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# The symmetry of least-energy solutions for semilinear elliptic equations

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## Abstract

In this paper we will apply the method of rotating planes (MRP) to investigate the radial and axial symmetry of the *least-energy* solutions for semilinear elliptic equations on the Dirichlet and Neumann problems, respectively. MRP is a variant of the famous method of moving planes. One of our main results is to consider the *least-energy* solutions of the following equation:

$$\begin{cases} \Delta u + K(x)u^p = 0, & x \in B_1, \\ u > 0 \text{ in } B_1, & u|_{\partial B_1} = 0, \end{cases} \qquad (*)$$

where $1 < p < \frac{n+2}{n-2}$ and $B_1$ is the unit ball of $\mathbb{R}^n$ with $n \geqslant 3$. Here $K(x) = K(|x|)$ is *not* assumed to be decreasing in $|x|$. In this paper, we prove that any *least-energy* solution of $(*)$ is axially symmetric with respect to some direction. Furthermore, when $p$ is close to $\frac{n+2}{n-2}$, under some reasonable condition of $K$, radial symmetry is shown for *least-energy* solutions. This is the example of the general phenomenon of the symmetry induced by *point-condensation*. A fine estimate for *least-energy* solution is required for the proof of symmetry of solutions. This estimate generalizes the result of Han (Ann. Inst. H. Poincaré Anal. Nonlinéaire 8 (1991) 159) to the case when $K(x)$ is nonconstant. In contrast to previous works for this kinds of estimates, we only assume that $K(x)$ is continuous.


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## 1. Introduction

Recently in the research area of nonlinear elliptic PDEs, there have been many works devoted to studying problems where solutions exhibit the "phenomenon of point-condensation". Two well-known examples are semilinear elliptic equations involving the Sobolev critical exponent and nonlinear elliptic equations with small diffusion coefficient. These works show that the concentration often induces the asymptotic symmetry. For example, spherical Harnack inequalities have been proved for blowup solutions to either mean field equations on compact Riemann surfaces or the scalar curvature equation. These spherical Harnack inequalities implies that blowup solutions usually are asymptotically symmetric. Similar results were proved for spike-layer solutions of singularly elliptic Neumann problem. See [CL1,CL2,L1,L2,NT1,NT2] for more precise statements. Naturally, when the underlying equation is invariant under a group of transformations, we would like to know whether solutions with point-condensation actually possess certain symmetry which is invariant under the action of some elements of the group. In [Ln1,Ln2], for the mean field equation on $S^2$, the second author first succeeded to prove the axial symmetry for solutions with two blowup points. In this article, we continue to study this problem.

In this paper, we first consider positive solutions of the following equation:

$$\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } B_1, \\ u|_{\partial B_1} = 0, \end{cases} \tag{1.1}$$

where $B_1$ is the unit ball in $\mathbb{R}^n$, $n \geqslant 2$, $\Delta$ is the Laplace operator and $f(r,t)$ is a $C^1$ function of both variables $r$ and $t$. The typical examples of $f$ are $K(|x|)u^p$ where $1 < p < \frac{n+2}{n-2}$ if $n \geqslant 3$, $1 < p$ if $n = 2$. When $K(r)$ is decreasing in $r$, the famous theorem by Gidas et al. [GNN1,GNN2] says that any positive solution $u(x)$ of (1.1) is radially symmetric. However, the radial symmetry of solutions generally fails if $K(r)$ does not decrease with respect to $r$ for all $r \leqslant 1$. In this paper, we want to show that certain symmetry still holds for *least-energy* solutions. The definition of the *least-energy* solutions of (1.1) is stated as follows. Consider the variational functional

$$J(u) = \int_{B_1} \left[ \frac{1}{2} |\nabla u|^2 - F(|x|, u^+) \right] dx \quad \text{in } H_0^1(B_1), \tag{1.2}$$

where $F(r,u) = \int_0^u f(r,s)\, ds$ and $u^+(x) = \max(0, u(x))$. For the nonlinear functional $J$, we set

$$c_* = \inf_{h \in \Gamma} \max_{0 \leqslant t \leqslant 1} J(h(t)), \tag{1.3}$$

where

$$\Gamma = \{ h \in C([0,1], H_0^1(B_1)) \mid h(0) = 0, h(1) = e \}$$

and $e \in H_0^1(B_1)$, $e \neq 0$ in $B_1$ with $J(e) = 0$. To guarantee the $c_*$ of (1.3) to be a critical value of $J$ by the mountain pass lemma, the nonlinear term $f$ is usually assumed to satisfy the following condition.

($f_a$) $f(r,t) = o(|t|)$ near $t = 0$ and $0 \leqslant r \leqslant 1$;
($f_b$) there exist constants $\theta \in (0, \frac{1}{2})$ and $U_0 > 0$ such that $0 < F(x,u) \equiv \int_0^u f(r,s)\,ds \leqslant \theta u f(x,u)$ for all $u \geqslant U_0$;
($f_c$) $|f(x,t)| \leqslant Ct^q$ for some $1 < q < \frac{n+2}{n-2}$ for large $t$ if $n \geqslant 3$ and $1 < q < +\infty$ if $n = 2$.

Using the above conditions ($f_a$)–($f_c$) and by the well-known mountain-pass lemma due to Ambrosetti and Rabinowitz (see [AR]), we can obtain that (1.2) possesses a positive critical point $u_*$ with its critical value $J(u^*)$ to be equal to $c_*$ of (1.3). Moreover, under the additional assumption that $f(r,t)/t$ is increasing in $t$, from the Lemma 3.1 in [NT1,NT2], $c_*$ does not depend on the choice of $e$ and is the least-positive critical value of $J$. Therefore, We call such $u_*$ to be a *least-energy* solution of Eq. (1.1). We remark that solutions of least energy can also be obtained by minimization of

$$\inf_{v \in H_0^1(\Omega)} \frac{\int |\nabla v|^2}{[(\int K(v^+)^{p+1})^+]^{\frac{2}{p+1}}},$$

where $f(x,u) = K(x)u^p$ with $\max_\Omega K > 0$.

Our first result is concerned with the axial symmetry of the *least-energy* solution of Eq. (1.1).

**Theorem 1.1.** *Suppose $f$ satisfies conditions* ($f_a$)–($f_c$) *and*

($f_d$) $\frac{\partial^2 f}{\partial t^2}(r,t) > 0$ *for* $t > 0$ *and for* $0 \leqslant r \leqslant 1$.

*Let $u$ be a least-energy solution of Eq.* (1.1) *and $P_0$ be a maximum point of $u$. Then the following conclusions hold.*

(i) *If $P_0 = O$ is the origin, then $u$ is radially symmetric.*
(ii) *If $P_0 \neq O$, then $u$ is axially symmetric with respect to $\overrightarrow{OP_0}$ and on each sphere $S_r = \{x\colon |x| = r\}$ for $0 < r < 1, u(x)$ is increasing as the angle of $\overrightarrow{Ox}$ and $\overrightarrow{OP_0}$ decreases. In particular, $u$ satisfies*

$$x_j \frac{\partial u}{\partial x_n}(x) - x_n \frac{\partial u}{\partial x_j}(x) > 0 \quad \textit{for } x_j > 0, \tag{1.4}$$

*where $P_0$ is assumed to locate on the positive $x_n$-axis.*

We note that, by condition ($f_d$), it is easy to see that $\frac{\partial}{\partial t}(t\frac{\partial f(r,t)}{\partial t} - f(r,t)) = t\frac{\partial^2 f(r,t)}{\partial t^2} \geqslant 0$ $\forall t > 0$ $\forall 0 \leqslant r \leqslant 1$ and hence $f(r,t)/t$ is increasing in $t$. So, by ($f_a$), we have

$f(r,t)\geqslant 0\ \forall t\geqslant 0\ \forall 0\leqslant r\leqslant 1$. In this paper, we will use the method of rotating planes to prove Theorem 1.1. The method of rotating planes is a variant of the famous method of moving planes (MMP). MMP was first invented by Alexandroff and later was used by Gidas–Ni–Nirenberg to prove the radial symmetry of positive solutions. See [BN,CGS,GNN1,GNN2] and the references therein. Recently, MMP was applied to prove spherical Harnack inequality for blowup solutions to either scalar curvature equation or mean field-type equations. See [CL1,CL2,L1,L2]. We note that MMP cannot be applied to the Neumann problem for semilinear elliptic equations. As far as the authors know, the result concerning the radial symmetry for the Neumann problem is very rare. Nevertheless, our next result shows that the method of rotating planes can be employed for the Neumann problem and the axial symmetry can be established by this method.

Our second result is about the axial symmetry of the *least-energy* solutions of the Neumann problem. We consider the following equation:

$$\begin{cases} d\Delta u - u + f(u) = 0 \quad \text{in } B_1, \\ u>0 \quad \text{in } B_1, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_1, \end{cases} \tag{1.5}$$

where $d$ is a positive parameter and $f$ satisfies conditions $(\mathrm{f_a})$–$(\mathrm{f_c})$.

The typical examples of $f$ are $u^p$ where $1<p<(n+2)/(n-2)$ if $n\geqslant 3$, and $1<p<+\infty$ if $n=2$. In this case, Eq. (1.5) is the steady-state problem for a chemotactic aggregation model with logarithmic sensitivity by Keller and Segel [KS]. It can also be considered as the shadow system of some reaction–diffusion system in chemotaxis, see e.g. [NT2].

Under conditions $(\mathrm{f_a})$–$(\mathrm{f_c})$, Theorem 2 and Proposition 2.2 in [LNT] guarantee that, for each $d>0$, (1.5) possesses a solution $u_d$ which is a critical point of the variational functional

$$J_d(u) = \frac{1}{2}\int_{B_1} (d|\nabla u|^2 + u^2)\,dx - \int_{B_1} F(u^+)\,dx \quad \forall u \in H^1(B_1), \tag{1.6}$$

where $u^+ = \max\{u(x), 0\}$, and its critical value $c_d = J_d(u_d)$ is proved to be equal to

$$c_d = \inf_{h\in\Gamma}\ \max_{0\leqslant t\leqslant 1}\ J(h(t)), \tag{1.7}$$

where $\Gamma$ is defined as before, and $c_d$ is independent of the choice of $e$ by the Lemma 3.1 in [NT1,NT2]. Such a critical point $u_d$ is called a *least-energy* solution of Eq. (1.5).

Our second result is in the following.

