |\alpha - a| - |a - F_{k-2}/F_k|. \tag{7.3}$$

By Theorem 43, $|\alpha - a| \geq a$. By Lemma 48, a is between F_{k-2}/F_k and F_{k-1}/F_{k+1}, so $|a - F_{k-2}/F_k| < |F_{k-2}/F_k - F_{k-1}/F_{k+1}| = 1/F_kF_{k+1}$ by Lemma 46. Substituting into Inequality (7.3) we have

$$|\alpha - F_{k-2}/F_k| > a - 1/F_kF_{k+1}.$$

Now if $k \geq 3$ then $a - 1/F_kF_{k+1} \geq a - 1/6 > 1/5$ so, from line (7.2), $|\beta + F_{k+1}/F_k| < 5/F_k^2$. If $k \geq 4$ then $a - 1/F_kF_{k+1} > a - 1/15 > 1/4$ so $|\beta + F_{k+1}/F_k| < 4/F_k^2$. If $k \geq 5$ then $a - 1/F_kF_{k+1} > a - 1/40 > 1/3$ so $|\beta + F_{k+1}/F_k| < 3/F_k^2$. □

Theorem 50. *Let α be any root of A_n other than a_1 or a_2. For $k \geq 3$ let $\beta = m_k^{-1}(\alpha)$. Then $|\beta + \phi| < 6/F_k^2$. If $k \geq 4$ then $|\beta + \phi| < 5/F_k^2$. If $k \geq 5$ then $|\beta + \phi| < 4/F_k^2$.*

Proof. By Lemma 48, ϕ is between F_{k+1}/F_k and F_{k+2}/F_{k+1}. We have

$$\begin{aligned} |\phi - F_{k+1}/F_k| &< |F_{k+1}/F_k - F_{k+2}/F_{k+1}| \\ &= 1/F_kF_{k+1} \\ &< 1/F_k^2 \end{aligned}$$

where the equality on the second line is due to Lemma 45. The assertions now follow from Theorem 49. □

Theorem 51. *Let α be any root of A_n other than a_1 or a_2. Let $k \geq 5$, and let $\beta = m_k^{-1}(\alpha)$. Then $\phi(1 - \phi^{-2k+6}) < |\beta| < \phi(1 + \phi^{-2k+6})$.*

Proof. By Theorem 50 and Lemma 32, $|\beta + \phi| < 4/F_k^2 \leq 4/\phi^{2k-4} < \phi^{-2k+7}$. Therefore $\phi - \phi^{-2k+7} < |\beta| < \phi + \phi^{-2k+7}$. □

Theorem 51 reflects the fact that $-\phi$ is an attractive fixed point of $t^{-1} \circ r$ with the local attraction rate $-1/\phi^2$.

8. The non-close roots and the transformed polynomials

We investigate the contribution of the roots of A_n that are different from a_1 and a_2 to the transformations B_k of A_n. Let $P_k(x) = p_{k,n-2}x^{n-2} + \cdots + p_{k,0}$ be the monic polynomial whose roots are those $m_k^{-1}(\alpha)$ such that α is a root of A_n other than a_1 or a_2. We show that, if k is large enough, $p_{k,n-2-i}$ is approximately $\binom{n-2}{i}\phi^i$.

Theorem 52. *Let m be a positive integer, and let $\beta_1, \ldots, \beta_m$ be nonzero complex numbers with $|arg(\beta_i)| < \pi/(4m)$ for all i, $1 \leq i \leq m$. Let $\beta = \beta_1 \cdots \beta_m$. Then $Re(\beta) > |\beta|/\sqrt{2}$.*

Proof. Let $\arg : \mathbb{C} - \{0\} \longrightarrow (-\pi, \pi]$ be the argument function (Knopp, 1952, for example), and let z_1 and z_2 be nonzero complex numbers. Then $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$ provided that $-\pi < \arg(z_1) + \arg(z_2) \leq \pi$. We clearly have $-\pi < \arg(\beta_1) + \cdots + \arg(\beta_k) \leq \pi$ for all k, $1 \leq k \leq m$. Hence $|\arg(\beta)| = |\arg(\beta_1) + \cdots + \arg(\beta_m)| \leq |\arg(\beta_1)| + \cdots + |\arg(\beta_m)| < \pi/4$. Thus $-\pi/4 < \arg(\beta) < \pi/4$ and, hence, $\mathrm{Re}(\beta) > |\beta|\cos(\pi/4) = |\beta|/\sqrt{2}$. □

Theorem 53. *Let α be any root of A_n other than a_1 or a_2. Let $k > 1.04 \ln n + 4.414$, and let $\beta = m_k^{-1}(\alpha)$. Then $|\beta + \phi| < \pi/(8n)$.*

Proof. Since $k \geq 5$, Theorem 50 and Lemma 32, respectively, yield $|\beta + \phi| < 4/F_k^2$ and $F_k^2 \geq \phi^{2k-4}$. Since the latter inequality implies $4/F_k^2 \leq 4/\phi^{2k-4}$, it suffices to show that $4/\phi^{2k-4} < \pi/(8n)$ or, equivalently, that $\phi^{-2k+4} < \pi/(32n)$. But that means showing $(2k-4)\ln \phi > \ln 32 + \ln n - \ln \pi$. Since $\ln \phi > 0.481$, $\ln 32 < 3.466$ and $\ln \pi > 1.144$, it suffices to show $0.481(2k-4) > \ln n + 2.322$ or, equivalently, $0.962k > \ln n + 4.246$. But the latter inequality can be obtained by multiplying both sides of the hypothesis $k > 1.04 \ln n + 4.414$ by 0.962. $\square$

Theorem 54. *Let α be any root of A_n other than a_1 or a_2. Let $k > 1.04 \ln n + 4.414$, and let $\beta = -m_k^{-1}(\alpha)$. Then $|arg(\beta)| < \pi/(4n)$.*

Proof. By Theorem 44, β is in the right half-plane. So, letting $x = |\arg(\beta)|$ we have $0 < x < \pi/2$. By Theorem 53, $|\beta - \phi| < \pi/(4n)$, so $|\mathrm{Im}(\beta)| < \pi/(4n)$. Moreover, $|\beta| \geq |\phi| - |\beta - \phi| > \phi - \pi/(4n) \geq \phi - \pi/20 > 5/4$. Hence,

$$\sin(x) = \frac{|\mathrm{Im}(\beta)|}{|\beta|} < \frac{\pi}{4n} \bigg/ \frac{5}{4} = \frac{\pi}{5n}. \tag{8.1}$$

Since $n \geq 5$ we have, in particular, $\sin(x) < \pi/25 < 1/2 = \sin(\pi/6)$. But the sine function on $(0, \pi/2)$ is monotonically increasing, so we have $x < \pi/6$.

Now note that, on $(0, \pi/6]$, $\sin(\xi)/\xi$ is a positive, monotone decreasing function: The function is positive since both $\sin(\xi)$ and ξ are positive; furthermore,

$$\frac{\sin(\xi)}{\xi} = 1 - \left(\frac{\xi^2}{3!} - \frac{\xi^4}{5!}\right) - \left(\frac{\xi^6}{7!} - \frac{\xi^8}{9!}\right) - \cdots,$$

where the parenthesized expressions are easily shown to be positive and monotone increasing on $(0, 1) \supset (0, \pi/6]$. Hence $\xi/\sin(\xi)$ is a monotone increasing function on $(0, \pi/6]$. In particular, $x/\sin(x) \leq (\pi/6)/\sin(\pi/6) = \pi/3 < 5/4$. So, $x < (5/4)\sin(x)$, and thus, using line (8.1), $x < \pi/(4n)$. $\square$

Theorem 55. *Let m be a positive integer, and let $\alpha_1, \ldots, \alpha_m$ be distinct roots of A_n, none equal to a_1 or a_2. Let $k > 1.04 \ln n + 4.414$. For any i, $1 \leq i \leq m$, let $\beta_i = -m_k^{-1}(\alpha_i)$. Let $\beta = \beta_1 \cdots \beta_m$. Then $Re(\beta) > |\beta|/\sqrt{2}$.*

Proof. By Theorem 54, $|\arg(\beta_i)| < \pi/(4n) < \pi/(4m)$ for all i since $m \leq n-2 < n$. Now the assertion follows from Theorem 52. $\square$

Theorem 56. *Let $k > 1.04 \ln n + 4.414$. Then*

$$\frac{1}{\sqrt{2}}\binom{n-2}{i}\phi^i(1-\phi^{-2k+6})^i < p_{k,n-2-i} < \binom{n-2}{i}\phi^i(1+\phi^{-2k+6})^i$$

for all i, $1 \leq i \leq n-2$.

