The underlying graph of a line digraph*

J.L. Villar

Departamento de Matemática Aplicada y Telemática, Universidad Politécnica de Catalunya, 08034 Barcelona, Spain

Received 5 June 1989
Revised 11 February 1991

Abstract


The recovery of a line digraph from its underlying graph is considered in this paper. The location of loops and digons are determined, and save for a few exceptions, the solution is unique up to -onverses. The relation between the automorphism groups of a line digraph and its underlying graph follows as a corollary. The results are applied to de Bruijn and Kautz digraphs.

1. Introduction

It is well known that by taking line digraph iterations of a given digraph, families of digraphs with good properties (such as high connectivity, small diameter and easy routing) can be obtained (see [6,7,13]). Most of these nice features are also shared by the underlying graphs of those digraphs. Two examples are the de Bruijn and Kautz digraphs. These digraphs, as well as their underlying graphs, have been widely studied by several authors (see for example [2,3,10,11]), since they are good models for interconnection networks.

Symmetry is an interesting feature of interconnection networks, since it leads to simpler algorithms. It can be described through the automorphism group of the corresponding graphs. In this context, an interesting problem posed by J. Bond is the determination of the automorphism group of undirected de Bruijn graphs. In this paper, some general results inspired by this problem are presented.

We begin by recalling a few basic concepts from graph theory. Let $G_0$ and $G$ be

* This work has been supported by CICYT (Comision Interministerial de Ciencia y Tecnologia) (Spain) under project 0173/86.
respectively a graph and a digraph. Let \( V(G_0) \) and \( E(G_0) \) denote respectively the vertex set and the edge set of the graph. Denote by \( \{x, y\} \) the edge of \( G_0 \) with end vertices \( x, y \). \( I_{G_0}(x) \) is the set of vertices adjacent to \( x \) in \( G_0 \). Let \( V(G) \) and \( A(G) \) denote respectively the vertex set and the arc set of the digraph. Denote by \( (x, y) \) or \( x \rightarrow y \) the arc of \( G \) with end vertices \( x, y \). Then it is said that \( x \) is adjacent to \( y \) and also that \( y \) is adjacent from \( x \). A loop at a vertex \( x \) of \( G \) is the arc \( (x, x) \), and a digon between two vertices \( x, y \) is the cycle \( x \rightarrow y \rightarrow x \). Denote by \( I^+_G(x) \) the set of vertices that are adjacent from \( x \) in \( G \) and \( I^-_G(x) \) the set of vertices adjacent to \( x \). An arc of \( G, x \rightarrow y \), is adjacent to another, \( z \rightarrow t \), iff \( y = z \). Denote by \( \delta_G(x) \) the minimum of the order of \( I^+_G(x) \) and \( I^-_G(x) \), and \( \delta_G \) the minimum of \( \delta_G(x) \) for all \( x \), that is the \textit{minimum degree} of \( G \). A digraph \( G \) is \textit{d-regular} iff \( d = |I^+_G(x)| = |I^-_G(x)| \), for every \( x \). The \textit{star} \( S_G(x) \) of a vertex \( x \) of \( G \) is defined to be the minimal subdigraph containing all arcs at \( x \). The \textit{converse} digraph of \( G \) is the digraph \( -G \) obtained by reversing the orientation of all its arcs. If \( F \) and \( G \) are subdigraphs of a given digraph \( G_0 \) iff \( G_0 = UG \). (Note that this definition allows the existence of loops and digons in \( G \).)

A \textit{simple} digraph is defined to be a digraph without parallel arcs (although digons are allowed), and hence with at most one loop at any vertex. A \textit{simple} graph is defined to be a graph with neither parallel edges nor loops.

The \textit{underlying graph} \( UG \) of a digraph \( G \) is defined to be the simple graph with the same vertex set as \( G \) and where two distinct vertices are adjacent in \( UG \) iff one is adjacent to the other in \( G \). Note that loops of \( G \) are ignored and that any digon of \( G \) transforms into just an edge. Conversely, a digraph \( G \) is an \textit{orientation} of a graph \( G_0 \) iff \( G_0 = UG \). (Note that this definition allows the existence of loops and digons in \( G \).)

