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# Centroid structures of $n$-Lie algebras

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**Abstract**

We concern the centroid $\Gamma(L)$ of an $n$-Lie algebra $L$ over a field $F$ of characteristic $p \geqslant 0$. Let $C(L)$ be the central derivations of $L$. We obtain the following results.

(1) If $p$ is not a factor of $n-1$, then $C(L)$ is the intersection of $\Gamma(L)$ and the derivation algebra of $L$.

(2) Let $B$ be a nonzero ideal of $L$ and invariant under $\Gamma(L)$. Then the vanishing ideal of $B$ is isomorphic to a subspace of $Hom(L/B, Z_L(B)$, where $Z_L(B)$ is the centralizer of $B$.

(3) Suppose $L = L_1 \bigoplus L_2$ with $L_1, L_2$ ideals of $L$. Then $\Gamma(L_1)$ and $\Gamma(L_2)$ are components of $\Gamma(L)$.

(4) If $L$ is a Heisenberg $n$-Lie algebra over an algebraically closed field of characteristic 0, then $\Gamma(L)$ is generated by central derivations and scalars, and $C(L)$ is made up of all the inner derivations of $L$.

(5) If dim $L \geqslant 2$ and $\Gamma(L)$ consists of scalars, then the centroid of the tensor product of an associative algebra and $L$ is the same as the tensor product of the centroids of the two algebras.


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## 1. Introduction

A centroid of an $n$-Lie algebra $L$ is closely related to the derivation algebra of $L$. Some basic properties of the centroid of $n$-Lie algebras are obtained in Bai and Meng [4]. This paper is a continuation of [4].

Filippov [10] classifies $n$-Lie algebras of dimension $n+1$ over an algebraically closed field $F$ of characteristic zero. In [11], Kasymov develops the structure and representation theory of $n$-Lie algebras. Ling [12] proves that all finite dimensional simple $n$-Lie algebras over $F$ are isomorphic to the vector product on $F^{n+1}$ for $n \geqslant 3$. In [15] Pozhidaev studies two classes of central simple $n$-Lie algebras. Bai and Meng [4] describes the centroid of $n$-Lie algebras; they also give some properties of strongly semisimple $n$-Lie algebras in [5]. Recently, the study on $n$-Lie algebras attracts more attention due largely to its close connection with the Nambu mechanics and geometries [8,16], Poisson and Jacobi manifolds [13], and Hamiltonian mechanics [14]. There are other results on representations and structures of $n$-Lie algebras [1,2,6,7,9,13,17].

The organization of the rest of this paper is as follows. Section 2 is for basic notions and facts on $n$-Lie algebras. Section 3 is devoted to the structures and properties of the centroid of $n$-Lie algebras. Section 4 describes the structures of the centroid of tensor product $n$-Lie algebras. Throughout this paper we consider $n$-Lie algebras with $n \geqslant 3$.

## 2. Fundamental notions

A vector space $L$ over a field $F$ with characteristic not equal to two is an $n$-Lie algebra if there is an $n$-ary multilinear operation $[, \dots, ]$ satisfying the following identities:

$$[x_1, \dots, x_n] = (-1)^{\tau(\sigma)}[x_{\sigma(1)}, \dots, x_{\sigma(n)}], \tag{2.1}$$

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^{n}[x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n], \tag{2.2}$$

where $\sigma$ runs over the symmetric group $S_n$ and the number $\tau(\sigma)$ equals 0 or 1 depending on the parity of the permutation $\sigma$. If $ch F = 2$, then (2.1) should be replaced by

$$[x_1, \dots, x_i, \dots, x_j, \dots, x_n] = 0 \quad \text{if } x_i = x_j \text{ for } 1 \leqslant i \neq j \leqslant n. \tag{2.3}$$

A derivation of an $n$-Lie algebra is a linear transformation $D$ of $L$ into itself satisfying

$$D([x_1, \dots, x_n]) = \sum_{i=1}^{n}[x_1, \dots, D(x_i), \dots, x_n] \tag{2.4}$$

for $x_1, \dots, x_n \in L$. Let $Der L$ be the set of all derivations of $L$. Then $Der L$ is a subalgebra of the general Lie algebra $gl(L)$ and called the derivation algebra of $L$.

The map ad $(x_1, \dots, x_{n-1}) : L \to L$ given by

$$\mathrm{ad}(x_1, \dots, x_{n-1})(x_n) = [x_1, \dots, x_{n-1}, x_n] \quad \text{for } x_n \in L$$

is referred to as a left multiplication defined by elements $x_1, \dots, x_{n-1} \in L$. It follows from identity (2.2) that ad $(x_1, \dots, x_{n-1})$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of $Der L$, which we denote $ad(L)$. Every derivation in $ad(L)$ is by definition an inner derivation.

