Computers and Mathematics with Applications 62 (2011) 3894–3901

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Approximate periodic solutions for the Helmholtz–Duffing equation

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ARTICLE INFO

Article history:
Received 15 May 2011
Received in revised form 17 September 2011
Accepted 19 September 2011

Keywords:
Helmholtz–Duffing oscillator
He's Energy Balance Method
He's Frequency Amplitude Formulation

ABSTRACT

Approximate periodic solutions for the Helmholtz–Duffing oscillator are obtained in this paper. He's Energy Balance Method (HEBM) and He's Frequency Amplitude Formulation (HFAF) are adopted as the solution methods. Oscillation natural frequencies are analytically analyzed. Error analysis is carried out and accuracy of the solution methods is evaluated.

0. Introduction

The nonlinear Helmholtz equation has received lots of attention especially in the last decade. The interest arises from large number of applications in the mathematical interpretation of the engineering problems such as ship dynamics, oscillation of the human eardrum, dynamics of a particle moving in a cubic potential and oscillations of one dimensional structural system with an initial curvature. Surveying the literature shows that there are varieties of approximate and numerical solution methods have been already proposed to solve the Helmholtz equation. The Harmonic balance method was employed by Thywle [1] to analyze the forced oscillation of a Helmholtz–Duffing oscillator. Based on the Jacobi elliptic function, an analytical solution was obtained by Cveticanin for a hardening and softening Helmholtz oscillator [2]. The symmetry breaking phenomenon for a general forced Helmholtz–Duffing oscillator was studied by Cao et al. [3]. More recently, the Homotopy perturbation method was employed by Guo and Leung [4,5] to obtain approximate solutions of a Helmholtz–Duffing oscillator. Varieties of variational and perturbative methods have been recently extended mostly by He for instance, one can refer to the He's Variational Approach (HVA) [6], He's Energy Balance Method (HEBM) [7–11], He's Frequency Amplitude Formulation (HFAF) [12–15] and other classical perturbative and non-perturbative techniques [16–37]. The approximate periodic solutions for the Helmholtz–Duffing oscillator are studied employing HEBM and HFAF in this paper. The collocation technique and also Galerkin–Petrov method are combined with the classical HEBM to achieve more accurate approximations. Recently, the Combined-Galerkin–Petrov method was used for the solving of the generalized nonlinear equation with fractional power [34]. Öziş and Yıldırım [34]at the first time have employed it for solving the nonlinear oscillators and found which this approach has better solutions than the classical EBM for the analyzing of the nonlinear oscillator with integer odd power.

The asymmetric nonlinear equation is separated into two auxiliary equations applicable in positive and negative directions. Analytical expressions are then shown for the natural frequency of the oscillation. Dynamic responses are

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.09.042

compared in time domain and also in phase-plane and accuracy of the approximate solutions is evaluated. Error analysis is then carried out and performances of the different solution techniques are compared.

1. Solution procedure

The Helmholtz–Duffing oscillator is considered in this section with the governing equation given by

$$\ddot{u} + u + (1-\alpha)\,u^2 + \alpha u^3 = 0, \tag{1}$$

$$u(0) = A, \qquad \dot{u}(0) = 0. \tag{2}$$

Since the behavior of an asymmetric nonlinear oscillator is different in positive and negative directions, the equation can be conveniently studied in two parts [4,5]

$$\ddot{u} + u + (1-\alpha)\,u^2 \operatorname{sgn}(u) + \alpha u^3 = 0, \quad \text{for } u \geq 0, \tag{3}$$

$$\ddot{u} + u - (1-\alpha)\,u^2 \operatorname{sgn}(u) + \alpha u^3 = 0, \quad \text{for } u \leq 0, \tag{4}$$

α is an asymmetric parameter representing the extend of asymmetry. For $\alpha = 0$ the equation governs motion of a Helmholtz oscillator and for $\alpha = 1$ it denotes a Duffing differential equation. The system is assumed to oscillate between an asymmetric limit zone $[-b, a]$, for positive and b. Both $u = a$ and $u = -b$ represent the turning points in which $\dot{u} = 0$, a and b are an unknown amplitude to be determined. By multiplying $\dot{u}$ on both sides of Eq. (1) and subsequently integration one can reach:

$$\frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 + \frac{1}{3}(1-\alpha)\,u^3 + \frac{1}{4}\alpha u^4 = C. \tag{5}$$

