



## Some existence results for a Paneitz type problem via the theory of critical points at infinity

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Received 15 March 2003

Available online 8 December 2004

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### Abstract

In this paper a fourth order equation involving critical growth is considered under the Navier boundary condition:  $\Delta^2 u = Ku^p$ ,  $u > 0$  in  $\Omega$ ,  $u = \Delta u = 0$  on  $\partial\Omega$ , where  $K$  is a positive function,  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^n$ ,  $n \geq 5$  and  $p + 1 = 2n/(n - 4)$ , is the critical Sobolev exponent. We give some topological conditions on  $K$  to ensure the existence of solution. Our methods involve the study of the critical points at infinity and their contribution to the topology of the level sets of the associated Euler–Lagrange functional.

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### Résumé

Dans cet article, nous considérons une équation d'ordre quatre ayant une croissance critique avec conditions de Navier au bord :  $\Delta^2 u = Ku^p$ ,  $u > 0$  dans  $\Omega$ ,  $u = \Delta u = 0$  sur  $\partial\Omega$ , où  $K$  est une fonction strictement positive,  $\Omega$  est un domaine borné régulier de  $\mathbb{R}^n$ ,  $n \geq 5$  et  $p + 1 = 2n/(n - 4)$ , est l'exposant critique de Sobolev. Nous donnons certaines conditions topologiques sur  $K$  pour assurer l'existence de solution. Notre approche est fondée sur l'étude des points critiques à l'infini et de leur contribution à la topologie des ensembles de niveau de la fonctionnelle d'Euler–Lagrange associée.

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MSC: 35J60; 35J65; 58E05

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*Keywords:* Critical points at infinity; Critical Sobolev exponent; Lack of compactness

## 1. Introduction and main results

In this paper we prove some existence results for the following nonlinear problem under the Navier boundary condition:

$$\begin{cases} \Delta^2 u = Ku^p, & u > 0 & \text{in } \Omega, \\ \Delta u = u = 0 & & \text{on } \partial\Omega, \end{cases} \quad (P)$$

where  $\Omega$  is a bounded smooth domain of  $\mathbb{R}^n$ ,  $n \geq 5$ ,  $p + 1 = 2n/(n - 4)$ , is the critical exponent of the embedding  $H^2 \cap H_0^1(\Omega)$  into  $L^{p+1}(\Omega)$  and  $K$  is a  $C^3$ -positive function in  $\overline{\Omega}$ .

This type of equation naturally arises from the study of conformal geometry. A well known example is the problem of prescribing the Paneitz curvature: given a function  $K$  defined in compact Riemannian manifold  $(M, g)$  of dimension  $n \geq 5$ , we ask whether there exists a metric  $\tilde{g}$  conformal to  $g$  such that  $K$  is the Paneitz curvature of the new metric  $\tilde{g}$  (for details one can see [9,10,14,17–20] and the references therein).

We observe that one of the main features of problem  $(P)$  is the lack of compactness, that is, the Euler–Lagrange functional  $J$  associated to  $(P)$  does not satisfy the Palais–Smale condition. This means that there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. Such a fact follows from the noncompactness of the embedding of  $H^2 \cap H_0^1(\Omega)$  into  $L^{p+1}(\Omega)$ . However, it is easy to see that a necessary condition for solving the problem  $(P)$  is that  $K$  has to be positive somewhere. Moreover, it turns out that there is at least another obstruction to solve the problem  $(P)$ , based on Kazdan–Warner type conditions, see [17]. Hence it is not expectable to solve problem  $(P)$  for all the functions  $K$ , thus a natural question arises: under which conditions on  $K$ ,  $(P)$  has a solution. Our aim in this paper is to give sufficient conditions on  $K$  such that  $(P)$  possesses a solution.

In the last years, several researches have been developed on the existence of solutions of fourth order elliptic equations with critical exponent on a domain of  $\mathbb{R}^n$ , see [11,12,15,16,21–23,26–28,31,32]. However, at the authors' knowledge, problem  $(P)$  has been considered for  $K \equiv 1$  only.

As we mentioned before,  $(P)$  is delicate from a variational viewpoint because of the failure of the Palais–Smale condition, more precisely because of the existence of critical points at infinity, that is orbits of the gradient flow of  $J$  along which  $J$  is bounded, its gradient goes to zero, and which do not converge [3]. In this article, we give a contribution in the same direction as in the papers [1,4,8] concerning the problem of prescribing the scalar curvature on closed manifolds. Precisely, we extend some topological and dynamical methods of the *Theory of critical points at infinity* (see [3]) to the framework of such higher order equations. To do such an extension, we perform a careful expansion of  $J$ , and its gradient near a neighborhood of highly concentrated functions. Then, we construct a special pseudogradient for the associated variational problem for which the

Palais–Smale condition is satisfied along the decreasing flow lines far from a finite number of such “singularities”. As a by product of the construction of our pseudogradient, we are able to characterize the critical points at infinity of our problem. Such a fine analysis of these critical points at infinity, which has its own interest, is highly nontrivial and plays a crucial role in the derivation of existence results. In our proofs, the main idea is to take advantage of the precise computation of the contribution of these critical points at infinity to the topology of the level sets of  $J$ ; the main argument being that, under our conditions on  $K$ , there remains some difference of topology which is not due to the critical points at infinity and therefore the existence of a critical point of  $J$ .

Our proofs go along the methods of Aubin and Bahri [1], Bahri [4] and Ben Ayed, Chtioui and Hammami [8]. However, in our case the presence of the boundary makes the analysis more involved: it turns out that the interaction of “bubbles” and the boundary creates a phenomenon of new type which is not present in the closed manifolds’ case. In addition, we have to prove the positivity of the critical point obtained by our process. It is known that in the framework of higher order equations such a proof is quite difficult in general (see [19], for example), and the way we handle it here is very simple compared with the literature, see Proposition 4.1 below.

In order to state our main results, we need to introduce some notation and the assumptions that we are using in our results. We denote by  $G$  the Green’s function and by  $H$  its regular part, that is for each  $x \in \Omega$ ,

$$\begin{cases} G(x, y) = |x - y|^{-(n-4)} - H(x, y) & \text{in } \Omega, \\ \Delta^2 H(x, \cdot) = 0 & \text{in } \Omega, \\ \Delta G(x, \cdot) = G(x, \cdot) = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, we state our assumptions.

(A<sub>0</sub>) Assume that, for each  $x \in \partial\Omega$ ,

$$\frac{\partial K(x)}{\partial \nu} < 0,$$

where  $\nu$  is the outward normal to  $\Omega$ .