Proof. The monic linear factors of P_k are the polynomials $x + \beta_j$ where $\beta_j = -m_k^{-1}(\alpha_j)$ and the α_j are the $n-2$ roots of A_n other than a_1 and a_2. Therefore $p_{k,n-2-i}$ is the sum of all $\binom{n-2}{i}$ products p_h of i of the β_j. By Theorem 51,

$$\phi(1-\phi^{-2k+6}) < |\beta_j| < \phi(1+\phi^{-2k+6})$$

for each j, so

$$\phi^i(1-\phi^{-2k+6})^i < |p_h| < \phi^i(1+\phi^{-2k+6})^i$$

for each product p_h. By Theorem 55, $\mathrm{Re}(p_h) > |p_h|/\sqrt{2}$ for each product p_h. Therefore

$$\begin{aligned}\mathrm{Re}(p_{k,n-2-i}) &= \mathrm{Re}\Big(\sum_h p_h\Big) = \sum_h \mathrm{Re}(p_h)\\ &> \sum_h \phi^i(1-\phi^{-2k+6})^i/\sqrt{2}\\ &= \frac{1}{\sqrt{2}}\binom{n-2}{i}\phi^i(1-\phi^{-2k+6})^i.\end{aligned}$$

Since the nonreal β_j occur in conjugate pairs, P_k is a real polynomial, that is, $\mathrm{Re}(p_{k,n-2-i}) = p_{k,n-2-i}$, proving the first inequality of the assertion. The second inequality follows from

$$\begin{aligned}p_{k,n-2-i} \le |p_{k,n-2-i}| \le \sum_h |p_h| &< \sum_h \phi^i(1+\phi^{-2k+6})^i\\ &= \binom{n-2}{i}\phi^i(1+\phi^{-2k+6})^i. \quad \square\end{aligned}$$

9. More on the two close real roots

We already have an upper bound for the distances of a_1 and a_2, the two close real roots of A_n, from $a = 1/\phi^2$. Now we derive a lower bound.

Theorem 57. *Let $h = a^{n/2+1}$, and let $\bar{h} = a^{1/2}h = a^{n/2+3/2}$. Then $a - h < a_1 < a - \bar{h}$ if $n \ge 5$, and $a + \bar{h} < a_2 < a + h$ if $n \ge 6$; if $n = 5$ then $a + h < a_2 < a + h + 0.002$.*

Proof. We will prove that $A_n(a-\bar{h}) > 0$ for $n \ge 5$. To do this we will prove that $(a-\bar{h})^n > 2B(a-\bar{h})^2$ for $n \ge 5$. First notice that $(a-\bar{h})^n = a^n(1-a^{n/2+1/2})^n$. Observe further that $1-a^{n/2+1/2} \ge 1-1/(3n)$ for $n \ge 5$ by induction. Moreover, $(1-1/(3n))^n$ is an increasing function of n that converges to $e^{-1/3}$ and, for $n = 5$, $(1-1/(3n))^n > 0.708$. So, $(a-\bar{h})^n > 0.708a^n = 0.708h^2/a^2 > 0.708 \cdot 6.854 \cdot h^2 > 4.852h^2$. On the other hand, $B(a-\bar{h})^2 = (3-2a+\bar{h})^2\bar{h}^2$. For $n \ge 5$, $\bar{h} \le a^4 < 0.022$; so, $B(a-\bar{h})^2 < (3-0.763+0.022)^2\bar{h}^2 < 5.104\bar{h}^2$ and $2B(a-\bar{h})^2 < 10.208\bar{h}^2 = 10.208ah^2 < 3.900h^2$. Thus, $(a-\bar{h})^n > 2B(a-\bar{h})^2$ and $A_n(a-\bar{h}) > 0$. By Lemma 16, also $A_n(a+\bar{h}) > 0$. By Theorems 13 and 14, $0 < x < 1$ and $A_n(x) > 0$ only if $a_1 < x < a_2$. Therefore, $a_1 < a-\bar{h}$ and $a+\bar{h} < a_2$. By Theorem 17, $a-h < a_1$ and, for $n \ge 6$, $a_2 < a+h$. This proves the first conclusion. The second was proved in Theorem 17. $\square$

Remark 58. The theorem can be used to show that the height of the CF-tree of A_n is at most $n/2+5$ without using the results of Section 5 that follow Theorem 26.

Let δ_1, δ_2 be such that $a_1 = (1+\delta_1)a$ and $a_2 = (1+\delta_2)a$. Note that $\delta_1 < 0$ and $\delta_2 > 0$.

Lemma 59. *Let $i = 1$ or $i = 2$, and let $n \ge 6$. Then $\phi^{-n-1} < |\delta_i| < \phi^{-n}$.*

Proof. By Theorem 57, $a+a^{n/2+3/2} < a_2 < a+a^{n/2+1}$. Equivalently, $\phi^{-2}+\phi^{-n-3} < a_2 < \phi^{-2}+\phi^{-n-2}$. By the definition of δ_2, $a_2 = (1+\delta_2)\phi^{-2}$. Substitution of this value for a_2 in the preceding inequality reveals that $\phi^{-n-1} < \delta_2 < \phi^{-n}$. A similar argument for a_1 shows that $\phi^{-n-1} < -\delta_1 < \phi^{-n}$. $\square$

Theorem 60. *If $n \ge 5$ then $0 < \delta_1 + \delta_2$. If $n \ge 6$ then $\delta_1 + \delta_2 < \phi^{-n-2}$.*

Proof. By definition, $a_1 = a + \delta_1 a$ and $A_n(a_1) = 0$ so, by Lemma 16, $A_n(a-\delta_1 a) > 0$. Since $A_n(x) < 0$ for $a_2 < x < 1$, this implies that $a - \delta_1 a < a_2 = a + \delta_2 a$, which implies $\delta_1 + \delta_2 > 0$. Now assume that $n \ge 6$. Then, by Lemma 59, $-\delta_1$ and δ_2 are both in the open interval (ϕ^{-n-1}, ϕ^{-n}) so that $\delta_1 + \delta_2 = |\delta_2 - (-\delta_1)| < \phi^{-n} - \phi^{-n-1} = \phi^{-n-2}$. $\square$

10. Linear fractional mappings of the close real roots

We now track the images of the two close real roots of A_n under the linear fractional mappings m_k^{-1} as k increases. So let $b_{k,1} = m_k^{-1}(a_1)$ and $b_{k,2} = m_k^{-1}(a_2)$ in this section and in Section 11. The two roots, a_1 and a_2, are close to a, and it turns out that $m_k^{-1}(a)$ is particularly easy to track.

Table 2
Proving Theorem 61. As k increases, the close real roots of B_k are centrifuged around $1/\phi$, the positive fixed point of $t^{-1} \circ r$. In each of the last five rows, $\{i, j\} = \{1, 2\}$.

Level (k)	The images of a_1, a_2 under m_k^{-1}								
0	$1/3$	$<$	$b_{0,1}$	$<$	$1/\phi^2$	$<$	$b_{0,2}$	$<$	$1/2$
1	1	$<$	$b_{1,2}$	$<$	ϕ	$<$	$b_{1,1}$	$<$	2
2	0	$<$	$b_{2,2}$	$<$	$1/\phi$	$<$	$b_{2,1}$	$<$	1
k	$1/2$	$<$	$b_{k,i}$	$<$	$1/\phi$	$<$	$b_{k,j}$	$<$	1
$k+1$	0	$<$	$b_{k+1,j}$	$<$	$1/\phi$	$<$	$b_{k+1,i}$	$<$	1
k'	$0 < b_{k',i}$	$<$	$1/2$	$<$	$1/\phi$	$<$	$b_{k',j}$	$<$	1
$k'+1$	0	$<$	$b_{k'+1,j}$	$<$	$1/\phi$	$<$	1	$<$	$b_{k'+1,i}$
$k'+2$	$b_{k'+2,i}$	$<$	0	$<$	$1/\phi$	$<$	$b_{k'+2,j}$		

Theorem 61. *Let h be the height of the CF tree of A_n, and let k be an integer, $0 \le k < h$. Then $b_{k,1}$ and $b_{k,2}$ are both positive.*

Proof. We have $b_{0,1} = a_1$ and $b_{0,2} = a_2$, so, by Theorem 14, $0 < b_{0,1} < a < b_{0,2}$. Since $a = \phi^{-2}$ and $n \ge 5$, we have $a^{n/2+1} = \phi^{-n-2} \le \phi^{-7}$. Hence, by Theorem 17, $b_{0,1} > a - \phi^{-7} > 0.381 - 0.035 > 1/3$ and $b_{0,2} < a + \phi^{-7} + 0.002 < 0.382 + 0.035 + 0.002 < 1/2$, and so

$$1/3 < b_{0,1} < 1/\phi^2 < b_{0,2} < 1/2.$$

Those inequalities appear in row "0" of Table 2. Row "1" shows the images of $b_{0,1}$ and $b_{0,2}$ under $m_1^{-1} = t^{-1} \circ r$; note that $(t^{-1} \circ r)(1/\phi^2) = \phi^2 - 1 = \phi$. Row "2" shows the images of $b_{1,1}$ and $b_{1,2}$ under t^{-1}; note that $t^{-1}(\phi) = 1/\phi$.

