On some cubic or quartic algebraic units

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A B S T R A C T

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to solve this problem for totally imaginary quartic units. Here, in Theorem 2, we give a simpler proof of their key result [PL, Theorem 2]. But first, we show in Theorem 1 that their method can also be used to give a simpler proof of [Lou06, Theorem 2], the key result for solving the case of a not totally real cubic algebraic unit. Finally, we mention the recent result [BHMMMS] which is an approach for generalizing these results to the case that the unit rank of the order \( \mathbb{Z}[\epsilon] \) is greater than 1.

**Theorem 1.** (See [Lou06, Lemma 2 and Theorem 2].) Let \( \epsilon > 1 \) be a real cubic algebraic unit which is not totally real. I.e., let \( \epsilon \) be the real root of \( \Pi_\epsilon(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X] \) with \( b \neq a \) and \( b \neq -a - 2 \) (\( \heartsuit \Pi_\epsilon(X) \) is \( \mathbb{Q} \)-irreducible), of negative discriminant \( -d_\epsilon < 0 \) (\( \heartsuit \Pi_\epsilon(X) \) has only one real root), and with \( b \leq a - 1 \) (\( \heartsuit \) the real root of \( \Pi_\epsilon(X) \) is greater than 1). Then,

\[
e^{3/2}/2 \leq d_\epsilon \leq 4(e^{3/4} + e^{-3/4})^4 \leq 64e^3.
\]

More precisely, it holds that \( d_\epsilon \geq 4e^{3/2} \), except for the following four cases:

<table>
<thead>
<tr>
<th>( \Pi_\epsilon(X) )</th>
<th>( d_\epsilon )</th>
<th>( e )</th>
<th>( d_\epsilon/e^{3/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X^3 - 5X^2 + 4X - 1 )</td>
<td>23</td>
<td>4.07959...</td>
<td>2.791...</td>
</tr>
<tr>
<td>( X^3 - 6X^2 - 5X - 1 )</td>
<td>31</td>
<td>6.76136...</td>
<td>1.763...</td>
</tr>
<tr>
<td>( X^3 - 7X^2 + 5X - 1 )</td>
<td>44</td>
<td>6.22226...</td>
<td>2.834...</td>
</tr>
<tr>
<td>( X^3 - 12X^2 - 7X - 1 )</td>
<td>23</td>
<td>12.56350...</td>
<td>0.516...</td>
</tr>
</tbody>
</table>

**Proof.** Let \( \epsilon' = e^{-1/2}e^{ix} \) and \( \epsilon'' = e^{-1/2}e^{-ix} \) be the non-real complex roots of \( \Pi_\epsilon(X) \) (use \( 1 = \epsilon |\epsilon'|^2 \)). Then, \( a = \epsilon + 2e^{-1/2} \cos \alpha, b = 2e^{1/2} \cos \alpha + e^{-1} \).

\[
d_\epsilon = - (\epsilon - \epsilon')^2 (\epsilon - \epsilon'')^2 (\epsilon' - \epsilon'')^2 \leq (\epsilon + e^{-1/2})^4 (2e^{-1/2})^2 = 4(e^{3/4} + e^{-3/4})^4
\]

and

\[
d_\epsilon = 4e^3 |1 - \epsilon'/\epsilon|^4 \sin^2 \alpha \geq 4e^3 (1 - e^{-3/2})^4 \sin^2 \alpha.
\]

Assume that \( \epsilon > 16.2 \). First, if \( \sin^2 \alpha \geq 2e^{-3/2} \), then \( d_\epsilon \geq 8(1 - e^{-3/2})^4e^{3/2} \), hence, \( d_\epsilon \geq 7e^{3/2} \). Second, assume that \( \sin^2 \alpha < 2e^{-3/2} \). Then,

\[-1 < -4e^{-1/2} - e^{-2} \leq 4a - b^2 = 4e \sin^2 \alpha + 4e^{-1/2} \cos \alpha - e^{-2} - 12e^{-1/2} \leq 3.
\]