(A<sub>1</sub>) We assume that  $K$  has only nondegenerate critical points  $y_0, y_1, \dots, y_s$  such that

$$K(y_0) \geq K(y_1) \geq \dots \geq K(y_l) > K(y_{l+1}) \geq \dots \geq K(y_s).$$

(A<sub>2</sub>) We assume that

$$\begin{aligned} -\frac{\Delta K(y_i)}{60K(y_i)} + H(y_i, y_i) &> 0 \quad \text{for } i \leq l \quad \text{and} \\ -\frac{\Delta K(y_i)}{60K(y_i)} + H(y_i, y_i) &< 0 \quad \text{for } i > l \quad (\text{if } n = 6), \end{aligned}$$

and

$$-\Delta K(y_i) > 0 \text{ for } i \leq l \quad \text{and} \quad -\Delta K(y_i) < 0 \text{ for } i > l \quad (\text{if } n \geq 7).$$

(A'<sub>2</sub>) We assume that

$$\begin{aligned} &-\frac{1}{60} \frac{\Delta K(y_i)}{K(y_i)} + H(y_i, y_i) < 0 \quad \text{for } i > l \text{ (if } n = 6) \quad \text{and} \\ &-\Delta K(y_i) < 0 \quad \text{for } i > l \text{ (if } n \geq 7). \end{aligned}$$

In addition, for every  $i \in \{1, \dots, l\}$  such that

$$-\frac{1}{60} \frac{\Delta K(y_i)}{K(y_i)} + H(y_i, y_i) \leq 0 \text{ (if } n = 6) \quad \text{and} \quad -\Delta K(y_i) \leq 0 \text{ (if } n \geq 7),$$

we assume that  $n - m + 3 \leq \text{index}(K, y_i) \leq n - 2$ , where  $\text{index}(K, y_i)$  is the Morse index of  $K$  at  $y_i$  and  $m$  is an integer defined in assumption (A<sub>3</sub>).

Now, let  $Z_K$  be a pseudogradient of  $K$  of Morse–Smale type (that is, the intersections of the stable and unstable manifolds of the critical points of  $K$  are transverse). Set:

$$X = \overline{\bigcup_{0 \leq i \leq l} W_s(y_i)},$$

where  $W_s(y)$  is the stable manifold of  $y$  for  $Z_K$ .

(A<sub>3</sub>) We assume that  $X$  is not contractible and denote by  $m$  the dimension of the first nontrivial reduced homological group of  $X$ .

(A<sub>4</sub>) We assume that there exists a positive constant  $\bar{c} < K(y_l)$  such that  $X$  is contractible in  $K^{\bar{c}} = \{x \in \Omega \mid K(x) \geq \bar{c}\}$ .

Now we are able to state our first results:

**Theorem 1.1.** *Let  $n \geq 6$ . Under the assumptions (A<sub>0</sub>), (A<sub>1</sub>), (A<sub>2</sub>), (A<sub>3</sub>) and (A<sub>4</sub>), there exists a constant  $c_0$  independent of  $K$  such that if  $K(y_0)/\bar{c} \leq 1 + c_0$ , then (P) has a solution.*

**Corollary 1.2.** *The solution obtained in Theorem 1.1 has an augmented Morse index  $\geq m$ .*

**Theorem 1.3.** *Let  $n \geq 7$ . Under the assumptions (A<sub>0</sub>), (A<sub>1</sub>), (A'<sub>2</sub>), (A<sub>3</sub>) and (A<sub>4</sub>), there exists a constant  $c_0$  independent of  $K$  such that if  $K(y_0)/\bar{c} \leq 1 + c_0$ , then (P) has a solution.*

**Remark 1.4.** (i) The assumption  $K(y_0)/\bar{c} \leq 1 + c_0$  allows basically to perform a single-bubble analysis.

(ii) To see how to construct an example of a function  $K$  satisfying our assumptions, we refer the interested reader to [2].

Next, we state another kind of existence results for problem (P) based on a topological invariant introduced by A. Bahri in [4]. In order to give our results in this direction, we need to fix some notation and state our assumptions.

We denote by  $W_s(y)$  and  $W_u(y)$  the stable and unstable manifolds of  $y$  for  $Z_K$ .

(A<sub>5</sub>) We assume that  $K$  has only nondegenerate critical points  $y_i$  satisfying  $\Delta K(y_i) \neq 0$  and  $W_s(y_i) \cap W_u(y_j) = \emptyset$  for any  $i$  such that  $-\Delta K(y_i) > 0$  and for any  $j$  such that

$$-\Delta K(y_j) < 0.$$

For  $k \in \{1, \dots, n - 1\}$ , we define  $X$  as

$$X = \overline{W_s(y_{i_0})},$$

where  $y_{i_0}$  satisfies

$$K(y_{i_0}) = \max\{K(y_i) \mid \text{index}(K, y_i) = n - k, -\Delta K(y_i) > 0\}.$$

(A<sub>6</sub>) We assume  $X$  without boundary.

We observe that assumption (A<sub>0</sub>) implies that  $X$  does not intersect the boundary  $\partial\Omega$  and therefore it is a compact set of  $\Omega$ .

Now, we denote by  $y_0$  the absolute maximum of  $K$ . Let us define the set  $C_{y_0}(X)$  as

$$C_{y_0}(X) = \{\alpha\delta_{y_0} + (1 - \alpha)\delta_x \mid \alpha \in [0, 1], x \in X\},$$

where  $\delta_x$  denotes the Dirac mass at  $x$ .

For  $\lambda$  large enough, we introduce a map  $f_\lambda : C_{y_0}(X) \rightarrow \Sigma^+ := \{u \in H^2 \cap H_0^1 \mid u > 0, \|u\|_2 = 1\}$ ,

$$\alpha\delta_{y_0} + (1 - \alpha)\delta_x \mapsto \frac{(\alpha/K(y_0))^{(n-4)/8} P\delta_{(y_0,\lambda)} + ((1 - \alpha)/K(x))^{(n-4)/8} P\delta_{(x,\lambda)}}{\|(\alpha/K(y_0))^{(n-4)/8} P\delta_{(y_0,\lambda)} + ((1 - \alpha)/K(x))^{(n-4)/8} P\delta_{(x,\lambda)}\|_2},$$

where  $\|u\|_2^2 = \int_\Omega |\Delta u|^2$ .