Regarding the remaining rows of the table we will say that *a row holds* if the inequalities of the row are satisfied for $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$. For example, row "$k + 1$" holds for $k = 1$. But row "$k + 1$" does not hold for all $k \ge 1$. Indeed, the row implies, by the Descartes rule, that $\text{var}(B_{k+1}) \ge 2$ which, by hypothesis, does not hold for $k = h - 1$. So let k' be minimal such that row "$k + 1$" does not hold for $k = k'$. Then $k' \ge 2$, and row "$k + 1$" holds for $k = k' - 1$. Hence,

$$\text{there are } i, j \text{ such that } \{i, j\} = \{1, 2\} \text{ and } 0 < b_{k',i} < 1/\phi < b_{k',j} < 1. \tag{10.1}$$

Now note that, for all k, $k \ge 2$, we have $m_{k+1}^{-1} = (t^{-1} \circ r) \circ m_k^{-1}$, so that row "$k$" implies row "$k + 1$". We conclude that row "k" does not hold for $k = k'$. Hence,

$$\text{there are no } i, j \text{ s. t. } \{i, j\} = \{1, 2\} \text{ and } 1/2 < b_{k',i} < 1/\phi < b_{k',j} < 1. \tag{10.2}$$

By (10.1) and (10.2) we have $0 < b_{k',i} \le 1/2$ and so, since $b_{k',i}$ is not rational, $0 < b_{k',i} < 1/2$. Hence row k' holds and, consequently, rows $k' + 1$ and $k' + 2$.

The table shows that $b_{k,1}$ and $b_{k,2}$ are positive for all k, $0 \le k < k' + 2$. To complete the proof we show $k' + 2 \ge h$ by showing that $\text{var}(B_{k'+2}) = 1$. Indeed, by row "$k' + 2$" and Theorem 44, $B_{k'+2}$ has exactly one positive root, so $\text{var}(B_{k'+2})$ is positive and odd by the Descartes rule. Moreover, by Theorem 19 and subadditivity, Theorem 21, $\text{var}(B_{k'+2}) \le 2$. Hence, $\text{var}(B_{k'+2}) = 1$. □

The next two lemmas are well-known. Benjamin and Quinn (2003, Corollaries 33 and 34) state them for positive k and use a different proof method.

Lemma 62. *Let k be an integer. Then $\phi^k = F_k\phi + F_{k-1}$.*

Proof. Induction on k using $\phi^2 = \phi + 1$ and the Fibonacci recurrence. □

Lemma 63. *Let k be an integer. Then $(-1)^{k+1}\phi^{-k} = F_k\phi - F_{k+1}$.*

Proof. Replace k by $-k$ in Lemma 62 to obtain $\phi^{-k} = F_{-k}\phi + F_{-k-1} = (-1)^{k+1}F_k\phi + (-1)^{k+2}F_{k+1}$. Multiplying by $(-1)^{k+1}$ completes the proof. □

Theorem 64. *Let $i = 1$ or $i = 2$. Then, for all k, $k \geq 2$,*

$$b_{k,i} - \frac{1}{\phi} = \frac{-1}{(-1)^k \phi^{-k+3} + F_k \phi \delta_i} \phi^{k+1} \delta_i.$$

Proof. Substituting $a_i = (1 + \delta_i)a$ for x in the formula for $m_k^{-1}(x)$ in Table 1,

$$\begin{aligned} b_{k,i} - \frac{1}{\phi} &= \frac{-F_{k+1}(1+\delta_i)\phi^{-2} + F_{k-1}}{F_k(1+\delta_i)\phi^{-2} - F_{k-2}} - \frac{1}{\phi} = \frac{-F_{k+1}(1+\delta_i) + F_{k-1}\phi^2}{F_k(1+\delta_i) - F_{k-2}\phi^2} - \frac{1}{\phi} \\ &= \frac{F_{k-1}\phi^3 - F_{k+1}(1+\delta_i)\phi - F_k(1+\delta_i) + F_{k-2}\phi^2}{F_k(1+\delta_i)\phi - F_{k-2}\phi^3} \\ &= \frac{F_k\phi^2 - F_k\phi - F_{k+1}\phi\delta_i - F_k(1+\delta_i)}{-F_{k-2}\phi^2 + F_{k-1}\phi + F_k\delta_i\phi} = \frac{-F_{k+1}\phi - F_k}{F_{k-3}\phi - F_{k-2} + F_k\phi\delta_i}\delta_i \\ &= \frac{-1}{(-1)^k \phi^{-k+3} + F_k\phi\delta_i}\phi^{k+1}\delta_i, \end{aligned}$$

where line 3 is obtained using $F_k = F_{k-1} + F_{k-2}$ and $\phi^2 = \phi + 1$, and the last line is obtained using Lemmas 62 and 63. □

Lemma 65. *If $k \geq 12$ then*

$$1.17080 < \frac{F_k}{\phi^{k-2}} < 1.17083.$$

Proof. The assertion clearly holds for $k = 12$ and $k = 13$. Now let j be an integer, $j \geq 12$, and assume that the assertion holds for $k = j$ and $k = j + 1$. Then

$$1.17080(\phi^{j-2} + \phi^{j-1}) < F_j + F_{j+1} < 1.17083(\phi^{j-2} + \phi^{j-1}).$$

Since $1 + \phi = \phi^2$, the left-hand side equals $1.17080\phi^j$, and the right-hand side equals $1.17083\phi^j$. The sum in the middle equals F_{j+2}. Hence the assertion holds for $k = j + 2$. □

Theorem 66. *Let $n \geq 28$ and $12 \leq k \leq n/2 - 2$ (and note that $\delta_1 < 0$ and $\delta_2 > 0$). Then, for even k,*

$$-\phi^{2k-2}\delta_1 < b_{k,1} - \frac{1}{\phi} < -1.026\phi^{2k-2}\delta_1, \tag{10.3}$$

$$-\phi^{2k-2}\delta_2 < b_{k,2} - \frac{1}{\phi} < -0.975\phi^{2k-2}\delta_2. \tag{10.4}$$

For odd k,

$$\phi^{2k-2}\delta_1 < b_{k,1} - \frac{1}{\phi} < 0.975\phi^{2k-2}\delta_1, \tag{10.5}$$

$$\phi^{2k-2}\delta_2 < b_{k,2} - \frac{1}{\phi} < 1.026\phi^{2k-2}\delta_2. \tag{10.6}$$

Proof. Let $i = 1$ or $i = 2$. Then $|F_k\phi\delta_i| > 0$, so

$$\frac{1}{\phi^{-k+3} + |F_k\phi\delta_i|} < \phi^{k-3} < \frac{1}{\phi^{-k+3} - |F_k\phi\delta_i|}. \tag{10.7}$$

By Lemma 65, $F_k\phi < 1.171\phi^{k-1}$ and, by Lemma 59, $|\delta_i| < \phi^{-n}$. Therefore, $|F_k\phi\delta_i| < 1.171\phi^{k-n-1}$. By hypothesis, $-n \leq -2k - 4$, so

$$|F_k\phi\delta_i| < 1.171\phi^{-k-5} = 1.171\phi^{-8}\phi^{-k+3} < 0.025\phi^{-k+3}.$$

