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Modeling with fractional difference equations

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ABSTRACT

In this paper, we develop some basics of discrete fractional calculus such as Leibniz rule and summation by parts formula. We define simplest discrete fractional calculus of variations problem and derive Euler–Lagrange equation. We introduce and solve Gompertz fractional difference equation for tumor growth models.

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1. Introduction

Fractional calculus is an emerging field recently drawing attention from both theoretical and applied disciplines. During the last two decades it has been successfully applied to problems in computational biology, medical sciences, economics, physics and several fields in engineering [7,8,11,15,17,18,21,23]. On the other hand, discrete fractional calculus is a very new area for scientists. In 1989, a pioneering work has been done by Miller and Ross in [19]. Here we aim to continue to develop the theory of discrete fractional calculus, in the direction of the papers by Atıcı and Eloe [3–6], and to improve modeling techniques with fractional order difference equations.

We refer to the books [20,22,24] for further reading on fractional calculus and fractional differential equations.

We start with some basic definitions and results so that this paper is self contained. Let a be any real number, α be any positive real number and $\sigma(s) = s + 1$. The α-th fractional sum (α-sum) of f is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} \big(t-\sigma(s)\big)^{(\alpha-1)} f(s). \tag{1.1}$$

Here f is defined for $s = a \pmod 1$ and $\Delta_a^{-\alpha} f$ is defined for $t = a + \alpha \pmod 1$; in particular, $\Delta_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$, where $\mathbb{N}_t = \{t, t+1, t+2, \dots\}$. We recall that the falling factorial is defined as $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$.

The following three results and their proofs can be found in [3] and [4].

Theorem 1.1. *Let f be a real-valued function defined on $\mathbb{N}_a$ and let $\mu, \nu > 0$. Then the following equality holds*

$$\Delta^{-\nu}\big[\Delta^{-\mu} f(t)\big] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}\big[\Delta^{-\nu} f(t)\big].$$

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Lemma 1.1. *Let $\mu \neq -1$ and assume $\mu+\nu+1$ is not a non-positive integer. Then*

$$\Delta^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{(\mu+\nu)}.$$

Theorem 1.2. *For any $\nu > 0$, the following equality holds*:

$$\Delta^{-\nu} \Delta f(t) = \Delta \Delta^{-\nu} f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a), \tag{1.2}$$

where f is defined on $\mathbb{N}_a$.

The μ-th fractional difference is defined as

$$\Delta^{\mu} u(t) = \Delta^{m-\nu} u(t) = \Delta^{m}\big(\Delta^{-\nu} u(t)\big),$$

where $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, and $-\nu = \mu - m$.

In Section 2, we give an alternate proof for Leibniz formula derived by Miller and Ross [19] for the fractional sum of the product of two functions

$$\Delta_0^{-\alpha}(fg)(t) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} \big[\Delta^k g(t)\big]\big[\Delta_0^{-(\alpha+k)} f(t+k)\big],$$

where f is defined on $\mathbb{N}_0$ and g is defined on $\mathbb{N}_\alpha \cup \mathbb{N}_0$. Then we derive another version of Leibniz formula in which both functions f and g are defined on the set of integers.

In Section 3, we introduce definitions of left and right fractional difference operators. Then we state and prove the summation by parts formula in discrete fractional calculus.

In Section 4, we introduce simplest discrete fractional calculus of variations problem and derive Euler–Lagrange equation. This section provides a good example where one can see that the summation by parts formula takes a place.

Finally, in Section 5, we construct a model to minimize growth of a tumor. We claim that the tumor size, as a function of time, fits well in Gompertz fractional difference equation. A series solution of the Gompertz equation is obtained by the method of successive approximations. We illustrate our findings with some graphs.

2. Leibniz formula

One question which may arise in discrete fractional calculus is whether the product rule for fractional difference operator is valid in a similar way that we have in discrete calculus. The answer to this question can be found in an early paper by Miller and Ross [19]. Here we give an alternate proof to Leibniz formula for α-sum with carefully stating the domains of the functions.

Lemma 2.1. *Let f be a real-valued function defined on $\mathbb{N}_0$ and g be a real-valued function defined on $\mathbb{N}_\alpha \cup \mathbb{N}_0$, where α is a real number between 0 and 1. Then the following equality holds*

$$\Delta_0^{-\alpha}(fg)(t) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} \big[\Delta^k g(t)\big]\big[\Delta_0^{-(\alpha+k)} f(t+k)\big],$$

where $t \equiv \alpha$ (mod 1) and

$$\binom{-\alpha}{k} = \frac{\Gamma(-\alpha+1)}{\Gamma(k+1)\Gamma(-\alpha-k+1)}.$$

Proof. By the definition of discrete fractional sum, we have

$$\Delta_0^{-\alpha}(fg)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \big(t-\sigma(s)\big)^{(\alpha-1)} f(s) g(s),$$

where $t \equiv \alpha$ (mod 1).