Then  $C_{y_0}(X)$  and  $f_\lambda(C_{y_0}(X))$  are manifolds in dimension  $k + 1$ , that is, their singularities arise in dimension  $k - 1$  and lower, see [4]. The codimension of  $W_s(y_0, y_{i_0})_\infty$  is equal to  $k + 1$ , then we can define the intersection number (modulo 2) of  $f_\lambda(C_{y_0}(X))$  with  $W_s(y_0, y_{i_0})_\infty$ :

$$\mu(y_{i_0}) = f_\lambda(C_{y_0}(X)) \cdot W_s(y_0, y_{i_0})_\infty,$$

where  $W_s(y_0, y_{i_0})_\infty$  is the stable manifold of the critical points at infinity  $(y_0, y_{i_0})_\infty$  for a decreasing pseudogradient for  $J$  which is transverse to  $f_\lambda(C_{y_0}(X))$ . Such a number is well defined see [4,25]. Observe that  $C_{y_0}(X)$  and  $f_\lambda(C_{y_0}(X))$  are contractible while  $X$  is not contractible.

(A<sub>7</sub>) Assume that  $2/K(y_0)^{(n-4)/4} < 1/K(y)^{(n-4)/4}$  for each critical point  $y$  of Morse index  $n - (k + 1)$  and satisfies  $-\Delta K(y) > 0$ .

We then have the following result:

**Theorem 1.5.** *Let  $n \geq 7$ . Under assumptions (A<sub>0</sub>), (A<sub>5</sub>), (A<sub>6</sub>) and (A<sub>7</sub>), if  $\mu(y_{i_0}) = 0$  then (P) has a solution of an augmented Morse index less than  $k + 1$ .*

Now, we give a more general statement than Theorem 1.5. For this purpose, we define  $X$  as

$$X = \overline{\bigcup_{y \in B} W_s(y)},$$

where  $B = \{y \in \Omega \mid \nabla K(y) = 0, -\Delta K(y) > 0\}$ . We denote by  $k$  the dimension of  $X$  and by  $B_k = \{y \in B \mid \text{index}(K, y) = n - k\}$ .

For  $y_i \in B_k$ , we define, for  $\lambda$  large enough, the intersection number (modulo 2):

$$\mu(y_i) = f_\lambda(C_{y_0}(X)) \cdot W_s(y_0, y_i)_\infty.$$

By the above arguments, this number is well defined, see [25].

Then, we have:

**Theorem 1.6.** *Let  $n \geq 7$ . Under assumptions (A<sub>0</sub>), (A<sub>5</sub>) and (A<sub>6</sub>), if  $\mu(y_i) = 0$  for each  $y_i \in B_k$ , then (P) has a solution of an augmented Morse index less than  $k + 1$ .*

The organization of the paper is the following. In Section 2, we set up the variational structure and recall some preliminaries. In Section 3, we give an expansion of the Euler functional associated to (P) and its gradient near potential critical points at infinity. In Section 4, we provide the proof of Theorem 1.1 and its corollary. In Section 5, we prove Theorem 1.3, while Section 6 is devoted to the proof of Theorems 1.5 and 1.6.

## 2. Preliminaries

In this section, we set up the variational structure and its mean features.

Problem (P) has a variational structure. The related functional is,

$$J(u) = \left( \int_{\Omega} K|u|^{2n/(n-4)} \right)^{-(n-4)/n},$$

defined on

$$\Sigma = \left\{ u \in H^2 \cap H_0^1(\Omega) \mid \|u\|_{H^2 \cap H_0^1(\Omega)}^2 := \|u\|_2^2 := \int_{\Omega} |\Delta u|^2 = 1 \right\}.$$

The positive critical points of  $J$  are solutions of  $(P)$ , up to a multiplicative constant.

Due to the non-compactness of the embedding  $H^2 \cap H_0^1(\Omega)$  into  $L^{p+1}(\Omega)$ , the functional  $J$  does not satisfy the Palais–Smale condition. An important result of Struwe [30] (see also [24] and [13]) describes the behavior of such sequences associated to second order equations of the type:

$$-\Delta u = u^{(n+2)/(n-2)}, \quad u > 0 \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

In [21], Gazzola, Grunau and Squassina proved the analogue of this result for problem  $(P)$ . To describe the sequences failing the Palais–Smale condition, we need to introduce some notation.

For  $a \in \Omega$  and  $\lambda > 0$ , let:

$$\delta_{(a,\lambda)}(x) = c_n \left( \frac{\lambda}{1 + \lambda^2|x - a|^2} \right)^{(n-4)/2}, \tag{2.2}$$

where  $c_n$  is a positive constant chosen so that  $\delta_{(a,\lambda)}$  is the family of solutions of the following problem (see [23]):

$$\Delta^2 u = u^{(n+4)/(n-4)}, \quad u > 0 \quad \text{in } \mathbb{R}^n. \tag{2.3}$$

For  $f \in H^2(\Omega)$ , we define the projection  $P$  by:

$$u = Pf \iff \Delta^2 u = \Delta^2 f \quad \text{in } \Omega, \quad u = \Delta u = 0 \quad \text{on } \partial\Omega. \tag{2.4}$$

We have the following proposition which is extracted from [11].

**Proposition 2.1** [11]. *Let  $a \in \Omega$ ,  $\lambda > 0$  and  $\varphi_{(a,\lambda)} = \delta_{(a,\lambda)} - P\delta_{(a,\lambda)}$ . We have:*

- (a)  $0 \leq \varphi_{(a,\lambda)} \leq \delta_{(a,\lambda)}$ ,
- (b)  $\varphi_{(a,\lambda)} = c_n \frac{H(a,\lambda)}{\lambda^{(n-4)/2}} + f_{(a,\lambda)}$ , where  $c_n$  is defined in (2.2) and  $f_{(a,\lambda)}$  satisfies:

$$f_{(a,\lambda)} = O\left(\frac{1}{\lambda^{n/2}d^{n-2}}\right), \quad \lambda \frac{\partial f_{(a,\lambda)}}{\partial \lambda} = O\left(\frac{1}{\lambda^{n/2}d^{n-2}}\right),$$

$$\frac{1}{\lambda} \frac{\partial f_{(a,\lambda)}}{\partial a} = O\left(\frac{1}{\lambda^{(n+2)/2}d^{n-1}}\right),$$

where  $d$  is the distance  $d(a, \partial\Omega)$ .