Hence, \( 4a = b^2, \cos \alpha < 0 \) (for otherwise \( 4a - b^2 \geq 4e^{-1/2} \cos \alpha - e^{-2} > 4e^{-1/2} \sqrt{1 - 2e^{-3/2} - e^{-2}} > 0 \), \( b = -2B \leq 0 \). \( \Pi_\epsilon(X) = X^3 - B^2X^2 - 2BX - 1 \) with \( B \geq 1 \) and \( d_\epsilon = 4B^3 + 27 \geq 4e^{3/2} \) (indeed, \( (1 + 2x)^3 \leq (1 + 27x/4)^2 \) for \( x \in [0, 1] \), hence \( (1 + 2/B^3)^3 \leq (1 + (27/4B^3)) \) and \( (B^2 + 2/B)^2 \leq (d_\epsilon /4)^2 \) for \( B \geq 1 \). Now, \( \Pi_\epsilon(B^2 + 2/B) = 3 + 8/B^3 > 0 \). Hence, \( \epsilon < B^2 + 2/B \) and \( \epsilon^3 \leq (d_\epsilon /4)^2 \). Therefore, \( d_\epsilon \geq 4e^{3/2} \) for \( \epsilon \geq 16.2 \). Finally, if \( 1 < \epsilon \leq 16.2 \), then \( 0 \leq a < e + 2 < 19 \) and \( |b| < 1 + 2\sqrt{a + 2} \). By computing approximations to \( \epsilon \) for the 211 such cubic polynomials with \( 0 \leq a \leq 18 \), we end up with the desired result. \( \square \)

**Theorem 2.** (See [PL, Theorem 2].) Let \( \epsilon \) be a totally imaginary quartic algebraic unit. Let \( \Pi_\epsilon(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X] \) be its minimal \( \mathbb{Q} \)-irreducible polynomial, of positive discriminant \( d_\epsilon > 0 \). We may assume that \( |c| \leq a \) (by changing \( \epsilon \) into \( -\epsilon, 1/\epsilon \) or \( -1/\epsilon \)) and that \( |\epsilon| \geq 1 \). Then,

\[
7|\epsilon|^4/10 \leq d_\epsilon \leq 16(|\epsilon| + |\epsilon|^{-1})^8 \leq 4096|\epsilon|^8.
\]
More precisely, it holds that \( d_\epsilon \geq 4|\epsilon|^4 \), except for the following three cases:

| \( \pi_\epsilon(X) \) | \( d_\epsilon \) | \( |\epsilon| \) | \( d_\epsilon /|\epsilon|^4 \) |
|---|---|---|---|
| \( X^2 - 5X^2 + 5X^2 + 3X + 1 \) | 229 | 2.75146 \ldots | 3.99557 \ldots |
| \( X^2 - 7X^2 + 14X^2 - 6X + 1 \) | 229 | 3.25705 \ldots | 2.03487 \ldots |
| \( X^4 - 13X^2 + 43X^2 - 5X + 1 \) | 1229 | 6.44362 \ldots | 0.71290 \ldots |

**Proof.** The result holds true if \( |\epsilon| = 1 \) (by [Was, Lemma 1.6]). From now on we assume that \( \rho = |\epsilon| > 1 \). Let \( \epsilon = \rho e^{i\alpha}, \bar{\epsilon}, \bar{\epsilon}' = \rho^{-1}e^{i\beta} \) and \( \bar{\epsilon}' \) be the four complex roots of \( \pi_\epsilon(X) \) (use \( |\epsilon|^2|\epsilon'|^2 = 1 \)). Then, \( a = 2\rho \cos \alpha + 2\rho^{-1} \cos \beta, b = \rho^2 + \rho^{-2} + 4(\cos \alpha)(\cos \beta), c = 2\rho^{-1} \cos \alpha + 2\rho \cos \beta \) and

\[
d_\epsilon = ((\epsilon - \bar{\epsilon})(\epsilon - \bar{\epsilon}')(\bar{\epsilon} - \bar{\epsilon}')(\epsilon - \bar{\epsilon}'))^2 \leq 16(\rho + \rho^{-1})^8.
\]

Now, \( \rho > 1 \) and \( |\epsilon| \leq 1 \) imply \( \cos \alpha \geq |\cos \beta| \) and

\[
d_\epsilon = 16(\sin \alpha)^2(\sin \beta)^2 \rho^8 |1 - \rho^{-2}e^{i(\beta - \alpha)}|^2 |1 - \rho^{-2}e^{i(\beta + \alpha)}|^2 \geq (4(\sin \alpha)^2(\rho^2 - 1)^2)^2.
\]

First, if \( \sin^2 \alpha \geq \frac{2}{5} \rho^{-2} \), then \( d_\epsilon \geq (8(1 - \rho^{-2})\rho^2/3)^2 \geq 4\rho^4 \) for \( \rho \geq 2 \).