**Theorem 1.2.** *Suppose conditions* $(\mathrm{f_a})$–$(\mathrm{f_d})$ *hold. Let* $u$ *be a least-energy solution of* (1.5) *and* $P_0$ *be a local maximum point of* $u$ *on* $\overline{B_1}$. *Then either* $u \equiv$ constant *in* $B_1$, *or*

*$P_0 \neq 0$ and u is axially symmetric with respect to $\overrightarrow{OP_0}$ and* (1.4) *holds where $P_0$ is assumed to locate on the positive $x_n$-axis.*

We give some remarks here. (I) If $f$ satisfies conditions $(f_a)$–$(f_d)$, then *any nonconstant least-energy solution u must be nonradial* because the origin cannot be a critical point of $u$. See the proof of Theorem 1.2. (II) Theorem 1.2 had been proved in [NT1,NT2, Proposition 5.1] when $P_0$ is assumed to locate on the boundary of $\bar{B}_1$ and $f(t)$ is an analytic function for $t>0$. After the paper is finished, a stronger version of Theorem 1.2, that is, $P_0$ must be on the boundary $\partial B_1$, has been proved in [Ln2].

Before stating our third result, we let $S_n$ be the best Sobolev constant, i.e., for any bounded domain $\Omega$ of $\mathbb{R}^n$ and for $n \geqslant 3$,

$$S_n = \inf_{v \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 \, dx}{\left(\int_\Omega |v|^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}}}. \tag{1.8}$$

It is well known that the best Sobolev constant is independent of $\Omega$ and is never achieved by an element in $H_0^1(\Omega)$. For a $C^1$ function $K$ with $\max_{B_1} K > 0$, we consider a *least-energy* solution $u$ of

$$\begin{cases} \Delta u + K(x) u^p = 0, & x \in B_1, \\ u > 0 \text{ in } B_1, & u|_{\partial B_1} = 0, \end{cases} \tag{1.9}$$

where $1 < p < \frac{n+2}{n-2}$. Suppose $u_i$ is a *least-energy* solution of (1.9) with $p = p_i \uparrow \frac{n+2}{n-2}$. It is easy to see that $u_i$ achieves the infinimum of the variational problem,

$$\begin{aligned} \frac{\int_{B_1} |\nabla u_i|^2 \, dx}{\left(\int_{B_1} K(x) u_i^{p_i+1} \, dx\right)^{\frac{2}{p_i+1}}} &= \inf_{v \in H_0^1(B_1)} \frac{\int_{B_1} |\nabla v|^2 \, dx}{\left[\left(\int_{B_1} K(x) (v^+)^{p_i+1} \, dx\right)^+\right]^{\frac{2}{p_i+1}}} \\ &= \left( \frac{1}{(\max_{\bar{B}_1} K)^{\frac{2}{p_i+1}}} S_n \right) (1 + o(1)) \end{aligned} \tag{1.10}$$

for $i$ sufficiently large. Let $P_i$ be the global maximum point of $u_i$. Since (1.8) is never achieved by some function in $H_0^1(B_1)$, we have

$$\lim_{i \to \infty} u_i(P_i) = +\infty \quad \text{and} \quad \lim_{i \to \infty} K(P_i) = \max_{\bar{B}_1} K(x). \tag{1.11}$$

Obviously, by (1.11), the necessary condition for $u_i$ to be radially symmetric is that $K(x) = K(|x|)$ and the origin is the maximum point of $K$. For the final result, we want to prove what is the sufficient condition of $K$ such that for any *least-energy* solution is radially symmetric. Suppose $K(x) = K(|x|)$ satisfies the following condition.

$$\text{There exists } r_0 \in (0,1] \text{ such that } K'(r) \leqslant 0 \quad \forall 0 \leqslant r \leqslant r_0$$

$$\text{and } \max(0, K(r)) < K(0) \quad \forall r_0 \leqslant r \leqslant 1. \tag{$\mathrm{K_a}$}$$

Note that $(\mathrm{K_a})$ could allow $K(r) \equiv K(0)$ for all small $r>0$. In this case, it is not evident that the maximum point $P_i$ of $u_i$ would tend to the origin. However, we have the following.

**Theorem 1.3.** *Suppose condition* $(\mathrm{K_a})$ *holds. Then there exists a small* $\varepsilon > 0$ *such that for any least-energy solution* $u$ *of* (1.9) *with* $0 < \frac{n+2}{n-2} - p \leqslant \varepsilon$, $u$ *is radially symmetric. Furthermore, if* $(\frac{rK'(r)}{K(r)})' \leqslant 0$ $\forall 0 \leqslant r \leqslant r_1$ *for some* $r_1 \leqslant r_0$, *then* (1.9) *possesses a unique least-energy solution when* $0 < \frac{n+2}{n-2} - p \leqslant \varepsilon$.

The proof of Theorem 1.3 is more complicated than previous theorems. Here, the concentration actually occurs for least-energy solutions as $p$ tends to $\frac{n+2}{n-2}$. In order to start the process of the method of rotating planes, we have to require some fine estimates for least-energy solutions, that is, we have to show that least-energy solutions always behaves "simply" near its blowup point. When $K(x) \equiv$ a positive constant, this was proved by Han [H]. However, for a nonconstant function $K(x)$, there is additional difficulty even by using Han's method. In the appendix, we give a proof which is simpler in conception even for the case when $K$ is a constant. Since the Pohozaev identity is *not* employed in our proof, we do not require any smoothness assumption on $K$.

We organize this paper as follows. In Section 2, we prove Theorem 1.1 by using the method of rotating planes. By the same method, the axial symmetry of the Neumann problem is established in Section 3. Here we emphasize that any nonconstant *least-energy* solution must be nonradially symmetric. Finally, we complete the proof of Theorem 1.3 in Section 4.

## 2. Maximum principle via the method of rotating planes

In this paper we will give the detail of the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let $u$ be a *least-energy* solution of Eq. (1.1). We divide the proof into the following steps.

*Step* 1: Let $T$ be any hyperplane which passes the origin $O$. We claim that one of the following conclusions holds:

(A) $u(x) = u(x^*)$ $\forall x \in B_1^+$,
(B) $u(x) > u(x^*)$ $\forall x \in B_1^+$,
(C) $u(x) < u(x^*)$ $\forall x \in B_1^+$, where $B_1^+$ is one of half-balls of $B_1 \backslash T$ and $x^*$ is the reflection point of $x$ with respect to the hyperplane $T$.

First, we will prove that

$$\begin{cases} \text{either } u(x) \geqslant u(x^*) & \forall x \in B_1^+ \\ \text{or } u(x) \leqslant u(x^*) & \forall x \in B_1^+. \end{cases} \tag{2.1}$$

Suppose the conclusions of (2.1) are not true. Then the following two sets are all nonempty:

$$\Omega_+ = \{x \in B_1^+ \mid u(x) > u(x^*)\},$$

$$\Omega_- = \{x \in B_1^+ \mid u(x) < u(x^*)\}. \tag{2.2}$$

Let

$$w(x) = u(x) - u(x^*) \quad \forall x \in B_1^+. \tag{2.3}$$

Then $w$ satisfies

$$\begin{cases} \Delta w + f_u(|x|, z(x))w = 0, & x \in B_1^+, \\ w|_{\partial B_1^+} = 0, \end{cases} \tag{2.4}$$

where $z(x)$ is between $u(x)$ and $u(x^*)$. Let

$$\Omega_-^* = \{x^* \mid x \in \Omega_-\}. \tag{2.5}$$

Set

$$v(x) = \begin{cases} w(x) & \text{if } x \in \Omega_+, \\ dw(x^*) & \text{if } x \in \Omega_-^*, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Choose the constant $d > 0$ such that

$$\int_{B_1} v(x)\phi_1(x)\,dx = 0, \tag{2.7}$$

where $\phi_1$ is the first eigenfunction of the following eigenvalue problem:

$$\begin{cases} \Delta \phi + f_u(|x|, u)\phi = -\lambda\phi, & x \in B_1, \\ \phi|_{\partial B_1} = 0. \end{cases} \tag{2.8}$$

Let $\lambda_2$ be the second eigenvalue of (2.8). By condition ($\mathrm{f_d}$), we easily have $\frac{\partial}{\partial t}(t\frac{\partial f(r,t)}{\partial t} - f(r,t)) = t\frac{\partial^2 f(r,t)}{\partial t^2} \geqslant 0 \ \forall t > 0 \ \forall 0 \leqslant r \leqslant 1$ and hence $f(r,t)/t$ is nondecreasing in $t$. Since $u$ is a *least-energy* solution of (1.1), by using the same method of the proof of Theorem 2.11 in [LN], we have

$$\lambda_2 \geqslant 0. \tag{2.9}$$

By condition $(\mathrm{f_d})$ and (2.2)–(2.6), it is easy to see that

$$\Delta v + f_u(|x|, u(x))v \begin{cases} \geqslant 0 & \forall x \in \Omega_+, \\ \leqslant 0 & \forall x \in \Omega_-^*, \\ = 0 & \text{otherwise.} \end{cases} \quad \not\equiv 0 \text{ in } B_1, \tag{2.10}$$

From (2.6), (2.7), (2.9), (2.10) and $v \not\equiv 0$, we obtain

$$\begin{aligned} 0 &> \int_{B_1} (-v(x)) \cdot [\Delta v(x) + f_u(|x|, u(x))v(x)]\, dx \\ &= \int_{B_1} [|Dv(x)|^2 - f_u(|x|, u(x))v^2(x)]\, dx \geqslant 0, \end{aligned} \tag{2.11}$$

a contradiction. This proves (2.1). By (2.1), we may assume $w(x) \geqslant 0$ for $x \in B_1^+$. Since $w(x)$ satisfies Eq. (2.4), by using the strong maximum principle, we have either $w(x) \equiv 0$ for $x \in B_1^+$ or $w(x) > 0$ for $x \in B_1^+$. Similarly, if $w(x) \leqslant 0$, we have either $w(x) \equiv 0$ on $B_1^+$ or $w(x) < 0$ on $B_1^+$ This finishes step 1.

*Step* 2: If $P_0 = O$, then we want to prove that $u$ is symmetric with respect to any linear hyperplane and then $u$ is radially symmetric. Without loss of generality, we assume that the hyperplane is $\{x_1 = 0\}$, that is, we want to prove $w(x) = u(x) - u(x^-) \equiv 0$ for $x_1 > 0$, where $x = (x_1, x')$ and $x^- = (-x_1, x')$. Suppose $w(x) \not\equiv 0$ for $x \in B_1^+ = \{x \in B_1 \mid x_1 > 0\}$. Then, by step 1, we may assume that $w(x) > 0\ \forall x \in B^+$. Then, from (2.4) and applying the Hopf lemma, we have

$$\frac{\partial w}{\partial(-x_1)}(O) = -2\frac{\partial u}{\partial x_1}(O) < 0. \tag{2.12}$$

However, since $O$ is the maximal point of $u$ in $B_1$, we have $\frac{\partial u}{\partial x_1}(O) = 0$, which yields a contradiction to (2.12). Thus, $w(x) \equiv 0$ and the radial symmetry of $u$ follows readily. We prove part (i) in Theorem 1.1.