(c)

$$|\varphi_{(a,\lambda)}|_{L^{2n/(n-4)}} = O\left(\frac{1}{(\lambda d)^{(n-4)/2}}\right), \quad \left| \lambda \frac{\partial \varphi_{(a,\lambda)}}{\partial \lambda} \right|_{L^{2n/(n-4)}} = O\left(\frac{1}{(\lambda d)^{(n-4)/2}}\right),$$

$$\|\varphi_{(a,\lambda)}\|_2 = O\left(\frac{1}{(\lambda d)^{(n-4)/2}}\right), \quad \left| \frac{1}{\lambda} \frac{\partial \varphi_{(a,\lambda)}}{\partial a} \right|_{L^{2n/(n-4)}} = O\left(\frac{1}{(\lambda d)^{(n-2)/2}}\right).$$

We now introduce the set of potential critical points at infinity.

For any  $\varepsilon > 0$  and  $p \in \mathbb{N}^*$ , let  $V(p, \varepsilon)$  be the subset of  $\Sigma$  of the following functions:  $u \in \Sigma$  such that there is  $(a_1, \dots, a_p) \in \Omega^p$ ,  $(\lambda_1, \dots, \lambda_p) \in (\varepsilon^{-1}, +\infty)^p$  and  $(\alpha_1, \dots, \alpha_p) \in (0, +\infty)^p$  such that

$$\left\| u - \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \right\|_2 < \varepsilon, \quad \lambda_i d(a_i, \partial\Omega) > \varepsilon^{-1};$$

$$\left| \frac{\alpha_i^{8/(n-4)} K(a_i)}{\alpha_j^{8/(n-4)} K(a_j)} - 1 \right| < \varepsilon, \quad \varepsilon_{ij} < \varepsilon \quad \text{for } i \neq j,$$

where

$$\varepsilon_{ij} = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{-(n-4)/2}. \quad (2.5)$$

The failure of the Palais–Smale condition can be described going along the ideas developed in [13,24,30]. Namely, we have:

**Proposition 2.2** [21]. *Assume that  $J$  has no critical point in  $\Sigma^+$ . Let  $(u_k) \in \Sigma^+$  be a sequence such that  $(\partial J(u_k))$  tends to zero and  $(J(u_k))$  is bounded. Then, after possibly having extracted a subsequence, there exist  $p \in \mathbb{N}^*$  and a sequence  $(\varepsilon_k)$ ,  $\varepsilon_k$  tends to zero, such that  $u_k \in V(p, \varepsilon_k)$ .*

Now, we consider the following minimization problem for a function  $u \in V(p, \varepsilon)$  with  $\varepsilon$  small:

$$\min \left\{ \left\| u - \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \right\|_2, \alpha_i > 0, \lambda_i > 0, a_i \in \Omega \right\}. \quad (2.6)$$

We then have the following proposition whose proof is similar, up to minor modifications, to the corresponding statement for the Laplacian operator in [5]. This proposition defines a parametrization of the set  $V(p, \varepsilon)$ .

**Proposition 2.3.** *For any  $p \in \mathbb{N}^*$ , there exists  $\varepsilon_p > 0$  such that, if  $\varepsilon < \varepsilon_p$  and  $u \in V(p, \varepsilon)$ , the minimization problem (2.6) has a unique solution  $(\alpha, a, \lambda)$  (up to permutation). In particular, we can write  $u \in V(p, \varepsilon)$  as follows:*

$$u = \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} + v,$$

where  $(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p, \lambda_1, \dots, \lambda_p)$  is the solution of (2.6) and  $v \in H^2(\Omega) \cap H_0^1(\Omega)$  such that



$$\begin{aligned} (v, P\delta_{(a_i, \lambda_i)})_2 &= (v, \partial P\delta_{(a_i, \lambda_i)}/\partial \lambda_i)_2 = 0, \\ (v, \partial P\delta_{(a_i, \lambda_i)}/\partial a_i)_2 &= 0 \quad \text{for } i = 1, \dots, p, \end{aligned} \tag{V_0}$$

where  $(u, w)_2 = \int_{\Omega} \Delta u \Delta w$ .

### 3. Expansion of the functional and its gradient

In this section, we will give a useful expansion of the functional  $J$  and its gradient in the potential set  $V(p, \varepsilon)$  for  $n \geq 6$ . In the sequel, for the sake of simplicity, we will write  $\delta_i$  instead of  $\delta_{(a_i, \lambda_i)}$ . We start by the expansion of  $J$ .

**Proposition 3.1.** *There exists  $\varepsilon_0 > 0$  such that for any  $u = \sum_{i=1}^p \alpha_i P\delta_i + v \in V(p, \varepsilon)$ ,  $\varepsilon < \varepsilon_0$ ,  $v$  satisfying (V<sub>0</sub>), we have:*

$$\begin{aligned} J(u) &= \frac{S_n^{4/n} \sum_{i=1}^p \alpha_i^2}{(\sum_{i=1}^p \alpha_i^{2n/(n-4)} K(a_i))^{(n-4)/n}} \\ &\times \left[ 1 + \frac{1}{S_n \sum_{i=1}^p K(a_i)^{(4-n)/4}} \left( -\frac{n-4}{n} c_3 \sum_{i=1}^p \frac{\Delta K(a_i)}{K(a_i)^{n/4} \lambda_i^2} \right. \right. \\ &+ c_2 \sum_{i=1}^p \frac{H(a_i, a_i)}{K(a_i)^{(n-4)/4} \lambda_i^{n-4}} - \frac{c_2}{(K(a_i)K(a_j))^{(n-4)/8}} \sum_{i \neq j} \left( \varepsilon_{ij} - \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-4)/2}} \right) \left. \right) \\ &- f(v) + \frac{1}{\sum_{i=1}^p \alpha_i^2 S_n} Q(v, v) + o\left( \sum \frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k d_k)^{n-4}} + \sum_{i \neq j} \varepsilon_{ij} + \|v\|_2^2 \right) \Big], \end{aligned}$$

where

$$\begin{aligned} Q(v, v) &= \|v\|_2^2 - \frac{n+4}{n-4} \sum_{i=1}^p \int_{\Omega} P\delta_i^{8/(n-4)} v^2, \\ f(v) &= \frac{2}{\sum_{i=1}^p \alpha_i^{2n/(n-4)} K(a_i) S_n} \int_{\Omega} K \left( \sum_{i=1}^p \alpha_i P\delta_i \right)^{(n+4)/(n-4)} v, \end{aligned}$$

$$S_n = \int_{\mathbb{R}^n} \frac{c_n^{2n/(n-4)} dy}{(1 + |y|^2)^n}, \quad c_2 = \int_{\mathbb{R}^n} \frac{c_n^{2n/(n-4)}}{(1 + |y|^2)^{(n+4)/2}} dy, \quad c_3 = \frac{c_n^{2n/(n-4)}}{2n} \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} dy,$$

and  $c_n$  is defined in (2.2). Observe that if  $n = 6$  we have  $c_2 = 20c_3$ .