By Taylor expansion of $g(s)$ [9, p. 40], we have

$$g(s) = \sum_{k=0}^{\infty} \frac{(s-t)^{(k)}}{k!} \Delta^k g(t) = \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k g(t)}{k!} \big(t-\sigma(s)+k\big)^{(k)}.$$

Thus by substituting Taylor series expansion of $g(s)$ in the sum, we have

$$\Delta_0^{-\alpha}(fg)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} f(s) \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k g(t)}{k!} \bigl(t-\sigma(s)+k\bigr)^{(k)}.$$

Since $(t-\sigma(s))^{(\alpha-1)}(t-\sigma(s)+k)^{(k)} = (t+k-\sigma(s))^{(\alpha+k-1)}$, we have

$$\Delta_0^{-\alpha}(fg)(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k g(t)}{k!} \sum_{s=0}^{(t+k)-(\alpha+k)} \bigl(t+k-\sigma(s)\bigr)^{(\alpha+k-1)} f(s).$$

Since $(-1)^k = \frac{\Gamma(-\alpha+1)\Gamma(\alpha)}{\Gamma(-\alpha-k+1)\Gamma(k+\alpha)}$ for any nonnegative integer k, the above expression on the right becomes

$$\sum_{k=0}^{\infty} \binom{-\alpha}{k} \bigl[\Delta^k g(t)\bigr]\bigl[\Delta_0^{-(\alpha+k)} f(t+k)\bigr].$$

This completes the proof of the lemma. □

Next we state and prove another version of Leibniz formula.

Lemma 2.2. *Let f and g be real-valued functions defined on the set of integers and α be any real number between* 0 *and* 1. *Then the following equality holds*

$$\Delta_0^{-\alpha}(fg)(t) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} \bigl[\nabla^k g(t-\alpha)\bigr]\bigl[\Delta_0^{-(\alpha+k)} f(t)\bigr],$$

where $t \equiv \alpha \pmod 1$ *and*

$$\binom{-\alpha}{k} = \frac{\Gamma(-\alpha+1)}{\Gamma(k+1)\Gamma(-\alpha-k+1)}.$$

Proof. By the definition of discrete fractional sum, we have

$$\Delta_0^{-\alpha}(fg)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} f(s)g(s).$$

By Taylor expansion of $g(s)$ (see [2]), we have

$$g(s) = \sum_{k=0}^{\infty} \frac{(s-t)^{\bar{k}}}{k!} \nabla^k g(t) = \sum_{k=0}^{\infty} (-1)^k (t-s)^{(k)} \frac{\nabla^k g(t)}{k!},$$

where $(s-t)^{\bar{k}} = \frac{\Gamma(s-t+k)}{\Gamma(s-t)}$ is the raising factorial power.

Thus by substituting Taylor series of $g(s)$ at $t-\alpha$ in the sum, we have

$$\Delta_0^{-\alpha}(fg)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} f(s) \sum_{k=0}^{\infty} (-1)^k \frac{\nabla^k g(t-\alpha)}{k!} (t-s-\alpha)^{(k)}.$$

Since $\sum_{s=t-\alpha-k+1}^{t-\alpha} (t-\alpha-s)^{(k)} = 0$ and $(t-\sigma(s))^{(\alpha-1)}(t-\alpha-s)^{(k)} = (t-\sigma(s))^{(\alpha+k-1)}$, we have

$$\Delta_0^{-\alpha}(fg)(t) = g(t-\alpha)\Delta^{-\alpha} f(t) + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \Gamma(\alpha+k) \frac{\nabla^k g(t-\alpha)}{k!} (-1)^k \Delta_0^{-(\alpha+k)} f(t).$$

Since $(-1)^k = \frac{\Gamma(-\alpha+1)\Gamma(\alpha)}{\Gamma(-\alpha-k+1)\Gamma(k+\alpha)}$ for any nonnegative integer k, the above expression on the right becomes

$$\sum_{k=0}^{\infty} \binom{-\alpha}{k} \bigl[\nabla^k g(t-\alpha)\bigr]\bigl[\Delta_0^{-(\alpha+k)} f(t)\bigr].$$

This completes the proof. □

Example 2.1. Let us derive the α-th order difference of the product $tf(t)$, where f and t are defined on $\mathbb{N}_0$ and on $\mathbb{N}_\alpha \cup \mathbb{N}_0$ respectively, and $0 < \alpha < 1$.