*Step* 3: If $P_0 \neq O$, then we will prove that part (ii) in Theorem 1.1 hold. Without loss of generality, we may assume $\overrightarrow{OP_0}$ is the positive $x_n$-axis. Let $P_0 = (0, \ldots, 0, t)$ and $T_0$ be the hyperplane $\{x_n = 0\}$. Then $u(P_0) \geqslant u(P_0^-)$, and from step 1, we obtain that:

$$\begin{aligned} &\text{either } u(x) = u(x^-) \quad \forall x \in B_1^+ \\ &\text{or } u(x) > u(x^-) \quad \forall x \in B_1^+, \end{aligned} \tag{2.13}$$

where $x = (x', x_n),\ x^- = (x', -x_n),\ B_1^+ = \{x \in B_1 \mid x_n > 0\}$. If the former case holds, then $u(P_0) = u(Q_0) = \max_{\bar{B}_1} u(x)$, where $Q_0 = (0, \ldots, 0, -t)$. Let $T$ be any hyperplane passing through the origin such that $P_0 \notin T$ and $B^+(T)$ be the half-ball of $B_1 \backslash T$ such that $P_0 \in B^+(T)$. Because $Q_0^{**} = Q_0 \notin B^+(T)$ and $u(P_0) \geqslant u(P_0^*)$ and $u(Q_0^*) \leqslant u(Q_0^{**})$, where $x^*$ is the reflection point of $x$ w.r.t. $T$, by step 1, we must have $u(x) = u(x^*)\ \forall x \in B^+(T)$. Since $T$ is any hyperplane, we conclude that $u$ is radially symmetric in this case. If the latter case is true, then we will prove that $u$ is axially symmetric with respect to $\overrightarrow{OP_0}$ and the second conclusion of part (ii) in Theorem 1.1 holds. Consider any two-dimensional plane which contains $P_0$. For the simplicity, let us assume that the plane is spanned by $e_1 = (1, 0, \dots, 0)$ and $e_n = (0, \dots, 0, 1)$. Let $l_\theta$ be the line having the angle $\theta$ with $x_1$-axis, and $v_\theta$, with $v_0 = e_n$, be the normal vector to the line in this plane. Set $T_\theta$ to be the $(n-1)$-dimensional linear hyperplane which passes the origin and has $v_\theta$ as the normal vector. Obviously, $T_\theta = \{(r_1 \cos\theta, x_2, \dots, x_{n-1}, r_1 \sin\theta) \mid x_j \in \mathbb{R} \text{ for } 2 \leqslant j \leqslant n-1 \text{ and } r_1 > 0\}$. Let $B_\theta$ be one of the half-balls of $B_1 \backslash T_\theta$ which contains $P_0$ for $0 \leqslant \theta < \frac{\pi}{2}$. Let $x^\theta$ be the reflection point of $x \in B_\theta$ w.r.t. $T_\theta$.

Set

$$w_\theta(x) = u(x) - u(x^\theta) \quad \forall x \in B_\theta. \tag{2.14}$$

Then $w_\theta$ satisfies

$$\begin{cases} \Delta w_\theta + c_\theta(x) w_\theta = 0 & \text{in } B_\theta, \\ w_\theta(x) = 0 & \text{on } \partial B_\theta, \end{cases} \tag{2.15}$$

where

$$c_\theta(x) = \frac{f(|x|, u(x)) - f(|x|, u(x^\theta))}{u(x) - u(x^*)}.$$

For $\theta = 0$, we have $w_0(x) > 0 \ \forall x \in B_0$. Set

$$\theta_0 = \sup\left\{\theta \middle| w_{\tilde\theta}(x) \geqslant 0 \ \forall x \in B_{\tilde\theta} \ \forall 0 \leqslant \tilde\theta \leqslant \theta \leqslant \frac{\pi}{2}\right\}. \tag{2.16}$$

We claim that $\theta_0 = \frac{\pi}{2}$. Suppose this is not true. Then, from step 1 and the definition of $\theta_0$, we have for $0 \leqslant \theta < \theta_0$,

$$\begin{aligned} & w_\theta(x) > 0 \text{ for } x \in B_\theta \quad \text{and} \quad \frac{\partial u}{\partial v_\theta}(x) < 0 \text{ for } x \in T_\theta, \\ & w_{\theta_0} \equiv 0 \quad \text{in } B_{\theta_0}. \end{aligned} \tag{2.17}$$

Let $P_0^*$ be the reflection point of $P_0$ w.r.t. $T_{\theta_0}$. Then $P_0^*$ is also a global maximum point. Since $w_0(x) > 0$ in $B_0$, we have $P_0^* \in T_{\theta_1}$ for some $\theta_1 \in (0, \theta_0)$ and $\nabla u(P_0^*) = 0$, which yields a contradictions to (2.17). Hence, we have

$$w_{\frac{\pi}{2}} \geqslant 0 \quad \text{in } B_{\frac{\pi}{2}}. \tag{2.18}$$

Similarly, using the above arguments, we can also obtain

$$w_{-\frac{\pi}{2}} \geqslant 0 \quad \text{in } B_{-\frac{\pi}{2}}. \tag{2.19}$$

From (2.18) and (2.19) we deduce that

$$w_{\frac{\pi}{2}} \equiv 0 \quad \text{in } B_{\frac{\pi}{2}}. \tag{2.20}$$

The axial symmetry follows readily from (2.20).

Let $x = (r_1 \cos \theta, x_2, \ldots, x_{n-1}, r_1 \sin \theta)$, $r_1 = (|x|^2 - x_2^2 - \cdots - x_{n-1}^2)^{1/2}$. Then, from

$$\frac{\partial u}{\partial \theta}(x) = \frac{-1}{2} \frac{\partial w_\theta}{\partial v_\theta}(x) > 0 \quad \forall x \in (B_1 \cap T_\theta) \ \forall \frac{-\pi}{2} < \theta < \frac{\pi}{2}, \tag{2.21}$$

where $v_\theta$ is the outnormal of $\Sigma_\theta$ on the boundary $T_\theta$, the monotonicity follows clearly. From (2.20) and (2.21), we easily obtain (1.4). This proves step 3 and completes the proof of Theorem 1.1. □

## 3. Axial symmetry for the Neumann problem

In this section, we will complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We divide the proof into the following steps.

*Step* 1: Let $u$ be the *least-energy* solution of (1.5). Consider the following eigenvalue problem:

$$\begin{cases} d\Delta\phi - \phi + f_u(|x|, u)\phi + \lambda\phi = 0, & x \in B_1, \\ \dfrac{\partial \phi}{\partial v} = 0 \quad \text{on } \partial B_1. \end{cases} \tag{3.1}$$

From conditions $(\mathrm{f_a})$–$(\mathrm{f_d})$ and Theorem 2.11 in [LN], we obtain that the second eigenvalue of (3.1) is nonnegative, i.e.,

$$\lambda_2(u) \geqslant 0. \tag{3.2}$$

Let $T$ be any hyperplane which contains the origin. Then, using the same arguments in step 1, we also obtain that one of the following conclusion holds.

$$\begin{cases} u(x) = u(x^*) \\ u(x) > u(x^*) \quad \text{for all } x \in B^+ \\ u(x) < u(x^*) \end{cases} \tag{3.3}$$

and

$$\text{the outnormal derivative } \frac{\partial w_0}{\partial v}(x) < 0 \text{ for } x \in T \backslash \partial B_1, \tag{3.4}$$

where $w_0(x) = u(x) - u(x^*)$ and $B^+$ is one of half-balls of $B_1$ which is divided by $T$ such that the maximum point $P_0 \in B^+$.

*Step* 2: We want to prove $u \equiv \text{constant}$ if $u(x)$ is radially symmetric. Let $x = (x_1, \dots, x_n)$ and $r = |x|$. If $u$ is radial symmetry, then $u$ satisfies

$$\begin{cases} d\left(u'' + \dfrac{n-1}{r}u'\right) - u + f(u) = 0, & r > 0, \\ u(0) = \alpha > 0, \quad u'(0) = 0 \end{cases} \tag{3.5}$$

and we have

$$\frac{\partial u}{\partial x_1} = u'(r)\frac{x_1}{r} \text{ in } B_1 \quad \text{and} \quad \frac{\partial u}{\partial x_1} = 0 \text{ on } \partial B_1. \tag{3.6}$$

Suppose $u \not\equiv \text{constant}$, then $\frac{\partial u}{\partial x_1} \not\equiv 0$. Let $w(r)$ be the first eigenfunction of (3.1). From (3.6), we easily have

$$\int_{B_1} w(|x|)\frac{\partial u}{\partial x_1}(|x|)\, dx = 0. \tag{3.7}$$

By (1.5) we obtain

$$\begin{cases} d\Delta\left(\dfrac{\partial u}{\partial x_1}\right) - \dfrac{\partial u}{\partial x_1} + f'(u)\dfrac{\partial u}{\partial x_1} = 0, \\ \dfrac{\partial u}{\partial x_1} = 0 \quad \text{on } \partial B_1. \end{cases} \tag{3.8}$$

Now using (3.2), (3.7) and (3.8), we obtain that

$$\begin{aligned} 0 \leqslant \lambda_2 &= \inf_{v \perp w, v \in H^1(B_1)} \int_{B_1} [d|\nabla v|^2 + v^2 - f'(u)v^2]\, dx \\ &= \int_{B_1} \left[ d\left|\nabla\left(\frac{\partial u}{\partial x_1}\right)\right|^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 - f'(u)\left(\frac{\partial u}{\partial x_1}\right)^2 \right] dx \\ &= 0. \end{aligned} \tag{3.9}$$

Since $\frac{\partial u}{\partial x_1}$ archives the infinimum of (3.9), we obtain that

$$\int_{\partial B_1} \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x_1}\right)\phi\, d\sigma = 0 \quad \forall \phi \in H^1(B_1) \text{ and } \phi \perp w. \tag{3.10}$$

Set $\phi = x_1\frac{\partial}{\partial x_1}(\frac{\partial u}{\partial x_1}) + x_2\frac{\partial}{\partial x_2}(\frac{\partial u}{\partial x_1})$. Since $\phi$ is odd in $x_1$, we have $\phi \perp w$. Then we obtain that $\frac{\partial}{\partial v}(\frac{\partial u}{\partial x_1}) = 0$ on $\partial B_1$ and $\frac{\partial u}{\partial x_1}$ is a solution of the Neumann problem (1.5). Hence we have $u''(1) = 0$ and, from Eq. (1.5), $u(1) = f(u(1))$. From Eq. (3.5) and the uniqueness of ODE, we finally obtain that $u \equiv u(1)$. This contradiction proves that $u \equiv \text{constant}$ if $u(x)$ is radially symmetric.