Substituting the kinematic conditions of the turning points one can reach to the following algebraic equation [4,5]:

$$\frac{1}{2}a^2 + \frac{1}{3}(1-\alpha)\,a^3 + \frac{1}{4}\alpha a^4 = \frac{1}{2}b^2 - \frac{1}{3}(1-\alpha)\,b^3 + \frac{1}{4}\alpha b^4. \tag{6}$$

Solving for b yields

$$b = \frac{1}{9a}(3a\alpha + 4 - 4a) + \frac{1}{9a}\Delta^{1/3} - \frac{2}{9a}\left(9a^2\alpha^2 + 6a\alpha - 6a\alpha^2 + 43\alpha - 8 - 8\alpha^2\right)\Delta^{-1/3}. \tag{7}$$

Where

$$\begin{aligned}\Delta ={}& 270a^2\alpha^2(1 + a\alpha - \alpha) - 72a\alpha\left(1+\alpha^2\right) - 516\alpha(1-\alpha) + 64\left(1-\alpha^3\right) + 630a\alpha^2 \\ &+ 54\alpha\left[-12\left(1+\alpha^2\right) + 16a\left(a + 1 + a\alpha^4 - \alpha^3\right) + 78a\left(1 + a^3a - a^3\alpha^2\right) - 8a^3\alpha\left(1-\alpha^3\right)\right. \\ &\left. - 172a^2\alpha\left(1+\alpha^2\right) - 120a\alpha(1-\alpha) + 9a^4\alpha^2\left(1 + 5a\alpha + 10\alpha + 3a^2\alpha^2 - 5a\alpha^2 + \alpha^2\right) + 447a^2\alpha^2\right]^{\frac{1}{2}}.\end{aligned} \tag{8}$$

1.1. Energy balance method

Since the nonlinear differential equation is asymmetric, it should be solved separately for two different parts of the sign axis i.e. initially for $u = a$ and afterward for $u = b$.

(A)

$$u = a \quad \text{and} \quad \dot{u} = 0.$$

The variational principle can be obtained in this case as:

$$J(u) = \int_0^t \left(-\frac{1}{2}\dot{u}^2 + \frac{u^2}{2} + \frac{(1-\alpha)\,u^3}{3}\operatorname{sgn}(u) + \frac{\alpha u^4}{4}\right)dt. \tag{9}$$

The Hamiltonian of Eq. (3), can be consequently obtained in the form of:

$$H = \frac{1}{2}\dot{u}^2 + \frac{u^2}{2} + \frac{(1-\alpha)\,u^3}{3}\operatorname{sgn}(u) + \frac{\alpha u^4}{4} = \frac{1}{2}a^2 + \frac{(1-\alpha)\,a^3}{3} + \frac{\alpha a^4}{4}. \tag{10}$$

In which a is the initial amplitude in positive direction. Employing the trial function of $u_a(t) = a\cos\omega t$ the following residual can be accordingly obtained.

$$\begin{aligned}R(t) ={}& \frac{1}{2}a^2\omega^2\sin^2\omega t + \frac{1}{2}a^2\cos^2\omega t + \left(\frac{(1-\alpha)\,a^3}{3}\cos^3\omega t\right)\operatorname{sgn}(a\cos\omega t) + \frac{\alpha a^4}{4}\cos^4\omega t \\ &- \left(\frac{1}{2}a^2 + \frac{(1-\alpha)\,a^3}{3} + \frac{\alpha a^4}{4}\right).\end{aligned} \tag{11}$$

Using the collocation method at $\omega t \to \frac{\pi}{4}$ one can reach:

$$\omega = \lim_{\omega t \to \frac{\pi}{4}} \frac{1}{\sin \omega t} \left(\left(1 - \cos^2 \omega t\right) + \frac{2}{3}\left(1 - \alpha\right) \alpha \left(1 - \left(\operatorname{sgn}\left(a \cos \omega t\right)\right) \cos^3 \omega t\right) + \frac{1}{2} \alpha a^2 \left(1 - \cos^4 \omega t\right) \right)^{\frac{1}{2}} \tag{12}$$

and finally the natural frequency is obtained as:

$$\omega_a = \sqrt{1 + \frac{\left(4 - \sqrt{2}\right)}{3}\left(1 - \alpha\right) a + \frac{3}{4} \alpha a^2}. \tag{13}$$