Given two disjoint sets \( A \) and \( B \), the \textit{complete bipartite} digraph with stable sets \( A \) and \( B \), \( \overline{K}(A, B) \), has \( A \cup B \) as vertex set and \( A \times B \) as arc set and \( A \cap B = \emptyset \). In the same way, the \textit{complete bipartite} graph \( K(A, B) \) is the underlying graph of \( \overline{K}(A, B) \). The \textit{line digraph}, \( \mathcal{L}(G) \), of a digraph \( G \) has as vertices the arcs of \( G \), and two vertices of \( \mathcal{L}(G) \) are adjacent iff their corresponding arcs are adjacent in \( G \). A necessary and sufficient condition for \( G \) to be a line digraph is Heuchenne's one, i.e., if \( (x, y) \), \( (x, z) \) and \( (t, y) \) are arcs of \( G \), then so is the arc \( (t, z) \). Other characterizations can be found in [9]. Note that locally the underlying graph of a line digraph looks like a complete bipartite graph and that line digraphs are necessarily simple digraphs. A \textit{restricted line digraph} is defined to be the line digraph of a simple digraph with minimum degree at least 2. Observe that any restricted line digraph also has minimum degree at least 2, and it has at most then one digon at any vertex. An orientation of a graph is called a \textit{line orientation} if it is the line digraph of a simple digraph.

An \textit{induced} subdigraph \( H \) of a digraph \( G \) is a subdigraph of \( G \) such that two vertices of \( H \) are adjacent in \( H \) iff they are adjacent in \( G \). Then, induced subdigraphs of a line digraph are also line digraphs. Furthermore, if \( H \) is an induced subdigraph of a restricted line digraph, \( H \) is a line digraph of a simple digraph.
Now, we state the problem studied in this paper. We ask whether it is possible to identify features of a digraph from its underlying graph. Since in general there are many digraphs with the same underlying graph, we restrict the question to restricted line digraphs. In this paper, it is shown that, except for a few cases, any connected graph can be oriented as a restricted line digraph in at most two ways, one being the converse of the other.

The definition of underlying graph leads us to the following question: given $UG$ can we determine where the loops and digons of $G$ are? Observe that loops of $G$ are not present in $UG$ and digons transform into edges. Then a priori any vertex in $UG$ can have a loop in $G$, and any edge can be oriented either as an arc or a digon. This question is answered in Section 2. Line orientations of complete bipartite graphs are also obtained.

The second problem is to characterize the underlying graph of restricted line digraphs. This characterization as well as the main result of the reconstruction of the original digraph $G$ from $UG$ are given in Section 3. From this it follows that the automorphism group of a restricted line digraph is related to the automorphism group of its underlying graph, and this result is applied to the undirected de Bruijn and Kautz graphs.

2. Local considerations

Let us analyze the meaning of a line orientation. Observe that the digraphs in Fig. 1(a) are forbidden subdigraphs of any line orientation (since they can only be obtained from parallel arcs). Furthermore, by Heuchenne's condition, the digraphs in Fig. 1(b) can only be induced subdigraphs of any line orientation if the dotted arcs are present. (This fact is widely used in almost all proofs throughout Sections 2 and 3.)

Now, let $G'$ be a simple digraph and $G = \mathcal{L}(G')$ its line digraph. Observe that any star $S_{G'}(x')$ without a loop induces a complete bipartite induced subdigraph in $G$, i.e., $K(A, B)$ where $A$ and $B$ are respectively the sets of arcs of $G'$ incident to and

![Forbidden subdigraphs in a line orientation.](image)
Fig. 2. Local behavior of $UG$, where $G = \gamma(G')$ is a restricted line digraph, around a vertex $x'$ of $G'$, (a) without any loop and (b) with a loop. (A dashed arc between the sets $P$, $Q$ means that each vertex of $P$ is adjacent to every vertex of $Q$.)

from $x'$ (see Fig. 2(a)). Thus, complete bipartite induced subgraphs play an important role in this work.

If there is a loop at a vertex $x'$ of $G'$, the induced subdigraph of $G$ corresponding to $S_{G'}(x')$ can be schematically represented as in Fig. 2(b) where $x$ is the vertex of $G$ corresponding to the loop at $x'$. Note that its underlying subgraph in $UG$ can be obtained as $F_{G}(A', B') = K(A', B') \oplus K(x, A' \cup B')$, where $A'$ and $B'$ are equal to $A$ and $B$ except that the loop at $x'$ is excluded. Then, we can define the graph $K_{L}(A, B)$ in the same way as $K(A, B)$ but allowing $|A \cap B| \leq 1$. The differences between subgraphs in Fig. 2 will allow us to find out where the loops of $G$ are.