It follows from Lemma 2.1 that

$$\Delta^\alpha\big(tf(t)\big) = \Delta\Delta^{-(1-\alpha)}\big(tf(t)\big) = \Delta\sum_{k=0}^{\infty}\binom{-1+\alpha}{k}\Delta^k t\Delta^{-(1-\alpha+k)}f(t+k),$$

for $t \equiv \alpha \pmod 1$.

Since $\Delta^k t = 0$ for $k \geqslant 2$, we have

$$\Delta^\alpha\big(tf(t)\big) = \Delta\big[t\Delta^{-(1-\alpha)}f(t) + (\alpha-1)\Delta^{-(2-\alpha)}f(t+1)\big],$$
$$\Delta^\alpha\big(tf(t)\big) = \Delta^{-(1-\alpha)}f(t+1) + t\Delta^\alpha f(t) + (\alpha-1)\Delta^{-(1-\alpha)}f(t+1),$$
$$\Delta^\alpha\big(tf(t)\big) = \alpha\Delta^{-(1-\alpha)}f(t+1) + t\Delta^\alpha f(t), \tag{2.1}$$

for $t \equiv \alpha \pmod 1$.

If we consider the domain of the function $g(t) = t$ as the set of integers, then it follows from Lemma 2.2 that

$$\Delta^\alpha\big(tf(t)\big) = \alpha\Delta^{-(1-\alpha)}f(t) + (t+\alpha)\Delta^\alpha f(t). \tag{2.2}$$

One immediate observation can be made. Since $\lim_{\alpha\to 1}\Delta^\alpha = \Delta$ (see [19]), we apply limit to each sides of the formulas in (2.1) and (2.2) to obtain the forward difference of the product $tf(t)$ in discrete calculus, namely,

$$\Delta\big(tf(t)\big) = f(t+1) + t\Delta f(t) = f(t) + (t+1)\Delta f(t).$$

3. Summation by parts formula

In order to obtain summation by parts formula in discrete fractional calculus, we shall start with defining left and right discrete fractional difference operators.

Definition 3.1. Let f be any real-valued function and α be a positive real number between 0 and 1. Then the left discrete fractional difference and the right discrete fractional difference operators are defined as follows, respectively,

$${}_t\Delta_a^\alpha f(t) = \Delta_t\Delta_a^{-(1-\alpha)}f(t) = \frac{1}{\Gamma(1-\alpha)}\Delta\sum_{s=a}^{t-1+\alpha}\big(t-\sigma(s)\big)^{(-\alpha)}f(s),$$

$t \equiv a + 1 - \alpha \pmod 1$,

$${}_b\Delta_t^\alpha f(t) = -\Delta_b\Delta_t^{-(1-\alpha)}f(t) = \frac{1}{\Gamma(1-\alpha)}(-\Delta)\sum_{s=t+1-\alpha}^{b}\big(s-\sigma(t)\big)^{(-\alpha)}f(s),$$

where $t \equiv b + \alpha - 1 \pmod 1$. We will use the symbol Δ^α for ${}_t\Delta_a^\alpha$ except otherwise stated.

Theorem 3.2. *Let F and G be real-valued functions and $0 < \alpha < 1$. If $F(b+\alpha-2) = 0$ and $F(a+\alpha-2) = 0$ or $G(a+\alpha-1) = 0$ and $G(b+\alpha-1) = 0$, then the following equality holds*

$$\sum_{s=a}^{b-1}F(s+\alpha-1)\,{}_s\Delta_{a+\alpha-1}^\alpha G(s) = \sum_{s=a}^{b-1}G(s+\alpha-1)\,{}_{b+\alpha-1}\Delta_s^\alpha F^\rho\big(s+2(\alpha-1)\big),$$

where $F^\rho = F\circ\rho$ with $\rho(t) = t-1$.

Proof. Using the definition of fractional difference on the left side of the equality, we have

$$\sum_{s=a}^{b-1}F(s+\alpha-1)\,{}_s\Delta_{a+\alpha-1}^\alpha G(s) = \sum_{s=a}^{b-1}F(s+\alpha-1)\Delta\Delta^{-(1-\alpha)}G(s).$$

Using summation by parts formula for Δ-operator, we have

$$\sum_{s=a}^{b-1}F(s+\alpha-1)\Delta_s\Delta_{a+\alpha-1}^{-(1-\alpha)}G(s) = F(s+\alpha-2)\,{}_s\Delta_{a+\alpha-1}^{-(1-\alpha)}G(s)\Big|_a^b - \sum_{s=a}^{b-1}{}_s\Delta_{a+\alpha-1}^{-(1-\alpha)}G(s)\Delta_s F(s+\alpha-2)$$
$$= \frac{-1}{\Gamma(1-\alpha)}\sum_{s=a}^{b-1}\sum_{\tau=a+\alpha-1}^{s-(1-\alpha)}\big(s-\sigma(\tau)\big)^{(-\alpha)}G(\tau)\Delta_s F(s+\alpha-2).$$