Without repeating the solution process, the first approximate oscillation frequency for the trial function of $u_b(t) = b \cos \omega t$ can be obtained as:

$$\omega_b = \sqrt{1 - \frac{\left(4 - \sqrt{2}\right)}{3}\left(1 - \alpha\right) b + \frac{3}{4} \alpha b^2} \tag{14}$$

effective in negative directions. In order to minimize the relative error in this section the approximate solution is obtained again by the Galerkin–Petrov (GP) method as one of the weighted-residual techniques. The first power of the response is taken into account to be the weighting function and the natural frequency can be accordingly obtained.

$$\int_0^{\frac{T}{4}} R(t) \cos \omega t dt = 0. \tag{15}$$

Substituting Eq. (11) into Eq. (15) gives:

$$\omega_a = \sqrt{1 + \left(2 - \frac{3\pi}{8}\right)\left(1 - \alpha\right) a + \frac{7}{10} \alpha a^2}. \tag{16}$$

Similarly for negative directions one can reach:

$$\omega_b = \sqrt{1 - \left(2 - \frac{3\pi}{8}\right)\left(1 - \alpha\right) b + \frac{7}{10} \alpha b^2}. \tag{17}$$

1.2. *He's frequency–amplitude formulation*

According to the standard procedure of the HFAF, the trial functions of $u_1(t) = a \cos t$ and $u_2(t) = a \cos \omega t$ are assumed in the positive direction. The frequency–amplitude formulation is consequently obtained:

$$\omega^2 = \frac{\omega_1^2 \tilde{R}_2 - \omega_2^2 \tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1}. \tag{18}$$

Substituting the trial functions into Eq. (3) results in the following residuals:

$$R_1\left(t_1\right) = \left(1 - \alpha\right)\left(\operatorname{sgn}\left(a \cos t\right)\right) \alpha^2 \cos^2 t + \alpha a^3 \cos^3 t \tag{19}$$

$$R_2\left(t_2\right) = a \cos \omega t \left(1 - \omega^2\right) + \left(1 - \alpha\right)\left(\operatorname{sgn}\left(a \cos \omega t\right)\right) a^2 \cos^2 \omega t + \alpha a^3 \cos^3 \omega. \tag{20}$$

The above residuals can be represented in the form of the following weighted residuals:

$$\tilde{R}_1 = \frac{4}{T_1} \int_0^{T_1/4} R_1 \cos t dt = \frac{4}{3\pi}\left(1 - \alpha\right) a^2 + \frac{3}{8} \alpha a^3 \tag{21}$$

$$\tilde{R}_2 = \frac{4}{T_2} \int_0^{T_2/4} R_2 \cos \omega t dt = \frac{a}{2}\left(1 - \omega^2\right) + \frac{4}{3\pi}\left(1 - \alpha\right) a^2 + \frac{3}{8} \alpha a^3. \tag{22}$$

Finally substituting Eqs. (21) and (22) into Eq. (18) yields:

$$\omega_a = \sqrt{1 + \frac{8}{3\pi}\left(1 - \alpha\right) a + \frac{3}{4} \alpha a^2}. \tag{23}$$

Similarly the same procedure for the negative directions gives:

$$\omega_b = \sqrt{1 - \frac{8}{3\pi}\left(1 - \alpha\right) b + \frac{3}{4} \alpha b^2}. \tag{24}$$

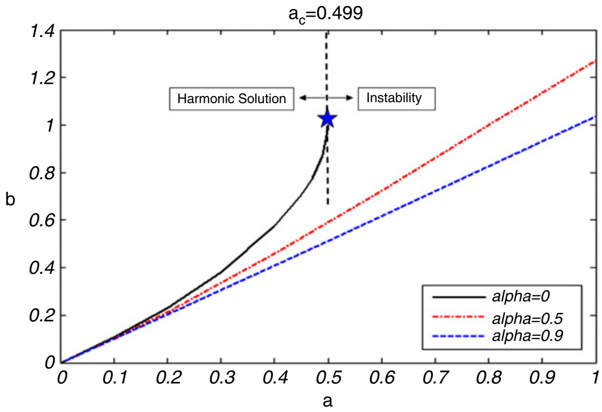

Fig. 1. b versus a for different asymmetric parameters.

