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# Construction of generators of quasi-interpolation operators of high approximation orders in spaces of polyharmonic splines

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ABSTRACT

The paper presents a simple procedure for the construction of quasi-interpolation operators in spaces of $m$-harmonic splines in $\mathbb{R}^d$, which reproduce polynomials of high degree. The procedure starts from a generator $\phi_0$, which is easy to derive but with corresponding quasi-interpolation operator reproducing only linear polynomials, and recursively defines generators $\phi_1, \phi_2, \ldots, \phi_{m-1}$ with corresponding quasi-interpolation operators reproducing polynomials of degree up to $3, 5, \ldots, 2m-1$ respectively. The construction of $\phi_j$ from $\phi_{j-1}$ is explicit, simple and independent of $m$. The special case $d = 1$ and the special cases $d = 2$, $m = 2, 3, 4$ are discussed in details.



## 1. Introduction

The space of $m$-harmonic splines $V_{2m}$, $2m > d$ is defined as the subspace of $S'(\mathbb{R}^d)$ (the space of $d$-dimensional tempered distributions)

$$V_{2m} = \left\{ g \in S'(\mathbb{R}^d) \bigcap C^{2m-d-1}(\mathbb{R}^d) : \Delta^m g = 0, \ \text{on}\ \mathbb{R}^d \setminus \mathbb{Z}^d \right\}, \tag{1}$$

where $\Delta$ is the Laplacian operator. It is well known (see e.g. [1]) that $V_{2m}$ contains $\Pi_{2m-1}$ (the space of polynomials defined on $\mathbb{R}^d$ of degree not exceeding $2m-1$), and that for any $n \le 2m-1$, it is possible to construct quasi-interpolation (QI) operators reproducing $\Pi_n$ (see e.g. [2,3]). For that, the generator $\phi \in V_{2m}$ of the QI operator

$$Q_\phi(x, h, f) = \sum_{l \in \mathbb{Z}^d} f(hl)\phi(x/h - l)$$

is required to decay fast enough at infinity, so that $Q_\phi(x, h, f)$ is well defined for $f$ growing at infinity not faster than a polynomial of degree $n$. Such a QI operator approximates smooth enough functions with $L^\infty$ error of order $h^{n+1}$ [2].

The known constructions of generators defining QI operators which reproduce $\Pi_n$ for $n$ large, are quite involved, and are different for different $m$ (see e.g. [2,4]), while simple generators, like the elementary polyharmonic $B$-splines [5], are easy to construct, but generates QI operators reproducing only $\Pi_1$.

In this paper, we present a simple procedure which starts from a simple generator $\phi_0$ and recursively defines generators $\phi_1, \phi_2, \ldots, \phi_{m-1}$ with corresponding QI operators reproducing $\Pi_3, \Pi_5, \ldots, \Pi_{2m-1}$ respectively. Our procedure defines $\phi_j$ as a linear combination of $\phi(\cdot)_{j-1}$ and $\phi_{j-1}(\cdot/2)$ with explicitly known simple coefficients which are independent of $m$.

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The general procedure is presented in Section 2, while in Section 3, the special case $d = 1$ with $\phi_0$ an odd degree symmetric $B$-spline with integer knots, and the special cases $d = 2$, $m = 2, 3, 4$ with two different $\phi_0$, are discussed.

We use in this paper the multi-index notation. In particular for $\alpha \in \mathbb{Z}^d$, $|\alpha| = \sum_{i=1}^{d} |\alpha_i|$. The approximation error is measured in the $L^\infty$ norm over $\mathbb{R}^d$.

It should be noted that our procedure can be extended to spaces defined by more general elliptic operators instead of just $\Delta^m$.

## 2. The construction

We start from a polyharmonic $B$-spline which is easy to construct. Known examples of polyharmonic $B$-splines are given in Section 3.

A function

$$\phi_0 : \mathbb{R}^d \to \mathbb{R}$$

is called $m$-harmonic $B$-spline for $2m > d$, if its Fourier transform has the form

$$\hat{\phi}_0(\omega) = e_{2m}(\omega)\hat{v}_m(\omega) \tag{2}$$

where

$$\hat{v}_m(\omega) = (-1)^m \|\omega\|^{-2m} \tag{3}$$

and $e_{2m}$ is a real, even trigonometric polynomial satisfying

$$e_{2m}(\omega) = (-1)^m \|\omega\|^{2m} + O(\|\omega\|^{2m+2}), \tag{4}$$

for $\omega$ in a neighborhood of the origin. Conditions (2)–(4) guarantee, as is shown in Lemma 1, that $\phi_0$ is decaying sufficiently fast and generates quasi-interpolation operator reproducing linear polynomials as do symmetric univariate $B$-splines.

