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Note

On a question regarding visibility of lattice points—III

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Abstract

For a positive integer m, let $\omega(m)$ denote the number of distinct prime factors of m. Let $h(n)$ be a function defined on the set of positive integers such that $h(n) \to \infty$ as $n \to \infty$ and let $E_n(h) = \{d\colon d$ is a positive integer, $d \leqslant n,\ \omega(d) \geqslant h(n)\}$. Writing $\Delta_n = \{(x, y) : x, y$ are integers, $1 \leqslant x,\ y \leqslant n\}$, in the present paper we show that one can give explicit description of a set $X_n \subset \Delta_n$ such that Δ_n is visible from X_n with at most $100|E_n(h)|^2$ exceptional points and for all sufficiently large n, one has

$$|X_n| \leqslant 800h(n)\log\log h(n).$$

As a corollary it follows that one can give explicit description of a set $Y_n \subset \Delta_n$ such that for large n's, Δ_n is visible except for at most $100\,n^2/(\log\log n)^2$ exceptional points from Y_n where Y_n satisfies

$$|Y_n| = O((\log\log n)(\log\log\log\log n)).$$

Keywords: Visibility; Lattice points

1. Introduction

Let $\Delta_n = \{(x, y) : x, y$ are integers, $1 \leqslant x, y \leqslant n\}$ be the set of integer lattice points in a particular square area in $\mathbb{R}^2$. If $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$ are two points in Δ_n, we say that α is visible from β if either $\alpha = \beta$ or there is no other lattice point in Δ_n on

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the line segment joining α and β. It is easy to observe that if $\alpha \neq \beta$, then $\alpha = (a_1, a_2)$ is visible from $\beta = (b_1, b_2)$ if and only if $(a_1 - b_1, a_2 - b_2) = 1$, where $(a_1 - b_1, a_2 - b_2)$ denotes the greatest common divisor of $a_1 - b_1$ and $a_2 - b_2$.

If A and B are subsets of Δ_n, one says that A is visible from B if each point of A is visible from some point of B.

Let $f(n)$ be defined by

$$f(n) = \min\{|S|:\ S \subset \Delta_n,\ \Delta_n \text{ is visible from } S\}.$$

Here and in what follows, for a finite set S, $|S|$ will denote the number of elements in S. Therefore, in other words, $f(n)$ is the least number of points that can be selected from Δ_n such that every point of Δ_n is visible from at least one of the selected points.

It was proved in [1] that

Theorem 1*. *For all sufficiently large n*

$$\frac{\log n}{2 \log\log n} < f(n) < 4 \log n. \tag{1}$$

The following result was also proved in [1].

Theorem 2*. *One can explicitly describe a set $S_n \subset \Delta_n$ such that Δ_n is visible from S_n and $|S_n| = O((\log n)^\alpha)$ where α has the property that the Jacobsthal function $g(n)$ satisfies $g(n) = O((\log n)^\alpha)$.*

Here, the Jacobsthal function $g(n)$ is defined to be the least integer with the property that among any $g(n)$ consecutive integers $a+1, \ldots, a+g(n)$, there is at least one which is relatively prime to n. Erdös et al. [4] had asked for a replacement of S_n in Theorem 2* by a set S_n' which would satisfy $|S_n'| = O(\log n)$ as is expected from Theorem 1* and it should be mentioned (see [2] or [3] for information regarding the order of $g(n)$) that even if the expected order of $g(n)$ is established, Abbot's explicit construction falls short of that target.

In [2], Adhikari and Balasubramanian gave an explicit construction of a set $S_n' \subset \Delta_n$ from which Δ_n is visible, where S_n' satisfies

$$|S_n'| = O\left(\frac{\log n \log\log\log n}{\log\log n}\right).$$

In [3], the corresponding problem for higher dimensions was solved up to a constant factor. For the case of dimension two, it remains an open question as to whether the order of $f(n)$ obtained in [2] can be improved or not.

In the present paper, we consider a slightly different question in the case of dimension two.

For a positive integer m, let $\omega(m)$ denote the number of distinct prime factors of m. Let $h(n)$ be a function defined on the set of positive integers such that $h(n) \to \infty$ as $n \to \infty$ and let $E_n(h) = \{d:\ d \text{ is a positive integer},\ d \leqslant n,\ \omega(d) \geqslant h(n)\}$.

We prove the following theorem.

Theorem 3. *One can give explicit description of a set $X_n \subset \Delta_n$ such that Δ_n is visible from X_n with at most $100|E_n(h)|^2$ exceptional points and for all sufficiently large n, one has*

$$|X_n| \leqslant 800h(n)\log\log h(n).$$

2. Proof of Theorem 3

Let n be sufficiently large and

$$s=[10h(n)], \quad t=[10\log\log h(n)], \quad t_0=\left[\frac{1}{10}t\right]+1.$$

Let

$$A_i=\{m:\ m \text{ is an integer, } (i-1)t+1<m\leqslant it\}, \quad \text{for } i=1,2,\ldots,$$

$$I=\{i:\ |A_i\cap E_n(h)|\geqslant t_0\}$$

and

$$B_i=\{it-a : a\in A_i\cap E_n(h)\}.$$

Suppose that $1\leqslant i\leqslant n/t$, $i\notin I$ and $1\leqslant j\leqslant n/s$. Then $|B_i|<t_0$ and $\omega(it-a)<h(n)$ if $1\leqslant a\leqslant t$ and $a\notin B_i$.