Next we may interchange the order of summation in the double sum to obtain

$$\frac{-1}{\Gamma(1-\alpha)}\sum_{\tau=a+\alpha-1}^{b-2+\alpha}\sum_{s=\tau+1-\alpha}^{b-1}\big(s-\sigma(\tau)\big)^{(-\alpha)}G(\tau)\Delta_s F(s+\alpha-2).$$

Let us call $u=\tau+1-\alpha$. Then the above expression becomes

$$\frac{-1}{\Gamma(1-\alpha)}\sum_{u=a}^{b-1}\sum_{s=u}^{b-1}(s-\alpha-u)^{(-\alpha)}G(u+\alpha-1)\Delta_s F(s+\alpha-2). \tag{3.1}$$

Now let us look at the following expression closely,

$$\frac{-1}{\Gamma(1-\alpha)}\sum_{s=u}^{b-1}(s-\alpha-u)^{(-\alpha)}\Delta_s F(s+\alpha-2). \tag{3.2}$$

Using summation by parts formula for Δ-operator, we have

$$\sum_{s=u}^{b-1}(s-\alpha-u)^{(-\alpha)}\Delta_s F(s+\alpha-2)$$
$$=\big[(s-\alpha-u-1)^{(-\alpha)}F(s+\alpha-2)\big]\big|_u^b-\sum_{s=u}^{b-1}\Delta_s(s-\alpha-u-1)^{(-\alpha)}F(s+\alpha-2).$$

Since $[(s-\alpha-u-1)^{(-\alpha)}F(s+\alpha-2)]|_u^b=0$, then

$$\frac{-1}{\Gamma(1-\alpha)}\sum_{s=u}^{b-1}(s-\alpha-u)^{(-\alpha)}\Delta_s F(s+\alpha-2)=\frac{1}{\Gamma(1-\alpha)}\sum_{s=u}^{b-1}\Delta_s(s-\alpha-u-1)^{(-\alpha)}F(s+\alpha-2)$$
$$=\frac{1}{\Gamma(1-\alpha)}(-\alpha)\sum_{s=u}^{b-1}(s-\alpha-u-1)^{(-1-\alpha)}F(s+\alpha-2).$$

It follows from [10, Theorem 8.50]

$$\Delta_u\sum_{s=u}^{b-1}(s-\alpha-u)^{(-\alpha)}F(s+\alpha-2)=\alpha\sum_{s=u}^{b-1}\big(s-\alpha-\sigma(u)\big)^{(-1-\alpha)}F(s+\alpha-2).$$

Hence the expression in (3.2) becomes

$$\frac{-1}{\Gamma(1-\alpha)}\sum_{s=u}^{b-1}(s-\alpha-u)^{(-\alpha)}\Delta_s F(s+\alpha-2)=\frac{-1}{\Gamma(1-\alpha)}\Delta_u\sum_{s=u}^{b-1}(s-\alpha-u)^{(-\alpha)}F(s+\alpha-2)$$
$$=\frac{-1}{\Gamma(1-\alpha)}\Delta_u\sum_{s=u}^{b-1}\big(s-\sigma(u+\alpha-1)\big)^{(-\alpha)}F^{\rho}(s+\alpha-1)$$
$$=\frac{1}{\Gamma(1-\alpha)}(-\Delta_u)\sum_{s=u-1+\alpha}^{b+\alpha-1}\big(s-\sigma(u+2\alpha-2)\big)^{(-\alpha)}F^{\rho}(s)$$
$$={}_{b+\alpha-1}\Delta_s^{\alpha}F^{\rho}\big(u+2(\alpha-1)\big).$$

Replacing this back in (3.1) we have the desired result. □

Remark 3.1. If we prove the equality in Theorem 3.2 starting from the right side of the identity, then we use the conditions $G(a+\alpha-1)=0$ and $G(b+\alpha-1)=0$.

4. Simplest variational problem in discrete fractional calculus

The calculus of variations is one of the old theories in mathematics, yet is very much alive and is still evolving. Besides its mathematical importance and its links to other branches of mathematics, such as geometry or differential equations, it is widely used in engineering, economics and biology to optimize several problems.

The theory of calculus of variations in fractional calculus has been first introduced by Agrawal [1], and then developed by other scientists [12,13].

Here we will demonstrate one application of the summation by parts formula (Theorem 3.2) which we derived in Section 3, and introduce the calculus of variations in discrete fractional calculus.

Let $F(t,u,v)$ be a real-valued function with continuous partial derivatives and $\mathfrak{D}$ be the set of all real-valued functions y defined on $[a+\alpha-1, b+\alpha-1] \subset \mathbb{N}_{a+\alpha-1}$ with $y(a+\alpha-1)=y_a$ and $y(b+\alpha-1)=y_b$.