In particular,

$$|F_k\phi\delta_i| < 0.025\phi^{-k+3} \tag{10.8}$$

and hence

$$|F_k\phi\delta_i| < \frac{0.025}{0.975}\phi^{-k+3}. \tag{10.9}$$

By Inequality (10.8), $\phi^{-k+3} - |F_k\phi\delta_i| > 0.975\phi^{-k+3}$, hence

$$\frac{1}{\phi^{-k+3} - |F_k\phi\delta_i|} < \frac{1}{0.975}\phi^{k-3} < 1.026\phi^{k-3}. \tag{10.10}$$

By Inequality (10.9), $0.975\phi^{k-3}|F_k\phi\delta_i| < 0.025$, hence $0.975\phi^{k-3}(\phi^{-k+3} + |F_k\phi\delta_i|) < 1$ and so

$$0.975\phi^{k-3} < \frac{1}{\phi^{-k+3} + |F_k\phi\delta_i|}. \tag{10.11}$$

Combining (10.7) and (10.10),

$$\phi^{k-3} < \frac{1}{\phi^{-k+3} - |F_k\phi\delta_i|} < 1.026\phi^{k-3}. \tag{10.12}$$

Combining (10.7) and (10.11),

$$0.975\phi^{k-3} < \frac{1}{\phi^{-k+3} + |F_k\phi\delta_i|} < \phi^{k-3}. \tag{10.13}$$

Now suppose that k is even. Then, by Theorem 64, and since $\delta_1 = -|\delta_1|$,

$$b_{k,1} - \frac{1}{\phi} = \frac{1}{\phi^{-k+3} - |F_k\phi\delta_1|}\phi^{k+1}|\delta_1|,$$

so, multiplying line (10.12) by $\phi^{k+1}|\delta_1|$ yields

$$\phi^{k-3}\phi^{k+1}|\delta_1| < b_{k,1} - \frac{1}{\phi} < 1.026\phi^{k-3}\phi^{k+1}|\delta_1|,$$

or, equivalently, Assertion (10.3). Again by Theorem 64, and since $\delta_2 = |\delta_2|$,

$$b_{k,2} - \frac{1}{\phi} = \frac{-1}{\phi^{-k+3} + |F_k\phi\delta_2|}\phi^{k+1}\delta_2,$$

so, multiplying line (10.13) by $\phi^{k+1}\delta_2$ yields

$$0.975\phi^{k-3}\phi^{k+1}\delta_2 < -\left(b_{k,2} - \frac{1}{\phi}\right) < \phi^{k-3}\phi^{k+1}\delta_2,$$

or, equivalently, Assertion (10.4).

Next suppose that k is odd. Then, by Theorem 64, and since $\delta_1 = -|\delta_1|$,

$$b_{k,1} - \frac{1}{\phi} = \frac{1}{\phi^{-k+3} - F_k\phi\delta_1}\phi^{k+1}\delta_1 = \frac{1}{\phi^{-k+3} + |F_k\phi\delta_1|}\phi^{k+1}\delta_1,$$

so, multiplying line (10.13) by $\phi^{k+1}\delta_1$ yields Assertion (10.5). Again by Theorem 64, and since $\delta_2 = |\delta_2|$,

$$b_{k,2} - \frac{1}{\phi} = \frac{1}{\phi^{-k+3} - |F_k\phi\delta_2|}\phi^{k+1}\delta_2,$$

so, multiplying line (10.12) by $\phi^{k+1}\delta_2$ yields Assertion (10.6). □

Theorem 66 reflects the fact that $1/\phi$ is a repulsive fixed point of $t^{-1} \circ r$ with the local repulsion rate $-\phi^2$.

11. The close real roots and the transformed polynomials

We investigate the contribution of the close real roots a_1 and a_2 of A_n to the transformations B_k of A_n. Let $Q_k(x)$ be the monic quadratic polynomial that has $b_{k,1}$ and $b_{k,2}$ as roots and that is a divisor of $B_k(x)$. Theorems 67–69 serve to compare $Q_k(x)$ to the polynomial $x^2 - (2/\phi)x + 1/\phi^2$.

Theorem 67. *Let $n \geq 28$ and $12 \leq k \leq n/2 - 2$. Then, for even k,*

$$-\phi^{-n+2k-4} < b_{k,1} + b_{k,2} - \frac{2}{\phi} < 0.051\phi^{-n+2k-2}.$$

For odd k,

$$0 < b_{k,1} + b_{k,2} - \frac{2}{\phi} < 1.109\phi^{-n+2k-4}.$$

Proof. First let k be even. In Theorem 66, add lines (10.3) and (10.4) to obtain

$$\begin{aligned} &-(\delta_1 + \delta_2)\phi^{2k-2} < b_{k,1} + b_{k,2} - 2/\phi \\ &\qquad < (-1.026\delta_1 - 0.975\delta_2)\phi^{2k-2} = -0.975(\delta_1 + \delta_2)\phi^{2k-2} - 0.051\delta_1\phi^{2k-2}. \end{aligned}$$

By Theorem 60, $\delta_1 + \delta_2 < \phi^{-n-2}$, so $-\phi^{-n+2k-4} < b_{k,1} + b_{k,2} - 2/\phi$. By the same theorem, $\delta_1 + \delta_2 > 0$ and, by Lemma 59, $-\delta_1 = |\delta_1| < \phi^{-n}$. Hence, $b_{k,1} + b_{k,2} - 2/\phi < 0.051\phi^{-n+2k-2}$.

Now let k be odd. In Theorem 66, add lines (10.5) and (10.6), again apply Theorem 60, and note that, by Lemma 59, $\delta_2 = |\delta_2| < \phi^{-n}$; this yields

$$\begin{aligned} 0 &< (\delta_1 + \delta_2)\phi^{2k-2} < b_{k,1} + b_{k,2} - 2/\phi \\ &< (0.975\delta_1 + 1.026\delta_2)\phi^{2k-2} < 0.975(\delta_1 + \delta_2)\phi^{2k-2} + 0.051\delta_2\phi^{2k-2} \\ &< (0.975 + 0.051\phi^2)\phi^{-n+2k-4} < 1.109\phi^{-n+2k-4}. \quad \square \end{aligned}$$

Theorem 68. *Let $n \geq 28$ and $12 \leq k \leq n/2 - 2$. Then*

$$-0.438\phi^{-n+2k-3} < b_{k,1}b_{k,2} - \frac{1}{\phi^2} < 0.424\phi^{-n+2k-3}.$$

Proof. By Theorem 66 and Lemma 59 we have, for even k,

$$\frac{1}{\phi} + \phi^{-n+2k-3} < \frac{1}{\phi} + \phi^{2k-2}|\delta_1| < b_{k,1} < \frac{1}{\phi} + 1.026\phi^{2k-2}|\delta_1| < \frac{1}{\phi} + 1.026\phi^{-n+2k-2}$$

and

$$\frac{1}{\phi} - \phi^{-n+2k-2} < \frac{1}{\phi} - \phi^{2k-2}\delta_2 < b_{k,2} < \frac{1}{\phi} - 0.975\phi^{2k-2}\delta_2 < \frac{1}{\phi} - 0.975\phi^{-n+2k-3}.$$

For odd k,

$$\frac{1}{\phi} - \phi^{-n+2k-2} < \frac{1}{\phi} - \phi^{2k-2}|\delta_1| < b_{k,1} < \frac{1}{\phi} - 0.975\phi^{2k-2}|\delta_1| < \frac{1}{\phi} - 0.975\phi^{-n+2k-3}$$

and

$$\frac{1}{\phi} + \phi^{-n+2k-3} < \frac{1}{\phi} + \phi^{2k-2}\delta_2 < b_{k,2} < \frac{1}{\phi} + 1.026\phi^{2k-2}\delta_2 < \frac{1}{\phi} + 1.026\phi^{-n+2k-2}.$$

So, for any k,

$$\begin{aligned} &\left(\frac{1}{\phi} + \phi^{-n+2k-3}\right)\left(\frac{1}{\phi} - \phi^{-n+2k-2}\right) < b_{k,1}b_{k,2} \\ &\qquad < \left(\frac{1}{\phi} - 0.975\phi^{-n+2k-3}\right)\left(\frac{1}{\phi} + 1.026\phi^{-n+2k-2}\right). \end{aligned}$$

The upper left side is equal to

$$\begin{aligned}\phi^{-2}-\phi^{-n+2k-3}+\phi^{-n+2k-4}-\phi^{-2n+4k-5} &= \phi^{-2}-\phi^{-n+2k-3}(1-\phi^{-1}+\phi^{-n+2k-2})\\ &\geq \phi^{-2}-\phi^{-n+2k-3}(1-\phi^{-1}+\phi^{-6})\\ &> \phi^{-2}-0.438\phi^{-n+2k-3},\end{aligned}$$

while the lower right side is strictly less than

$$\begin{aligned}\phi^{-2}+\phi^{-n+2k-3}(1.026-0.975/\phi) &< \phi^{-2}+\phi^{-n+2k-3}(1.026-0.602)\\ &= \phi^{-2}+0.424\phi^{-n+2k-3}. \quad \square\end{aligned}$$

The polynomial Q_k was defined as the monic quadratic polynomial with roots $b_{k,1}$ and $b_{k,2}$. Now let c_k and d_k be such that $Q_k(x) = x^2 - c_k x + d_k$. Then $c_k = b_{k,1} + b_{k,2}$ and $d_k = b_{k,1}b_{k,2}$.