**Proof.** On one hand, Proposition 2.1 implies:

$$\|P\delta\|_2^2 = S_n - c_2 \frac{H(a, a)}{\lambda^{n-4}} + O\left(\frac{1}{(\lambda d)^{n-2}}\right), \quad (3.1)$$

$$\begin{aligned} \int_{\Omega} K P \delta^{2n/(n-4)} &= K(a)S_n + c_3 \frac{\Delta K(a)}{\lambda^2} - \frac{2n}{n-4} c_2 K(a) \frac{H(a, a)}{\lambda^{n-4}} \\ &+ O\left(\frac{1}{\lambda^3} + \frac{1}{(\lambda d)^{n-2}}\right). \end{aligned} \quad (3.2)$$

On the other hand, a computation similar to the one performed in [3] shows that, for  $i \neq j$ , we have:

$$\begin{aligned} \int_{\mathbb{R}^n} \delta_i^{(n+4)/(n-4)} \delta_j &= c_2 \varepsilon_{ij} + O(\varepsilon_{ij}^{(n-2)/(n-4)}), \\ \int_{\mathbb{R}^n} (\delta_i \delta_j)^{n/(n-4)} &= O(\varepsilon_{ij}^{n/(n-4)} \log(\varepsilon_{ij}^{-1})). \end{aligned} \quad (3.3)$$

Thus, we derive that

$$(P\delta_i, P\delta_j)_2 = c_2 \left( \varepsilon_{ij} - \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-4)/2}} \right) + O\left( \varepsilon_{ij}^{(n-2)/(n-4)} + \sum_{k=i, j} \frac{1}{(\lambda_k d_k)^{n-2}} \right), \quad (3.4)$$

$$\int_{\Omega} K P \delta_i^{(n+4)/(n-4)} P \delta_j = K(a_i) (P\delta_i, P\delta_j)_2 + o\left( \sum \frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k d_k)^{n-4}} + \varepsilon_{ij} \right) \quad (3.5)$$

and

$$\int K \left( \sum_{i=1}^p \alpha_i P \delta_i \right)^{8/(n-4)} v^2 = \sum_{i=1}^p \alpha_i^{8/(n-4)} K(a_i) \int P \delta_i^{8/(n-4)} v^2 + o(\|v\|_2^2). \quad (3.6)$$

Combining (3.1)–(3.6) and the fact that  $\alpha_i^{8/(n-4)} K(a_i) / (\alpha_j^{8/(n-4)} K(a_j)) = 1 + o(1)$ , our result follows.  $\square$

Now, let us recall that the quadratic form  $Q(v, v)$  defined in Proposition 3.1 is positive definite (see [9]). Thus we have the following proposition which deals with the  $v$ -part of  $u$ .

**Proposition 3.2** (see [9]). *There exists a  $C^1$ -map which, to each  $(\alpha, a, \lambda)$  satisfying  $\sum_{i=1}^p \alpha_i P \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$ , with  $\varepsilon$  small enough, associates  $\bar{v} = \bar{v}(\alpha, a, \lambda)$  satisfying  $(V_0)$*

such that  $\bar{v}$  is unique, minimizing  $J(\sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} + v)$  with respect to  $v$  satisfying  $(V_0)$ , and we have the following estimate:

$$\begin{aligned} \|\bar{v}\|_2 \leq c|f| &= O\left(\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) \\ &+ \begin{cases} O(\sum \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} + 1/(\lambda_i d_i)^{n-4}) & \text{if } n < 12, \\ O(\sum \varepsilon_{ij}^{(n+4)/(2(n-4))} (\log \varepsilon_{ij}^{-1})^{(n+4)/2n} \\ + (\log \lambda_i d_i)^{(n+4)/2n} / (\lambda_i d_i)^{(n+4)/2}) & \text{if } n \geq 12. \end{cases} \end{aligned}$$

Now regarding the gradient of  $J$  which we will denote by  $\partial J$ , we have the following expansions:

**Proposition 3.3.** For  $u = \sum_{i=1}^p \alpha_i P\delta_i \in V(p, \varepsilon)$ , we have the following expansion:

$$\begin{aligned} \left(\partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right)_2 &= 2J(u) \left[ \frac{n-4}{n} c_3 \alpha_i \frac{\Delta K(a_i)}{K(a_i) \lambda_i^2} - \frac{n-4}{2} c_2 \alpha_i \frac{H(a_i, a_i)}{\lambda_i^{n-4}} (1 + o(1)) \right. \\ &\quad \left. - c_2 \sum_{j \neq i} \alpha_j \left( \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-4)/2}} \right) (1 + o(1)) \right] \\ &+ o\left(\sum \frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k d_k)^{n-3}} + \sum_{k \neq r} \varepsilon_{kr}^{(n-3)/(n-4)}\right). \end{aligned}$$

**Proof.** We have:

$$\begin{aligned} \left(\partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right)_2 &= 2J(u) \left[ \sum \alpha_j \left( P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right)_2 \right. \\ &\quad \left. - J(u)^{n/(n-4)} \int K \left( \sum \alpha_j P\delta_j \right)^{(n+4)/(n-4)} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right]. \quad (3.7) \end{aligned}$$

Observe that

$$\begin{aligned} \left(\sum \alpha_j P\delta_j\right)^{(n+4)/(n-4)} &= \sum (\alpha_j P\delta_j)^{(n+4)/(n-4)} + \frac{n+4}{n-4} \sum_{j \neq i} (\alpha_i P\delta_i)^{8/(n-4)} \alpha_j P\delta_j \\ &+ O\left(\sum_{j \neq i} P\delta_j^{8/(n-4)} P\delta_i \chi_{P\delta_i \leq \sum_{j \neq i} P\delta_j}\right) \\ &+ \sum_{j \neq i} P\delta_i^{(12-n)/(n-4)} P\delta_j^2 \chi_{P\delta_j \leq P\delta_i} \\ &+ \sum_{k \neq j, k, j \neq i} P\delta_j^{8/(n-4)} P\delta_k. \quad (3.8) \end{aligned}$$