In the sequel we assume that $2m > d$.

**Lemma 1.** *Under conditions (2)–(4)*

$$|\phi_0(x)| \le \frac{C}{\|x\|^{d+2}}, \quad \|x\| \to \infty, \tag{5}$$

*and the sum*

$$\sum_{l \in \mathbb{Z}^d} f(l)\phi_0(x - l) \tag{6}$$

*is well defined for any $f$ growing at infinity not faster than a linear polynomial. Moreover if $f$ is a polynomial of degree not exceeding one, the sum (6) equals $f$.*

**Proof.** Let $\alpha \in \mathbb{Z}_+^d$ and consider $D^\alpha \hat{\phi}_0$. Under the assumptions on $\hat{\phi}_0$, we obtain for $\|\omega\|$ large enough

$$|D^\alpha \hat{\phi}_0(\omega)| \le C\|\omega\|^{-2m}. \tag{7}$$

Since $2m > d$, $D^\alpha \hat{\phi}_0$ is summable in a neighborhood of infinity.

Now we check the behavior of $D^\alpha \hat{\phi}_0$ in the neighborhood of the origin, $U(0)$. We know from (2)–(4) that

$$\hat{\phi}_0(\omega) = 1 + \sum_{i=1}^{\infty} h_{2i}^{[0]}(\omega) = 1 + h_2^{[0]}(\omega) + O(\|\omega\|^4), \quad \omega \in U(0), \tag{8}$$

where $h_{2i}^{[0]}$ is a rational homogeneous function of degree $2i$ of the form

$$h_{2i}^{[0]}(\omega) = \frac{p_{2m+2i}^{[0]}(\omega)}{\|\omega\|^{2m}},$$

with $p_{2m+2i}^{[0]}$ a homogeneous polynomial of degree $2m+2i$. Observe that $D^\alpha h_{2i}^{[0]}$ is a homogeneous function of degree $2i - |\alpha|$. Thus $D^\alpha \hat{\phi}_0$ is integrable in $U(0)$ whenever $2 - |\alpha| + d - 1 > -1$. Since $|\alpha|$ and $d$ are integers, we can conclude that $D^\alpha \hat{\phi}_0 \in L^1(\mathbb{R}^d)$ for $|\alpha| \le d + 1$. From this, it follows that $\phi_0(x) = o(\|x\|^{-d-1})$, $\|x\| \to \infty$ (see e.g [6, p. 26]), and since $\phi_0$ admits a series expansion away from $U(0)$, we have

$$\phi_0(x) = O(\|x\|^{-d-2}), \quad \|x\| \to \infty.$$

With this decay the sum (6) is well defined for $f$ as above. To show that the sum equals $f$ when $f \in \Pi_1$, we have to show that $\phi_0$ satisfies the Strang–Fix conditions of order one (see e.g. [2]), namely that

$$\hat{\phi}_0(0) = 1, \qquad D^\alpha \hat{\phi}_0(0) = 0, \quad |\alpha| = 1, \tag{9}$$

$$D^\alpha \hat{\phi}_0(2k\pi) = 0, \quad k \in \mathbb{Z}^d \setminus 0, \ |\alpha| = 0, 1. \tag{10}$$

From (4) and (3) it is easy to see that (9) holds. Using (4) we conclude that $D^\alpha e_{2m}(0) = 0$ for $|\alpha| < 2m$. This together with the $2\pi\mathbb{Z}^d$ periodicity of $e_{2m}$ leads to (10). □

It follows directly from the last lemma and [2] that

**Corollary 2.** *For $f$ with bounded derivatives of order 1–3, the sum*

$$Q_0(x, h, f) = \sum_{l \in \mathbb{Z}^d} f(hl)\phi_0(x/h - l),$$

*approximates $f$ in $\mathbb{R}^d$ with $L^\infty$ error of order $h^2$.*

To obtain higher approximation order than in Corollary 2 the function $\phi_0$ should be replaced by a function, with a stronger decay rate as $\|x\| \to \infty$, generating a QI operator with higher degree of polynomial reproduction. Here, we provide quasi-interpolation operators with approximation error of order $h^{2j}$, $j = 1, \ldots, m$ by a simple procedure based on $\phi_0$.