Now suppose $u \not\equiv \text{constant}$. Let $P_0$ be a maximum point of $u$ on $\bar{B}_1$. If $P_0 = O$, then, from the above step 1 and using the same arguments in step 1 of the proof of

Theorem 1.1, we obtain that $u$ is radially symmetric. By the above step 2, $u \equiv$ constant in this case, a contradiction. Hence $P_0 \neq O$ if $u$ is nonconstant.

*Step* 3: We claim that $u$ is axially symmetric w.r.t. $\overrightarrow{OP_0}$ and (1.4) holds.

Without loss of generality, we may assume $\overrightarrow{OP_0}$ is the positive $x_n$-axis and $T_0$ is the hyperplane $x_n = 0$. Consider any two-dimensional plane where $P_0$ is contained. For the simplicity, we assume the plane is spanned by $e_1 = (1, 0, \dots, 0)$ and $e_n = (0, \dots, 0, 1)$. Let $l_\theta$ be the line having the angle $\theta$ with $x_1$-axis, and $v_\theta$ be the normal vector to the line in this plane. Set $T_\theta$ to be the $(n-1)$-dimensional linear hyperplane which passes the origin and has $v_\theta$ as the normal vector. Let $B_\theta$ be one of the half-balls of $B_1$ which is divided by $T_\theta$ and $P = (0, \dots, 0, t) \in B_\theta$ $\forall 0 \leqslant \theta < \frac{\pi}{2}$. Let $\Sigma_\theta$ denote the component of $B_\theta \backslash T_\theta$ and $x^\theta$ be the reflection point of $x$ w.r.t. $T_\theta$.

Set

$$w_\theta(x) = u(x) - u(x^\theta) \quad \forall x \in \Sigma_\theta. \tag{3.11}$$

Clearly, $w_\theta$ satisfies

$$\begin{cases} d\Delta w_\theta + (c_\theta(x) - 1) w_\theta = 0, \\ w_\theta(x) = 0 \text{ on } T_\theta \quad \text{and} \quad \dfrac{\partial w_\theta}{\partial v} = 0 \text{ on } \partial B_1 \cup \Sigma_\theta, \end{cases} \tag{3.12}$$

where

$$c_\theta(x) = \frac{f(u(x)) - f(u(x^\theta))}{u(x) - u(x^*)}.$$

We want to prove

$$w_\theta(x) > 0 \quad \text{for } x \in \Sigma_\theta \text{ and } 0 \leqslant \theta < \frac{\pi}{2} \tag{3.13}$$

and

$$w_{\frac{\pi}{2}} \equiv 0. \tag{3.14}$$

After (3.13) and (3.14) are established, the axial symmetry and the monotonicity follow readily. For $\theta = 0$, from step 1 we obtain that: either $w_0 \equiv 0$ in $B_1^+$ or $w_0(x) > 0$ $\forall x \in B_1^+$, where $B_1 = \{x \in B_1 \mid x_n > 0\}$. If the former case holds, then using the above step 2 and first part of step 3 in the proof of Theorem 1.1, we can conclude that $u$ is constant. This contradicts with $u$ is a *least-energy* solution of (1.5). If the second case is true, then we set

$$\theta_0 = \sup\left\{\theta | w_{\tilde{\theta}}(x) \geqslant 0 \ \forall x \in \Sigma_{\tilde{\theta}} \ \forall 0 \leqslant \tilde{\theta} \leqslant \theta \leqslant \frac{\pi}{2}\right\}. \tag{3.15}$$

Following the standard argument of the method of moving planes, we can prove $\theta_0 = \frac{\pi}{2}$. Since the present case is the Neumann problem and the boundary of $\partial\Sigma_\theta$ is not smooth, we should briefly scatch the proof for the sake of completeness.

Suppose $\theta_0 < \pi/2$. Then, by the continuity, $w_{\theta_0}(x) \geqslant 0$ for $x \in \Sigma_{\theta_0}$. By the above step 1, we have $w_{\theta_0}(x) > 0$ for $x \in \bar{\Sigma}_{\theta_0}/\partial B_1$. By the definition of $\theta_0$, there is a sequence of $\theta_j > \theta_0$ with $\lim_{j\to\infty} \theta_j = \theta_0$ such that

$$w_{\theta_j}(x_j) = \inf_{\bar{\Sigma}_{\theta_j}} w_{\theta_j}(x) < 0.$$

By passing to a subsequence, $x_0 = \lim_{j\to\infty} x_j$ satisfies $w_{\theta_0}(x_0) = 0$ and $\nabla w_{\theta_0}(x_0) = 0$. Hence we have $x_0 \in T_{\theta_0} \cap \partial B_1$. Since $w_{\theta_0} \equiv 0$ on $T_{\theta_0}$, $D_{e_i} D_{e_j} w_{\theta_0}(x_0) < 0$ for any tangent vector $e_i, e_j$ on $T_{\theta_0}$. Since $\frac{\partial w_{\theta_0}}{\partial v}(x_0) = 0$, $D_{\hat{e}_i} \frac{\partial w_{\theta_0}}{\partial v}(x_0) = 0$ for any tangent vector $\hat{e}_i$ of $\partial B_1$ at $x_0$. Let $\{e_1, \ldots, e_{n-1}\}$ be the base of the normal to the plane $T_{\theta_0}$ such that $e_{n-1}$ is the normal of $\partial B_1$ at $x_0$, and $e_n$ be the normal to $T_{\theta_0}$. Then we have $D_{e_i e_j} w_{\theta_0}(x_0) = 0$ $\forall 1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. Thus, the Hessian of $w_{\theta_0}$ at $x_0$ is completely zero, which yields a contradiction to Lemma S in [GNN2]. Therefore, $\theta_0 = \frac{\pi}{2}$, the axial symmetry follows readily.

The monotonicity and (1.4) follow from $\frac{\partial w_\theta}{\partial v} < 0$ for $x \in T_\theta$ where $v$ is the outnormal of $\Sigma_\theta$ on the boundary $T_\theta$. This proves the results of the case $P_0 \neq O$ of Theorem 1.2. The proof of Theorem 1.2 is complete. $\square$

## 4. Radial symmetry near the critical exponent

**Proof of Theorem 1.3.** Suppose that the conclusion of the first part of Theorem 1.3 does not hold. Then there exists a sequence of least-energy solution $u_i$ of (1.9) with $p = p_i \uparrow \frac{n+2}{n-2}$ such that $u_i(x)$ is not radially symmetric. Let $P_i$ be a maximum point of $u_i$. If $P_i$ is the origin, then we can prove that $u_i(x)$ is radially symmetric. For the detail of the argument, see the end of the proof of Lemma 4.1. Under the assumption that $u_i(x)$ is nonradial, we have $P_i \neq O$. We first want to prove $u_i$ is axially symmetric with respect to $\overrightarrow{OP_i}$. Note that by (1.10), $u_i$ satisfies

$$\frac{\int_{B_1} |\nabla u_i|^2 \, dx}{(\int_{B_1} K(x) u_i^{p_i+1} \, dx)^{\frac{2}{p_i+1}}} = \frac{S_n}{(\max_{B_1} K)^{\frac{n-2}{n}}} (1 + o(1)). \tag{4.1}$$

In the following, the axial symmetry is established for solution $u_i$ satisfying (4.1). Note that even for least-energy solutions, Theorem 1.1 cannot be applied for our present situation, because $K(|x|)$ in *not* assumed to be positive in the whole ball $B_1$.

**Lemma 4.1.** *Suppose $u_i$ is a solution of* (1.9) *with $p = p_i$ and $K \in C(\bar{B}_1)$, $K(x) = K(|x|)$ and $\max_{B_1} K > 0$. Assume* (4.1) *holds and $P_i$ is a global maximum point of $u_i$. Then $u_i$ is axially symmetric with respect to the axis $OP_i^{\leftrightarrow}$.*

**Proof.** Let $P_i$ be a maximum point of $u_i$ and assume $P_i \neq O$ first. Since the Sobolev constant is never achieved in $H_0^1(B_1)$, by (4.1), we have

$$K(P_i) \to \max_{B_1} K \quad \text{and} \quad u_i(P_i) \to +\infty.$$

Without loss of generality, we may assume $P_i = (0, \dots, 0, t_i)$ for some $t_i > 0$ and $\max_{B_1} K(x) = 1$.

As in the proof of Theorem 1.1, we want to show

$$w_i(x) = u_i(x) - u_i(x^-) > 0 \quad \text{for } x_n > 0, \tag{4.2}$$

where $x^- = (x_1, \dots, x_{n-1}, -x_n)$. To prove (4.2), instead of (2.9) for Theorem 1.1, we claim that

$$\text{There exists a constant } c > 0 \text{ such that}$$

$$u_i(x) \leqslant c \ U_{\lambda_i}(x - P_i) \quad \text{for } x \in B_1, \tag{4.3}$$

where $U_{\lambda_i}(x) = \left(\frac{\lambda_i}{\lambda_i^2 + \frac{|x|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$ and $\lambda_i^{-1} = (u_i(P_i))^{\frac{2}{n-2}}$.

Recall that for any $\lambda > 0$, $U_\lambda(x)$ satisfies

$$\begin{cases} \Delta U_\lambda(x) + U_\lambda^{\frac{n+2}{n-2}}(x) = 0 \text{ in } \mathbb{R}^n, \text{ and} \\ U_\lambda(0) = \max_{\mathbb{R}^n} U_\lambda(x). \end{cases} \tag{4.4}$$

Eq. (4.3) was proved in [H] for the case $K(x) \equiv$ a positive constant. However, it is unclear whether the argument in [H] can be applied to the present case where $K(x)$ is only assumed to be continuous. For the sake of completeness, an alternative proof will be presented in the appendix of this paper. For the moment, let us assume that (4.3) holds and we return to the proof of (4.2). Clearly, $w_i(x)$ satisfies

$$\begin{cases} \Delta w_i(x) + b_i(x) w_i = 0 \quad \text{for } B_1^+ = \{x \in B_1 \mid x_n > 0\}, \\ w_i(x) = 0 \quad \text{on } \partial B_1^+, \end{cases} \tag{4.5}$$

where

$$b_i(x) = K(x) \frac{u_i^{p_i}(x) - u_i^{p_i}(x^-)}{u_i(x) - u_i(x^-)}.$$

Suppose $\Omega_i = \{x \in B_1^+ \mid w_i(x) < 0\}$ is a nonempty set. We want to prove

$$|x^- - P_i|^{\frac{n-2}{2}} u_i(P_i) \to +\infty \quad \text{for } x \in \Omega_i \tag{4.6}$$

as $i \to +\infty$. If there is a sequence $x_i \in \Omega_i$ such that (4.6) fails. Since

$$|P_i|^{\frac{n-2}{2}} u_i(P_i) < |x^- - P_i|^{\frac{n-2}{2}} u_i(P_i),$$

$|P_i|^{\frac{n-2}{2}} u_i(P_i)$ is bounded. By passing to a subsequence, we let $\xi_0 = \lim_{i \to +\infty} t_i (u_i(P_i))^{\frac{2}{n-2}}$ and $q_0 = (0, \ldots, 0, \xi_0)$. By rescaling $u_i$, we set

$$V_i(y) = M_i^{-1} u_i(M_i^{\frac{-2}{n-2}} y),$$

where $M_i = u_i(P_i)$. By elliptic estimates, $V_i(y)$ is bounded in $C^2_{\text{loc}}(\mathbb{R}^n)$, thus, by passing to a subsequence, $V_i$ converges to $U_1(y - q_0)$ in $C^2_{\text{loc}}(\bar{\mathbb{R}}^n_+)$, where $U_1(y) = (1 + \frac{|y|^2}{n(n-2)})^{-\frac{n-2}{2}}$. Note that $U_1(y)$ is the solution of (4.4) with $U_1(0) = 1$.