Using Proposition 2.1, a computation similar to the one performed in [3] and [29] shows that

$$\left( P\delta, \lambda \frac{\partial P\delta}{\partial \lambda} \right)_2 = \frac{n-4}{2} c_2 \frac{H(a, a)}{\lambda^{n-4}} + O\left( \frac{1}{(\lambda d)^{n-2}} \right), \quad (3.9)$$

$$\begin{aligned} \int_{\mathbb{R}^n} K P \delta^{(n+4)/(n-4)} \lambda \frac{\partial P\delta}{\partial \lambda} &= -\frac{n-4}{n} c_3 \frac{\Delta K(a)}{\lambda^2} + (n-4) c_2 K(a) \frac{H(a, a)}{\lambda^{n-4}} \\ &+ O\left( \frac{1}{\lambda^3} + \frac{1}{(\lambda d)^{n-2}} \right). \end{aligned}$$

For  $i \neq j$ , we have:

$$\int_{\mathbb{R}^n} \delta_i^{(n+4)/(n-4)} \lambda_j \frac{\partial \delta_j}{\partial \lambda_j} = c_2 \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + O(\varepsilon_{ij}^{(n-2)/(n-4)}), \quad (3.10)$$

$$\begin{aligned} \left( P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right)_2 &= c_2 \left( \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-4}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-4)/2}} \right) \\ &+ O\left( \sum_{k=i,j} \frac{1}{(\lambda_k d_k)^{n-2}} + \varepsilon_{ij}^{(n-2)/(n-4)} \right), \end{aligned} \quad (3.11)$$

$$\begin{aligned} \int K P \delta_j^{(n+4)/(n-4)} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} &= K(a_j) \left( P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right)_2 + O\left( \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} \left( \frac{1}{\lambda_j} + \frac{1}{(\lambda_j d_j)^4} \right) \right) \\ &+ \begin{cases} O(\varepsilon_{ij}^{n/(n-4)} \log \varepsilon_{ij}^{-1} + \log(\lambda_j d_j) / (\lambda_j d_j)^n) & \text{if } n \geq 8, \\ O(\varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} / (\lambda_j d_j)^{n-4}), & \text{if } n < 8, \end{cases} \end{aligned} \quad (3.12)$$

$$\begin{aligned} \int K P \delta_j \lambda_i \frac{\partial (P\delta_i)^{(n+4)/(n-4)}}{\partial \lambda_i} &= K(a_i) \left( P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right)_2 + O\left( \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} \left( \frac{1}{\lambda_i} + \frac{1}{(\lambda_i d_i)^4} \right) \right) \\ &+ \begin{cases} O(\varepsilon_{ij}^{n/(n-4)} \log \varepsilon_{ij}^{-1} + \log(\lambda_i d_i) / (\lambda_i d_i)^n) & \text{if } n \geq 8, \\ O(\varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} / (\lambda_i d_i)^{n-4}) & \text{if } n < 8. \end{cases} \end{aligned} \quad (3.13)$$

Now, it is easy to check,

$$|\lambda_i \partial P \delta_i / \partial \lambda_i| \leq c \delta_i, \quad P \delta_k \leq \delta_k \quad \text{and}$$

$$J(u)^{n/(n-4)} \alpha_j^{8/(n-4)} K(a_j) = 1 + o(1) \quad \forall j = 1, \dots, p. \tag{3.14}$$

Combining (3.7)–(3.14), we easily derive our proposition.  $\square$

**Proposition 3.4.** *For  $u = \sum_{i=1}^p \alpha_i P \delta_i$  belonging to  $V(p, \varepsilon)$ , we have the following expansion:*

$$\begin{aligned} \left( \partial J(u), \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right)_2 &= 2J(u) \left[ -c_4 \alpha_i^{(n+4)/(n-4)} J(u)^{n/(n-4)} \frac{\nabla K(a_i)}{\lambda_i} (1 + o(1)) \right. \\ &\quad \left. + \frac{c_2}{2} \frac{\alpha_i}{\lambda_i^{n-3}} \frac{\partial H(a_i, a_i)}{\partial a_i} (1 + o(1)) \right. \\ &\quad \left. + O\left( \frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{n-2}} + \sum_{j \neq i} \varepsilon_{ij} \right) \right]. \end{aligned}$$

We can improve this expansion and we obtain:

$$\begin{aligned} \left( \partial J(u), \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right)_2 &= 2J(u) \left[ -c_4 \alpha_i^{(n+4)/(n-4)} J(u)^{n/(n-4)} \frac{\nabla K(a_i)}{\lambda_i} (1 + o(1)) \right. \\ &\quad \left. + \frac{c_2}{2} \frac{\alpha_i}{\lambda_i^{n-3}} \frac{\partial H(a_i, a_i)}{\partial a_i} \right. \\ &\quad \left. + c_2 \sum_{j \neq i} \alpha_j \left( \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{1}{(\lambda_i \lambda_j)^{(n-4)/2}} \frac{1}{\lambda_i} \frac{\partial H(a_i, a_j)}{\partial a_i} \right) \right. \\ &\quad \left. \times \left( 1 - J(u)^{n/(n-4)} \sum_{k=i,j} \alpha_k^{8/(n-4)} K(a_k) \right) \right] \\ &\quad + O\left( \frac{1}{\lambda_i^2} + \sum_{j \neq i} \lambda_j |a_i - a_j| \varepsilon_{ij}^{(n-1)/(n-4)} \right) \\ &\quad + o\left( \sum_k \frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k d_k)^{n-3}} + \sum_{k \neq j} \varepsilon_{kj}^{(n-3)/(n-4)} \right). \end{aligned}$$

**Proof.** As in the proof of Proposition 3.3, we get (3.7) but with  $\lambda_i \partial P \delta_i / \partial \lambda_i$  changed by  $\lambda_i^{-1} \partial P \delta_i / \partial a_i$ .

Now, using Proposition 2.1, we observe (see [3] and [29]):

$$\left( P \delta, \frac{1}{\lambda} \frac{\partial P \delta}{\partial a} \right)_2 = -\frac{c_2}{2\lambda^{n-3}} \frac{\partial H}{\partial a}(a, a) + O\left( \frac{1}{(\lambda d)^{n-2}} \right), \tag{3.15}$$

$$\int K P \delta^{(n+4)/(n-4)} \frac{1}{\lambda} \frac{\partial P \delta}{\partial a} = -K(a) \frac{c_2}{\lambda^{n-3}} \frac{\partial H}{\partial a}(a, a) + c_4 \frac{\nabla K(a)}{\lambda} (1 + o(1)) \\ + O\left(\frac{1}{\lambda^2} + \frac{1}{(\lambda d)^{n-2}}\right),$$

where  $c_4$  is a positive constant.