Starting from $\phi_0$ we construct new generators

$$\phi_1, \phi_2, \ldots, \phi_{m-1}$$

of QI operators reproducing polynomials up to degree $3, 5, \ldots, 2m-1$ respectively.
The construction is done recursively.
For $j = 1, 2, \ldots, m-1$, we define

$$\phi_j(x) = a_j\phi_{j-1}(x) + b_j\phi_{j-1}(x/2), \quad x \in \mathbb{R}^d, \tag{11}$$

and choose the coefficients $a_j$, $b_j$ so that

$$\hat{\phi}_j(\omega) = 1 + \sum_{i=j+1}^{\infty} h_{2i}^{[j]}(\omega), \quad \omega \in U(0), \tag{12}$$

with $h_{2i}^{[j]}$ a homogeneous function of order $2i$. This is possible since

$$\hat{\phi}_j(\omega) = a_j\hat{\phi}_{j-1}(\omega) + 2^d b_j\hat{\phi}_{j-1}(2\omega), \tag{13}$$

and by our inductive hypothesis

$$\hat{\phi}_{j-1}(\omega) = 1 + \sum_{i=j}^{\infty} h_{2i}^{[j-1]}(\omega), \quad \omega \in U(0). \tag{14}$$

Note that by (8) $\phi_0$ satisfies (14).
It is easy to see that the coefficients $a_j$, $b_j$ in (13) are the solution of the system

$$\begin{cases} a_j + 2^d b_j = 1 \\ a_j + 2^{d+2j} b_j = 0, \end{cases} \tag{15}$$

that is

$$a_j = \frac{2^{2j}}{2^{2j}-1}, \qquad b_j = -\frac{1}{2^d(2^{2j}-1)}, \quad j = 1, \ldots, m-1. \tag{16}$$

Note that the coefficients $a_j$, $b_j$ do not depend on $\phi_0$ and on $m$.

By construction $\hat{\phi}_j(\omega)$ has the form

$$\hat{\phi}_j(\omega) = e_{2m}^{[j]}(\omega)\hat{v}_m(\omega), \tag{17}$$

where

$$e_{2m}^{[j]}(\omega) = a_j e_{2m}^{[j-1]}(\omega) + 2^{d-2m} b_j e_{2m}^{[j-1]}(2\omega), \quad j = 1, \ldots, m-1, \tag{18}$$

and $e^{[0]}_{2m}(\omega) = \hat{e}_{2m}(\omega)$. Note also that $e^{[j]}_{2m}$ is a symmetric trigonometric polynomial. Hence it follows from (14) and (17) that

$$e^{[j]}_{2m}(\omega) = \|\omega\|^{2m} + \sum_{i=j+1}^{\infty} p^{[j]}_{2m+2i}(\omega), \quad \omega \in U(0), \tag{19}$$

with $p^{[j]}_{2m+2i}$ a homogeneous polynomial of degree $2m + 2i$.

Next we prove that $\phi_j$ has a correct decay at infinity needed for the quasi-interpolation based on it to be well defined for polynomials of degree up to $2j + 1$, and that these polynomials are reproduced by this quasi-interpolation.

**Proposition 3.** *For $j = 1, 2, \ldots, m-1$*

$$\left|\phi_j(x)\right| \le \frac{C}{\|x\|^{d+2j+2}}, \quad \|x\| \to \infty. \tag{20}$$

*Here $C$ is a generic constant.*

**Proof.** First let us consider the decay of $\hat{\phi}_j$ and its derivatives near infinity. By (11) it is the same as that of $\hat{\phi}_{j-1}$, and therefore by recursion, as that of $\hat{\phi}_0$. Thus in view of (7), we obtain

$$|D^\alpha \hat{\phi}_j(\omega)| \le C\|\omega\|^{-2m}, \quad \alpha \in \mathbb{Z}^d, \ \omega \notin U(0). \tag{21}$$

Since $2m > d$, $D^\alpha \hat{\phi}_j$ is always summable in a neighborhood of infinity.
Next we consider $\hat{\phi}_j(\omega)$, $\omega \in U(0)$. By (12)

$$\hat{\phi}_j(\omega) = 1 + h^{[j]}_{2j+2}(\omega) + O(\|\omega\|^{2j+4}), \quad \omega \in U(0) \tag{22}$$

and by (19)

$$h^{[j]}_{2j+2}(\omega) = \frac{p^{[j]}_{2m+2j+2}(\omega)}{\|\omega\|^{2m}},$$

with $p^{[j]}_{2m+2j+2}$ a homogeneous polynomial of degree $2m+2j+2$. Thus by arguments similar to those in the proof of Lemma 1, we obtain the claim of the proposition. □