2.1. Complete bipartite subgraphs

Two natural orientations of $K(A, B)$ are the two complete bipartite digraphs, namely $\tilde{K}(A, B)$ and $\tilde{K}(B, A)$. Note that they are both line orientations.

Let us find all possible line orientations of $K(A, B)$. If $|A|, |B| \geq 2$, $K(A, B)$ has two families of line orientations, namely $\tilde{K}^{*}(A, B)$ and $\tilde{K}^{*}(B, A)$, defined by transforming a (possibly empty) set of arcs of the complete bipartite digraph into digons. Note that there can be at most one digon at each vertex. Call all such orientations normal. Orientations $\tilde{K}_{L}(A, B)$ and $\tilde{K}_{L}^{*}(A, B)$ are defined in the same way as $\tilde{K}(A, B)$ and $\tilde{K}^{*}(A, B)$. Each normal orientation of $K(A, B)$ or $K_{L}(A, B)$ comes from the line digraph of a star. There exists only one line orientation of $K(A, B)$ that is not normal, namely the one in Fig. 3. Call it the 4-cycle orientation. To justify this statement, take any 4-vertex bipartite subgraph of $K(A, B)$. One may routinely verify
that no line orientations of this subgraph are possible other than those mentioned and the 4-cycle is only possible if \(|A| = |B| = 2\).

Note that \(K_4(A,B)\) is always a subdigraph of any \(\tilde{K^*_4}(A,B)\) orientation, but it is not a subdigraph of the 4-cycle orientation. In fact, the existence of the arcs \(x \rightarrow y\) and \(x \rightarrow z\), \(x \in A\), in a line orientation suffices to assure that \(K_4(A,B)\) is a subdigraph of it, and the same occurs with the arcs \(y \rightarrow x\) and \(z \rightarrow x\), \(x \in B\).

2.2. Loops

Let \(G\) be a restricted line digraph. \(F_x(A,B) = K(A,B) \oplus K(x,A \cup B), x \in V(UG), A, B \subseteq V(UG)\), denotes any induced subgraph of \(UG\) such that:

1. \(A \cap B = \emptyset, x \notin A \cup B\),
2. \(I_{UG}(x) = A \cup B\),
3. \(A, B \neq \emptyset\)

(see Fig. 2(b)).

**Lemma 2.1.** If \(x \in V(UG)\) has a loop in the restricted line digraph \(G\), then there exists \(F_x(A,B)\) in \(UG\), with \(A = \Gamma_G^-(x) \setminus x\) and \(B = \Gamma_G^+(x) \setminus x\).

**Proof.** (1) \(A \cap B \neq \emptyset\) is not possible for otherwise there is a digon and a loop at the same vertex. By definition, \(x\) is excluded from \(A \cup B\).

(2) \(I_{UG}(x) = (\Gamma_G^-(x) \cup \Gamma_G^+(x)) \setminus x = A \cup B\).

(3) This follows from \(\delta_G(x) \geq 2\).

Then, by (1), \(K(x,A \cup B)\) is a subgraph of \(UG\). Since \(\forall a \in A, x \in \Gamma_G^+(a) \cap \Gamma_G^-(x)\), by the definition of line digraph \(\Gamma_G^+(a) = \Gamma_G^+(x) = B \cup x\). In the same way, \(\forall b \in B, \Gamma_G^-(b) = A \cup x\). Therefore, \(K(A,B)\) is an induced subgraph of \(UG\).

**Lemma 2.2.** The only line orientation \(H\) of a subgraph \(F_x(A,B)\) with \(\delta_H(x) \geq 2\), that has no loop at \(x\) is the de Bruijn digraph \(B(2,2)\) in Fig. 4.

**Proof.** Observe that since \(K(A,B)\) is a complete bipartite induced subgraph of \(UG\), it must be oriented as a line digraph of a simple digraph.