Theorem 69. *Let $n \geq 28$ and $12 \leq k \leq n/2-2$. Then $|c_k - 2/\phi| < 1.109\phi^{-n+2k-4}$ and $|d_k - \phi^{-2}| < 0.438\phi^{-n+2k-3}$. In particular, $1.212 < c_k < 1.261$ and $0.365 < d_k < 0.398$.*

Proof. The first assertion is immediate from Theorem 67 if k is odd. It also follows from Theorem 67 if k is even since $\phi^{-n+2k-4} < 1.109\phi^{-n+2k-4}$ and also

$$0.051\phi^{-n+2k-2} = 0.051\phi^2\phi^{-n+2k-4} < 0.134\phi^{-n+2k-4} < 1.109\phi^{-n+2k-4}.$$

The second assertion is immediate from Theorem 68.

The hypothesis $k \leq n/2-2$ is equivalent to $-n+2k-4 \leq -8$ and to $-n+2k-3 \leq -7$. Thus the two proven assertions imply

$$\begin{aligned}c_k &> 2/\phi - 1.109\phi^{-n+2k-4} \geq 2/\phi - 1.109\phi^{-8} > 1.236-0.024 = 1.212,\\ c_k &< 2/\phi + 1.109\phi^{-n+2k-4} \leq 2/\phi + 1.109\phi^{-8} < 1.237+0.024 = 1.261,\\ d_k &> \phi^{-2} - 0.438\phi^{-n+2k-3} \geq \phi^{-2} - 0.438\phi^{-7} > 0.381-0.016 = 0.365,\\ d_k &< \phi^{-2} + 0.438\phi^{-n+2k-3} \leq \phi^{-2} + 0.438\phi^{-7} < 0.382+0.016 = 0.398. \quad \square\end{aligned}$$

12. The transformed polynomials

The polynomial Q_k represents the contribution of the two close roots of A_n to the polynomial transformation B_k of A_n; the polynomial P_k represents the contribution of the non-close roots of A_n. We combine those contributions by letting $R_k(x) = r_{k,n}x^n + r_{k,n-1}x^{n-1} + \cdots + r_{k,0} = P_k(x)Q_k(x)$ so that $R_k(x)$ is the monic associate of the integer polynomial $B_k(x)$. We show that the low-order and high-order coefficients of the polynomial B_k are large negative integers.

Theorem 70. *Let n and k be integers such that $n \geq 28$, $12 \leq k \leq n/2-2$ and $k > 1.04\ln n + 4.414$. Then $r_{k,n-1} > (n-2)/\sqrt{2}$ and $r_{k,n-2} > 21(n-2)$.*

Proof. To show the first assertion note that $r_{k,n-1} = p_{k,n-3} - c_k$. By Theorem 56, $p_{k,n-3} > (n-2)(\phi - \phi^{-2k+7})/\sqrt{2} > (n-2)(\phi-\phi^{-17})/\sqrt{2} > 1.14(n-2)$. By Theorem 69, and since $n-2 \geq 26$, $c_k < 1.261 < 0.05(n-2)$. So $r_{k,n-1} > 1.09(n-2) > (n-2)/\sqrt{2}$.

For the second assertion note that $r_{k,n-2} = p_{k,n-4} - c_k p_{k,n-3} + d_k$. But, by Theorem 69, $d_k > 0$, so $r_{k,n-2} > p_{k,n-4} - c_k p_{k,n-3}$. We complete the proof with the sequence of inequalities

$$\begin{aligned}p_{k,n-4} - c_k p_{k,n-3} &> \frac{1}{\sqrt{2}}\binom{n-2}{2}\phi^2(1-\phi^{-2k+6})^2 - c_k(n-2)\phi(1+\phi^{-2k+6})\\ &> (n-2)\phi\left(\frac{0.999}{2\sqrt{2}}(n-3)\phi - 1.001c_k\right)\\ &> (n-2)\phi\left(\frac{0.999}{2.829}\cdot 25\cdot 1.618 - 1.263\right)\\ &> (14.284-1.263)\phi(n-2)\\ &> 21(n-2)\end{aligned}$$

where the first inequality is due to Theorem 56. The second inequality holds since $\phi^{-2k+6} < 0.0002$ by the hypothesis that $k \geq 12$. For the third inequality we use $n-3 \geq 25$ and, again, $c_k < 1.261$. □

Lemma 71. *Let u, v be real numbers, $u > 0$ and $v > -1$. Then the function*

$$x \longmapsto \left(\frac{1-x}{1+x}\right)^{ux^v}$$

is monotone decreasing on $(0, 1)$.

Proof. The function that maps x to

$$\begin{aligned} ux^v \ln\left(\frac{1-x}{1+x}\right) &= ux^v(\ln(1-x) - \ln(1+x)) \\ &= ux^v\left(\left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots\right) - \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots\right)\right) \\ &= -2ux^v\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots\right) \\ &= -2u\left(x^{1+v} + \frac{1}{3}x^{3+v} + \frac{1}{5}x^{5+v} + \cdots\right) \end{aligned}$$

is clearly a decreasing function on $(0, 1)$. But the exponential of a decreasing function is also a decreasing function. □

Lemma 72. *Let $n > 1$, and let $k \geq 2.08 \ln n + 3$. Then*

$$\frac{1-\phi^{-2k+6}}{1+\phi^{-2k+6}} \geq \frac{1-1/n^2}{1+1/n^2}.$$

Proof. We have $\ln(\phi^{2(k-3)}) = (k-3)\ln(\phi^2) > 0.962(k-3) \geq 0.962 \cdot 2.08 \cdot \ln(n) > 2\ln n = \ln(n^2)$. So $\phi^{2(k-3)} > n^2$ and, hence, $\phi^{-2k+6} < 1/n^2$. But $n > 1$, so both $1/n^2$ and ϕ^{-2k+6} are in the interval $(0, 1)$, and Lemma 71 can be applied with $u = 1$ and $v = 0$. □

Theorem 73. *Let n and k be integers such that $n \geq 28$, $12 \leq k \leq n/2 - 2$ and $k \geq 2.08 \ln n + 3$. Then $r_{k,n-i} > d_k p_{k,n-i}$ for all i, $3 \leq i \leq 0.45n$.*

Proof. Since $R_k = P_k Q_k$, we have $r_{k,n-i} = p_{k,n-2-i} - c_k p_{k,n-1-i} + d_k p_{k,n-i}$, so it suffices to show that $p_{k,n-2-i} \geq c_k p_{k,n-1-i}$. We first note that, by Theorem 56, $p_{k,n-1-i} = p_{k,n-2-(i-1)} > 0$. Indeed, the hypotheses of the theorem are satisfied since $k \geq 2.08 \ln n + 3 \geq 1.04 \ln n + (1.04 \ln 28 + 3) > 1.04 \ln n + 4.414$. So, after dividing by $p_{k,n-1-i}$ it suffices to show that $p_{k,n-2-i}/p_{k,n-1-i} \geq c_k$.