**Proposition 4.** *For $j = 1, \ldots, m-1$, the sum*

$$\sum_{l\in\mathbb{Z}^d} f(l)\phi_j(x-l), \tag{23}$$

*is well defined for any $f$ growing at infinity not faster than a polynomial of degree $2j + 1$. Moreover if $f \in \Pi_{2j+1}$, the sum (23) equals $f$.*

**Proof.** Since $\phi_j$ satisfies (20), the sum (23) is well defined. To show that the sum equals $f$ when $f \in \Pi_{2j+1}$, we have to show that $\phi_j$ satisfies the Strang–Fix conditions of order $2j + 1$, (see e.g. [2]), namely

$$\hat{\phi}_j(0) = 1, \qquad D^\alpha \hat{\phi}_j(0) = 0, \quad 1 \le |\alpha| \le 2j+1, \tag{24}$$

$$D^\alpha \hat{\phi}_j(2k\pi) = 0, \quad k \in \mathbb{Z}^d \setminus 0, \ |\alpha| \le 2j+1. \tag{25}$$

By (22) we obtain $\hat{\phi}_j(0) = 1$. Since $D^\alpha h^{[j]}_{2i}$ is a homogeneous function of degree $2i - |\alpha|$, we get from (22)

$$D^\alpha \hat{\phi}_j(0) = 0, \quad \forall \alpha \in \mathbb{Z}^d_+, \ 1 \le |\alpha| \le 2j+1. \tag{26}$$

Now, by (19)

$$D^\alpha e^{[j]}_{2m}(0) = 0, \quad \forall \alpha \in \mathbb{Z}^d_+, \ |\alpha| \le 2m-1, \tag{27}$$

and since $e^{[j]}_{2m}$ is $2\pi\mathbb{Z}^d$-periodic and $\hat{v}_m(\omega)$ is finite for $\omega \notin U(0)$, we obtain

$$D^\alpha \hat{\phi}_j(2k\pi) = 0, \quad \forall \alpha \in \mathbb{Z}^d_+, \ |\alpha| \le 2m-1, \ k \in \mathbb{Z}^d \setminus 0. \quad \square \tag{28}$$

It follows directly from Propositions 3 and 4 and [2], that

**Corollary 5.** *For $j = 1, \ldots, m-1$, the quasi-interpolant*

$$Q_j(x, h, f) = \sum_{l\in\mathbb{Z}^d} f(hl)\phi_j(x/h - l),$$

*approximates $f$ having bounded derivatives of order $2j+1, 2j+2, 2j+3$, with error of order $h^{2j+2}$.*

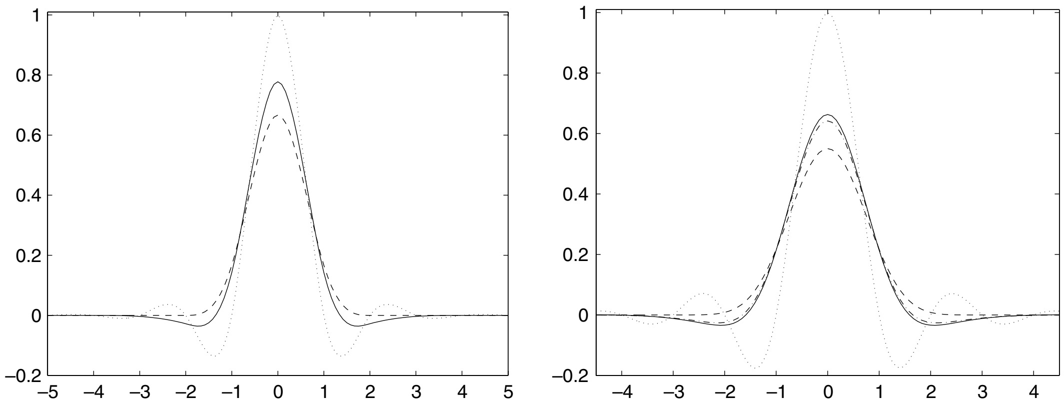


**Fig. 1.** Left: $m = 2$, the cubic symmetric *B*-spline $\phi_0$ (dashed line), $\phi_1$ (solid line), the cubic cardinal Lagrange spline (dotted line). Right: $m = 3$, the quintic symmetric *B*-spline $\phi_0$ (dashed line), $\phi_1$ (dashdot line), $\phi_2$ (solid line), the quintic cardinal Lagrange spline (dotted line).