We consider the following functional

$$J[y]=\sum_{t=a}^{b-1} F\big(t+\alpha-1, y(t+\alpha-1), {}_t\Delta^{\alpha}_{a+\alpha-1}y(t)\big),$$

where α is a real number between 0 and 1. Here our aim is to optimize this functional assuming that it has an extremum. This is called the "simplest variational problem" in discrete fractional calculus. To develop the necessary conditions for the extremum, let us assume that $y^*(t)$ is the desired function such that

$$y(t)=y^*(t)+\epsilon\eta(t), \quad \epsilon\in\mathbb{R},$$

where $\eta\in\mathfrak{A}=\{w\,|\,w(t)$ is a real-valued function defined on $[a+\alpha-1, b+\alpha-1]$ with $\eta(a+\alpha-1)=\eta(b+\alpha-1)=0\}$.

Since ${}_t\Delta^{\alpha}_{a+\alpha-1}$ is a linear operator, we have ${}_t\Delta^{\alpha}_{a+\alpha-1}y(t)={}_t\Delta^{\alpha}_{a+\alpha-1}y^*(t)+\epsilon\,{}_t\Delta^{\alpha}_{a+\alpha-1}\eta(t)$.

If we substitute this expression into the functional, we have

$$J\big(y^*+\epsilon\eta\big)=\sum_{t=a}^{b-1} F\big(t+\alpha-1, y^*(t+\alpha-1)+\epsilon\eta(t+\alpha-1), {}_t\Delta^{\alpha}_{a+\alpha-1}y^*(t)+\epsilon\,{}_t\Delta^{\alpha}_{a+\alpha-1}\eta(t)\big).$$

Then differentiating $J(y^*+\epsilon\eta)$ with respect to ϵ, we have

$$\begin{aligned}\frac{\partial J\big(y^*+\epsilon\eta\big)}{\partial\epsilon}=&\sum_{t=a}^{b-1}\big[F_u\big(t+\alpha-1, y(t+\alpha-1), {}_t\Delta^{\alpha}_{a+\alpha-1}y(t)\big)\eta(t+\alpha-1)\\&+F_v\big(t+\alpha-1, y(t+\alpha-1), {}_t\Delta^{\alpha}_{a+\alpha-1}y(t)\big)\,{}_t\Delta^{\alpha}_{a+\alpha-1}\eta(t)\big],\end{aligned}$$

where F_u and F_v are partial derivatives of $F(\cdot,u,v)$ with respect to u and v respectively.

Therefore for the functional $J[y]$ to have an extremum at $y=y^*(t)$, the following must hold

$$\frac{\partial J(y^*+\epsilon\eta)}{\partial\epsilon}\bigg|_{\epsilon=0}=\sum_{t=a}^{b-1}F_u\eta(t+\alpha-1)+F_{v\,t}\Delta^{\alpha}_{a+\alpha-1}\eta(t)=0. \tag{4.1}$$

Using summation by parts formula in discrete fractional calculus (Theorem 3.2), we have

$$\sum_{t=a}^{b-1}F_v(t+\alpha-1)\,{}_t\Delta^{\alpha}_{a+\alpha-1}\eta(t)=\sum_{t=a}^{b-1}\eta(t+\alpha-1)\,{}_{b+\alpha-1}\Delta^{\alpha}_t F^{\rho}_v\big(t+2(\alpha-1)\big),$$

where $F_v(\tau)=F_v(\tau, y(\tau), {}_t\Delta^{\alpha}_{a+\alpha-1}y(\tau-\alpha+1))$.

Hence (4.1) becomes

$$\sum_{t=a}^{b-1}\big[F_u\big(t+\alpha-1, y(t+\alpha-1), {}_t\Delta^{\alpha}_{a+\alpha-1}y(t)\big)+{}_{b+\alpha-1}\Delta^{\alpha}_t F^{\rho}_v\big(t+2(\alpha-1)\big)\big]\eta(t+\alpha-1)=0.$$

Since $\eta(t+\alpha-1)$ is arbitrary, we have

$$F_u\big(t+\alpha-1, y(t+\alpha-1), {}_t\Delta^{\alpha}_{a+\alpha-1}y(t)\big)+{}_{b+\alpha-1}\Delta^{\alpha}_t F^{\rho}_v\big(t+2(\alpha-1)\big)=0,$$

on $[a+1, b-1]$.

We just proved the following theorem.

Theorem 4.1. *If the simplest variational problem has a local extremum at $y^*(t)$, then $y^*(t)$ satisfies the Euler–Lagrange equation*

$$F_u\big(t+\alpha-1, y(t+\alpha-1), {}_t\Delta^{\alpha}_{a+\alpha-1}y(t)\big)+{}_{b+\alpha-1}\Delta^{\alpha}_t F^{\rho}_v\big(t+2(\alpha-1)\big)=0,$$

for $t\in[a+1, b-1]$.