If \(|A|, |B| \geq 2\), then except for the 4-cycle orientation, one of the two complete bipartite digraphs, \(\tilde{K}(A,B)\) and \(\tilde{K}(B,A)\), must be a subdigraph of \(H\). Suppose for simplicity that \(\tilde{K}(A,B)\) is. Since there cannot be loops at any vertex of \(A \cup B\), for

![Fig. 3. 4-cycle orientation of \(K(A,B)\), \(|A| = |B| = 2\).](image-url)
any $a \in A$ and $b \in B$ the only line orientation of $H$ not having a loop at $x$ has the arcs $b \to x \to a \to b$, and it is inconsistent with the absence of parallel arcs in $G'$.

It is routine to verify that no such orientation can be obtained with the 4-cycle orientation of $K(A, B)$. (Case $|A| = |B| = 2$.)

If $A$ or $B$ has just one vertex, say $A = \{a_1\}$, then the other set must have at least two vertices. It is not difficult to show that there must be a digon connecting $x$ and $a_1$, since, from the minimum degree requirement, forbidden subgraphs must result. It is then routine to show that there must be a loop at each vertex of $B$ and that $|B| = 2$ (see Fig. 4). \[\square\]

Lemmas 2.1 and 2.2 provide us a means to find the loops of $G$ in $UG$: The subgraphs $F_x(A, B)$ of $UG$ are their tracc. Then

**Proposition 2.3.** If $UG$ is the underlying graph of a restricted line digraph, then the vertices that have loops are determined.

**Proof.** The existence of a $F_x(A, B)$ subgraph in $UG$ is necessary for the existence of a loop at $x$ in $G$. Furthermore, it is sufficient except for the case in Fig. 4. Observe that if the existence of a loop at $x$ is ambiguous, then it must be the same at $b_1$ and at $b_2$. Then, the subdigraphs $F_{b_1}(A_1, B_1)$ and $F_{b_2}(A_2, B_2)$, corresponding to the, possibly existing, loops at $b_1$ and $b_2$, are also exceptional, and so on. In this case, the existence of more than one digon at the vertex $a$ can be easily shown. \[\square\]

Once the vertices with loops are fixed, the orientation of the edges incident to them are determined.

**Lemma 2.4.** Let $G$ be a restricted line digraph. There are only two orientations of the edges of $UG$ incident to any vertex that has a loop in $G$, and one is the converse of the other.

**Proof.** Let $x$ be a vertex with a loop in $G$. Then no edge of $UG$ adjacent to $x$ can be oriented as a digon.
The underlying graph of a line digraph

If \( a \in A \) and \( b \in B \) (in \( F'_x(A, B) \)), then \( a \cdot x \rightarrow b \) and \( a \cdot x \rightarrow b \) are forbidden, and hence only two situations are possible: \( A = \Gamma^+_G(x) \backslash x \) and \( B = \Gamma^-_G(x) \backslash x \) or the converse \( A = \Gamma^-_G(x) \backslash x \) and \( B = \Gamma^+_G(x) \backslash x \).

2.3. Digons

Now consider a digon \( x' \rightarrow y' \rightarrow x' \) in \( G' \), a simple digraph with \( \delta_G \geq 2 \). Another digon \( x \rightarrow y \rightarrow x \) is induced in \( G = \mathcal{P}(G') \). The subgraph in \( UG \) induced by \( S_G(x') \oplus S_G(y') \) is schematically represented in Fig. 5, with \( A = \Gamma_G^-(x), B = \Gamma_G^+(y), C = \Gamma_G^-(y), D = \Gamma_G^+(x) \).

A new class of subgraphs of \( UG \) is defined. Let \( H_{x,y}(A, B, C, D) = K_L(A, B) \oplus K_L(C, D), x, y \in V(UG), A, B, C, D \subseteq V(UG), \) denote any subgraph of \( UG \) such that:

1. \( K_L(A, B) \) and \( K_L(C, D) \) are induced subgraphs of \( UG \),
2. \( A \cap C = B \cap D = \emptyset, A \cap D = \{y\}, B \cap C = \{x\} \),
3. \( \Gamma_U(x) = A \cup D, \Gamma_U(y) = B \cup C \),
4. except for the edge \( \{x, y\} \), there are no edges of \( UG \) in either \( K(A, C) \) or \( K(B, D) \),
5. \( |A|, |B|, |C|, |D| \geq 2 \)

(see Fig. 5, where \( \mathcal{A}' = A \setminus y, \) and so on).