## 3. Examples

In this section we investigate some special cases of our construction.

### 3.1. The case $d = 1$: starting from symmetric odd degree uniform B-splines

First we illustrate in Fig. 1, $\phi_0$, the generators $\phi_j$, and the cardinal Lagrange interpolant in the space of splines of degree 3, 5 with integer knots. Here $\phi_0$ is the symmetric *B*-spline in these spaces.

It is easy to observe that $\phi_j$ for general $m$, is a spline of degree $2m - 1$ with integer knots and support $[-2^j m, 2^j m]$. We show it by induction. For $j = 0$, $\phi_0$ is the symmetric *B*-spline of degree $2m - 1$ with integer knots and support $[-m, m]$. If $\phi_{j-1}$ is a spline of degree $2m - 1$, has integer knots and support $[-2^{j-1}m, 2^{j-1}m]$, then $\phi_j(\cdot) = a_j\phi_{j-1}(\cdot) + b_j\phi_{j-1}\left(\frac{\cdot}{2}\right)$ is also a spline of degree $2m - 1$ with support determined by that of $\phi_{j-1}\left(\frac{\cdot}{2}\right)$, which is double the support of $\phi_{j-1}$, namely $[-2^j m, 2^j m]$. Since the knots of $\phi_{j-1}$ are integers and those of $\phi_{j-1}\left(\frac{\cdot}{2}\right)$ are even integers, the knots of $\phi_j$ are integers.

### 3.2. The case $d = 2$: starting from the m-harmonic B-splines

In this subsection we discuss in details some examples of new generators arising from known two-dimensional polyharmonic *B*-splines of order $m$, the elementary and the isotropic polyharmonic *B*-splines [7]. For the latter we give also the errors in approximating two well-known test functions by the different quasi-interpolation operators in case $m = 3$.

#### 3.2.1. Starting from the elementary polyharmonic B-spline

For the elementary polyharmonic *B*-spline of order $m > 1$, the trigonometric polynomial of (2) is (see [5])

$$e_{2m}(\omega_1, \omega_2) = \left(-4\sin^2\frac{\omega_1}{2} - 4\sin^2\frac{\omega_2}{2}\right)^m. \tag{29}$$

In this case, we have for any $m > 1$

$$\hat{\phi}_0^{[E]}(\omega) = 1 + h_2^{[0]}(\omega) + O(\|\omega\|^4), \quad \omega \in U(0), \tag{30}$$

$$|\phi_0^{[E]}(x)| \le \frac{C}{\|x\|^4}, \quad x \notin U(0), \tag{31}$$

and the generated quasi-interpolation operators satisfy for all $m > 1$

$$Q_0^{[E]}(x, h, p) = p, \quad p \in \Pi_1. \tag{32}$$

Thus $Q_0^{[E]}(x, h, p)$ for all $m > 1$ provide approximation error of order $h^2$. Here and after $Q_j^{[E]}$ stands for the quasi-interpolation operator based on $\phi_j^{[E]}$, where $\{\phi_j^{[E]}\}$ are the new generators obtained from $\phi_0^{[E]}$.

**Table 1**
The new generator based on $\phi_0^{[E]}$, $m = 2$.

| $j$ | $\hat{\phi}_j^{[E]}, \omega \in U(0)$ | $\lvert\phi_j^{[E]}\rvert \le C\lVert x\rVert^{-\beta}, x \notin U(0)$ | $Q_j^{[E]}(x, h, p) = p$ |
|---|---|---|---|
| 0 | $1 + h_2^{[0]} + O(\lVert\omega\rVert^4)$ | $\beta = 4$ | $p \in \Pi_1$ |
| 1 | $1 + h_4^{[1]} + O(\lVert\omega\rVert^6)$ | $\beta = 6$ | $p \in \Pi_3$ |