5. Gompertz fractional difference equation

Tumor growth, a special relationship between tumor size and time, is of special interest since growth estimation is very critical in clinical practice. Three different terms are typically used to model growth behavior in biology: exponential, logistic and sigmoidal. Tumor growth, however, is best described by sigmoidal functions. In 1825, Benjamin Gompertz introduced the Gompertz function, a sigmoid function, which is found to be applicable to various growth phenomena, in particular tumor growth (see [14]). The Gompertz difference equation describes the growth models and these models can be studied on the basis of the parameters a (growth rate) and b (exponential rate of growth deceleration) in the recursive formulation of the Gompertz law of growth [8].

The Gompertz difference equation in [8] is given by

$$\ln G(t+1) = a + b\ln G(t).$$

Next we introduce Gompertz fractional difference equation

$$\Delta^{\alpha} \ln G(t-\alpha+1) = (b-1)\ln G(t) + a. \tag{5.1}$$

For simplicity if we replace $\ln G(t) = y(t)$, we obtain

$$\Delta_0^{\alpha} y(t-\alpha+1) = (b-1)y(t) + a. \tag{5.2}$$

Next we are concerned with the following optimization problem

$$J[y] = \min \sum_{t=0}^{T-1} U\big(y(t+\alpha-1)\big)$$

with a constraint

$$\Delta_0^{\alpha} y(t-\alpha+1) = (b-1)y(t) + a, \quad y(0) = c$$

or

$$\Delta_{\alpha-1}^{\alpha} y(t) = (b-1)y(t+\alpha-1) + a$$

where $y(t)$ is the size of tumor and U is a function with continuous partial derivatives.

We have

$$J[y] = \sum_{t=0}^{T-1} \big\{U\big(y(t+\alpha-1)\big) + \lambda(t+\alpha-1)\big(\Delta_{\alpha-1}^{\alpha} y(t) - (b-1)y(t+\alpha-1) - a\big)\big\}.$$

It follows from Theorem 4.1, Euler–Lagrange equations with respect to λ and y are

$$U_y\big(y(t+\alpha-1)\big) - \lambda(t+\alpha-1)(b-1) + {}_{T+\alpha-1}\Delta_t^{\alpha} \lambda^{\rho}\big(t+2(\alpha-1)\big) = 0, \tag{5.3}$$

or

$$U_y\big(y(t+\alpha-1)\big) - \lambda(t+\alpha-1)(b-1) + \frac{1}{\Gamma(1-\alpha)}(-\Delta) \sum_{s=t+\alpha-1}^{T+\alpha-1} \big(s-\sigma\big(t+(2\alpha-1)\big)\big)^{(-\alpha)} \lambda\big(\rho(s)\big),$$

and

$$\Delta_{\alpha-1}^{\alpha} y(t) - (b-1)y(t+\alpha-1) - a = 0, \tag{5.4}$$

where $t \in [1, T-1]$.

At this point, to the best of the authors' knowledge, there is no known or published numerical methods of solving the above systems of Eqs. (5.3)–(5.4). Therefore we shall focus on proving the existence and uniqueness result for the Gompertz fractional difference equation (5.2) with an initial condition $y(0) = c$.

Consider the following fractional difference equation with an initial condition

$$\Delta_0^{\alpha} y(t-\alpha+1) = f\big(t, y(t)\big), \quad t = 0, 1, 2, \ldots, \tag{5.5}$$

$$y(0) = c \tag{5.6}$$

where $\alpha \in (0, 1]$, f is a real-valued function, and c is a real number.

Applying $\Delta^{-\alpha}$ operator to both side of Eq. (5.5) and with $t+\alpha-1$ shift at the same time, we obtain

$$\Delta^{-\alpha}\Delta^{\alpha} y(t) = \Delta^{-\alpha} f\big(t+\alpha-1, y(t+\alpha-1)\big),$$

where $t = 1, 2, \ldots.$

Applying Theorem 1.2 to the left-hand side of the above equation, we get

$$\Delta^{-\alpha}\Delta^{\alpha}y(t)=\Delta^{-\alpha}\Delta\Delta^{-(1-\alpha)}y(t)=\Delta\Delta^{-\alpha}\Delta^{-(1-\alpha)}y(t)-\frac{(t+\alpha-1)^{(\alpha-1)}y(0)}{\Gamma(\alpha)}.$$

Hence we have

$$y(t)=\frac{(t+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}c+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-1}\bigl(t+\alpha-1-\sigma(s)\bigr)^{(\alpha-1)}f\bigl(s,y(s)\bigr), \tag{5.7}$$

for $t=0,1,2,\ldots.$

The recursive iteration to the sum equation implies that (5.7) represents the unique solution of the IVP.