Let $L_1, L_2, \dots, L_n$ be subalgebras of an $n$-Lie algebra $L$. Denote by $[L_1, L_2, \dots, L_n]$ the subalgebra of $L$ generated by all vectors $[x_1, x_2, \dots, x_n]$, where $x_i \in L_i$, for $i = 1, 2, \dots, n$. The subalgebra $L^1 = [L, L, \ldots, L]$ is called the derived algebra of $L$. If $L^1 = 0$, then $L$ is called an abelian $n$-Lie algebra.

A one-dimensional central extension of an abelian $n$-Lie algebra is called the Heisenberg $n$-Lie algebra. Bai and Meng [3] introduced methods to construct Heisenberg $n$-Lie algebras.

An ideal $I$ of an $n$-Lie algebra $L$ is a subspace of $L$ such that $[I, L, \ldots, L] \subseteq I$. If $[I, I, L, \ldots, L] = 0$, then $I$ is referred to as an abelian ideal. If $L^1 \neq 0$ and $L$ has no ideals except for 0 and itself, then $L$ is by definition a simple $n$-Lie algebra. An ideal $I$ is perfect, if $I^1 = [I, \ldots, I] = I$. An $n$-Lie algebra is said to be decomposable if there are nonzero ideals $L_1, L_2$ such that

$$L = L_1 \oplus L_2$$

and $[L_1, L_2, L, \ldots, L] = 0$. Otherwise, we say that $L$ is indecomposable. Clearly if $L$ is a simple $n$-Lie algebra then $L$ is indecomposable. For a subalgebra $H$ of $L$, let

$$Z_L(H) = \{x \in L | [x, H, L, \ldots, L] = 0\} \tag{2.5}$$

denote the centralizer of $H$ in $L$. If $H$ is an ideal of $L$, then so is $Z_L(H)$. In particular, if $H = L$, write $Z(L) = Z_L(L)$. We refer to $Z(L)$ as the center of $L$. Clearly $Z(L)$ is an abelian ideal of $L$.

**Definition 2.1.** Let $End(L)$ be the endomorphism algebra of $L$. Then the following subalgebra of $End(L)$

$$\Gamma(L) = \{\varphi \in End(L) | \varphi([x_1, x_2, \ldots, x_n]) = [\varphi(x_1), x_2, \ldots, x_n] \text{ for } x_1, \ldots, x_n \in L\}$$

is called the centroid of $L$.

It follows from Theorem 2.1 in [4] that $\Gamma(L)$ is an associative algebra with the unit element $id_L$. By identity (2.1) or (2.3), for every $\varphi \in \Gamma(L)$ and $x_1, \ldots, x_n \in L$, we have

$$\varphi([x_1, \ldots, x_n]) = [x_1, \ldots, \varphi(x_i), \ldots, x_n] \quad \text{for } 1 \leqslant i \leqslant n \tag{2.6}$$

and $\varphi$ and $ad(x_1, \ldots, x_{n-1})$ commute. An $n$-Lie algebra is called a central $n$-Lie algebra if its centroid is isomorphic to $F$, the base field.

**Example 2.2.** Let $L$ be a 4-dimensional 3-Lie algebra over an algebraically closed field of characteristic 0 with the multiplication table on a basis $e_1, e_2, e_3, e_4$ of $L$ given by

$$[e_2, e_3, e_4] = e_1$$

and other multiplications are zero. Let $\phi(e_1) = ae_1$ and $\phi(e_i) = a_i e_1 + ae_i$ for $a, a_i \in F, i = 2, 3, 4$. Then $\phi \in \Gamma(L)$.

## 3. Centroid of $n$-Lie algebras

In this section we study the structure of the centroid of $n$-Lie algebras over a field $F$ of characteristic $p \geqslant 0$.

**Proposition 3.1.** *Let $L$ be an $n$-Lie algebra over $F$ and $B$ a subset of $L$. Then*

(1) *$Z_L(B)$ is invariant under $\Gamma(L)$.*
(2) *Every perfect ideal of $L$ is invariant under $\Gamma(L)$.*

**Proof.** For any $\varphi \in \Gamma(L)$, $x \in Z_L(B)$, by (2.5) we have

$$0 = \varphi([x, B, L, \dots, L]) = [\varphi(x), B, L, \dots, L].$$

Therefore $\varphi(x) \in Z_L(B)$, which implies that $Z_L(B)$ is invariant under $\Gamma(L)$.

To show (2) let $B$ be a perfect ideal of $L$. Then $B = B^1$, and so for any $x \in B$ there exist $x_1^i, x_2^i, \dots, x_n^i \in B$ with $0 < i < \infty$ such that $x = \sum_i [x_1^i, x_2^i, \dots, x_n^i]$. If $\varphi \in \Gamma(L)$, then

$$\varphi(x) = \varphi\left(\sum_i [x_1^i, x_2^i, \dots, x_n^i]\right) = \sum_i [\varphi(x_1^i), x_2^i, \dots, x_n^i] \in B.$$

This shows that $B$ is invariant under $\Gamma(L)$. □

**Definition 3.2.** Let $L$ be an $n$-Lie algebra over $F$ and $\varphi \in End(L)$. Then $\varphi$ is called a central derivation, if $\varphi(L) \subseteq Z(L)$ and $\varphi(L^1) = 0$.