Second, assume that \( \sin^2 \alpha < \frac{2}{5} \rho^{-2} \). Since \( \rho \geq 1 \), we have

\[
a = 2\rho \sqrt{1 - \sin^2 \alpha + 2\rho^{-1} \cos \beta} \geq 2\rho \sqrt{1 - 3\rho^{-2}/3 - 2\rho^{-1}} \geq 2\rho - 3\rho^{-1}
\]

and, for \( \rho \geq 2\sqrt{3} \), we have

\[-8 < 4b - a^2 = 4(\sin \alpha)^2 \rho^2 + 4(\sin \beta)^2 \rho^{-2} + 8(\cos \alpha)(\cos \beta) < \frac{8}{3} + 4\rho^{-2} + 8 \leq 11 \]

(\( \epsilon \) is totally imaginary, hence \( |\cos \alpha| \neq 1 \) and \( |\cos \beta| \neq 1 \)). Since \( 4b - a^2 \equiv 0, 3 \) (mod 4), we obtain \( 4b - a^2 \in \{-5, -4, -1, 0, 3, 4, 7, 8\} \). There are two cases.

First case, \( a = 2m \) is even, with \( m \geq 0 \), and \( b = m^2 + j \) with \( j \in \{-1, 0, 1, 2\} \). Since \( \pi_\epsilon(m) = jm^2 - cm + 1 \) is positive (for \( \pi_\epsilon(X) \) has no real root), it is greater than or equal to 1, and we have \( -2m \leq c \leq jm \) and \( m \geq 1 \). Since \( X^4 - 2mX^3 + (m^2 + 2)X^2 - 2mX + 1 = (X^2 - mX + 1)^2 \) is Q-reducible, we have \( -2m = A \leq c \leq B := \min jm, 2m - 1 \) for \( m \geq 1 \). We fix \( m \geq 1 \) and \( j \in \{-1, 0, 1, 2\} \), and let \( c \) vary. Numerical investigations suggest that \( d_\epsilon = d_\epsilon(c) \) as a function of \( c \) has four real roots close to \( -2m, jm, 2m \) and \( 4m^2/27 + jm/3 \). Hence, we write \( d_\epsilon(c) = -27\Delta(m, j, c) + P(m, j, c) \), where

\[
\Delta(m, j, c) = (c + 2m)(c - jm)(c - (2m - 1))\left(c - \frac{4m^3}{27} - \frac{jm}{3} - 1\right),
\]

and \( P(m, j, c) = -\alpha(m, j)c^2 - \beta(m, j)c + \delta(m, j) \) with

\[
\alpha(m, j) = 4m^3 - (j^2 + 12)m^2 + 9(j - 6)m + 4j^3 - 144j + 27.
\]

We have \( \Delta(m, j, c) \leq 0 \) for \( A \leq c \leq B \) and \( m \geq 1 \), and \( \Delta(m, j, A) = \Delta(m, j, B) = 0 \). Hence \( d_\epsilon(c) \geq P(m, j, c) \) for \( A \leq c \leq B \), \( P(m, j, A) = d_\epsilon(A) \) and \( P(m, j, B) = d_\epsilon(B) \). Since \( \alpha(m, j) \) is positive for \( j \in \{-1, 0, 1, 2\} \) and \( m \geq 7 \) (note that in (4), we subtracted 1 to the third approximation of the roots of \( d_\epsilon(c) \) and added 1 to the fourth to ensure that \( \alpha(m, j) \) be positive), we obtain:

\[
d_\epsilon(c) \geq \min_{A \leq c \leq B} P(m, j, c) = \min\{P(m, j, A), P(m, j, B)\} = \min\{d_\epsilon(A), d_\epsilon(B)\}
\]
for $m \geq 7$. (This is the lower bound proved at the bottom of page 1341 and top of page 1342 of [PL]. However, our proof is easier to check than theirs. You just have to use any software for mathematics, e.g. Maple, to check that if $\Delta(m, j, c)$ is as in (4), then $P(m, j, c) := d_ε(c) + 27 \Delta(m, j, c)$, is a quadratic polynomial in $c$ whose leading term is of the form $-\alpha(m, j)c^2$ with $\alpha(m, j) > 0$.)

Hence, for $m \geq 7$ and $ρ \geq 2\sqrt{3}$, we obtain

$$d_ε(c) \geq \min(d_ε(-2m), d_ε(\min(jm, 2m - 1)))$$

$$= \begin{cases} d_ε(-m) = 9(m^4 + 4m^2 + 16) & \text{if } j = -1, \\ d_ε(0) = 16(m^4 + 16) & \text{if } j = 0, \\ d_ε(m) = 9(m^4 - 4m^2 + 16) & \text{if } j = 1, \\ d_ε(2m - 1) = 16m^4 - 4m^3 - 128m^2 + 144m + 229 & \text{if } j = 2, \\ \geq 9(m^4 - 4m^2 + 16). \end{cases}$$

Using $ρ \leq ρ \leq 3ρ^{-1}/2 + 3/2 \leq a/2 + 3/2 = m + 3/2$, by (3), we obtain $d_ε \geq 9(m^4 - 4m^2 + 16) \geq 4ρ^4$ for $m \geq 8$.