Lemma 2.5. If the digon \( x \rightarrow y \rightarrow x \) is in \( G \), then \( UG \) contains \( H_{x,y}(A, B, C, D) \) in \( UG \), where \( A = \Gamma^-_G(x), B = \Gamma^+_G(y), C = \Gamma^+_G(y) \) and \( D = \Gamma^-_G(x) \).

Proof. (1) \( \forall a \in A, x \in \Gamma_G^+(a) = \Gamma_G^+(a) = \Gamma_G^-(y) = B \), and so on. Furthermore, \( |A \cap B|, |C \cap D| \leq 1 \) since there are no parallel arcs in \( G' \).

(2) \( A \cap C \neq \emptyset \) or \( B \cap D \neq \emptyset \) forces the existence of two loops: one at \( x \) and the other at \( y \).

(3) \( \Gamma_U(x) = \Gamma^-_G(x) \cup \Gamma^+_G(x) = A \cup D \) and \( \Gamma_U(y) = \Gamma^-_G(y) \cup \Gamma^+_G(y) = B \cup C \).

(4) If there is an arc from \( a \in A \) to \( c \in C \) in \( G \), then \( c \in \Gamma^-_G(a) = B \), but \( B \cap C = \{x\} \). In the same way, \( a \in A \cap D = \{y\} \). Likewise, one can show that \( \{x, y\} \) is the only edge in \( K(B, D) \).

(5) This follows since \( \delta_G(x), \delta_G(y) \geq 2 \).

Fig. 5. Local behaviour of \( UG \), where \( G = \mathcal{P}(G') \) is a restricted line digraph, around a digon of \( G' \).
532 J.L. Villar

Fig. 6. Exception of Lemma 2.6.

Lemma 2.6. The only line orientations of a subgraph $H_{x,y}(A,B,C,D)$ with $\delta_H(x), \delta_H(y) \geq 2$ and without the digon $x \rightarrow y \rightarrow x$ are those that contain the digraph in Fig. 6.

Proof. Exceptions must be constructed from line orientations of $K_1(A,B)$ and $K_1(C,D)$ with the same orientation of the edge $\{x, y\}$, just as an arc. So $K_1^+(A,B) \oplus K_1^+(C,D)$ and its converse do not fulfil the previous condition. Moreover, $K_1^+(A,B) \oplus K_1^+(D,C)$ and its converse are not valid orientations since then, $K(A \cup D, B \cup C)$ would be a subgraph of $UG$. Therefore, in the exceptions to this lemma exactly one of $K_1(A,B)$ and $K_1(C,D)$ must be a 4-cycle (since $\delta_C(x) \geq 2$, one of them must be normal). For the same reason, $\delta_G(x) \geq 2$ and $\delta_G(y) \geq 2$, the normal part must have two digons, one at $x$ and the other at $y$. This is just the definition of the subdigraph in Fig. 6. \(\square\)

Note that, at this stage, it cannot be determined which edges are oriented as digons, but only which vertices are incident to them.

Proposition 2.7. If $UG$ is the underlying graph of a restricted line digraph, then the vertices which are on a digon in $G$ are determined.

Proof. This is a corollary of Lemma 2.4. Note that in the only case where there is no digon between $x$ and $y$, the existence of other digons at these vertices is assured. \(\square\)

The ambiguity is not about the existence of digons but about their location. No ambiguity exists if the minimum distance between digons in $G$ is at least 2 or if the minimum degree in $G$ of the vertices on a digon is at least 3 (since Lemma 2.5 assures the existence of a nonexceptional $H_{x,y}(A,B,C,D)$ digraph in the sense of Lemma 2.6).

Now, let $V_d$ be the set of vertices on digons. If $G$ is a restricted line digraph, then there exists a subset of edges of the subgraph of $UG$ induced by $V_d$ such that each vertex of $V_d$ is on exactly one digon. If there exists only one such subset, then there is no ambiguity as to which edges are oriented as digons.

Now, we give a final lemma about the orientation of edges near a digon.
Lemma 2.8. Let $G_0$ be the underlying graph of a restricted line digraph. If an edge \( \{x, y\} \) in $G_0$ is determined to be oriented as a digon, then there is only one way, up to converses, to orient the edges incident to either $x$ or $y$.