The set of all central derivations of $L$ is denoted by $C(L)$. It is a simple fact that $C(L) \subseteq \Gamma(L)$. Indeed, $C(L)$ is an ideal of $\Gamma(L)$. A more precise relationship is summarized as follows.

**Proposition 3.3.** *If the characteristic of $F$ is 0 or not a factor of $n - 1$. Then*

$$C(L) = \Gamma(L) \cap Der L. \tag{3.1}$$

**Proof.** If $\varphi \in \Gamma(L) \cap Der L$ then by virtue of (2.4) and (2.6) we have $\varphi(L^1) = 0$ and $\varphi(L) \subseteq Z(L)$ where the assumption that the characteristic of $F$ is 0 or not a factor of $n - 1$ is used. It follows easily that $\Gamma(L) \cap Der L \subseteq C(L)$.

To show the inverse inclusion let $\varphi \in C(L)$. Then

$$0 = \varphi([x_1, \dots, x_n]) = [x_1, \dots, \varphi(x_i), \dots, x_n] \quad \text{for } 1 \leqslant i \leqslant n$$

and thus $\varphi \in \Gamma(L) \cap Der L$. This implies $\Gamma(L) \cap Der L = C(L)$. □

If $B$ is a $\Gamma(L)$-invariant ideal of $L$ let

$$V(B) = \{\varphi \in \Gamma(L) | \varphi(B) = 0\}$$

be its vanishing ideal. Let $Hom(L/B, Z_L(B))$ be the vector space of all linear maps from $L/B$ to $Z_L(B)$ over $F$. Define

$$T(B) = \{f \in Hom(L/B, Z_L(B)) | f([\bar{x}_1, \dots, \bar{x}_n]) = [x_1, \dots, f(\bar{x}_i), \dots, x_n]\}, \tag{3.2}$$

where $\bar{x}_i \in L/B$ and $i = 1, \dots, n$. Then $T(B)$ is a subspace of $Hom(L/B, Z_L(B))$.

**Theorem 3.4.** *Let $B$ be a nonzero $\Gamma(L)$-invariant ideal of $L$ over $F$. Then*

(1) $V(B) \cong T(B)$ *as vector spaces.*
(2) *If $\Gamma(B) = Fid_B$, then $\Gamma(L) = Fid_L \oplus V(B)$* as vector spaces.

**Proof.** It is easily seen that $V(B)$ is an ideal of the associative algebra $\Gamma(L)$. To prove (1) consider the following map $\alpha : V(B) \to T(B)$ given by

$$\alpha(\varphi)(\bar{y}) = \varphi(y),$$

where $\varphi \in V(B)$ and $\bar{y} = y + B \in L/B$. The map $\alpha$ is well defined. For if $\bar{y} = \bar{y}_1$, then $y - y_1 \in B$, and so $\varphi(y - y_1) = 0$. It follows easily that $\alpha$ is injective. We now show that $\alpha$ is onto. For every $f \in T(B)$, set

$$\varphi_f : L \to L, \quad \varphi_f(x) = f(\bar{x}) \quad \text{for all } x \in L.$$

It follows from identity (3.2) that, for all $x_1, \ldots, x_n \in L$,

$$\begin{aligned}\varphi_f([x_1, \ldots, x_n]) &= f([\bar{x}_1, \ldots, \bar{x}_n]) = [x_1, \ldots, f(\bar{x}_i), \ldots, x_n]\\ &= [x_1, \ldots, \varphi_f(x_i), \ldots, x_n].\end{aligned}$$

Thus $\varphi_f \in \Gamma(L)$, and so $\varphi_f \in V(B)$ since $\varphi_f(B) = 0$. But $\alpha(\varphi_f) = f$ implies that $\alpha$ is onto. It fairly easy to see that $\alpha$ preserves operations on vector spaces $L/B$ and $Z_L(B)$. This proves (1).

We now prove (2). If $\Gamma(B) = Fid_B$, then for all $\varphi \in \Gamma(L)$, $\varphi|_B = \lambda id_B$, for some $\lambda \in F$. If $\varphi \neq \lambda id_L$, let $\psi(x) = \lambda x$, for all $x \in L$. Then $\psi \in \Gamma(L)$ and $\varphi - \psi \in V(B)$. Clearly $\varphi = \psi + (\varphi - \psi)$. Furthermore, $Fid_L \cap V(B) = 0$, and so (2) is proved. □

**Corollary 3.5.** *If the characteristic of $F$ is* 0 *or not a factor of $n-1$, then the following is true*

$$\begin{aligned}C(L) &= \{\varphi \in Der(L) | Im\varphi \subseteq Z(L)\}\\ &= V(L^1) \cong T(L^1).\end{aligned}$$

**Proof.** Similar proof to that of Proposition 3.3 yields $C(L) = V(L^1) = \{\varphi \in Der(L) | Im\varphi \subseteq Z(L)\}$. It follows from Theorem 3.4 that $C(L) \cong T(L^1)$. □