Next we obtain a solution for Eq. (5.2) with an initial value condition $y(0)=c$.

Replacing $f(s,y(s))=(b-1)y(s)+a$ in (5.7), the solution of the IVP (5.5)–(5.6) is

$$y(t)=\frac{(t+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}c+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-1}\bigl(t+\alpha-1-\sigma(s)\bigr)^{(\alpha-1)}\bigl[(b-1)y(s)+a\bigr].$$

Next, we employ the method of successive approximations. Set

$$\begin{aligned}y_0(t)&=\frac{(t+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}c+\Delta^{-\alpha}a(t+\alpha-1)^{(0)},\\ y_m(t)&=(b-1)\Delta^{-\alpha}y_{m-1}(t+\alpha-1),\quad m=1,2,\ldots.\end{aligned}$$

Apply the power rule (Lemma 1.1) to see that

$$\begin{aligned}y_1(t)&=(b-1)\Delta^{-\alpha}y_0(t+\alpha-1)\\&=c(b-1)\frac{(t+2\alpha-2)^{(2\alpha-1)}}{\Gamma(2\alpha)}+(b-1)\Delta^{-2\alpha}a(t+2\alpha-2)^{(0)}\end{aligned}$$

and

$$\begin{aligned}y_2(t)&=(b-1)\Delta^{-\alpha}y_1(t+\alpha-1)\\&=c(b-1)^2\frac{(t+3\alpha-3)^{(3\alpha-1)}}{\Gamma(3\alpha)}+(b-1)^2\Delta^{-3\alpha}a(t+3\alpha-3)^{(0)}.\end{aligned}$$

With repeated applications of the power rule, it follows inductively that

$$y_m(t)=c(b-1)^m\frac{(t+(m+1)(\alpha-1))^{((m+1)\alpha-1)}}{\Gamma((m+1)\alpha)}+(b-1)^m a\frac{(t+(m+1)(\alpha-1))^{((m+1)\alpha)}}{\Gamma((m+1)\alpha+1)}.$$

$\sum_{m=0}^{\infty}y_m$ converges to the unique solution of the initial value problem

$$\Delta_0^{\alpha}y(t-\alpha+1)-(b-1)y(t)=a,\quad y(0)=c.$$

Hence we have

$$y(t)=c\sum_{m=0}^{\infty}(b-1)^m\frac{(t+(m+1)(\alpha-1))^{((m+1)\alpha-1)}}{\Gamma((m+1)\alpha)}+a\sum_{m=0}^{\infty}(b-1)^m\frac{(t+(m+1)(\alpha-1))^{((m+1)\alpha)}}{\Gamma((m+1)\alpha+1)}.$$

One immediate observation can be made. Set $\alpha=1$. Then

$$y(t)=c\sum_{m=0}^{\infty}(b-1)^m\frac{t^{(m)}}{\Gamma(m+1)}+a\sum_{m=0}^{\infty}(b-1)^m\frac{t^{(m+1)}}{\Gamma(m+2)}.$$

Since the IVP with $\alpha=1$ (see [16]) has the unique solution $y(t)=cb^t+\frac{a}{b-1}b^t-\frac{a}{b-1}$, we obtain the equality $b^t=\sum_{i=0}^{\infty}\frac{(b-1)^i}{i!}t^{(i)}$ which appears as a special case of [9, Lemma 4.4] for a time scale $\mathbb{T}=\mathbb{Z}$, the set of integers.

The solution of Eq. (5.1) is $G(t)=e^{y(t)}$.

In biomedical research, growth comparisons are very important especially for tumor growth. These comparisons are made according to growth stimulation or growth inhibition.

There are two ways to measure growth alterations which are typically used in literature: Assays with time as the independent variable and the changing tumor size as the dependent variable or assays with tumor size as the independent variable and the time to reach a given size as the dependent variable. Since the experiments are limited in time these assays generally measure alterations of growth rates. Thus, growth stimulation or growth inhibition refers to an increase or a decrease of the growth rate (see Fig. 1 for $\alpha=1$, $\alpha=0.94$ and $\alpha=0.83$).