Thus, using p to denote prime numbers, we estimate the following sum which counts the number of pairs (a,b), $1\leqslant a\leqslant t$, $1\leqslant b\leqslant s$, $a\notin B_i$ for which the gcd $((it-a),(js-b))>1$. We have

$$\begin{aligned}
&\sum_{\substack{a=1\\ a\notin B_i}}^{t}\sum_{b=1}^{s}\sum_{((it-a),(js-b))>1}1\\
&\quad\leqslant \sum_{\substack{a=1\\ a\notin B_i}}^{t}\sum_{b=1}^{s}\sum_{p}\sum_{\substack{p|(it-a)\\ p|(js-b)}}1\\
&\quad\leqslant \left(\sum_{p\leqslant s}\sum_{\substack{a=1,\ a\notin B_i\\ p|(it-a)}}^{t}\sum_{\substack{b=1\\ p|(js-b)}}^{s}1\right)+\left(\sum_{p>s}\sum_{\substack{a=1,\ a\notin B_i\\ p|(it-a)}}^{t}1\right)\\
&\quad\leqslant \sum_{p\leqslant s}\left(\frac{t}{p}+1\right)\left(\frac{s}{p}+1\right)+\sum_{p>s}\sum_{\substack{a=1,\ a\notin B_i\\ p|(it-a)}}^{t}1
\end{aligned}$$

$$\leqslant \sum_{p\leqslant s}\left(\frac{t}{p}+1\right)\left(\frac{s}{p}+1\right)+\sum_{a=1,\ a\notin B_i}^{t}\ \sum_{p|(it-a)} 1$$

(removing the restriction $p>s$ on p in the last term)

$$\leqslant ts\sum_{p}\frac{1}{p^2}+(t+s)\sum_{p\leqslant s}\frac{1}{p}+s+(t-|B_i|)h(n)$$

$$<\frac{7}{10}ts+(t+s)(\log\log s+O(1))+(t-|B_i|)h(n)$$

$$<(t-|B_i|)s \text{ for all sufficiently large } n. \tag{2}$$

(It should be noted that we have used the trivial estimate $\sum_p 1/p^2<(\sum_{n=1}^{\infty} 1/n^2)-1=(\pi^2/6)-1<\frac{7}{10}$. However, it is known [6] that $\sum_p 1/p^2=0.452247\ldots$.)

Since the number of pairs (a,b), $1\leqslant a\leqslant t$, $1\leqslant b\leqslant s$, $a\notin B_i$ for which $((it-a),(js-b))>1$ is seen to be strictly less than $(t-|B_i|)s$, there exist integers a_{ij},b_{ij}, $1\leqslant a_{ij}\leqslant t, 1\leqslant b_{ij}\leqslant s$, $a_{ij}\notin B_i$ such that $(it-a_{ij},js-b_{ij})=1$.

Now, let a lattice point $(x,y)\in\Delta_n$ be given. Then there are non-negative integers u and v such that

$$ut<x\leqslant(u+1)t,\quad vt<y\leqslant(v+1)t.$$

We consider the following two cases.

Case I. $u\notin I$.

Let j be the integer such that $js<y\leqslant(j+1)s$. If $u=0$ or $j=0$, then observing that a lattice point is visible from any lattice point on an adjacent line, we get that (x,y) is visible from $\{(a,b)\,|\,1\leqslant a\leqslant 2t,\ 1\leqslant b\leqslant 2s\}$. If $u\geqslant 1$ and $j\geqslant 1$, then by our observation following (2), we have (x,y) is visible from the point $(x-ut+a_{uj},\,y-js+b_{uj})$, where

$$1\leqslant x-ut+a_{uj}\leqslant t+a_{uj}\leqslant 2t\quad\text{and}\quad 1\leqslant y-js+b_{uj}\leqslant s+b_{uj}\leqslant 2s$$

remembering that the condition $(a_1-b_1,a_2-b_2)=1$ implies that (a_1,a_2) is visible from (b_1,b_2).

Case II. $v\notin I$.

By symmetry, employing an argument similar to that in Case I, in this case we get (x,y) is visible from $\{(a,b)\,|\,1\leqslant a\leqslant 2s,\ 1\leqslant b\leqslant 2t\}$.

Thus, the exceptional points not visible from $\{(a,b)\,|\,1\leqslant a\leqslant 2t,\ 1\leqslant b\leqslant 2s\}\cup\{(a,b)\,|\,1\leqslant a\leqslant 2s,\ 1\leqslant b\leqslant 2t\}$ are among those points for which both $u\in I$ and $v\in I$ and since for a particular (u,v), there correspond t^2 points, the total number of such points is at most $t^2|I|^2$.

Since $t_0|I|\leqslant|E_n(h)|$, the number of exceptional points is at most

$$\left(\frac{t}{t_0}\right)^2|E_n(h)|^2<100|E_n(h)|^2.$$

This completes the proof of the theorem.

Corollary. *One can give explicit description of a set* $Y_n \subset \Delta_n$ *such that for large* n*'s,* Δ_n *is visible except for at most* $100\,n^2/(\log\log n)^2$ *exceptional points from* Y_n *where* Y_n *satisfies*

$$|Y_n| = O((\log\log n)(\log\log\log\log n)).$$

Proof. We take $h(n) = 2\log\log n$. A well known theorem of Hardy and Ramanujan [5] (or see [7, p. 306]), says that

$$|E_n(h)| \leqslant \frac{n}{\log\log n},$$

where $E_n(h)$ is as defined in Section 1 before the statement of Theorem 3.

Therefore, the corollary follows from Theorem 3.

Remark. If we take $h(n) = 2(\log n)/(\log\log n)$, then $E_n(h)$ is empty for sufficiently large n. Therefore, our Theorem 3 gives an explicit description of a set Z_n such that Δ_n is visible from Z_n where

$$|Z_n| \leqslant 1600\frac{(\log n)(\log\log\log n)}{\log\log n}.$$

Thus we have obtained the result of Adhikari and Balasubramanian [2], which was stated in the introduction, with an explicit O-constant.

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