**Theorem 3.6.** *Let $L$ be an $n$-Lie algebra. Then $\varphi D$ is a derivation for $\varphi \in \Gamma(L)$, $D \in Der L$.*

**Proof.** If $x_1, \ldots, x_n \in L$ then

$$\begin{aligned}\varphi D([x_1, \ldots, x_n]) &= \sum_{i=1}^{n} \varphi([x_1, \ldots, D(x_i), \ldots, x_n])\\ &= \sum_{i=1}^{n} [x_1, \ldots, \varphi D(x_i), \ldots, x_n].\end{aligned}$$

Thus $\varphi D$ is a derivation. □

**Theorem 3.7.** *Let $L$ be an $n$-Lie algebra. Then for any $D \in Der L$ and $\varphi \in \Gamma(L)$,*

(1) *$Der L$ is contained in the normalizer of $\Gamma(L)$ in $gl(L)$.*
(2) *$D\varphi$ is contained in $\Gamma(L)$ if and only if $\varphi D$ is a central derivation of $L$.*
(3) *$D\varphi$ is a derivation of $L$ if and only if $[D, \varphi]$ is a central derivation of $L$.*

**Proof.** For any $D \in Der L$, $\varphi \in \Gamma(L)$ and $x_1, \ldots, x_n \in L$

$$\begin{aligned}D\varphi([x_1, \ldots, x_n]) &= [D\varphi(x_1), \ldots, x_i, \ldots, x_n] + \sum_{i=2}^{n} [\varphi(x_1), \ldots, D(x_i), \ldots, x_n]\\ &= [D\varphi(x_1), \ldots, x_i, \ldots, x_n] + \sum_{i=2}^{n} [x_1, \ldots, \varphi D(x_i), \ldots, x_n]\\ &= [D\varphi(x_1), \ldots, x_i, \ldots, x_n] + \varphi D([x_1, \ldots, x_n])\\ &\quad - [\varphi D(x_1), \ldots, x_i, \ldots, x_n].\end{aligned}$$

Then we get

$$(D\varphi - \varphi D)([x_1, \ldots, x_n]) = [(D\varphi - \varphi D)(x_1), \ldots, x_n],$$

that is, $[D, \varphi] = D\varphi - \varphi D \in \Gamma(L)$. This proves (1). From Theorem 3.6 and (1), $D\varphi$ is an element of $\Gamma(L)$ if and only if $\varphi D \in Der L \cap \Gamma(L)$. Thanks to Proposition 3.3, we get the result (2). It follows from (1), Proposition 3.3 and Theorem 3.6 that (3) holds. □

Now we study the relationship between the centroid of a decomposable $n$-Lie algebra and the centroid of its factors.

**Theorem 3.8.** *Suppose that $L$ is an $n$-Lie algebra over $F$ and $L = L_1 \bigoplus L_2$ with $L_1, L_2$ being ideals of $L$. Then*

$$\Gamma(L) \cong \Gamma(L_1) \oplus \Gamma(L_2) \oplus C_1 \oplus C_2 \text{ as vector spaces}, \tag{3.3}$$

*where*

$$C_i = \{\varphi \in Hom(L_i, L_j) | \varphi(L_i) \subseteq Z(L_j) \text{ and } \varphi(L_i^1) = 0 \text{ for } 1 \leqslant i \neq j \leqslant 2\}.$$

**Proof.** Let $\pi_i : L \to L_i$ be canonical projections for $i = 1, 2$. Then $\pi_1, \pi_2 \in \Gamma(L)$ and $\pi_1 + \pi_2 = id_L$. So we have for $\varphi \in \Gamma(L)$,

$$\varphi = \pi_1\varphi\pi_1 + \pi_1\varphi\pi_2 + \pi_2\varphi\pi_1 + \pi_2\varphi\pi_2. \tag{3.4}$$

Note that $\pi_i\varphi\pi_j \in \Gamma(L)$ for $i, j = 1, 2$. We claim

$$\Gamma(L) = \pi_1\Gamma(L)\pi_1 \oplus \pi_1\Gamma(L)\pi_2 \oplus \pi_2\Gamma(L)\pi_1 \oplus \pi_2\Gamma(L)\pi_2 \text{ as vector spaces.} \tag{3.5}$$

It suffices to show that $\pi_1\Gamma(L)\pi_1 \cap \pi_1\Gamma(L)\pi_2 = 0$ (other cases are similar). For any $\varphi \in \pi_1\Gamma(L)\pi_1 \cap \pi_1\Gamma(L)\pi_2$, there exist $f_i \in \Gamma(L), i = 1, 2$ such that $\varphi = \pi_1 f_1\pi_1 = \pi_1 f_2\pi_2$. Then $\varphi(x) = \pi_1 f_2\pi_2(x) = \pi_1 f_2\pi_2(\pi_2(x)) = \pi_1 f_1\pi_1(\pi_2(x)) = \pi_1 f_1(0) = 0$, for all $x \in A$, and so $\varphi = 0$.