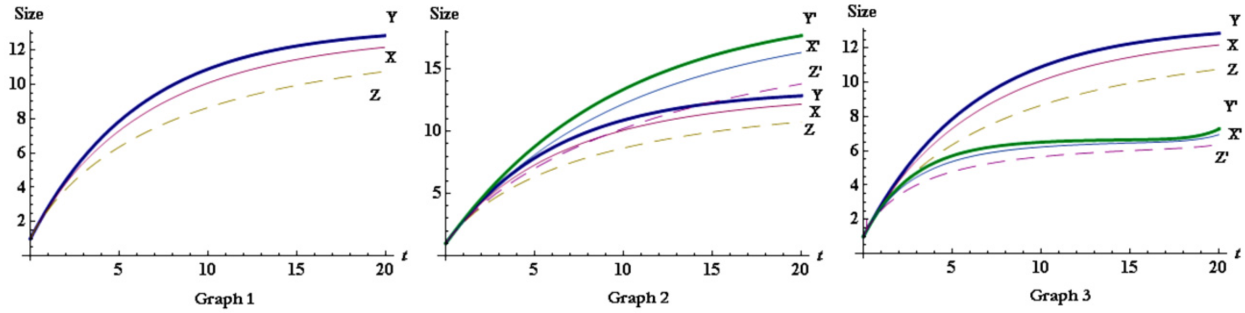

Fig. 1. Alterations of growth (Graph 1: Control growth ($a = 2$, $b = 0.85$, $c = 1$, Y: $\alpha = 1$, X: $\alpha = 0.94$, Z: $\alpha = 0.83$), Graph 2: Stimulation ($a = 2$, $b = 0.9$, $c = 1$, Y': $\alpha = 1$, X': $\alpha = 0.94$, Z': $\alpha = 0.83$), Graph 3: Inhibition ($a = 2$, $b = 0.7$, $c = 1$, Y': $\alpha = 1$, X': $\alpha = 0.94$, Z': $\alpha = 0.83$)).

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References

[1] O.P. Agrawal, Formulation of Euler–Lagrange equations for fractional variational problems, J. Math. Anal. Appl. 272 (2002) 368–379.
[2] D.R. Anderson, Taylor polynomials for nabla dynamic equations on time scales, Panamer. Math. J. 12 (4) (2002) 17–27.
[3] F.M. Atıcı, P.W. Eloe, A transform method in discrete fractional calculus, Int. J. Difference Equ. 2 (2) (2007) 165–176.
[4] F.M. Atıcı, P.W. Eloe, Initial value problems in discrete fractional calculus, Proc. Amer. Math. Soc. 137 (3) (2009) 981–989.
[5] F.M. Atıcı, P.W. Eloe, Discrete fractional calculus with the nabla operator, Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I (3) (2009) 1–12.
[6] F.M. Atıcı, P.W. Eloe, Two-point boundary value problems for finite fractional difference equations, J. Difference Equ. Appl., doi:10.1080/10236190903029241, in press.
[7] I.D. Bassukas, Comparative Gompertzian analysis of alterations of tumor growth patterns, Cancer Res. 54 (1994) 4385–4392.
[8] I.D. Bassukas, B. Maurer Schultze, The recursion formula of the Gompertz function: A simple method for the estimation and comparison of tumor growth curves, Growth Dev. Aging 52 (1988) 113–122.
[9] M. Bohner, G. Guseinov, The convolution on time scales, Abstr. Appl. Anal. (2007), Art. ID58373, 24 pp.
[10] M. Bohner, A.C. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, 2001.
[11] C. Coussot, Fractional derivative models and their use in the characterization of hydropolymer and in-vivo breast tissue viscoelasticity, Master thesis, University of Illinois at Urbana-Champain, 2008.
[12] G.S.F. Frederico, D.F.M. Torres, A formulation of Noether's theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. 334 (2007) 834–846.
[13] G.S.F. Frederico, D.F.M. Torres, Fractional conservation laws in optimal control theory, Nonlinear Dynam. 53 (2008) 215–222.
[14] B. Gompertz, On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies, Philos. Trans. R. Soc. Lond. 115 (1825) 513–585.
[15] G. Jumarie, Stock exchange fractional dynamics defined as fractional exponential growth driven by (usual) Gaussian white noise. Application to fractional Black–Scholes equations, Insurance Math. Econom. 42 (2008) 271–287.
[16] W.G. Kelley, A.C. Peterson, Difference Equations: An Introduction with Applications, Academic Press, 2001.
[17] R.L. Magin, Fractional Calculus in Bioengineering, Begell House, 2006.
[18] R. Maronski, Optimal strategy in chemotheraphy for a Gompertzian model of cancer growth, Acta Bioeng. Biomech. 10 (2) (2008) 81–84.
[19] K.S. Miller, B. Ross, Fractional difference calculus, in: Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications, Nihon University, Koriyama, Japan, May 1988, in: Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1989, pp. 139–152.
[20] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, Inc., New York, 1993.
[21] L.A. Norton, Gompertzian model of human breast cancer growth, Cancer Res. 48 (1988) 7067–7071.
[22] I. Podlubny, Fractional Differential Equations, Academic Press, 1999.
[23] J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado, Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, 2007.
[24] G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.