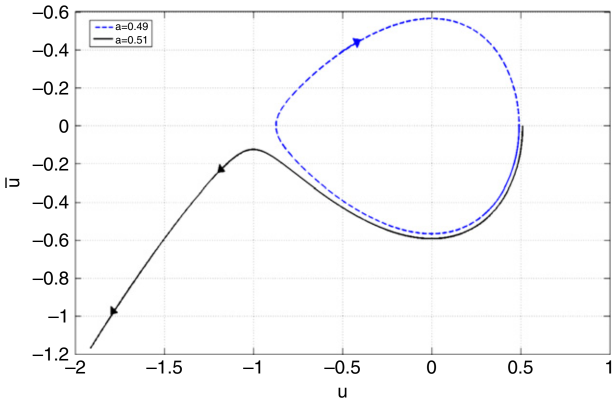

Fig. 2. Phase-plane trajectories for a periodic and an unstable solution ($\alpha = 0$).

2. Numerical results

The approximate oscillation periods found by three methods of HEBM_{C}, HEBM_{GP} and HFAF are compared with the exact periods presented in elliptic integral forms [4,5]

$$T_e = \int_0^a \frac{2dx}{\sqrt{a^2 - x^2 + \frac{2}{3}(1-\alpha)\left(a^3 - x^3\right) + \frac{1}{2}\alpha\left(a^4 - x^4\right)}} + \int_0^a \frac{2dx}{\sqrt{b^2 - x^2 - \frac{2}{3}(1-\alpha)\left(b^3 - x^3\right) + \frac{1}{2}\alpha\left(b^4 - x^4\right)}}. \tag{25}$$

The above elliptic integrals are numerically computed to reach the appropriate reference basis for any error analysis. Series of numerical simulations are then carried out and accuracy and performance of the three methods are evaluated for varieties of asymmetric parameter α and different initial amplitudes. At the first stage, different roots of Eq. (6) are numerically obtained for a range of the asymmetric parameters and the results are illustrated in Fig. 1. As it is seen, for a nonzero asymmetric parameter, the harmonic response is globally stable for any initial amplitude. Furthermore, it is found that in case of a vanishing asymmetric parameter ($\alpha = 0$) a critical value of $a_c = 0.499$ is obtained to be the margin of the stability region. In other words, for any initial amplitude larger than a_c, no real value for b can be found and accordingly no harmonic response exists. This fact can be recognized by the phase-plane trajectories numerically obtained for two initial amplitudes in vicinity of a_c. As it is seen in Fig. 2, for any initial amplitude smaller than a_c, periodic solutions do exist but the trajectories are monotonically approaching infinity for any initial amplitude larger than a_c. Figs. 3–5 illustrate the time histories of oscillation for different α and initial amplitudes. It is seen that the approximate solutions are well correlated with the exact ones even for large amplitudes. The natural period of the oscillatory system is illustrated in Fig. 6 for different amplitudes and asymmetric parameters. The relative error and accuracy of the four different methods are compared in Tables 1 and 2. It is seen that for a pure Helmholtz equation the HEBM-Collocation method is more accurate than others. When the coefficient of quadratic term is vanishing and accordingly the equation approaches the Duffing equation, the

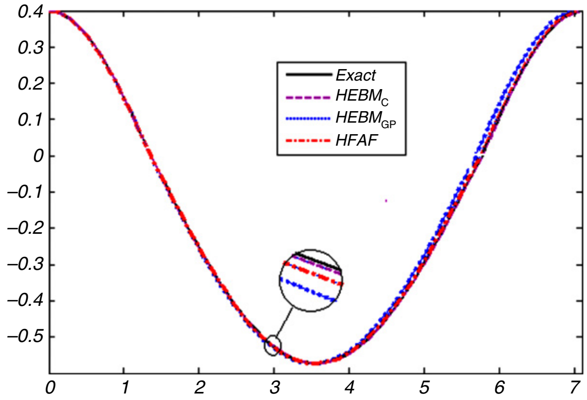

Fig. 3. A comparison between the HEBM_C, HEBM_GP and HFAM solutions ($a = 0.4$, $\alpha = 0$).