Applying Theorem 56 again,

$$\begin{aligned} \frac{p_{k,n-2-i}}{p_{k,n-1-i}} &> \binom{n-2}{i}\phi^i(1-\phi^{-2k+6})^i \Big/ \sqrt{2}\binom{n-2}{i-1}\phi^{i-1}(1+\phi^{-2k+6})^{i-1} \\ &= \frac{n-i-1}{i} \cdot \frac{\phi}{\sqrt{2}} \cdot (1-\phi^{-2k+6}) \cdot \left(\frac{1-\phi^{-2k+6}}{1+\phi^{-2k+6}}\right)^{i-1}. \end{aligned}$$

We will analyze the four factors of the latter expression from left to right. Since $i \leq 0.45n$ we have $(n-i-1)/i \geq (0.55n-1)/(0.45n) = 0.55/0.45 - 1/(0.45n) \geq 0.55/0.45 - 1/(0.45 \cdot 28) > 1.2222 - 0.0794 = 1.1428$. Furthermore, $\phi/\sqrt{2} > 1.144$ and, since $k \geq 12$, $1 - \phi^{-2k+6} > 0.9998$. Using $i - 1 < n/2$, Lemma 72, and Lemma 71 with $u = 1/2$, $v = -1/2$ and $x = n^{-2}$,

$$\begin{aligned} \left(\tfrac{1-\phi^{-2k+6}}{1+\phi^{-2k+6}}\right)^{i-1} &> \left(\tfrac{1-\phi^{-2k+6}}{1+\phi^{-2k+6}}\right)^{n/2} \geq \left(\tfrac{1-1/n^2}{1+1/n^2}\right)^{n/2} \\ &\geq \left(\tfrac{1-1/28^2}{1+1/28^2}\right)^{14} > 0.9649. \end{aligned}$$

In summary,

$$\frac{p_{k,n-2-i}}{p_{k,n-1-i}} > 1.1428 \cdot 1.144 \cdot 0.9998 \cdot 0.9649 > 1.261.$$

But $1.261 > c_k$ by Theorem 69. □

Theorem 74. *Let n and k be integers such that $n \geq 28$, $12 \leq k \leq n/2-2$ and $k \geq 2.08 \ln n + 3$. Then $r_{k,i} > 1$ for all i, $0 \leq i \leq n/10$.*

Proof. We will apply Theorems 56 and 69. The hypotheses of Theorem 69 are clearly satisfied, and the hypotheses of Theorem 56 are satisfied since $k \geq 2.08 \ln n + 3 \geq 1.04 \ln n + (1.04 \ln 28 + 3) > 1.04 \ln n + 4.414$. We begin by rewriting the inequalities given by Theorem 56 as follows.

$$\frac{1}{\sqrt{2}}\binom{n-2}{i}\phi^{n-2-i}(1-\phi^{-2k+6})^{n-2-i} < p_{k,i} < \binom{n-2}{i}\phi^{n-2-i}(1+\phi^{-2k+6})^{n-2-i}$$

for all i such that $0 \leq i < n-2$.

We first discuss $r_{k,0} = d_k p_{k,0}$. By Theorem 56 rewritten, $p_{k,0} > \phi^{n-2}(1-\phi^{-2k+6})^{n-2}/\sqrt{2}$. Since $k \geq 12$, $(1-\phi^{-2k+6}) > 0.999$ and therefore $\phi(1-\phi^{-2k+6}) > 1.616$. So $p_{k,0} > 1.616^{26}/\sqrt{2}$. By Theorem 69, $d_k > 0.365$, so $r_{k,0} = d_k p_{k,0} > 0.365 \cdot 1.616^{26}/\sqrt{2} > 1$.

Now we discuss $r_{k,1} = d_k p_{k,1} - c_k p_{k,0}$ and, for $2 \leq i \leq n/10$, $r_{k,i} = p_{k,i-2} - c_k p_{k,i-1} + d_k p_{k,i}$. We shall consider the ratio $d_k p_{k,i}/c_k p_{k,i-1}$ for all i, $1 \leq i \leq n/10$. By Theorem 56 rewritten,

$$\frac{d_k p_{k,i}}{c_k p_{k,i-1}} > d_k \, \frac{n-i-1}{i}\left(\frac{1-\phi^{-2k+6}}{1+\phi^{-2k+6}}\right)^{n-2-i} \Big/ c_k\sqrt{2}\phi(1+\phi^{-2k+6}).$$

Again, we have $d_k > 0.365$. Using Lemma 72, $n-2-i < n$, and Lemma 71 with $u=1$, $v=-1/2$ and $x = n^{-2}$,

$$\begin{aligned}\left(\tfrac{1-\phi^{-2k+6}}{1+\phi^{-2k+6}}\right)^{n-2-i} &\geq \left(\tfrac{1-1/n^2}{1+1/n^2}\right)^{n-2-i} > \left(\tfrac{1-1/n^2}{1+1/n^2}\right)^{n} \\ &\geq \left(\tfrac{1-1/28^2}{1+1/28^2}\right)^{28} > 0.931.\end{aligned}$$

By Theorem 69, $c_k < 1.261$. Also, $\sqrt{2} < 1.4143$, $\phi < 1.6181$, and $1+\phi^{-2k+6} \leq 1+\phi^{-18} < 1.0002$. Evaluating,

$$\frac{d_k p_{k,i}}{c_k p_{k,i-1}} > \frac{0.365 \cdot 0.931}{1.261 \cdot 1.4143 \cdot 1.6181 \cdot 1.0002} \cdot \frac{n-i-1}{i} > 0.1177 \cdot \frac{n-i-1}{i}.$$

For $i=1$ we obtain $d_k p_{k,1}/c_k p_{k,0} > 0.1177(n-2) \geq 0.1177 \cdot 26 > 2$ and therefore $r_{k,1} > c_k p_{k,0}$. But $c_k p_{k,0} > p_{k,0}$ by Theorem 69, and we already determined that $p_{k,0} > 1$. So $r_{k,1} > 1$. For i such that $2 \leq i \leq n/10$ we have

$$\frac{n-i-1}{i} \geq \frac{n-i}{i} - \frac{1}{2} \geq 9 - \frac{1}{2} = 8.5,$$

so $d_k p_{k,i}/c_k p_{k,i-1} > 0.1177 \cdot 8.5 > 1$. Therefore, $d_k p_{k,i} > c_k p_{k,i-1}$ and, hence, $r_{k,i} > p_{k,i-2}$. By Theorem 56 rewritten, $p_{k,i-2} > (\phi(1-\phi^{-2k+6}))^{n-i}/\sqrt{2}$. We already determined that $\phi(1-\phi^{-2k+6}) > 1.616$. So,

$$p_{k,i-2} > \frac{1.616^{n-i}}{\sqrt{2}} \geq \frac{1.616^{0.9n}}{\sqrt{2}} > \frac{1.54^n}{\sqrt{2}} > 1,$$

and thus $r_{k,i} > 1$. □

In the remainder of this section, and in Section 13, let the polynomial B_k be given as $B_k(x) = b_{k,n}x^n + \cdots + b_{k,0}$.

Theorem 75. *Let n and k be integers, $n \geq 5$ and $2 \leq k \leq n/2-2$. Then $b_{k,n}$ and $b_{k,0}$ are both negative and $b_{k+1,n} = b_{k,0}$.*

Proof. Using the entry for A_n in Table 1 we obtain $B_1(x) = (\mathrm{T} \circ \mathrm{R})(A_n(x)) = \mathrm{T}(-2x^n + 12x^{n-1} - 22x^{n-2} + 12x^{n-3} - 2x^{n-4} + 1)$. Since T preserves leading coefficients, we have $b_{1,n} = -2$. Since $b_{1,0}$ is the sum of the coefficients of $\mathrm{R}(A_n)$, we have $b_{1,0} = -1$. The leading coefficient of B_2 is the same as the leading coefficient of B_1 since $B_2 = \mathrm{T}(B_1)$. So $b_{2,n} = -2$.

Now let $2 \le k \le n/2 - 2$. Then the product of the negatives of the roots of B_k is positive. Indeed, by Theorem 61, the two close real roots are positive. By Theorem 44, all the other roots of B_k are in the left half-plane. Of those roots, the real roots are negative and the nonreal roots occur in conjugate pairs. It follows that the product of the negatives of the roots of B_k is positive. Therefore, $b_{k,n}$ and $b_{k,0}$ have the same sign whenever $2 \le k \le n/2 - 2$.