We also observe, for  $i \neq j$ ,

$$\int_{\mathbb{R}^n} \delta_i^{(n+4)/(n-4)} \frac{1}{\lambda_j} \frac{\partial \delta_j}{\partial a_j} = c_2 \frac{1}{\lambda_j} \frac{\partial \varepsilon_{ij}}{\partial a_j} + O(\lambda_i |a_i - a_j| \varepsilon_{ij}^{(n-1)/(n-4)}), \quad (3.16)$$

$$\left( P \delta_j, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right)_2 = c_2 \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{c_2}{(\lambda_i \lambda_j)^{(n-4)/2}} \frac{1}{\lambda_i} \frac{\partial H}{\partial a_i}(a_i, a_j) \\ + O\left(\sum_{k=i,j} \frac{1}{(\lambda_k d_k)^{n-2}} + \varepsilon_{ij}^{(n-1)/(n-4)} \lambda_j |a_i - a_j|\right), \quad (3.17)$$

$$\int K P \delta_j^{(n+4)/(n-4)} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \\ = K(a_j) \left( P \delta_j, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right)_2 + O\left(\varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} \left(\frac{1}{\lambda_j} + \frac{1}{(\lambda_j d_j)^4}\right)\right) \\ + \begin{cases} O(\varepsilon_{ij}^{n/(n-4)} \log \varepsilon_{ij}^{-1} + \log(\lambda_j d_j)/(\lambda_j d_j)^n) & \text{if } n \geq 8, \\ O(\varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} / (\lambda_j d_j)^{n-4}) & \text{if } n < 8, \end{cases} \quad (3.18)$$

$$\int K P \delta_j \frac{1}{\lambda_i} \frac{\partial (P \delta_i)^{(n+4)/(n-4)}}{\partial a_i} \\ = K(a_i) \left( P \delta_j, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \right)_2 + O\left(\varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} \left(\frac{1}{\lambda_i} + \frac{1}{(\lambda_i d_i)^4}\right)\right) \\ + \begin{cases} O(\varepsilon_{ij}^{n/(n-4)} \log \varepsilon_{ij}^{-1} + \log(\lambda_i d_i)/(\lambda_i d_i)^n) & \text{if } n \geq 8, \\ O(\varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{(n-4)/n} / (\lambda_i d_i)^{n-4}) & \text{if } n < 8. \end{cases} \quad (3.19)$$

Using (3.15)–(3.19), the proposition follows.  $\square$

#### 4. Proof of Theorem 1.1 and its corollary

First, we prove the following technical result which will be useful to prove the positivity of the solution that we will find.

**Proposition 4.1.** *There exists a positive constant  $\varepsilon_0$  such that, if  $u \in H^2(\Omega)$  is a solution of the following equation,*

$$\Delta^2 u = K|u|^{8/(n-4)}u \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega,$$

and satisfying  $|u^-|_{L^{2n/(n-4)}} < \varepsilon_0$ , then  $u$  has to be positive.

**Proof.** First, we observe that  $K(u^-)^{(n+4)/(n-4)} \in L^{2n/(n+4)}$ , where  $u^- = \max(0, -u)$ .

Now, let us introduce  $w$  satisfying:

$$\Delta^2 w = -K(u^-)^{(n+4)/(n-4)} \text{ in } \Omega, \quad w = \Delta w = 0 \text{ on } \partial\Omega. \tag{4.1}$$

Using a regularity argument, we derive that  $w \in H^2 \cap H_0^1(\Omega)$ . Furthermore, the maximum principle implies that  $w \leq 0$ . Now, multiplying Eq. (4.1) by  $w$  and integrating on  $\Omega$ , we derive that

$$\|w\|_2^2 = \int_{\Omega} \Delta^2 w \cdot w = - \int_{\Omega} K(u^-)^{(n+4)/(n-4)} w \leq c \|w\|_2 |u^-|_{L^{2n/(n-4)}}^{(n+4)/(n-4)}. \tag{4.2}$$

Thus, either  $\|w\|_2 = 0$  and it follows that  $u^- = 0$  or  $\|w\|_2 \neq 0$  and therefore

$$\|w\|_2 \leq c |u^-|_{L^{2n/(n-4)}}^{(n+4)/(n-4)}. \tag{4.3}$$

On the other hand, we have:

$$\int_{\Omega} \Delta^2 w \cdot u = \int_{\Omega} K(u^-)^{2n/(n-4)} \geq c |u^-|_{L^{2n/(n-4)}}^{2n/(n-4)}. \tag{4.4}$$

Furthermore we obtain:

$$\begin{aligned} \int_{\Omega} \Delta^2 w \cdot u &= \int_{\Omega} w \cdot \Delta^2 u = \int_{\Omega} K|u|^{8/(n-4)}uw \\ &= - \int_{u \leq 0} K(u^-)^{(n+4)/(n-4)}w + \int_{u \geq 0} K(u^+)^{(n+4)/(n-4)}w \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\leq \int_{u \leq 0} -K(u^-)^{(n+4)/(n-4)}w = \int_{\Omega} -K(u^-)^{(n+4)/(n-4)}w \\ &= \int_{\Omega} \Delta^2 w \cdot w = \|w\|_2^2. \end{aligned} \tag{4.6}$$

Thus,

$$|u^-|_{L^{2n/(n-4)}}^{2n/(n-4)} \leq c \|w\|_2^2 \leq c |u^-|_{L^{2n/(n-4)}}^{2(n+4)/(n-4)}. \tag{4.7}$$

Then, for  $|u^-|_{L^{2n/(n-4)}}$  small enough, we derive a contradiction and therefore the case  $\|w\|_2 \neq 0$  cannot occur, so  $\|w\|_2$  has to be equal to zero and therefore  $u^- = 0$ . Thus the result follows.  $\square$

Now, we provide the characterization of the critical points at infinity of  $J$  in the case where we have only one mass. We recall that the critical points at infinity are the orbits of the gradient flow of  $J$  which remain in  $V(p, \varepsilon(s))$ , where  $\varepsilon(s)$  is some function such that  $\varepsilon(s)$  tends to zero when  $s$  tends to  $+\infty$  (see [3]).