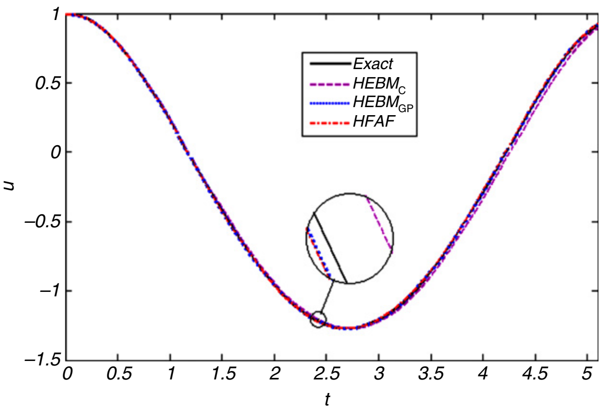

Fig. 4. A comparison between the HEBM_C, HEBM_GP and HFAM solutions ($a = 1$, $\alpha = 0.5$).

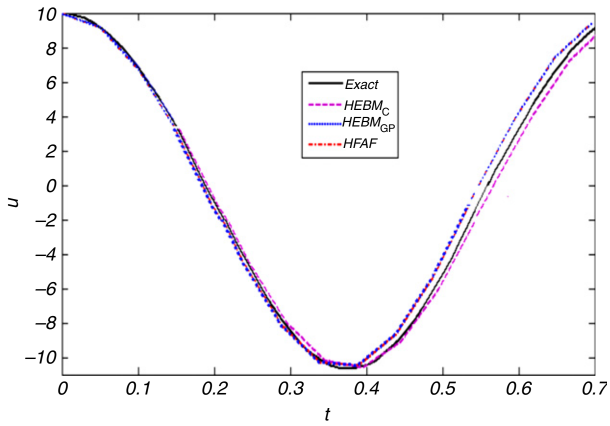

Fig. 5. A comparison between the HEBM_C, HEBM_GP and HFAM solutions ($a = 1$, $\alpha = 0.9$).

HEBM-Galerkin–Petrov combinational method has a better performance. The phase-plane trajectories of the time responses are plotted in Figs. 7 and 8 for two different cases. It is found that the drift in trajectories is enhanced when the governing equation approaches further to the Helmholtz equation.

3. Conclusion

Approximate periodic solutions for the Helmholtz–Duffing oscillator were analytically obtained using HEBM and HFAF. As a combinational method, the Galerkin–Petrov technique was combined with the classical HEBM. Periodic solutions

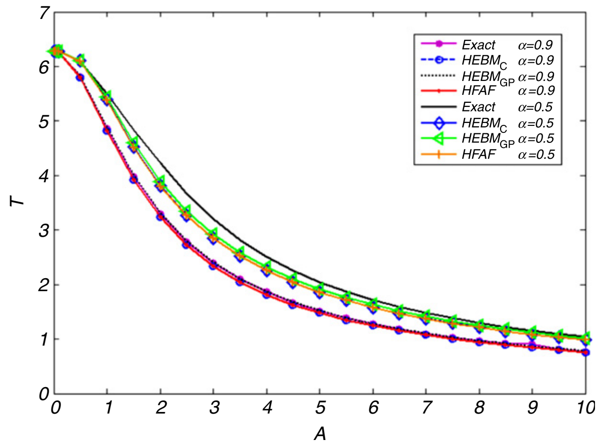

Fig. 6. Natural periods for different values of nonlinearity and oscillation amplitudes.

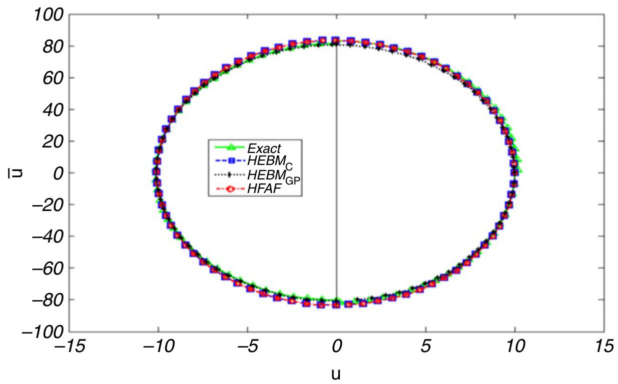

Fig. 7. Phase-plane trajectories of a periodic solution ($a = 10$, $\alpha = 0.9$).