**Table 2**
The new generators based on $\phi_0^{[E]}$, $m = 4$.

| $j$ | $\hat{\phi}_j^{[E]}, \omega \in U(0)$ | $\lvert\phi_j^{[E]}\rvert \le C\lVert x\rVert^{-\beta}, x \notin U(0)$ | $Q_j^{[E]}(x, h, p) = p$ |
|---|---|---|---|
| 0 | $1 + h_2^{[0]} + O(\lVert\omega\rVert^4)$ | $\beta = 4$ | $p \in \Pi_1$ |
| 1 | $1 + h_4^{[1]} + O(\lVert\omega\rVert^6)$ | $\beta = 6$ | $p \in \Pi_3$ |
| 2 | $1 + h_6^{[2]} + O(\lVert\omega\rVert^8)$ | $\beta = 8$ | $p \in \Pi_5$ |
| 3 | $1 + h_8^{[3]} + O(\lVert\omega\rVert^{10})$ | $\beta = 10$ | $p \in \Pi_7$ |

For $m = 3$ the construction of Section 2 yields

$$\hat{\phi}_1^{[E]}(\omega) = 1 + h_4^{[1]}(\omega) + O(\|\omega\|^6), \quad \omega \in U(0), \tag{33}$$

$$|\phi_1^{[E]}(x)| \le \frac{C}{\|x\|^6}, \quad x \notin U(0), \tag{34}$$

$$Q_1^{[E]}(x, h, p) = p, \quad p \in \Pi_3, \tag{35}$$

and

$$\hat{\phi}_2^{[E]}(\omega) = 1 + h_6^{[2]}(\omega) + O(\|\omega\|^8), \quad \omega \in U(0), \tag{36}$$

$$|\phi_2^{[E]}(x)| \le \frac{C}{\|x\|^8}, \quad x \notin U(0), \tag{37}$$

$$Q_2^{[E]}(x, h, p) = p, \quad p \in \Pi_5. \tag{38}$$

In Tables 1 and 2 we summarize the cases $m = 2$ and $m = 4$.

#### 3.2.2. *Starting from the isotropic polyharmonic B-splines*

The trigonometric polynomial $e_{2m}$ associated with the isotropic polyharmonic *B*-spline of order $m > 1$ is (see e.g [8])

$$e_{2m}(\omega_1, \omega_2) = \left[-\frac{2}{3}\left(1 + 4\sin^2\frac{\omega_1}{2} + 4\sin^2\frac{\omega_2}{2} + \cos\omega_1\cos\omega_2\right)\right]^m. \tag{39}$$

In this case, for any $m > 1$, we have a closer to radial behavior of $\hat{\phi}_0^{[I]}(\omega)$ for $\omega$ near zero, in fact

$$\hat{\phi}_0^{[I]}(\omega) = 1 - \frac{m}{12}\|\omega\|^2 + O(\|\omega\|^4), \quad \omega \in U(0) \tag{40}$$

implying a faster decay with respect to the elementary polyharmonic *B*-spline,

$$|\phi_0^{[I]}(x)| \le \frac{C}{\|x\|^6}, \quad x \notin U(0). \tag{41}$$

Yet the polynomial reproduction is as in the case of the elementary polyharmonic *B*-splines, namely

$$Q_0^{[I]}(x, h, p) = p, \quad p \in \Pi_1, \tag{42}$$

for all $m > 1$. Here, similarly to the previous case we denote by $Q_j^{[I]}$ the quasi-interpolation operator based on $\phi_j^{[I]}$.

In case $m = 3$ the construction of Section 2 yields

$$\hat{\phi}_1^{[I]}(\omega) = 1 + p_4^{[1]}(\omega) + O(\|\omega\|^6), \quad \omega \in U(0), \tag{43}$$