Two cases are considered separately. If $\xi_0 > 0$, then $U_1(y - q_0) > U_1(y^- - q_0)$ for $y \in \mathbb{R}^n_+ = \{y \mid y_n > 0\}$. Set $y_i = M_i^{\frac{-2}{n-2}} x_i$, where $x_i \in \Omega_i$ is the sequence such that $|P_i - x_i^-|^{\frac{n-2}{2}} u_i(P_i) < +\infty$. Thus,

$$\begin{aligned}
|x_i| u_i^{\frac{2}{n-2}}(P_i) &\leqslant |x_i - P_i| u_i^{\frac{2}{n-2}}(P_i) + |P_i| u_i(P_i)^{\frac{2}{n-2}} \\
&\leqslant |x_i^- - P_i| u_i(P_i)^{\frac{2}{n-2}} + |P_i| u_i(P_i)^{\frac{2}{n-2}} \\
&\leqslant C
\end{aligned}$$

for some constant $C$. Then $|y_i|$ is bounded. Assume $y_0 = \lim_{i \to +\infty} y_i$. Since $V_i(y_i) - V_i(y_i^-) = M_i^{-1} w_i(x_i) < 0$, we have

$$U_1(y_0 - q_0) - U_1(y_0^- - q_0) = \lim_{i \to +\infty} M_i^{-1} w_i(x_i) \leqslant 0,$$

which implies $y_{0,n} = 0$, here $y_{0,n}$ is the $y_n$-coordinate of $y_0$. Since $V_i(y) - V_i(y^-) = 0$ on $y_n = 0$, there exists $\eta_i = (y_{i,1}, \ldots, y_{i,n-1}, \eta_{i,n})$ with $\eta_{i,n} \in (0, y_{i,n})$ such that $\frac{\partial}{\partial y_n}(V_i(\eta_i) - V_i(\eta_i^-)) \leqslant 0$ by the mean value theorem. By passing to the limit, it yields

$$\begin{aligned}
0 &\geqslant \frac{\partial}{\partial y_n}(U_1(y - \xi_0) - U_1(y^- - \xi_0))|_{y=y_0} \\
&= 2\frac{\partial U_1}{\partial y_n}(y_0 - \xi_0) = \frac{2}{n} \frac{\xi_0}{(1 + \frac{|y_0 - \xi_0|^2}{n(n-2)})^n} > 0
\end{aligned}$$

a contradiction. Hence, we have $\xi_0 = 0$.

Now we assume $\xi_0 = 0$. To prove (4.6), as the previous step, it suffices to show that

$$\tilde{w}_i(y) = N_i^{-1} w_i(M_i^{\frac{-2}{n-2}} y)$$

converges to a positive function in $C^2_{\mathrm{loc}}(\bar{\mathbb{R}}^n_+)$, where

$$N_i = \max_{B_1^+} |w_i(x)| = |w_i(z_i)|.$$

We first claim that

$$|z_i| M_i^{\frac{p_i-1}{2}} \leqslant c \tag{4.7}$$

for some positive constant $c > 0$. Assume (4.7) does not hold. First, we assume $r_i \to 0$ as $i \to +\infty$. Set $r_i = |z_i|$ and rescale $w_i$ by

$$\tilde{w}_i(y) = N_i^{-1} w_i(r_i y)$$

and $\tilde{w}_i$ satisfies

$$\begin{cases} \Delta \tilde{w}_i(y) + r_i^2 b_i(r_i y) \tilde{w}_i = 0, \quad \text{for } |y| \leqslant \dfrac{1}{r_i}, \\ \sup_{|y|=1} |\tilde{w}_i(y)| = 1. \end{cases}$$

By (4.3), for any compact set of $\bar{\mathbb{R}}^n_+ \backslash \{0\}$, we have

$$\begin{aligned} |r_i^2 b_i(r_i y)| &\leqslant c_1 r_i^2 |u_i(r_i y) + u_i(r_i y^-)|^{p_i - 1} \\ &\leqslant o(1) |y|^{-2}. \end{aligned}$$

Here, we have used the assumption $\lim_{i\to+\infty} r_i M_i^{\frac{p_i-1}{2}} = +\infty$ and the fact that $\lim_{i\to+\infty} M_i^{\frac{n+2}{n-2 \cdot p_i}} = 1$. For the proof of $\lim_{i\to+\infty} M_i^{\frac{n+2}{N-2-p_i}} = 1$, see step 1 in the proof of A.2 in the appendix.

Hence, $r_i^2 b_i(r_i y)$ converges to 0 uniformly in any compact set of $\bar{\mathbb{R}}^n_+ \backslash \{0\}$. By elliptic estimates and due to the assumption $r_i \to 0$, $\tilde{w}_i(y)$ converges to a harmonic function $h(y)$ in $C^2_{\mathrm{loc}}(\bar{\mathbb{R}}^n_+ \backslash \{0\})$, where $h$ satisfies.

$$|h(y)| \leqslant 1 \quad \text{and} \quad \sup_{|y|=1} |h(y)| = 1. \tag{4.8}$$

By the regularity theorem for bounded harmonic functions, $h(y)$ is smooth at 0. Note that $h(y) \equiv 0$ for $y_n = 0$. By the Liouville theorem, $h(y) \equiv 0$ in $\mathbb{R}^n_+$, which yields a contradiction to the second identity of (4.8). Hence (4.7) is established, in the first case.

Secondly, we assume $|z_i| \geqslant \delta_0 > 0$. Then the argument of contradiction in the above yields that there exists $r_1 > 0$ such that

$$\sup_{|x| \leqslant r_1} |w_i(x)| = o(1)N_i. \tag{4.9}$$

Let $\lambda > 0$ and $\psi(x) > 0$ be the eigenvalue and the eigenfunction of $\Delta$ in $B_2$ with the Dirichlet problem and set

$$\bar{w}_i(x) = \frac{w_i(x)}{\psi(x)} \quad \text{for } x \in B_1.$$

By a direct computation, $\bar{w}_i(x)$ satisfies

$$\Delta \bar{w}_i(x) + 2\nabla \log \psi(x) \cdot \nabla \bar{w}_i(x) + (b_i(x) - \lambda)\bar{w}_i(x) = 0$$

for $x \in B_1$. Let $\bar{x}_i$ be the maximum of $|\bar{w}_i(x)|$. By (4.9), we have $|\bar{x}_i| \geqslant r_1$. Clearly, (4.3) yields $|b_i(\bar{x}_i)| = O(1)M_i^{\frac{-4}{n-2}}$. Applying the maximum principle at $\bar{x}_i$, we have

$$0 \geqslant \Delta \bar{w}_i(\bar{x}_i) = (\lambda - b_i(\bar{x}_i))\bar{w}_i(\bar{x}_i) > 0,$$

a contradiction, where $|\bar{w}_i(\bar{x}_i)| = \bar{w}_i(\bar{x}_i)$ is assumed. Thus, $|z_i| \geqslant \delta_0$ is impossible. This proves (4.7).

Rescale $w_i$ again by

$$\tilde{w}_i(y) = N_i^{-1} w_i(M_i^{\frac{-2}{n-2}} y).$$

It is easy to see that by passing to a subsequence, $\tilde{w}_i$ converges to $w$ in $C^2_{\mathrm{loc}}(\bar{\mathbb{R}}^n_+)$, where $w$ satisfies

$$\begin{cases} \Delta w + \dfrac{n+2}{n-2} U_1^{\frac{4}{n-2}}(y) w = 0 & y \in \mathbb{R}^n_+, \\ w(y) = 0 & \text{on } y_n = 0. \end{cases} \tag{4.10}$$

Readily from (4.7), $w(y)$ is a bounded nonzero function. Thus,

$$w(y) = c \frac{\partial U_1(y)}{\partial y_n} \quad \text{for some constant } c \neq 0.$$

From the explicit expression of $U_1(y)$, we have $w(y)\neq 0$ for any $y\in\mathbb{R}^n_+$ and $\frac{\partial w}{\partial y_n}(y)\neq 0$ for $y_n=0$. Let $\xi_i = M_i^{\frac{2}{n-2}}P_i$. We have

$$\begin{aligned}\frac{\partial\tilde{w}_i(\xi_i^-)}{\partial y_n} &= N_i^{-1}M_i^{\frac{-2}{n-2}}\left(\frac{\partial w_i}{\partial x_n}\right)(P_i^-)\\ &= N_i^{-1}M_i^{\frac{-2}{n-2}}\left(\frac{\partial}{\partial x_n}u_i(P_i^-)-\frac{\partial}{\partial x_n}u_i(P_i)\right)\\ &= N_i^{-1}\left\{\frac{\partial}{\partial y_n}V_i(\xi_i^-)-\frac{\partial}{\partial y_n}V_i(\xi_i)\right\}\\ &= N_i^{-1}\frac{\partial^2 V_i}{\partial y_n^2}(\eta_i)(-2\xi_{i,n}),\end{aligned}\tag{4.11}$$

where $\eta_i\in(\xi_i^-,\xi_i)$. Since $\xi_i\to 0$ and

$$0>\frac{\partial^2 U_1}{\partial y_n^2}(0)=\lim_{i\to+\infty}\frac{\partial^2 V_i}{\partial y_n^2}(\eta_i).$$

Eq. (4.10) yields

$$\frac{\partial w(0)}{\partial y_n}=\lim_{i\to\infty}N_i^{-1}\frac{\partial^2 V_i(\eta_i)}{\partial y_n^2}(-2\xi_{i,n})\geqslant 0.$$

Hence, $\frac{\partial w}{\partial y_n}(y)>0$ for $y_n=0$ and we conclude that $w(y)>0$ in $\mathbb{R}^n_+$. Now suppose $x_i\in\Omega_i$. Because, $\bar{w}_i$, the scaling of $w_i$, converges to a positive function in $\mathbb{R}^n_+$, with the negative outnormal derivative on $\partial\mathbb{R}^n_+$, we conclude (4.6) holds.