Let

$$\Gamma(L)_{ij} = \pi_i\Gamma(L)\pi_j, \quad i, j = 1, 2.$$

We now prove

$$\Gamma(L)_{11} \cong \Gamma(L_1), \quad \Gamma(L)_{22} \cong \Gamma(L_2), \quad \Gamma(L)_{12} \cong C_2, \quad \Gamma(L)_{21} \cong C_1.$$

Since $\varphi(L_2) = 0$ for $\varphi \in \Gamma(L)_{11}$, we have $\varphi|_{L_1} \in \Gamma(L_1)$. On the other hand, one can regard $\Gamma(L_1)$ as a subalgebra of $\Gamma(L)$ by extending any $\varphi_0 \in \Gamma(L_1)$ on $L_2$ being equal to zero, that is

$$\varphi_0(x_1) = \varphi_0(x_1), \quad \varphi_0(x_2) = 0 \quad \text{for all } x_1 \in L_1, x_2 \in L_2.$$

Then $\varphi_0 \in \Gamma(L)$ and $\varphi_0 \in \Gamma(L)_{11}$. Therefore $\Gamma(L)_{11} \cong \Gamma(L_1)$ with isomorphism

$$\sigma : \Gamma(L)_{11} \to \Gamma(L_1), \quad \sigma(\varphi) = \varphi|_{L_1} \quad \text{for all } \varphi \in \Gamma(L)_{11}.$$

Similarly, we have $\Gamma(L)_{22} \cong \Gamma(L_2)$.

Next, we prove $\Gamma(L)_{12} \cong C_2$. If $\varphi \in \Gamma(L)_{12}$ there exists $\varphi_0$ in $\Gamma(L)$ such that $\varphi = \pi_1\varphi_0\pi_2$. For $x_k = x_k^1 + x_k^2 \in L$ where $x_k^i \in L_i, i = 1, 2$ and $k = 1, \ldots, n$ we have

$$\begin{aligned}
\varphi([x_1, x_2, \ldots, x_n]) &= \pi_1\varphi_0\pi_2([x_1, x_2, \ldots, x_n]) \\
&= \pi_1\varphi_0([x_1^2, x_2^2, \ldots, x_n^2]) \\
&= \pi_1([\varphi_0(x_1^2), x_2^2, \ldots, x_n^2]) \\
&= 0
\end{aligned}$$

and

$$[\varphi(x_1), x_2, \ldots, x_n] = \varphi([x_1, x_2, \ldots, x_n]) = 0.$$

Then $\varphi(L) \subseteq Z(L)$ and $\varphi(L^1) = 0$. It follows that $\varphi|_{L_2}(L_2) \subseteq Z(L_1)$ and $\varphi|_{L_2}(L_2^1) = 0$ and so $\varphi|_{L_2} \in C_2$.

Conversely for $\varphi \in C_2$, expanding $\varphi$ on $L$ (also denoted by $\varphi$) by $\varphi(L_1) = 0$, we have $\pi_1\varphi\pi_2 = \varphi$ and $\varphi \in \Gamma(L)_{12}$. This proves that $\Gamma(L)_{12}$ is isomorphic to $C_2$ with the following isomorphism $\tau : \Gamma(L)_{12} \to C_2$,

$$\tau(\varphi) = \varphi|_{L_2} \quad \text{for all } \varphi \in \Gamma(L)_{21}.$$

Similarly, we can prove $\Gamma(L)_{21} \cong C_1$. Summarizing the above discussion we get

$$\Gamma(L) \cong \Gamma(L_1) \oplus \Gamma(L_2) \oplus C_1 \oplus C_2.$$

The proof is completed. □

A generalized version of the above theorem is stated below without proof.

**Theorem 3.9.** *Suppose $L$ is an $n$-Lie algebra over $F$ with a decomposition of ideals*

$$L = L_1 \oplus \cdots \oplus L_m.$$

*Then we have*

$$\Gamma(L) \cong \Gamma(L_1) \oplus \cdots \oplus \Gamma(L_m) \oplus (\oplus_{1 \leqslant i \neq j \leqslant m} C_{ij}) \text{ as vector spaces},$$

*where*

$$C_{ij} = \{\varphi \in Hom(L_i, L_j) | \varphi(L_i) \subseteq Z(L_j) \text{ and } \varphi(L_i^1) = 0 \text{ for } 1 \leqslant i \neq j \leqslant m\}.$$

In the following we study the centroid of $n$-Lie algebras over a field $F$ of characteristic zero. Let $\{c_k\}$ be a basis of the central derivations $C(L)$ and $\{\varphi_j\}$ a maximal subset of $\Gamma(L)$ such that $\{\varphi_j|_{[L,\ldots,L]}\}$ is linear independent. Then we have the following result.