$$|\phi_1^{[I]}(x)| \le \frac{C}{\|x\|^8}, \quad x \notin U(0), \tag{44}$$

$$Q_1^{[I]}(x, h, p) = p, \quad p \in \Pi_3, \tag{45}$$

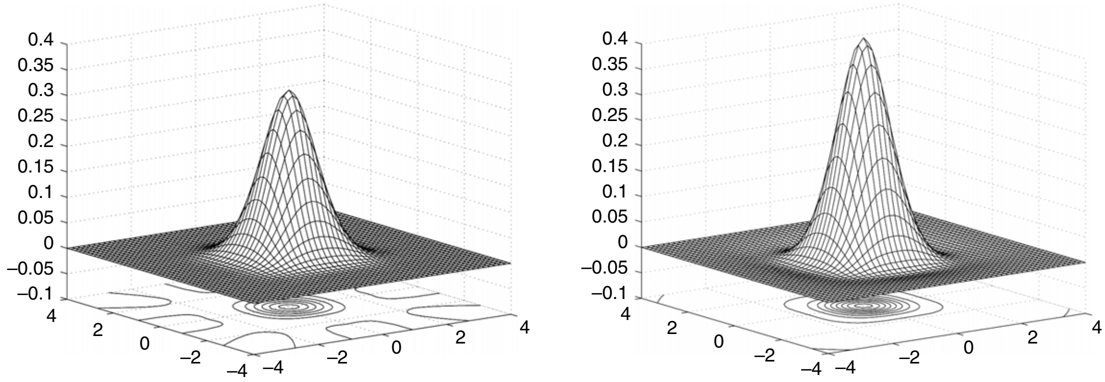


**Fig. 2.** The isotropic polyharmonic $B$-spline $\phi_0^{[I]}$ (left) and the generator $\phi_2^{[I]}$ (right) for $m = 3$.

**Table 3**
The new generator based on $\phi_0^{[I]}$, $m = 2$.

| j | $\hat{\phi}_j^{[I]}, \omega \in U(0)$ | $\lvert\phi_j^{[I]}\rvert \le C\lVert x\rVert^{-\beta},\ x \notin U(0)$ | $Q_j^{[I]}(x,h,p) = p$ |
|---|---|---|---|
| 0 | $1 - \frac{1}{6}\lVert\omega\rVert^2 + O(\lVert\omega\rVert^4)$ | $\beta = 6$ | $p \in \Pi_1$ |
| 1 | $1+p_4^{[1]}+h_6^{[1]}+O(\lVert\omega\rVert^8)$ | $\beta = 8$ | $p \in \Pi_3$ |

**Table 4**
The new generators based on $\phi_0^{[I]}$, $m = 4$.

| j | $\hat{\phi}_j^{[I]}, \omega \in U(0)$ | $\lvert\phi_j^{[I]}\rvert \le C\lVert x\rVert^{-\beta},\ x \notin U(0)$ | $Q_j^{[I]}(x,h,p) = p$ |
|---|---|---|---|
| 0 | $1 - \frac{3}{4}\lVert\omega\rVert^2 + O(\lVert\omega\rVert^4)$ | $\beta = 6$ | $p \in \Pi_1$ |
| 1 | $1+p_4^{[1]}+h_6^{[1]}+O(\lVert\omega\rVert^8)+O(\lVert\omega\rVert^8)$ | $\beta = 8$ | $p \in \Pi_3$ |
| 2 | $1 + h_6^{[2]} + p_8^{[2]} + O(\lVert\omega\rVert^{10})$ | $\beta = 8$ | $p \in \Pi_5$ |
| 3 | $1 + p_8^{[3]} + h_{10}^{[3]} + O(\lVert\omega\rVert^{12})$ | $\beta = 12$ | $p \in \Pi_7$ |

and

$$\hat{\phi}_2^{[I]}(\omega) = 1 + h_6^{[2]}(\omega) + O(\lVert\omega\rVert^8), \quad \omega \in U(0), \tag{46}$$

$$\lvert\phi_2^{[I]}(x)\rvert \le \frac{C}{\lVert x\rVert^8}, \quad x \notin U(0), \tag{47}$$

$$Q_2^{[I]}(x,h,p) = p, \quad p \in \Pi_5. \tag{48}$$

In Fig. 2 we show $\phi_0^{[I]}$ and the new generator $\phi_2^{[I]}$. It is easy to observe from Fig. 2 that the new generator $\phi_2^{[I]}(x)$, when compared with $\phi_0^{[I]}$, is more concentrated near the origin and has a higher maximum at the origin.

In Tables 3 and 4, we summarize the cases $m = 2$ and $m = 4$ which show together with the case $m = 3$ that, in general, the isotropic polyharmonic $B$-splines provide generators decaying faster than those generated from the elementary polyharmonic ones.

We conclude by showing the errors in approximating two smooth test functions defined on $[0,1]^2$, by the different quasi-interpolation operators $Q_j^{[I]}$, $j = 0, 1, 2$, corresponding to the case $m = 3$. We have considered the well-known Franke's function (F1) and the functions F5 [9],

$$\mathrm{F5}(x,y) = \frac{1}{3}\exp\left(-20.25((x-0.5)^2 + (y-0.5)^2)\right), \quad x, y \in [0,1]^2.$$

Both test functions are depicted in Fig. 3.