Proof. No edge adjacent to either $x$ or $y$ can be oriented as a digon and then, since the subgraphs $K_L(A, B)$ and $K_L(C, D)$ of $H_{x,y}(A, B, C, D)$ must be normally oriented, $K_L(A, B) \oplus K_L(C, D)$ and its converse are the only line orientations. \( \square \)

3. Characterization of $U \mathcal{L}(G')$

Having analyzed local properties, we turn to a global characterization. The following theorem characterizes underlying graphs of restricted line digraphs. The original version of this result applies to the characterization of line digraphs with minimum degree at least 1 and it can be found in [8].

Theorem 3.1. A graph $G_0$ is the underlying graph of a restricted line digraph iff there exist two partitions of $V(G_0)$, $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ such that:

1. $|A_i|, |B_i| \geq 2$, $|A_i \cap B_i| \leq 1 \forall i, j$.
2. $E(G_0) = \bigcup_{i \in I} E(K_L(A_i, B_i))$.

Proof. ($\Rightarrow$) Let $G_0$ be the underlying graph of $G = \mathcal{L}(G')$, a restricted line digraph. Let $I = V(G')$ and let $A_i$ and $B_i$ respectively be the arcs of $G'$ incident to and from the vertex $i$. (Arcs of $G'$ are vertices of $G_0$.)

1. This holds since $\delta_G \geq 2$ and $G'$ has no parallel arcs.
2. Clearly, each arc of $A_i$ is adjacent to every arc of $B_j$, and then

$$\bigcup_{i \in I} E(K_L(A_i, B_i)) \subseteq E(G_0).$$

Conversely, if $\{x, y\}$ is an edge of $G_0$, the corresponding arcs of $G'$ have a common vertex $j$, i.e., the start vertex of $x$ and the end vertex of $y$, or vice versa. Then, $x \in A_j$ and $y \in B_j$, or vice versa.

($\Leftarrow$) Now, let us suppose that there exist two such partitions $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ (see Fig. 7). Consider the digraph $H$ whose vertex set is $I$ and a vertex $i$ is adjacent to another one $j$ iff $B_i \cap A_j \neq \emptyset$.

A natural map, $\pi$, between $V(G_0)$ and $A(H)$ is:

$$\pi : A(H) \rightarrow V(G_0),$$

$$(i, j) \rightarrow B_i \cap A_j.$$

- $\pi$ is a bijection: $\forall x \in V(G_0)$, $\exists! i, j \in I$, $x \in B_i \cap A_j$ and then $\pi(i, j) = x$. Thus, a digraph $G$ isomorphic to $\mathcal{L}(H)$ can be induced by $\pi$.
- $\pi$ is a graph isomorphism between $U \mathcal{L}(H)$ and $G_0$: $\forall x, y \in V(G_0)$, say $x = \pi(i, j)$ and $y = \pi(k, l)$, $\{x, y\} \in E(G_0) \Rightarrow j = k$ or $i = l$. 
Fig. 7. Example of \(\{A_i\}_{i \in I}, \{B_i\}_{i \in I}\) partitions of a graph and one of its orientations as a restricted line digraph.

Hence, \(\{x,y\} \in E(K_L(A_i,B_i))\), i.e., \(\tau\) preserves the adjacencies of \(\mathcal{U}(H)\). Conversely, \(\forall \{x,y\} \in E(G_0), \exists h \in I, \{x,y\} \in E(K_L(A_h,B_h))\). Thus, either \(j = k = h\) or \(!=i=h\), and \(\pi^{-1}(x)\) is adjacent to \(\pi^{-1}(y)\) in \(\mathcal{U}(H)\) or vice versa. In both cases, \(\{\pi^{-1}(x),\pi^{-1}(y)\}\) is an edge of \(\mathcal{U}(H)\). □

The present aim is to show the uniqueness of the above partitions.

Let \(G_0 = UG\), where \(G = \tau(G')\) is a restricted line digraph. Then, each subgraph \(K_L(A_i,B_i), i \in I\), defined as above, is an induced subgraph of \(G_0\). Furthermore, the corresponding subdigraph in \(G\) is an induced line digraph of a simple digraph.

Suppose that the location of the digons in \(G\) is determined from \(G_0 = UG\) (e.g. if the minimum distance between digons is at least two). Then, there are only two line orientations of each \(K_L(A_i,B_i)\) subgraph, except for the 4-cycle case, that is, if \(|A_i| = |B_i| = 2, A_i \cap B_i = \emptyset\), and there are no digons in \(K_L(A_i,B_i)\).