**Theorem 3.10.** *Let $\Psi$ denote the subspace of $\Gamma(L)$ spanned by $\{\varphi_j\}$. Then $\{c_k, \varphi_j\}$ is a basis of $\Gamma(L)$ and $\Gamma(L) = \Psi \oplus C(L)$ as vector spaces.*

**Proof.** Since $\{\varphi_j|_{[L,\ldots,L]}\}$ is linear independent, $\{\varphi_j\}$ is linear independent in $\Gamma(L)$. By definition of $\{c_k, \varphi_j\}$, the $\{c_k, \varphi_j\}$ is independent in $\Gamma(L)$.

For $\varphi \in \Gamma(L)$ since $\{\varphi_j|_{[L,\ldots,L]}\}$ is a basis of vector space $\{\varphi|_{[L,\ldots,L]} | \varphi \in \Gamma(L)\}$, there exist $l_s \in F$, $s \in J$ (a finite set of positive integers) such that

$$\varphi|_{[L,\ldots,L]} = \sum_{s \in J} l_s \varphi_s|_{[L,\ldots,L]}.$$

We then have

$$\left(\varphi - \sum_{s \in J} l_s \varphi_s\right)|_{[L,\ldots,L]} = 0.$$

If $y_1, \ldots, y_n \in L$ then

$$0 = \left(\varphi - \sum_{s \in J} l_s \varphi_s\right)([y_1, \ldots, y_n]) = \left[\left(\varphi - \sum_{s \in J} l_s \varphi_s\right)(y_1), y_2, \ldots, y_n\right].$$

It follows that $\left(\varphi - \sum_{s\in J} l_s\varphi_s\right)(L) \subseteq Z(L)$ and $\varphi - \sum_{s\in J} l_s\varphi_s$ is a central derivation. So there exist $r_i \in F$, $i \in I$ (a finite set of positive integers) such that

$$\varphi - \sum_{s\in J} l_s\varphi_s = \sum_{i\in I} r_i c_i.$$

Therefore $\varphi = \sum_{s\in J} l_s\varphi_s + \sum_{i\in I} r_i c_i$. The proof is completed. □

**Lemma 3.11** [4, Theorem 2.5]. *Let $L$ be an indecomposable $n$-Lie algebra over an algebraically closed field $F$ of characteristic zero and $N$ the nilradical of $\Gamma(L)$. Then*

$$\Gamma(L) = Fid_L \oplus N.$$

**Definition 3.12.** Let $L$ be an indecomposable $n$-Lie algebra over a field $F$. Then $\Gamma(L)$ is small if $\Gamma(L)$ is generated by central derivations and the scalars.

The centroid of a decomposable $n$-Lie algebra is small if the centroid of every maximal indecomposable ideal is small. The centroid of an $m$-dimensional abelian $n$-Lie algebra $L$ is regarded as small, in which case $\Gamma(L) = gl(m, F)$.

**Theorem 3.13.** *If $L$ is a Heisenberg $n$-Lie algebra over an algebraically closed field $F$ of characteristic* 0, *then $\Gamma(L)$ is small and the central derivations $C(L) = ad(L)$.*

**Proof.** Since $L$ is indecomposable, it follows from Lemma 3.11 that $\Gamma(L) = Fid_L \oplus N$ where $N$ is the nilradical of $\Gamma(L)$. If $\varphi \in N$ then there exists a natural number $k$ such that $\varphi^k = 0$. Because $L$ is a Heisenberg $n$-Lie algebra, we have

$$[L, \ldots, L] = Fc,$$

where $c$ is the center element of $L$. There exist $x_1, \ldots, x_n \in L$ such that $[x_1, \ldots, x_n] = c$. Thus

$$\varphi[x_1, \ldots, x_n] = \varphi(c) = [\varphi(x_1), x_2, \ldots, x_n] = \lambda c \quad \text{for some } \lambda \in F.$$

By $\varphi^k(c) = \lambda^k c = 0$ we have $\lambda = 0$ and $\varphi([L, \ldots, L]) = 0$. Since, for $y_1, \ldots, y_n \in L$, $[\varphi(y_1), y_2, \ldots, y_n] = \varphi([y_1, \ldots, y_n]) = 0$, we see $\varphi(L) \subseteq Z(L)$. It implies that $\varphi$ is a central derivation. This proves that $\Gamma(L)$ is small.

Now suppose $\{e_1, \ldots, e_m, c\}$ is a basis of $L$ and $\varphi$ is a central derivation of $L$. From Definition 3.2 we have

$$\varphi \begin{pmatrix} e_1 \\ \vdots \\ e_m \\ c \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & a_1 \\ 0 & \cdots & 0 & a_2 \\ & \cdots & & \cdots \\ 0 & \cdots & 0 & a_m \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_m \\ c \end{pmatrix}. \tag{3.6}$$

It implies that $\dim C(L) \leqslant m$.