In Table 5 we show the maximum absolute approximation errors for F1 and F5. The errors were computed using a $33 \times 33$ uniform grid of evaluation points in $[0.1, 0.9]^2$. We have also computed the approximation errors obtained by the interpolant

$$I_m(x,h,f) = \sum_{l \in \mathbb{Z}^2} f(hl) L_m(x/h - l), \tag{49}$$

where $L_m$ is the cardinal Lagrange polyharmonic spline of order $m$, and by the QI operator $Q_{m,m-1}^{[HL]}$, generated by the high-level $m$-harmonic $B$-spline of level $m-1$ [4]. Note that both $L_m$ and $Q_{m,m-1}^{[HL]}$ reproduces $\Pi_{2m-1}$ [1,4]. All errors were computed

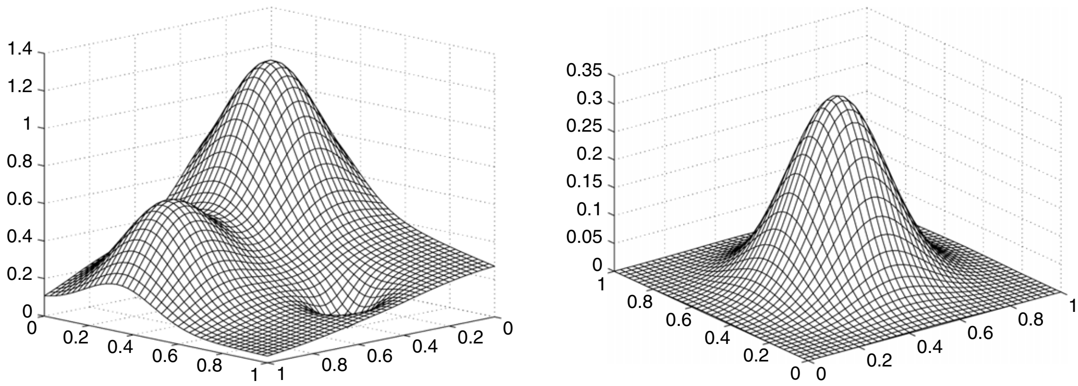


**Fig. 3.** The test functions F1 (left) and F5 (right).

**Table 5**
Maximum approximation errors by quasi-interpolation and interpolation operators.

| $m = 3, h = 0.01$ | $Q_0^{[I]}$ | $Q_1^{[I]}$ | $Q_2^{[I]}$ | $I_3$ | $Q_{3,2}^{[HL]}$ |
|---|---|---|---|---|---|
| F1 | 1.6e−3 | 4.5e−5 | 1.6e−5 | 1.2e−5 | 1.8e−4 |
| F5 | 6.7e−4 | 5.4e−6 | 4.3e−7 | 3.1e−7 | 4.9e−6 |

for $h = 0.01$. We can see from Table 5 that, as expected, when going from $Q_0^{[I]}$ to $Q_2^{[I]}$, the error is reduced and reaches the same order as that of the interpolation error, which in this specific example is smaller than the error by $Q_{3,2}^{[HL]}$.

## References

[1] W.R. Madych, S.A. Nelson, Polyharmonic cardinal splines, Journal of Approximation Theory 60 (1990) 141–156.
[2] N. Dyn, I.R.H. Jackson, D. Levin, A. Ron, On multivariate approximation by integer translates of basis function, Israel Journal of Mathematics 74 (1992) 95–130.
[3] N. Dyn, Approximation by translates of a radial function, in: J.R. Higgins, R.L. Sten (Eds.), Sampling Theory in Fourier and Signal Analysis: Advanced Topics, Oxford University Press, 1999, pp. 187–208.
[4] C. Rabut, High level $m$-harmonic cardinal B-splines, Numerical Algorithms 2 (1992) 63–84.
[5] C. Rabut, Elementary polyharmonic cardinal B-splines, Numerical Algorithms 2 (1992) 39–62.
[6] K. Vo-Khac, Distribution, Analyse de Fourier, Opérateurs aux Dériveés Partielles, in: Tome, vol. 2, Vuibert, Paris, 1972.
[7] C. Rabut, M. Rossini, Polyharmonic multiresolution analysis: an overview and some new results, Numerical Algorithms 48 (1–3) (2008) 135–160.
[8] M. Rossini, On the construction of polyharmonic B-splines, Journal of Computational and Applied Mathematics 221 (2008) 437–446.
[9] R.J. Renka, R. Brown, Algorithm 792: accuracy tests of ACM algoritm for interpolation of scattered data in the plane, ACM Transactions on Mathematical Software 25 (1999) 78–94.