Second case, $a = 2m + 1$ is odd, with $m \geq 0$, and $b = m^2 + m + j$ with $j \in \{-1, 0, 1, 2\}$. Since $Π_ε(m + 1/2) = (m + 1/2)^2(j - 1/4) - c(m + 1/2) + 1$ is positive, we have $8c < (2m + 1)(4j - 1) + 16/(2m + 1)$. Hence, $8c \leq (2m + 1)(4j - 1)$ for $m \geq 8$, hence $8c \leq (2m + 1)(4j - 1) - 1$ and $-2m + 1 = A ≤ c \leq B := (j - 1/4)(m + 1/2) - 1/8 \leq 2m$ for $-1 \leq j \leq 2$ and $m \geq 8$. Now, as in the proof of the first case, we write $d_ε(c) = -27A(m, j, c) + P(m, j, c)$, where

$$Δ(m, j, c) = (c - A)(c - B)(c - (2m + 1/8)) \left( c - \frac{4m + \frac{1}{2}^3}{27} - \frac{(j - \frac{1}{4})(m + \frac{1}{2})}{3} - 1 \right)$$

which yields $d_ε(c) \geq P(m, j, c)$ for $A \leq c \leq B$, $d_ε(A) = P(m, j, A)$ and $d_ε(B) = P(m, j, B)$, and where $P(m, j, c) = -\alpha(m, j)c^2 - β(m, j)c + δ(m, j)$ with

$$\alpha(m, j) = 4m^3 - \left( j^2 - \frac{j}{2} + \frac{97}{16} \right)m^2 - \left( j^2 - \frac{49j}{8} + \frac{1847}{32} \right)m + 4j^3 - \frac{13j^2}{4} - \frac{2245j}{16} + \frac{2121}{64}.$$

Since $\alpha(m, j)$ is positive for $j \in \{-1, 0, 1, 2\}$ and $m \geq 6$, we obtain

$$d_ε(c) \geq \min_{A \leq c \leq B} P(m, j, c) = \min(P(m, j, A), P(m, j, B)) = \min(d_ε(A), d_ε(B))$$

for $m \geq 8$, and

$$d_ε(c) \geq \min(d_ε(-2m - 1), d_ε\left(\left( j - \frac{1}{4}\right)(m + \frac{1}{2}) - \frac{1}{8}\right)) \geq 9(m^4 + 2m^3 + 11m^2 + 10m + 13)$$

for $m \geq 18$. (Look at the four cases $j \in \{-1, 0, 1, 2\}$. In these four cases, $d_ε(B)$ is a quintic polynomial in $m$.) Using $ρ \leq ρ - 3ρ^{-1}/2 + 3/2 \leq a/2 + 3/2 = m + 2$, by (3), we obtain $d_ε \geq 9(m^4 + 2m^3 + 11m^2 + 10m + 13) \geq 4ρ^4$ for $m \geq 18$.

Finally, since $|c| \leq a < \sqrt{4b + 24}$ and $-1 \leq b \leq ρ^2 + ρ^{-2} + 4$, it is easy to list all the possible polynomials $Π_ε(X)$ for which $ρ \leq 2\sqrt{3}$ and to check that Theorem 2 holds true for these polynomials. Since $1/ρ^2 < 1$ is the only real root in $(0, 1)$ of the sextic polynomial

$$Q_ε(X) = X^6 - bX^5 + (ac - 1)X^4 - (a^2 - 2b + c^2)X^2 + (ac - 1)X^2 - bX + 1$$
(see [Lou08, Proof of Lemma 13]) (and since $Q_ε(0) = 1 > 0$, but $Q_ε(1) = -(a - c)^2 < 0$ for $c \neq a$), using the dichotomy method, it is easy to compute numerical approximations to $\rho$ and to check that Theorem 2 holds true for all the polynomials $Π_ε(X)$ with either (i) $a = 2m$, $b = m^2 + j$, $0 \leq m \leq 7$ and $-1 \leq j \leq 2$, or (ii) $a = 2m + 1$, $b = m^2 + m + j$, $0 \leq m < 18$ and $-1 \leq j \leq 2$. □

References