Thanks to the properties of Heisenberg $n$-Lie algebras, we know $C(L)$ contains $ad(L)$. On the other hand $ad(L)$ is in the space of linear transformations of L and has dimension $m$ since its annihilator in $L$ has dimension 1. Hence $ad(L) = C(L)$. □

## 4. Centroid of tensor product $n$-Lie algebras

Let $A$ be a commutative associative algebra over $F$. The centroid $\Gamma(A)$ of $A$ is by definition

$$\Gamma(A) = \{f \in End(A) | f(ab) = f(a)b = af(b) \text{ for all } a, b \in A\}.$$

Then $\Gamma(A)$ is an associative subalgebra of $End(A)$. If $L$ is an $n$-Lie algebra over $F$, let $A \otimes L$ be the tensor product over $F$ of the underlying vector spaces $A$ and $L$. Then $A \otimes L$ is an $n$-Lie algebra over $F$ with respect to the following $n$-ary multilinear operation

$$[a_1 \otimes x_1, \ldots, a_n \otimes x_n] = (a_1 \cdots a_n) \otimes [x_1, \ldots, x_n], \tag{4.1}$$

where $a_i \in A, x_i \in L$, and $i = 1, \ldots, n$. This $n$-Lie algebra $A \otimes L$ is called the tensor product $n$-Lie algebra of $A$ and $L$. For $f \in End(A), \varphi \in End(L)$, let

$$f \otimes \varphi : A \otimes L \to A \otimes L$$

be given by

$$f \otimes \varphi(a \otimes x) = f(a) \otimes \varphi(x), \text{ for } a \in A, x \in L.$$

Then $f \otimes \varphi \in End(A \otimes L)$. By the above notation we have following result.

**Lemma 4.1.** $\Gamma(A \otimes L) \supseteq \Gamma(A) \otimes \Gamma(L)$.

**Proof.** For every $f \otimes \varphi \in \Gamma(A) \otimes \Gamma(L), a_i \otimes x_i \in A \otimes L, i = 1, \ldots, n$, we have

$$\begin{aligned}(f \otimes \varphi)([a_1 \otimes x_1, \ldots, a_n \otimes x_n]) &= (f \otimes \varphi)((a_1 \cdots a_n) \otimes [x_1, \ldots, x_n]) \\ &= f(a_1 \cdots a_n) \otimes \varphi([x_1, \ldots, x_n]) \\ &= (f(a_1)a_2 \cdots a_n) \otimes [\varphi(x_1), x_2, \ldots, x_n], \\ &= [(f(a_1) \otimes \varphi(x_1), a_2 \otimes x_2, \ldots, a_n \otimes x_n] \\ &= [f \otimes \varphi(a_1 \otimes x_1), a_2 \otimes x_2, \ldots, a_n \otimes x_n].\end{aligned}$$

Therefore $f \otimes \varphi \in \Gamma(A \otimes L)$. □

If $A$ is a commutative associative algebra with the unit element 1, then $\Gamma(A) \cong A$ with the map $\sigma : \Gamma(A) \to A$ given by

$$\sigma(f) = f(1) \quad \text{for all } f \in \Gamma(A).$$

In the rest of the paper we suppose $L$ is an $n$-Lie algebra over an algebraically closed field $F$ of characteristic zero and $A$ is a unital commutative associative algebra over $F$.

**Proposition 4.2.** *Using the notation of Theorem 3.10, we get*

$$\Gamma(A \otimes L) \supseteq A \otimes \Psi + End(A) \otimes C(L).$$

**Proof.** By Lemma 4.1 we have $A \otimes \Psi \subseteq \Gamma(A \otimes L)$. For every $f \in End(A), c \in C(L)$,

$$\begin{aligned}&(f \otimes c)([a_1 \otimes x_1, a_2 \otimes x_2, \ldots, a_n \otimes x_n]) \\ &\quad = (f \otimes c)((a_1 a_2 \cdots a_n) \otimes [x_1, x_2, \ldots, x_n]) \\ &\quad = f(a_1 a_2 \cdots a_n) \otimes c([x_1, x_2, \ldots, x_n]) = f(a_1 a_2 \cdots a_n) \otimes 0 = 0;\end{aligned}$$

$$\begin{aligned}&[(f \otimes c)(a_1 \otimes x_1), a_2 \otimes x_2, \ldots, a_n \otimes x_n]\\ &\quad = [f(a_1) \otimes c(x_1), a_2 \otimes x_2, \ldots, a_n \otimes x_n]\\ &\quad = (f(a_1)a_2 \cdots a_n) \otimes [c(x_1), x_2, \ldots, x_n] = (f(a_1)a_2 \cdots a_n) \otimes 0\\ &\quad = (f \otimes c)([a_1 \otimes x_1, a_2 \otimes x_2, \ldots, a_n \otimes x_n]).\end{aligned}$$

Therefore, $f \otimes c \in \Gamma(A \otimes L)$. □

**Theorem 4.3.** *If* $\dim L \geqslant 2$ *and* $\Gamma(L) = Fid$, *then* $\Gamma(A \otimes L) = \Gamma(A) \otimes \Gamma(L) \cong A$.