By (4.6), we have for $x\in\Omega_i$,

$$b_i(x)\leqslant\frac{n+2}{n-2}u_i^{\frac{4}{n-2}}(x^-)\leqslant o(1)|x^- - P_i|^{-2}.\tag{4.12}$$

For $x\in\Omega_i$, we set

$$\bar{w}_i(x) = -w_i(x)|x-P_i|^{-\alpha},$$

where $0<\alpha<\frac{n-2}{2}$. By a straightforward computation, $\bar{w}_i(x)$ satisfies

$$\begin{aligned}&\Delta\bar{w}_i(x)+2(\nabla\log|x-P_i|^{\alpha}\cdot\nabla\bar{w}_i(x))\\ &\quad+\ (b_i(x)-\alpha(n-2-\alpha)|x-P_i|^{-2})\bar{w}_i(x)=0.\end{aligned}\tag{4.13}$$

Note that $\bar{w}_i(x)=0$ on $\partial\Omega_i$ and $P_i\notin\Omega_i$. Let $\bar{w}_i(x)$ achieves its maximum in $\bar{\Omega}_i$ at $\bar{x}_i$. Then by the maximum principle and (4.11), (4.12) yields

$$\begin{aligned}0&=\Delta\bar{w}_i(\bar{x}_i)+(b_i(\bar{x}_i)-\alpha(n-2-\alpha)|\bar{x}-P_i|^{-2})\bar{w}_i(\bar{x}_i)\\ &\leqslant(b_i(\bar{x}_i)-\alpha(n-2-\alpha)|\bar{x}-P_i|^{-2})\bar{w}_i(\bar{x}_i)<0\end{aligned}$$

when $i$ is sufficiently large. Therefore, we have proved $w_i(x)>0$ in $B_1^+$ for $i$ large.

Once (4.2) is proved, we can apply the method of rotating planes and Alexandroff Maximum Principle in [BN,HL] to conclude that $u_i(x)$ is axially symmetric with respect to $x_n$-axis and the monotonicity

$$x_j \frac{\partial u_i(x)}{\partial x_n} - x_n \frac{\partial u_i}{\partial x_j}(x) > 0 \tag{4.14}$$

holds for $x_j > 0$ and $1 \leqslant j \leqslant n-1$. This ends the proof of Lemma 4.1 for the case $P_i \neq O$.

When $P_i = O$, we want to prove $w_i(x) \equiv 0$ on $B_1^+$. Suppose not. Then let $z_i$ be the maximum point of $|w_i(x)|$, and as the same proof of (4.7), we have

$$|z_i| M_i^{\frac{p_i - 1}{2}} \leqslant c$$

for some constant $c$. Set $\tilde{w}_i(y) = N_i^{-1} w_i(x)$, where $N_i = w_i(z_i)$ and $x = M_i^{\frac{-2}{n-2}} y$. It is easy to see that $\tilde{w}_i(y)$ converges to a nonzero limit $w(y)$ in $C^2_{\text{loc}}(\bar{\mathbb{R}}^n_+)$, where $w(y)$ is a solution of (4.9). Thus, $w(y) = c\frac{\partial U_1}{\partial y_n}$ for some $c \neq 0$. However, $\nabla \tilde{w}_i(0) = 0$ because the origin is a maximum point of $u_i$. Especially,

$$0 = \frac{\partial w}{\partial y_n}(0) = c \frac{\partial^2 U_1}{\partial y_n^2}(0)$$

yields $c = 0$, a contradiction. Therefore, we conclude $w_i(x) \equiv 0$, that is, $u_i(x)$ is symmetric with respect to $x_n$. Of course, we can prove the symmetry of $u_i$ with respect to any hyperplane passing the origin by the same argument. Hence $u_i(x)$ is radially symmetric if $P_i = O$. This completely proves Lemma 4.1. $\square$

Now we return to the proof of Theorem 1.3. now suppose $P_i \neq 0$. Without loss of generality, we may assume $P_i = (0, \dots, 0, t_i)$ for some $t_i > 0$. Set

$$\phi_i(x) = x_1 \frac{\partial u_i}{\partial x_n}(x) - x_n \frac{\partial u_i}{\partial x_1}(x) > 0. \tag{4.15}$$

Then $\phi_i(x) > 0$ in $B_1^+ = \{x \in B_1 \mid x_1 > 0\}$, and $\phi_i$ satisfies

$$\begin{cases} \Delta \phi_i + p_i K(|x|) u_i^{p_i - 1} \phi_i = 0 & \text{in } B_1^+, \\ \phi_i = 0 & \text{on } \partial B_1^+. \end{cases} \tag{4.16}$$

Since $u_i^{p_i-1}(x)$ uniformly converges to zero in any compact set of $\bar{B}_1^+ \backslash \{P_i\}$, by the Harnack inequality, $(\max_{|x| = r_0} \phi_i(x))^{-1} \phi_i(x)$ converges to a harmonic function $h$ in $C^2_{\text{loc}}(\bar{B}_1^+ \backslash \{P_i\})$, where $r_0$ is the positive number in condition $(\mathrm{K_a})$. Since $h(x) = 0$ for $x \in \partial B_1^+ \backslash \{P_0\}$ where $P_0 = \lim_{i \to +\infty} P_i$, we have $h(x)$ has a nonremovable singularity at $P_0$. Otherwise, $h(x) \equiv 0$ on $B_1^+$, which contradicts to the fact that $\max_{|x| = r_0} h(x) = 1$.

Thus, $\frac{\partial h(x)}{\partial \nu} < 0$ for $x \in \partial B_1^+ \setminus \{x_1 = 0\}$. Hence we have

$$-\frac{\partial \phi_i(x)}{\partial \nu} \geqslant c_1 \left( \max_{|x| \geqslant r_0} |\phi_i(x)| \right) \tag{4.17}$$

for $x \in \partial B_1^+$, $x_1 \geqslant \frac{1}{2}$. and for a positive constant $c_1$.

From (1.9), we have

$$\begin{cases} \Delta \left( \dfrac{\partial u_i}{\partial x_1} \right) + p_i K(|x|) u_i^{p_i - 1} \left( \dfrac{\partial u_i}{\partial x_1} \right) = -K'(|x|) \dfrac{x_1}{r} u_i^{p_i} & \text{in } B_1^+, \\ \dfrac{\partial u_i}{\partial x_1} = 0 \quad \text{on } x_1 = 0. \end{cases} \tag{4.18}$$

By the boundary condition of $u_i, -\frac{\partial u_i}{\partial x_1} > 0$ for $x \in \partial B_1^+ \setminus \{x_1 = 0\}$. Since by (4.3), $M_i u_i(x)$ converges to $c\ G(x, P_0)$ for some $c > 0$ where $G(x, P_0)$ is the Green function with the singularity at $P_0$, we have

$$-\frac{\partial u_i(x)}{\partial x_1} \geqslant c_1 M_i^{-1} \quad \text{for } x \in \partial B_1^+ \text{ and } x_1 \geqslant \frac{1}{2}. \tag{4.19}$$

By (4.16), (4.18) and $(\mathrm{K_a})$, we obtain

$$\begin{aligned} \int_{\partial B_1^+ \setminus \{x_1 = 0\}} \frac{\partial u_i}{\partial x_1} \frac{\partial \phi_i}{\partial \nu} dS &= - \int_{B_1^+} \left[ \phi_i \Delta \left( \frac{\partial u_i}{\partial x_1} \right) - \frac{\partial u_i}{\partial x_1} \Delta \phi_i \right] dx \\ &= \int_{B_1^+ \cap \{|x| \leqslant r_0\}} \left[ K'(|x|) \frac{x_1}{|x|} u_i^{p_i} \phi_i \right] dx \\ &\quad + \int_{B_1^+ \cap \{|x| \geqslant r_0\}} \left[ K'(|x|) \frac{x_1}{|x|} u_i^{p_i} \phi_i \right] dx \\ &\leqslant \int_{B_1^+ \cap \{|x| \geqslant r_0\}} \left[ K'(|x|) \frac{x_1}{|x|} u_i^{p_i} \phi_i \right] dx \\ &\leqslant C \max_{B_1^+ \cap \{|x| \geqslant r_0\}} \phi_i(x) \int_{B_1^+ \cap \{|x| \geqslant r_0\}} u_i^{p_i} \, dx. \end{aligned}$$

By (4.17) and (4.19), we get

$$C_1 \, M_i^{-1} \leqslant C_2 \, M_i^{-p_i} \quad \text{for } i \text{ sufficiently large,}$$

a contradiction. This proves $P_i = O$ for $i$ large.

For the uniqueness part of Theorem 1.3, we reduce (1.9) in an ODE. Here, $K(r)$ is continuously extended for all $r \in [0, \infty)$. Following conventional notations, for any fixed $p$, we denote $u(r;\alpha)$ to be the unique radial solution of

$$\begin{cases} u''(r) + \dfrac{n-1}{r}u'(r) + K(r)u^p = 0, \quad r>0, \\ u(0;\alpha) = \alpha > 0, \quad u'(0;\alpha) = 0. \end{cases} \tag{4.20}$$

For $p \in (1, \frac{n+2}{n-2})$, we set

$$\alpha(p) = \inf\{u(0) \mid u(r) \text{ is a least-energy solution of (4.19)}$$
$$\text{in the class of radial functions of } H_0^1(B_1)\}.$$

We claim that there exists a small $\epsilon > 0$ such that for the solution $u(r;\alpha)$ with $\alpha \geqslant \alpha(p)$ and $0 < \frac{n+2}{n-2} - p \leqslant \epsilon$, there is a $R(\alpha) \in (0, \infty)$ satisfying

$$\begin{cases} u(r;\alpha) > 0 \quad \text{for } r \in [0, R(\alpha)) \text{ and} \\ u(R(\alpha), \alpha) = 0. \end{cases} \tag{4.21}$$

Furthermore, $R(\alpha)$ *decreases* with respect to $\alpha$ whenever $\alpha \geqslant \alpha(p)$. Since $R(\alpha(p)) = 1$, the uniqueness follows readily from the claim.