Given the orientation of a \(K_L(A_i,B_i)\) subgraph, the orientation of any other adjacent to it is given in the next lemma.

**Lemma 3.2.** If \(\tilde{K}_L(A_i,B_j)\) is a subdigraph of \(G\) and \(A_i \cap B_j = \{x\}\), then one of the following statements holds:

1. \(\tilde{K}_L(A_j,B_j)\) is a subdigraph of \(G\).

2. \(A_j \cap B_i = \{y\}\), and \(H_{x,y}(A_j,B_j,A_i,B_i)\) is a subgraph of \(UG\) corresponding to the exception 1 case in Lemma 2.6 (and then, \(K_L(A_j,B_j)\) is oriented as the 4-cycle).

**Proof.** Let \(z \in A_i \setminus B_j\) and \(x' \in A_j \setminus B_i\). If \(x\) is adjacent to \(z\), then so is \(x'\). But the edge \(\{x',z\}\) cannot exist in \(G_0\). Therefore, \(A_i \setminus B_j \subseteq \mathcal{U}(x)\).

\(\tilde{K}_L(A_j,B_j)\) is not a subdigraph of \(G\) only if \(|A_i \setminus B_j| = 1\) and \(y \in \mathcal{U}(x)\), where \(y \in A_i \cap B_j\). But this case is just the exceptional case of \(H_{x,y}(A_j,B_j,A_i,B_i)\). □
Lemma 3.3. If $\overline{K}_L(A_i, B_i)$ is a subdigraph of $G$ and $A_i \cap B_i = \{y\}$, then one of the following statements holds:
(1) $\overline{K}_L(A_i, B_i)$ is a subdigraph of $G$.
(2) $A_i \cap B_i = \{x\}$ and $H_{x,y}(A_j, B_j, A_i, B_i)$ is a subgraph of $UG$ corresponding to the exceptional case in Lemma 2.4 (and then, $K_L(A_j, B_j)$ is oriented as the 4-cycle).

Proof. As in Lemma 3.2. □

Next, we have the final result that assures the reconstruction of a restricted line digraph from its underlying graph, when the orientation of one edge which is not a digon is known.

Theorem 3.4. Let $G_0 = UG$ be a connected graph, with $G$ a restricted line digraph. If digon location is determined and there is a subgraph $K_L(A_i, B_i)$ normally oriented (e.g. $|A_i| + |B_i| \geq 5$ or there is a loop or a digon in $G$), then the only orientations of $G_0$ as a restricted line digraph are $G$ and $-G$ (that is, $G$ is unique up to conventions).

Proof. Consider the vertex set $V_0^* = \bigcup_{j \in J} (A_j \cup B_j)$, where $K_L(A_j, B_j)$, $j \in J$, are the subdigraphs that can be oriented from $K_L(A_i, B_i)$. Let $G'_0$ be the subgraph of $G_0$ induced by $V_0^*$. Then $E(G'_0) = \bigcup_{j \in J} E(K_L(A_j, B_j))$. If there exists an edge $(x, y) \in E(K_L(A_k, B_k))$, $k \in J$ and $x, y \in V_0^*$, then there exist $p, q \in J$ such that $x \in A_p \cup B_p$, $y \in A_q \cup B_q$, and by Lemmas 3.2 and 3.3, $K_L(A_k, B_k)$ can be oriented either from $K_L(A_p, B_p)$ or from $K_L(A_q, B_q)$. Thus $k \in J$.

For the same reason, there cannot exist any arc $(x, y)$ of $G_0$ where $x \in V_0^*$ and $y \in V_0^*$. Hence, since $G_0$ is connected $G'_0 = G_0$ and $J = I$. □

Note that if $G_0$ can be oriented as a restricted line digraph $G$, then it can also be oriented as $-G$, which is clearly also a restricted line digraph. Thus, the existence of two (possibly isomorphic) orientations of $G_0 = UG$ is assured, and both are line digraphs of orientations of the same graph.

Corollary 3.5. If digon location is determined and a $K_L(A_i, B_i)$ subgraph is oriented as a 4-cycle, then all $K_L(A_j, B_j)$ subgraphs in $G_0$ are oriented in the same way, too. Such an orientation must be a 2-regular digraph with girth at least 3.