**Proof.** By Lemma 4.1 it suffices to prove

$$\Gamma(A \otimes L) \subseteq \Gamma(A) \otimes \Gamma(L) \cong A.$$

Suppose $\{m_i\}$ is a basis of $A$. Then for every $\varphi \in \Gamma(A \otimes L), a \in A$, there exists a set of transformations $\{\eta_i(a, -)\}$ in $End(L)$ such that for $x \in L$,

$$\varphi(a \otimes x) = \sum_i m_i \otimes \eta_i(a, x), \tag{4.2}$$

where in the summation only finite number of summands are not equal to zero, that is for every $x \in L$ there exist at most finite $\eta_i(a, -)$, such that $\eta_i(a, x) \neq 0$. Now we prove $\eta_i(a, -) \in \Gamma(L)$. Notice that

$$\begin{aligned}\varphi([a \otimes x_1, 1 \otimes x_2, \ldots, 1 \otimes x_n]) &= [\varphi(a \otimes x_1), 1 \otimes x_2, \ldots, 1 \otimes x_n]\\ &= \left[\sum_i m_i \otimes \eta_i(a, x_1), 1 \otimes x_2, \ldots, 1 \otimes x_n\right]\\ &= \sum_i m_i \otimes [\eta_i(a, x_1), x_2, \ldots, x_n]\end{aligned}$$

and

$$\begin{aligned}\varphi([a \otimes x_1, 1 \otimes x_2, \ldots, 1 \otimes x_n]) &= \varphi(a \otimes [x_1, x_2, \ldots, x_n])\\ &= \sum_i m_i \otimes \eta_i(a, [x_1, x_2, \ldots, x_n])\end{aligned}$$

for any $x_1, x_2, \ldots, x_n \in L$. Therefore $[\eta_i(a, x_1), x_2, \ldots, x_n] = \eta_i(a, [x_1, x_2, \ldots, x_n])$ and $\eta_i(a, -) \in \Gamma(L)$. It follows from $\Gamma(L) = Fid$ that $\eta_i(a, x) = \lambda_i(a)x$. By (4.2), for $\varphi \in \Gamma(A \otimes L), a \in A$ there exists a finite set $J$ of positive integers such that if $i \notin J$ then $\eta_i(a, -) = 0$. Then we have

$$\varphi(a \otimes x) = \sum_{i \in J} \lambda_i(a) m_i \otimes x \quad \text{for all } x \in A.$$

Let $\rho : A \to A$ given by $\rho(a) = \sum \lambda_i(a) m_i$ for all $a \in A$. Then $\varphi(a \otimes x) = \rho(a) \otimes x$. Since

$$\begin{aligned}\varphi([a \otimes x_1, 1 \otimes x_2, \ldots, 1 \otimes x_n]) &= \varphi(a \otimes [x_1, x_2, \ldots, x_n])\\ &= \rho(a) \otimes [x_1, x_2, \ldots, x_n]\end{aligned}$$

and

$$\begin{aligned}\varphi([a \otimes x_1, 1 \otimes x_2, \ldots, 1 \otimes x_n]) &= [a \otimes x_1, \varphi(1 \otimes x_2), \ldots, 1 \otimes x_n]\\ &= [a \otimes x_1, \rho(1) \otimes x_2, \ldots, 1 \otimes x_n]\end{aligned}$$

$$= \rho(1)a \otimes [x_1, x_2, \dots, x_n]$$

for $x_1, x_2, \dots, x_n \in L, a \in A$, we get

$$\rho(a) \otimes [x_1, x_2, \dots, x_n] = \rho(1)a \otimes [x_1, x_2, \dots, x_n].$$

As $L$ is not an abelian $n$-Lie algebra, we see $\rho(a) = a\rho(1)$ for $a \in A$. Therefore

$$\varphi(a \otimes x) = (\rho(1) \otimes id_L)(a \otimes x), \quad \text{for all } a \in A, x \in L.$$

This shows
$\Gamma(A \otimes L) = \Gamma(A) \otimes \Gamma(L) \cong A.$ □

**Remark 4.4.** Theorem 4.3 does not hold if $A$ is not unital.

**Example 4.5.** Let $F[t] = \{f(t) = \sum_{i=0}^{m} a_i t^i | a_i \in F, 0 \leqslant i \leqslant m, 0 \leqslant m < \infty\}$ be the polynomial ring over an algebraically closed field $F$ of characteristic zero. Then

$$B = t^m F[t], \quad m > 0$$

is a subalgebra of $F[t]$ not containing the unit element. Let

$$L = Fe_1 + \cdots + Fe_{n+1}$$

be an $n + 1$ dimensional simple $n$-Lie algebra over $F$. Thanks to Theorem 2.6 in [4] we have $\Gamma(L) = Fid_L$. A direct computation yields that $f(t)id_B \otimes id_L \in \Gamma(B \otimes L)$ for $f(t) \in F[t]$. Therefore

$$\Gamma(B \otimes L) \cong F[t] \neq B.$$

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