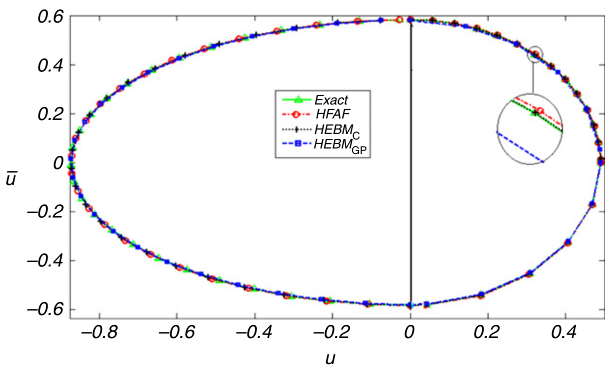

Fig. 8. Phase-plane trajectories of a periodic solution ($a = 0.49$, $\alpha = 0$).

and natural frequencies were analytically studied. Effects of the asymmetric parameters and the initial amplitude on the natural frequencies were investigated in a parametric study. In a fully asymmetric equation, it was found that for any initial amplitude larger than a critical value, no harmonic response can be predictable. This fact was numerically proved and the value of 0.499 was adopted as the critical amplitude. A series of numerical simulations were carried out and the accuracy and performance of the proposed methods were found to be quite satisfactory even for large amplitudes of oscillation. Error analysis was also carried out and it was found that the combinational methods based on the HEBM-Galerkin–Petrov and the HEBM-Collocation lead to more accurate solutions respectively in large and small asymmetric parameters.

Table 1
A comparison between the approximate natural frequencies for $\alpha = 0$.

a	ω_e (exact)	$\omega_{\text{EBM(Collocation)}}$ (relative error %)	$\omega_{\text{HFAF}}$and HPM [4,5] (relative error %)	$\omega_{\text{EBM(Galerkin–Petrov)}}$ (relative error %)
0.1	0.995498178382306	0.995439596570945 (0.005884672883732)	0.995554489461756 (0.005656572826770)	0.995786107091698 (0.028923077474613)
0.2	0.980003298375702	0.979772648320943 (0.023535640659672)	0.980290441237757 (0.029300193431155)	0.981332911455861 (0.135674347460222)
0.3	0.947782010710780	0.947361797968418 (0.044336433653922)	0.948755903642522 (0.102754950055583)	0.951554411027461 (0.398024047096155)
0.4	0.881904164446708	0.882071200882383 (0.018940429403708)	0.885431191029714 (0.399933090827289)	0.892129132672090 (1.159419428730933)
0.49	0.682361789125212	0.702288882401665 (2.920312009557213)	0.713040991802438 (4.496031748283573)	0.733841706350206 (7.544372801265886)

Table 2
A comparison between the approximate natural frequencies for $\alpha = 0.9$.

a	ω_e (exact)	$\omega_{\text{EBM(Collocation)}}$ (relative error %)	$\omega_{\text{HFAF}}$and HPM [4,5] (relative error %)	$\omega_{\text{EBM(Galerkin–Petrov)}}$ (relative error %)
0.01	1.000033361131667	1.000033350555938 (1.0575376374 e−006)	1.000033350555938 (6.0755263445 e−007)	1.000031131040324 (2.2300169472 e−004)
0.1	1.003349649755321	1.003350010889308 (3.5992835235 e−005)	1.003351061321545 (1.4068537563 e−004)	1.003127380145567 (0.022152756998387)
0.5	1.082091820228084	1.082564049913178 (0.043640444948038)	1.082583415822174 (0.045430118304276)	1.077255647678826 (0.446928112647411)
1.0	1.297822417600536	1.302410575147559 (0.353527376688801)	1.302447856718409 (0.356400001660069)	1.284442991939774 (1.030913434636009)
5.0	4.173610227906132	4.255776250029260 (1.968703775300775)	4.255793877545816 (1.969126131860062)	4.119545701194236 (1.295389932447535)
10	8.131090204531180	8.306126340078357 (2.152677330398275)	8.306135462265289 (2.152789519375420)	8.028620448485054 (1.260221611968139)
50	40.229559160163646	41.121720273229229 (2.217675589020683)	41.121722121299847 (2.217680182833432)	39.728192483385911 (1.246264406680850)
100	80.409950769772820	82.194867079860416 (2.219770430152495)	82.194868004528885 (2.219771580095333)	79.408204157824315 (1.245799310108613)

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