To prove the claim, we let

$$\phi(r,\alpha) := \frac{\partial u}{\partial \alpha}(r;\alpha). \tag{4.22}$$

We claim

$$\phi(R(\alpha), \alpha) < 0 \quad \text{for } \alpha \geqslant \alpha(p) \text{ and } 0 < \frac{n+2}{n-2} - p \leqslant \epsilon. \tag{4.23}$$

Suppose (4.23) holds. By differentiating (4.20) with respect to $\alpha$, we have

$$u'(R(\alpha), \alpha)\frac{\partial R(\alpha)}{\partial \alpha} + \phi(R(\alpha), \alpha) = 0.$$

Since $u'(R(\alpha), \alpha) < 0$, (4.23) yields $\frac{\partial R(\alpha)}{\partial \alpha} < 0$. Obviously, (4.21) and the decrease of $R(\alpha)$ follows. Thus, it suffices for us to show (4.23).

Recall that $\phi$ satisfies the linearized equation

$$\begin{cases} \phi'' + \dfrac{n-1}{r}\phi' + pK(r)u^{p-1}\phi = 0, \quad 0 < r < R(\alpha), \\ \phi(0;\alpha) = 1 \quad \text{and} \quad \phi'(0,\alpha) = 0. \end{cases} \tag{4.24}$$

By the choice of $\alpha(p)$, we see that $\phi(r;\alpha(p))$ changes sign only once and $\phi(R(\alpha(p)), \alpha(p)) \leqslant 0$. Now suppose (4.23) fails for any small $\epsilon > 0$. Then there is a sequence of $\alpha_i \to +\infty$ as $i \to +\infty$ such that $\phi_i(r) := \phi(r;\alpha_i)$ of (4.24) with $p_i \uparrow \frac{n+2}{n-2}$ and $\phi_i(r)$ *changes sign only once* and $\phi_i(R_i) = 0$, where $R_i = R(\alpha_i)$. Let $r_i$ be the first zero of $\phi_i$. Then $\phi_i(r) > 0$ for $r \in (0, r_i)$ and $\phi_i(r) < 0$ for $r \in (r_i, R_i)$. Clearly,

$$R_i \leqslant 1 \quad \text{and} \quad \phi_i'(R_i) > 0. \tag{4.25}$$

For the simplicity of notations, we let $u_i(r) \equiv u(r;\alpha_i)$ denote the solution of (4.20) with $p = p_i$.

To yield a contradiction, we set

$$w_i(r) = ru_i'(r) + \frac{2}{p_i - 1} u_i(r). \tag{4.26}$$

Then, from (4.20), $w_i$ satisfies

$$\begin{cases} w_i'' + \dfrac{n-1}{r} w_i' + p_i K(r) u^{p_i - 1} w = -rK'(r) u^{p_i}, \\ w_i(R_i) = R_i u_i'(R_i) < 0. \end{cases} \tag{4.27}$$

By (4.24) and (4.27), we get

$$\begin{aligned} \int_0^{R_i} r^{n-1}(rK'(r)) u_i^{p_i} \phi_i \, dr &= \int_{B_1} [w_i \Delta\phi_i - \phi_i \Delta w_i] \, dx \\ &= R_i^{n-1} w_i(R_i) \phi_i'(R_i) < 0. \end{aligned} \tag{4.28}$$

Here (4.25) is used. Recall that $r_i$ is the first zero of $\phi_i$. By scaling in (4.24), we easily have $r_i \to 0$ as $i \to +\infty$. Let

$$C_i = \frac{-r_i K'(r_i)}{K(r_i)}. \tag{4.29}$$

From the condition $(\frac{rK'(r)}{K(r)})' \leqslant 0$ for $0 \leqslant r \leqslant r_0$ and (4.29), we have

$$C_i K(r) + rK'(r) \begin{cases} \geqslant 0 & \text{if } 0 \leqslant r < r_i \leqslant r_0, \\ \leqslant 0 & \text{if } r_i \leqslant r \leqslant r_0. \end{cases} \tag{4.30}$$

Two cases are discussed separately.

*Case* 1: If $R_i \leqslant r_0$, then, from (4.30), we obtain

$$0 \leqslant \int_0^{R_i} r^{n-1}(C_i K(r) + rK'(r)) u_i^{p_i} \phi_i \, dr = R_i^{n-1} w_i(R_i) \phi_i'(R_i) < 0.$$

This proves (4.23) in this case.

*Case* 2: If $R_i > r_0$, then

$$\begin{aligned} 0 < -R_i^{n-1} w_i(R_i) \phi_i'(R_i) &= -\int_0^{R_i} r^{n-1}(C_i K(r) + rK'(r)) u_i^{p_i} \phi_i \, dr \\ &= \left(-\int_0^{r_0}\right) + \left(-\int_{r_0}^{R_i}\right) = (\mathrm{I}) + (\mathrm{II}). \end{aligned} \tag{4.31}$$

Since the first term (I) in (4.31) is negative, we obtain

$$R_i^n |u_i'(R_i)| |\phi_i'(R_i)| \leqslant \int_{r_0}^{R_i} r^{n-1}(C_i K(r) + rK'(r)) u_i^{p_i} \phi_i \, dr. \tag{4.32}$$

By using the same arguments of (4.3), (4.17) and (4.19), we can easily obtain

$$|u_i'(R_i)| \sim u_i(r_0) \sim \alpha_i^{-1}, \quad \phi_i'(R_i) \sim \phi_i(r_0) \text{ and } C_i \text{ is small.} \tag{4.33}$$

Hence, (4.32) yields

$$\alpha_i^{-1} \leqslant c\alpha_i^{-p_i},$$

a contradiction. This ends the proof of the claim (4.23), and the uniqueness follows. Hence we have finished the proof of Theorem 1.3. □

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**Appendix**

In this appendix, we consider a sequence of solutions $u_i$ of

$$\begin{cases} \Delta u_i + K(x) u_i^{p_i} = 0 & \text{in } B_2 = \{|x| < 2\}, \\ u_i = 0 & \text{on } \partial B_2, \end{cases}$$

such that

$$u_i(P_i) = \max_{\bar{B}_1} u_i(x) \to +\infty, \quad P_i \to P_0 \quad \text{for some } |P_0| < 1 \text{ and}$$

$$\int_{B_2} K(x) u_i^{p_i+1} \, dx = \left( \frac{S_n}{K(P_0)^{\frac{n-2}{n}}} \right)^{\frac{n}{2}} (1 + o(1)), \tag{A.1}$$

where $K(x) \in C(\bar{B}_2)$ and $K(P_0) > 0$ and $p_i \uparrow \frac{n+2}{n-2}$. We want to prove that there exists a constant $c > 0$ such that

$$u_i(x) \leqslant c \left( \frac{M_i}{1 + \frac{K(P_0)}{n(n-2)} M_i^2 |x - P_i|^2} \right)^{\frac{n-2}{2}} \quad \text{for } |x| \leqslant 1, \tag{A.2}$$

where $M_i = u_i^{\frac{2}{n-2}}(P_i)$.

We note that when $K(x) \equiv$ a positive constant, (A.2) was proved by Han [H]. Here we will present a proof of (A.2), which is simpler than [H] even for the case of constant $K$. This proof does not employ the Pohozaev identity. Thus, the smooth assumption of $K$ is not required.

**Proof of A.2.** We divide the proof into several steps.

*Step* 1: $\lim_{i\to+\infty} M_i^{\sigma_i} = 1$, where $\sigma_i = \frac{n+2}{n-2} - p_i$.

Rescaling $u_i$ by

$$U_i(y) = M_i^{-1} u_i(P_i + M_i^{-\frac{p_i-1}{2}} y). \tag{A.3}$$

Then $U_i$ satisfies

$$\Delta U_i(y) + K_i(y) U_i^{p_i}(y) = 0 \quad \text{for } |y| \leqslant M_i^{\frac{p_i-1}{2}},$$

where $K_i(y) = K(P_i + M_i^{-\frac{2}{n-2}} y)$. By elliptic estimates, $U_i(y)$ converges to $U(y)$ in $C^2_{\text{loc}}(\mathbb{R}^n)$, where $U(y)$ is the solution of

$$\begin{cases} \Delta U(y) + K(P_0) U^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \\ U(0) = \max_{\mathbb{R}^n} U(y) = 1. \end{cases} \tag{A.4}$$

Then by a theorem of Caffarelli–Gidas–Spruck [CGS], we have $U(y) = (1 + \frac{K(P_0)}{n(n-2)}|y|^2)^{-\frac{n-2}{2}}$ and

$$\int_{\mathbb{R}^n} K(P_0) U^{\frac{2n}{n-2}}(y)\,dy = \left(\frac{S_n}{K(P_0)^{\frac{n-2}{n}}}\right)^{\frac{n}{2}}.$$

Choose $R_i \to +\infty$ as $i \to +\infty$ such that $U_i(y)$ converges to $U(y)$ uniformly for $|y| \leqslant R_i$. Then

$$\begin{aligned} \left(\frac{S_n}{K(P_0)^{\frac{n-2}{n}}}\right)^{\frac{n}{2}} (1+o(1)) &\geqslant \int_{|P_i - x| \leqslant R_i M_i^{-\frac{p_i-1}{2}}} K(x) u_i^{p_i+1}(x)\,dx \\ &= M_i^{\frac{n-2}{2}\sigma_i} \int_{|y| \leqslant R_i} K_i(y) U_i^{p_i+1}(y)\,dy \\ &= M_i^{\frac{n-2}{2}\sigma_i} \left(\frac{S_n}{K(P_0)^{\frac{n-2}{n}}}\right)^{\frac{n}{2}} (1+o(1)). \end{aligned}$$

Therefore, $\lim_{i\to+\infty} M_i^{\frac{n-2}{2}\sigma_i} \leqslant 1$. Step 1 follows readily.

Set

$$m_i = \inf_{|x|\leqslant 1} u_i(x).$$

To Prove (A.2), we have to compare $m_i$ and $M_i^{-1}$. First, we claim

*Step* 2: there exists a constant $c$ such that

$$M_i^{-1} \leqslant c m_i.$$

Consider $G(x) = M_i^{-1}(|P_i - x|^{2-n} - 1)$ for $|P_i - x| \leqslant 1$. Note that by rescaling (A.3) and step 1, we have

$$u_i(x) \geqslant c M_i \quad \text{for } |x - P_i| = M_i^{-\frac{2}{n-2}}.$$

Since $u_i(x)$ is superharmonic, by the maximum principle,

$$cG(x) \leqslant u_i(x).$$

In particular,

$$u_i(x) \geqslant c M_i^{-1} \quad \text{for } |x| = \tfrac{1}{2},$$

where step 2 follows immediately.