In particular, $b_{2,n}$ and $b_{2,0}$ are both negative. This is the basis of a proof by induction on k. For the induction step observe that, if $b_{k,0} < 0$ for some k, $2 \le k \le n/2 - 3$, then $b_{k+1,n} = b_{k,0} < 0$ since $B_{k+1} = (\mathrm{T} \circ \mathrm{R})(B_k)$ and T preserves leading coefficients. □

Theorem 76. *Let $n \ge 28$, let k_1 be the least integer that is at least 12 and at least $2.08 \ln n + 3$, and let k_2 be the greatest integer that is at most $n/2 - 2$. Let $\bar{\phi} = \phi(1 - \phi^{-18})$. Then $b_{k,i} \le -\bar{\phi}^{(n-5)(k-k_1)}$ for all k, $k_1 \le k \le k_2$ and all i, $0.55n \le i \le n$.*

Proof. Let k be such that $k_1 \le k \le k_2$. Then $k \ge 2.08 \ln n + 3 \ge 1.04 \ln n + 1.04 \ln n + 3 \ge 1.04 \ln n + 1.04 \ln 28 + 3 > 1.04 \ln n + 4.414$, so Theorems 56, 70 and 73 are applicable. Since R_k is monic and of degree n, $r_{k,n} = 1$. By Theorem 70, $r_{k,n-1} > 1$ and $r_{k,n-2} > 1$. By Theorem 73, $r_{k,n-i} > d_k p_{k,n-i}$ for all i, $3 \le i \le 0.45n$. By Theorem 56, $p_{k,n-2-i} > \binom{n-2}{i}\bar{\phi}^i/\sqrt{2}$ for all i, $1 \le i \le n-2$, hence $p_{k,n-i} > \binom{n-2}{i-2}\bar{\phi}^{i-2}/\sqrt{2}$ for all i, $3 \le i \le n$. But $\bar{\phi} > 1$ and $\binom{n-2}{i-2} \ge n-2$ for all i, $3 \le i \le 0.45n$, and so $r_{k,n-i} > (n-2)d_k/\sqrt{2} \ge 26 d_k/\sqrt{2}$. By Theorem 69, $d_k > 0.365$. So $r_{k,n-i} > 26 \cdot 0.365/\sqrt{2} > 1$ for all i, $3 \le i \le 0.45n$. So far we have shown that $r_{k,i} \ge 1$ for all k, $k_1 \le k \le k_2$, and for all i, $0.55n \le i \le n$.

Let k again be such that $k_1 \le k \le k_2$. Since R_k is the monic associate of B_k we have $b_{k,i} = b_{k,n} \cdot r_{k,i}$ for all i, $0 \le i \le n$. By Theorem 75, $b_{k,n}$ is negative. Since $b_{k,n}$ is also an integer, we have $b_{k,n} \le -1$. Hence, $b_{k,i} = b_{k,n} \cdot r_{k,i} \le b_{k,n} \le -1$ for all i, $0.55n \le i \le n$. This proves the assertion for $k = k_1$.

We now perform an induction step. Assume that, for some k such that $k_1 \le k < k_2$ we have $b_{k,i} \le -\bar{\phi}^{(n-5)(k-k_1)}$ for all i, $0.55n \le i \le n$. Then, in particular, $b_{k,n} \le -\bar{\phi}^{(n-5)(k-k_1)}$. By Theorem 56, $p_{k,0} > \bar{\phi}^{n-2}/\sqrt{2}$. Since $R_k = P_k \cdot Q_k$, we have $r_{k,0} = p_{k,0} d_k > \bar{\phi}^{n-2} d_k/\sqrt{2}$. As before, $d_k > 0.365$, so $d_k/\sqrt{2} > 0.258 > \phi^{-3} > \bar{\phi}^{-3}$. Hence, $r_{k,0} > \bar{\phi}^{n-5}$, and thus $b_{k,0} = b_{k,n} \cdot r_{k,0} < -\bar{\phi}^{(n-5)(k-k_1)}\bar{\phi}^{n-5} = -\bar{\phi}^{(n-5)((k+1)-k_1)}$. By Theorem 75, $b_{k+1,n} = b_{k,0}$. So, we have $b_{k+1,i} = b_{k+1,n} \cdot r_{k+1,i} \le -\bar{\phi}^{(n-5)((k+1)-k_1)}$ for all i, $0.55n \le i \le n$, completing the induction step. Hence the assertion holds for all k, $k_1 \le k \le k_2$. □

Theorem 77. *Let $n \ge 28$, let k_1 be the least integer that is at least 12 and at least $2.08 \ln n + 3$, and let k_2 be the greatest integer that is at most $n/2 - 2$. Let $\bar{\phi} = \phi(1 - \phi^{-18})$. Then $b_{k,i} < -\bar{\phi}^{(n-5)(k-k_1)}$ for all k, $k_1 \le k \le k_2$ and all i, $0 \le i \le n/10$.*

Proof. Let k be such that $k_1 \le k \le k_2$. For all i, $0 \le i \le n/10$, we have

$$\begin{aligned} b_{k,i} &= b_{k,n} \cdot r_{k,i} \\ &< b_{k,n} \\ &\le -\bar{\phi}^{(n-5)(k-k_1)} \end{aligned}$$

where the equality holds since R_k is the monic associate of B_k. The first inequality holds since, by Theorem 75, $b_{k,n}$ is negative and, by Theorem 74, $r_{k,i} > 1$. The second inequality holds by Theorem 76. □

We note that the coefficients of B_k cannot be reduced by extracting a common factor. Indeed, A_n is monic and, hence, primitive so that the following results apply.

Lemma 78. *Let A be a primitive polynomial. Then $\mathrm{T}(A)$ is primitive.*

$$
\begin{array}{ccccccc}
 & & 0 & 0 & 0 & 0 \\
 & & \downarrow & \downarrow & \downarrow & \downarrow \\
a_n & \rightarrow & a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\
a_{n-1} & \rightarrow & a_{1,0} & a_{1,1} & a_{1,2} & \\
a_{n-2} & \rightarrow & a_{2,0} & a_{2,1} & & \\
a_{n-3} & \rightarrow & a_{3,0} & & &
\end{array}
$$

Fig. 13.1. Computing $\mathrm{T}(A)$ using additions. The coefficients of A are in the leftmost column with the leading coefficient at the top. Each element $a_{i,j}$, $(i,j) \in I_n$, is the sum of its upper and left neighbors. The antidiagonal contains the coefficients of $\mathrm{T}(A)$ with the leading coefficient at the top, see Theorem 81.

Proof. For all polynomials A let $\mathrm{T}_{-1}(A)(x) = A(x-1)$. Note that, for all polynomials A and B, $\mathrm{T}_{-1}(A \cdot B) = \mathrm{T}_{-1}(A) \cdot \mathrm{T}_{-1}(B)$. Hence, if p is a real number and B is a polynomial then $\mathrm{T}_{-1}(p \cdot B) = p \cdot \mathrm{T}_{-1}(B)$. Assume now that $\mathrm{T}(A)$ is not primitive. Then there is an integer p, $p > 1$, and an integer polynomial B such that $\mathrm{T}(A) = p \cdot B$. Then $A = \mathrm{T}_{-1}(p \cdot B) = p \cdot \mathrm{T}_{-1}(B)$. But $\mathrm{T}_{-1}(B)$ is an integer polynomial, so A is not primitive. □

Theorem 79. *Let A be a primitive polynomial. Then all polynomials in the CF-tree of A are primitive.*

Proof. Every polynomial in the CF-tree of A is obtained from A by applying a sequence of translations T and reciprocal transformations R. Both transformations preserve primitivity, T by Lemma 78, and R since R preserves the set of non-zero coefficients. □

13. A lower bound for the computing time

We define classical translation by 1 by prescribing the arithmetic operations that are to be performed (Johnson et al., 2005). We then derive a lower bound for the time required to translate the polynomials $\mathrm{R}(B_k)$. When the CF-method is applied to the polynomials A_n, the total time for those translations dominates n^5.

Definition 80. Let n be a nonnegative integer, and let A be a polynomial, $A(x) = a_n x^n + \cdots + a_1 x + a_0$. Let $I_n = \{(i,j) : i,j \geq 0 \text{ and } i+j \leq n\}$. For any $k \in \{0,\ldots,n\}$ and any $(i,j) \in I_n$, let

$$
\begin{aligned}
a_{-1,k} &= 0, \\
a_{k,-1} &= a_{n-k}, \\
a_{i,j} &= a_{i,j-1} + a_{i-1,j}.
\end{aligned}
$$

Fig. 13.1 illustrates the definition.