Proof. If $K_L(A_i, B_i)$ is oriented as the 4-cycle, then the conditions in Theorem 3.4 must fail, i.e., $|A_j| = |B_j| = 2$ and there are no digons and no loops. □

Note that in the case of Corollary 3.5, there can exist different $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ partitions as we can see in the example in Fig. 8. This can also occur if digon location is not determined.
In the following result, the relation between the automorphism groups of a restricted line digraph and its underlying graph is given.

**Corollary 3.6.** Let $G_0$ be a graph satisfying the conditions in Theorem 3.4, so that $G_0 = UG$ for a restricted line digraph $G$. Then $|\text{Aut } G_0| = 2 |\text{Aut } G|$ if $G$ is isomorphic to its converse, and $\text{Aut } G_0 = \text{Aut } G$ otherwise.

**Proof.** Let $\mathcal{O}(G_0)$ denote the set of all possible orientations of $G_0$, and let $\mathcal{O}_\phi(G_0)$ be the subset of $\mathcal{O}(G_0)$ formed by those orientations which are restricted line digraphs. Observe that the automorphism group of $G_0$, $\text{Aut } G_0$, acts as a permutation group over $\mathcal{O}(G_0)$ in the natural way: $(x, y) \in A(\phi G) \Rightarrow (\phi^{-1} x, \phi^{-1} y) \in A(G)$ where $\phi \in \text{Aut } G_0$ and $G \in \mathcal{O}(G_0)$. (Note that $\phi G$ is isomorphic to $G$.)

Let $[H], H \in \mathcal{O}(G_0)$, be the set of orientations of $G_0$ isomorphic to $H$ (i.e., the orbit of $H$). If $H$ is a restricted line digraph, then so are all orientations in $[H]$. Clearly, $\text{Aut } H$ is the stabilizer of $H$, and

$$|\text{Aut } G_0| = |[H]| |\text{Aut } H|.$$

Finally, since $G_0$ fulfills the conditions in Theorem 3.4, $\mathcal{O}_\phi(G_0) = \{G, -G\}$. Therefore, if $G$ is isomorphic to its converse, then $|[G]| = 2$, and $|[G]| = 1$ otherwise. \(\square\)
Let us apply the above results to two well-known families of digraphs, namely de Bruijn and Kautz digraphs (see [5, 12]). Both families can be defined respectively as line digraph iterations of the complete symmetrical digraph $F_s$ on $s$ vertices and the loopless one $K_s$ (see [7]).

$$K(d,n) = \mathcal{L}^{n-1}(K_{d+1}) = \text{Kautz digraph of degree } d \text{ and diameter } n.$$  

$$B(d,n) = \mathcal{L}^{n-1}(F_d) = \text{de Bruijn digraph of degree } d \text{ and diameter } n.$$  

Now, the final result gives the answer to the original problem proposed by J. Bond (in fact, the result about de Bruijn graphs was conjectured by him).

**Corollary 3.7.**

$$\text{Aut } UK(d-1,n) \cong \text{Aut } UB(d,n) \cong \mathbb{Z}_2 \times S_d.$$  

**Proof.** Observe that if $n \geq 2$ and $d \geq 2$, then $K(d,n)$ and $B(d,n)$ are restricted line digraphs and they have digons. In both cases, the minimum distance between digons is $n-1$. Moreover, digon location is determined since:

- If $n=3$, the minimum distance between digons is at least 2.
- If $n=2$ and $d \geq 3$, there are no vertices with minimum degree 2.
- $B(2,2)$ has only one digon.
- In $K(2,2)$ a 4-cycle orientation of any subdigraph is not possible, so an exceptional orientation of a $H_{x,y}(A,B,C,D)$ subgraph cannot occur.

Theorem 3.4 then assures that the undirected Kautz and de Bruijn graphs have exactly two (possibly isomorphic) orientations as restricted line digraphs, i.e., Kautz and de Bruijn digraphs and their converses. Any Kautz and de Bruijn digraph is isomorphic to its converse, and it can be shown that this isomorphism commutes with any of their automorphisms. Finally, the automorphism group of Kautz and de Bruijn digraphs are:

$$\text{Aut } K(d-1,n) \cong \text{Aut } B(d,n) \cong S_d$$

since those digraphs are iterated line digraphs of vertex transitive digraphs of $d$ vertices, and the automorphism group of a strong digraph is isomorphic to the automorphism group of its line digraph (see [1]).

**References**