The spherical Harnack inequality is very important in the study of the blowup behavior of $u_i$. Usually, this is a difficult step to prove. However, by the energy assumption (A.1), we can prove

*Step* 3: There exists a constant $c > 0$ such that

$$u_i(x)|x - P_i|^{\frac{2}{p_i-1}} \leqslant c \quad \text{for } |x| \leqslant 1. \tag{A.5}$$

Because, if $\lim_{i\to+\infty} \sup_{\bar{B}_1} (u_i(x)|x - P_i|^{\frac{n}{p_i-1}}) = +\infty$, then there is a local maximum point $Q_i$ of $u_i(x)$ such that the rescaling of $u_i$ with the center $Q_i$,

$$\tilde{U}_i(y) = \tilde{M}_i^{-1} u_i(Q_i + \tilde{M}_i^{-\frac{p_i-1}{2}} y) \quad \text{with } \tilde{M}_i = u_i(Q_i)$$

converges to $U(y)$ of (A.4), where $|Q_i - P_i|^{\frac{n-2}{2}} M_i \to +\infty$ as $i \to +\infty$. Thus, $u_i$ possesses at least two bubbles, a contradiction to (A.1). The existence of $Q_i$ can be proved by employing the method of localizing blowup points by R. Schoen. Since the method is well-known now, we refer the proof to [CL1,CL2].

By (A.5), we have the spherical Harnack inequality,

$$\begin{cases} u_i(x) \leqslant \bar{u}_i(x - P_i) \text{ and} \\ |\nabla u(x)| \leqslant c|x - P_i|^{-1}\bar{u}_i(|x - P_i|), \end{cases} \tag{A.6}$$

where $\bar{u}_i(r)$ is the average of $u_i$ over the sphere $|x - P_i| = r$. Set

$$v_i(t) = \bar{u}_i(r)r^{\frac{n-2}{2}} \quad \text{with } r = e^t.$$

By a straightforward computation, $v_i(t)$ satisfies

$$v_i''(t) - \left(\frac{n-2}{2}\right)^2 v_i(t) + \hat{K}_i(t)v_i^{\frac{n+2}{n-2}} = 0, \tag{A.7}$$

where

$$\hat{K}_i(t) = \left(⨍_{|x-P_i|=t} K(x)u_i^{-\sigma_i}u_i^{\frac{n+2}{n-2}}(x)\, d\sigma\right)(\bar{u}^{\frac{n+2}{n-2}}(r))^{-1}$$

and ($⨍$ denotes the average of integration over the sphere $|x - P_i| = r$. By steps 1 and 2, we have $u_i^{\sigma_i}(x)$ uniformly converges to 1 for $|x| \leqslant 1$. Therefore, $0 < c_1 \leqslant \hat{K}_i(t) \leqslant c_2$ for $t \leqslant 0$. By rescaling (A.3), we see that $v_i(t)$ has a first local maximum at $t = t_i = -\frac{2}{n-2}\log M_i + c_0$ for some constant $c_0$. Let $s_i > t_i$ be the first local minimum point unless $v_i(t)$ is decreasing for $t_i \leqslant t \leqslant 0$. In the latter case, we set $s_i = 0$.

*Step* 4: If $s_i < 0$, then $v_i(t)$ is increasing for $s_i < t \leqslant 0$.

If not, then $v_i(t)$ has a local maximum at some point $\hat{s}_i \in (s_i, 0]$. By (A.7), $v_i(\hat{s}_i) \geqslant c > 0$ for some constant $c > 0$. By the spherical Harnack inequality, $|v_i'(t)| \leqslant c_1$. Thus, there exists $\delta_0 > 0$ such that

$$v_i(t) \geqslant \frac{c}{2} \quad \text{if } |t - \hat{s}_i| \leqslant \delta_0.$$

Therefore,

$$\int_{T_i} u_i^{\frac{2n}{n-2}}(x)\, dx \geqslant c_1 > 0, \tag{A.8}$$

where $T_i = \{x \mid e^{\hat{s}_i - \delta_0} \leqslant |x - P_i| \leqslant e^{\hat{s}_i + \delta_0}\}$. However,

$$\int_{|P_i - x| \leqslant e^{s_i}} u_i^{\frac{2n}{n-2}}(x)\, dx = \left(\frac{S_n}{K(P_0)^{\frac{n-2}{n}}}\right)^{\frac{n}{2}}(1 + o(1)).$$

Together with (A.8), it yields a contradiction to (A.1).

*Step* 5: There exists $T_0 \leqslant 0$ such that $s_i \geqslant T_0$. Furthermore,

$$u_i(x) \leqslant c M_i^{-1} |x - P_i|^{2-n} \quad \text{for } M_i^{-\frac{2}{n-2}} \leqslant |x - P_i| \leqslant e^{T_0}. \tag{A.9}$$

To prove step 5, we recall an ODE result from [CL2,CL3]. See Lemma 5.1 in [CL2] or Lemma 3.2 in [CL3]. Assume $\varepsilon_0$ to be a fixed small positive number. By rescaling as in (A.3), there is a unique $\hat{t}_i = t_i + c(\varepsilon_0) > t_i$ such that $v_i(t)$ is decreasing for $t_i \leqslant t \leqslant \hat{t}_i$ and $v_i(\hat{t}_i) = \varepsilon_0$. If $\varepsilon_0$ is small enough, then by (A.7), $v_i(t)$ has no critical point for $t \in (\hat{t}_i, s_i)$, where we recall that $s_i$ is the first minimum point after $t_i$.

**Lemma A.** *There exists a constant c such that the following statements hold*:

(1) *For* $\hat{t}_i \leqslant t_0 \leqslant t_1 \leqslant s_i$, $v_i$ *satisfies*

$$t_1 - t_0 \leqslant \frac{2}{n-2} \log \frac{v_i(t_0)}{v_i(t_1)} + c_1 \text{ and}$$
$$s_i - t_0 \geqslant \frac{2}{n-2} \log \frac{v_i(t_0)}{v_i(s_i)}.$$

(2) *For* $s_i \leqslant t \leqslant 0$,

$$(t - s_i) - c_1 \leqslant \frac{2}{n-2} \log \frac{v_i(t)}{v_i(s_i)} \leqslant (t - s_i).$$

From (2), we have for $t \geqslant s_i$,

$$\bar{u}_i(e^t) = v_i(t) e^{-\frac{n-2}{2}t} \geqslant c_2 e^{-\frac{n-2}{2}s_i} v_i(s_i) = c_2 \bar{u}_i(e^{s_i}). \tag{A.10}$$

Since $\bar{u}_i(r)$ is decreasing in $r$, by (A.10) together with the spherical Harnack inequality, we have for some positive constant $c_3$,

$$m_i \sim u_i(x) \sim \min_{|x-P_i|=e^{s_i}} u_i(x) \quad \text{for } |x - P_i| \geqslant s_i. \tag{A.11}$$

From the first inequality of (1) of Lemma A, we have

$$u_i(x) \leqslant c_4 \bar{u}_i(r_i) \left( \frac{r_i}{|x|} \right)^{n-2} \leqslant c_4 M_i^{-1} |x|^{2-n} \tag{A.12}$$

for $e^{t_i} = r_i \leqslant |x - P_i| \leqslant e^{s_i}$ where $\hat{t}_i = t_i + c(\epsilon_0)$, $t_i = -\frac{2}{n-2} \log M_i$, and $\bar{u}_i(r_i) \sim M_i$ are used. The second inequality of (1) in Lemma A implies

$$\bar{u}_i(e^t) \geqslant c_5 M_i^{-1} (e^{s_i})^{2-n}$$

for $t_i \leqslant t \leqslant s_i$. Thus, together with (A.12) and (A.11), we have

$$m_i \sim M_i^{-1}(e^{s_i})^{2-n}. \tag{A.13}$$

Now suppose $s_i \to -\infty$. Then by (A.13),

$$m_i M_i \to +\infty \quad \text{as } i \to +\infty. \tag{A.14}$$

Since $u_i(x)/m_i$ is uniformly bounded in $C^2_{\mathrm{loc}}(B_2 \setminus \{P_0\})$, by passing to a subsequence, $u_i(x)/m_i$ converges to a positive harmonic function $h(x)$ in $C^2_{\mathrm{loc}}(B_2 \setminus \{P_0\})$. For any $\delta > 0$,

$$\begin{aligned} -\int_{|x-P_0|=\delta} \frac{\partial h}{\partial \nu}(x)\, d\sigma &= -\lim_{i \to +\infty} \int_{|x-P_0| \leqslant \delta} \Delta(u_i(x)/m_i)\, dx \\ &= \frac{1}{m_i} \int_{|x-P_0| \leqslant \delta} K(x) u_i^{p_i}\, dx. \end{aligned}$$

To estimate the right-hand side, we decompose the domain into three parts: For any large $R > 0$,

$$\frac{1}{m_i} \int_{|x-P_i| \leqslant M_i^{-\frac{2}{n-2}} R} K(x) u_i^{p_i}\, dx \leqslant \frac{c}{m_i M_i} \int_{|y| \leqslant R} U_i^{\frac{n+2}{n-2}}(y)\, dy \to 0$$

by (A.14). By using (A.12), we have

$$\begin{aligned} &\frac{1}{m_i} \int_{M_i^{\frac{-2}{n-2}} R \leqslant |x-P_i| \leqslant e^{s_i}} K(x) u_i^{p_i}\, dx \\ &\quad \leqslant \frac{c}{m_i} M_i^{-\frac{n+2}{n-2}} \int_{|x-P_i| \geqslant M_i^{-\frac{2}{n-2}} R} |x - P_i|^{-(n+2)}\, dx \\ &\quad \leqslant \frac{c}{m_i} M_i^{-\frac{n+2}{n-2}} (M_i^{-\frac{2}{n-2}} R)^{-2} \\ &\quad = \frac{c_1}{m_i M_i} R^{-2} \end{aligned}$$

$\to 0$ by (A.14) again. By (A.11), the last term can be estimated by

$$\frac{1}{m_i} \int_{|x-P_i| \geqslant e^{s_i}} K u_i^{p_i}\, dx \leqslant c_1 m_i^{-\frac{4}{n-2}} \to 0.$$

Thus,

$$\int_{|x-P_0|=\delta} \frac{\partial h}{\partial \nu}\, d\sigma = 0 \quad \text{for any } \delta > 0,$$

which implies $h$ is smooth at 0. Since $h(x)$ vanishes on the boundary of $B_2$, $h(x) \equiv 0$ on $B_2$, which contradicts to $\inf_{\bar{B}_1} h(x) = 1$. Hence step 5 is proved. Clearly, (A.2) is equivalent to (A.9). Therefore, (A.2) is proved completely. $\square$

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