Theorem 81.

$$
A(x+1) = \sum_{h=0}^{n} a_{n-h,h} x^h.
$$

Proof. The assertion clearly holds for $n = 0$; so we may assume $n > 0$. For every $k \in \{0,\ldots,n\}$ let $A_k(x) = \sum_{h=0}^{k} a_{k-h,h} x^h$. The coefficients of the polynomial A_k reside on the k-th antidiagonal of the matrix in Fig. 13.1 with the leading coefficient at the top. Then, for all $k \in \{0,\ldots,n-1\}$, we have $A_{k+1}(x) = (x+1)A_k(x) + a_{n-(k+1)}$. Now an easy induction on k shows that $A_k(x) = \sum_{h=0}^{k} a_{n-k+h}(x+1)^h$ for all $k \in \{0,\ldots,n\}$. In particular, $A_n(x) = \sum_{h=0}^{n} a_h (x+1)^h = A(x+1)$. □

By Theorem 81, the coefficients of $\mathrm{T}(A)$ can be computed from the coefficients of A using only additions. No explicit additions are needed to compute the top row in Fig. 13.1, the elements $a_{0,0},\ldots,a_{0,n}$.

Definition 82. A method that computes the coefficients of $\mathrm{T}(A)$ from the coefficients of A is called *classical translation by* 1 if, in the notation of Definition 80, the method performs the additions $a_{i,j} = a_{i,j-1} + a_{i-1,j}$ for the pairs (i,j) such that $1 \leq i \leq n$ and $0 \leq j \leq n$ and $i+j \leq n$.

Note that Definition 82 calls for $n(n+1)/2$ additions.

Theorem 83. *For all* $(i,j) \in I_n$,

$$a_{i,j} = \binom{i+j}{j} a_n + \binom{i+j-1}{j} a_{n-1} + \cdots + \binom{j}{j} a_{n-i}.$$

Proof. The assertion follows from Definition 80 by induction on $i+j$. □

We analyze computing times using the integer length function L and the dominance relation between functions; both concepts are well known (Collins, 1974).

Theorem 84. *Let c be a constant,* $0 < c \le 1$. *For any nonnegative integer n, let b be a negative integer and let* $A(x) = a_n x^n + \cdots + a_1 x + a_0$ *be an integer polynomial such that* $a_{n-i} \le b$, *for all i,* $0 \le i \le cn$. *Then the time to translate A by* 1 *using the classical method dominates* $n^2\,\mathrm{L}(b)$.

Proof. Using Theorem 83, we have, for all $(i,j) \in I_n$, $0 \le i \le cn$,

$$\begin{aligned} a_{i,j} &= \binom{i+j}{j} a_n + \binom{i+j-1}{j} a_{n-1} + \cdots + \binom{j}{j} a_{n-i} \\ &\le \left(\binom{i+j}{j} + \binom{i+j-1}{j} + \cdots + \binom{j}{j} \right) b \\ &= \binom{i+j+1}{j+1} b \\ &\le b, \end{aligned}$$

and hence $\mathrm{L}(a_{i,j}) \ge \mathrm{L}(b)$. According to Definition 82, the classical computation of $\mathrm{T}(A)$ requires the computation of $a_{i,j} = a_{i,j-1} + a_{i-1,j}$ for all pairs $(i,j) \in I_n$ such that $1 \le i \le cn$. But the number of those pairs dominates n^2. So the total computing time for all those additions dominates $n^2\,\mathrm{L}(b)$. □

Theorem 85. *Let* $n \ge 28$, *let* k_1 *be the least integer that is at least* 12 *and at least* $2.08 \ln n + 3$, *and let* k_2 *be the greatest integer that is at most* $n/2 - 2$. *Let k be such that* $k_1 \le k \le k_2$. *Then the time to compute the polynomial* B_{k+1} *from the polynomial* $\mathrm{R}(B_k)$ *using classical translation by* 1 *dominates* $n^3(k - k_1)$.

Proof. Let $\bar{\phi} = \phi(1 - \phi^{-18})$ and $b = \left\lfloor -\bar{\phi}^{(n-5)(k-k_1)} \right\rfloor$. By Theorem 77, $b_{k,i} \le b$ for all i, $0 \le i \le n/10$. But the coefficients of $\mathrm{R}(B_k)$ are the coefficients of B_k in reverse order. So the coefficient of x^{n-i} in $\mathrm{R}(B_k)$ is less than b for $0 \le i \le n/10$. Now Theorem 84, applied with $c = 1/10$, $A = \mathrm{R}(B_k)$, and b, yields that the time to translate $\mathrm{R}(B_k)$ dominates $n^2\,\mathrm{L}(b)$. But $\mathrm{L}(b)$ dominates $n(k - k_1)$. □

Theorem 86. *Let* $n \ge 28$, *let* k_1 *be the least integer that is at least* 12 *and at least* $2.08 \ln n + 3$, *and let* k_2 *be the greatest integer that is at most* $n/2 - 2$. *Let k be such that* $k_1 \le k \le k_2$. *Then the time to translate the polynomial* B_k *using classical translation by* 1 *dominates* $n^3(k - k_1)$.

Proof. Let $\bar{\phi} = \phi(1 - \phi^{-18})$ and $b = \left\lfloor -\bar{\phi}^{(n-5)(k-k_1)} \right\rfloor$. For all i, $0 \le i \le 0.45n$, we have $0.55n \le n - i \le n$ and so, by Theorem 76, $b_{k,n-i} \le b$. Now Theorem 84, applied with $c = 0.45$, $A = B_k$, and b, yields that the time to translate B_k dominates $n^2\,\mathrm{L}(b)$. But $\mathrm{L}(b)$ dominates $n(k - k_1)$. □

Theorem 87. *The time required for* A_n *by the CF-method with classical translation by* 1 *dominates* n^5.

Proof. We may assume $n \ge 100$. Then the height of the CF-tree of A_n is $\lfloor n/2 \rfloor + 2$ by Theorem 40. In particular, the method computes the polynomials B_k for all k, $k_1 \le k \le k_2$ where k_1 is the least integer greater than $2.08 \ln n + 3$ and $k_2 = \lfloor n/2 \rfloor - 2$. Since $n \ge 100$ we have $k_1 \ge 13$ and $k_2 \ge 48$ and

$$\begin{aligned} (k_2 - k_1)/n &\ge ((n/2 - 5/2) - (2.08 \ln n + 4))/n \\ &= 1/2 - 5/(2n) - (2.08 \ln n)/n - 4/n \\ &\ge 1/2 - 5/200 - 0.0208 \ln 100 - 4/100 \\ &> 1/3 \end{aligned}$$

where the second inequality holds since the expression to the right of the equal sign is an increasing function of n. Then, by Theorem 85, the total time to compute the polynomials B_{k+1} from the polynomials B_k for all k, $k_1 \le k \le k_2$, dominates $n^3 \sum_{k=k_1+1}^{k_2} (k - k_1) = n^3 \sum_{j=1}^{k_2-k_1} j > n^3 \lfloor n/3 \rfloor (\lfloor n/3 \rfloor + 1)/2$ which dominates n^5. □

The lower bound also applies when the CF-method uses root bounds in the way of Procedure CF in Sharma's paper (2008, Section 2). Table 2 and the proof of Theorem 61 show how that method will work if one assumes the computation of ideal polynomial root bounds. The ideal polynomial root bound is defined as the floor of the smallest positive root. According to the table, the ideal polynomial root bound is 0 at nodes ϵ and 21τ where $\tau \in \{2\}^*$ and $2 \leq |21\tau| \leq h - 1$. At node "2" the ideal polynomial root bound is 1. Hence Procedure CF computes, for all k, $2 \leq k \leq h - 1$, the polynomial B_{k+1} from the polynomial $\mathrm{R}(B_k)$ in the same way as the CF-method that we analyzed. Therefore, Theorems 85 and 87 apply also to Procedure CF. The theorems also apply to earlier statements of the CF-method with root bounds (Akritas, 1978, 1980; Akritas et al., 2007; Tsigaridas and Emiris, 2008) for the same reasons.

The time required for A_n by the CF-method is dominated by n^5 if classical translation by 1 is used. If either of the asymptotically fast methods E and F in the paper by von zur Gathen and Gerhard (1997) is used, the computing time is dominated by $n^4(\log n)^k$ for some positive integer k; it is not clear whether such a function would be codominant with the computing time.

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