**Proposition 4.2.** *Let  $n \geq 7$  and assume that  $(A_0)$  holds. Then there exists a pseudogradient  $Y_1$  such that the following holds:*

(1) *there exists a constant  $c > 0$  independent of  $u = \alpha\delta_{(a,\lambda)} \in V(1, \varepsilon)$  such that*

$$(-\partial J(u), Y_1)_2 \geq c \left( \frac{1}{\lambda^2} + \frac{|\nabla K(a)|}{\lambda} + \frac{1}{(\lambda d)^{n-3}} \right);$$

(2)  $\left( -\partial J(u + \bar{v}), Y_1 + \frac{\partial \bar{v}}{\partial(\alpha, a, \lambda)}(Y_1) \right)_2 \geq c \left( \frac{1}{\lambda^2} + \frac{|\nabla K(a)|}{\lambda} + \frac{1}{(\lambda d)^{n-3}} \right);$

(3)  $Y_1$  is bounded and the only case where  $\lambda$  increases along  $Y_1$  is when  $a$  is close to a critical point  $y$  of  $K$  with  $-\Delta K(y) > 0$ . Furthermore the distance to the boundary only increases if it is small enough.

**Proof.** Using  $(A_0)$  and the fact that the boundary of  $\Omega$  is a compact set, then there exist two positive constants  $c$  and  $d_0$  such that for each  $x$  satisfying  $d_x \leq d_0$  we have  $\nabla K(x) \cdot \nu_x < -c$  where  $\nu_x$  is the outward normal to  $\Omega_{d_x} = \{z \in \Omega \mid d_z = d(z, \partial\Omega) > d_x\}$ . The construction will depend on  $a$  and  $\lambda$ . We distinguish three cases:

1st case: If  $a$  is near the boundary, that is  $d_a \leq d_0$ , we define:

$$W_1 = -\frac{1}{\lambda} \frac{\partial P\delta_{(a,\lambda)}}{\partial a} \nu_a.$$

2nd case: If  $d_a \geq d_0$  and  $|\nabla K(a)| \geq C_2/\lambda$  where  $C_2$  is a large positive constant. In this case, we define:

$$W_2 = \frac{1}{\lambda} \frac{\partial P\delta_{(a,\lambda)}}{\partial a} \frac{\nabla K(a)}{|\nabla K(a)|}.$$

3rd case: If  $|\nabla K(a)| \leq 2C_2/\lambda$ , thus  $a$  is near a critical point  $y$  of  $K$ . Then we define:

$$W_3 = (\text{sign}(-\Delta K(y))) \lambda \frac{\partial P\delta_{(a,\lambda)}}{\partial \lambda}.$$

In all cases, using Propositions 3.3 and 3.4, we derive that

$$(-\partial J(u), W_i)_2 \geq c \left( \frac{1}{\lambda^2} + \frac{1}{(\lambda d)^{n-3}} + \frac{|\nabla K(a)|}{\lambda} \right).$$



The pseudogradient  $Y_1$  will be a convex combination of  $W_1$ ,  $W_2$  and  $W_3$ . Thus the proof of claim (1) is completed. The proof of claim (2) follows from the estimate of  $\bar{v}$  as in [4] and [7]. The proof of claim (3) follows from the construction of the vector field  $Y_1$ .  $\square$

**Proposition 4.3.** *Assume that  $J$  does not have any critical points in  $\Sigma^+$  and assume that  $(A_0)$  and  $(A_2)$  hold. Then the only critical points at infinity of  $J$  in  $V(1, \varepsilon)$ , for  $\varepsilon$  small enough, correspond to  $P\delta_{(y, +\infty)}$  where  $y$  is a critical point of  $K$  with  $-\Delta K(y) > 0$  if  $n \geq 7$  and with  $-\Delta K(y)/(60K(y)) + H(y, y) > 0$  if  $n = 6$ . Moreover, such a critical point at infinity has a Morse index equal to  $n - \text{index}(K, y)$ .*

**Proof.** First, we recall that the 6-dimension case of such a proposition has already been proved in [11], so we need to prove our result for  $n \geq 7$ .

Now, from Proposition 4.2, we know that the only region where  $\lambda$  increases along the pseudogradient  $Y_1$ , defined in Proposition 4.2, is the region where  $a$  is near a critical point  $y$  of  $K$  with  $-\Delta K(y) > 0$ . Arguing as in [4] and [7], we can easily derive from Proposition 4.2, the following normal form:

If  $a$  is near a critical point  $y$  of  $K$  with  $-\Delta K(y) > 0$ , we can find a change of variables  $(a, \lambda) \rightarrow (\bar{a}, \bar{\lambda})$  such that

$$J(P\delta_{(a,\lambda)} + \bar{v}) = \Psi(\bar{a}, \bar{\lambda}) := \frac{S_n^{4/n}}{K(\bar{a})^{(n-4)/n}} \left( 1 - \frac{(c - \eta)}{\bar{\lambda}^2} \frac{\Delta K(y)}{K(y)^{n/4}} \right), \tag{4.8}$$

where  $c$  is a constant which depends only on  $n$  and  $\eta$  is a small positive constant.

This yields a split of variables  $a$  and  $\lambda$ , thus it follows that if  $a = y$ , only  $\lambda$  can move. In order to decrease the functional  $J$ , we have to increase  $\lambda$ , thus we find a critical point at infinity only in this case and our result follows.  $\square$

Now, we are ready to prove Theorem 1.1 and its corollary.

**Proof of Theorem 1.1.** Arguing by contradiction, we suppose that  $J$  has no critical points in  $\Sigma^+$ . It follows from Propositions 3.1 and 4.3, that under the assumptions of Theorem 1.1, the critical points at infinity of  $J$  under the level  $c_1 = (S_n)^{4/n} (K(y_l))^{(4-n)/n} + \varepsilon$ , for  $\varepsilon$  small enough, are in one-to-one correspondence with the critical points of  $K$   $y_0, y_1, \dots, y_l$ . The unstable manifold at infinity of such critical points at infinity,  $W_u(y_0)_\infty, \dots, W_u(y_l)_\infty$  can be described, using (4.8), as the product of  $W_s(y_0), \dots, W_s(y_l)$  (for a pseudogradient of  $K$ ) by  $[A, +\infty[$  domain of the variable  $\lambda$ , for some positive number  $A$  large enough.

Let  $\eta$  be a small positive constant and let:

$$V_\eta(\Sigma^+) = \{u \in \Sigma \mid J(u)^{(2n-4)/(n-4)} e^{2J(u)} |u|_{L^{2n/(n-4)}}^{-8/(n-4)} < \eta\}. \tag{4.9}$$

Since  $J$  has no critical points in  $\Sigma^+$ , it follows that  $J_{c_1} = \{u \in V_\eta(\Sigma^+) \mid J(u) \leq c_1\}$  retracts by deformation on  $X_\infty = \bigcup_{0 \leq j \leq l} W_u(y_j)_\infty$  (see Sections 7 and 8 of [6]) which can be parametrized as we said before by  $X \times [A, +\infty[$ .

On the other hand, we have  $X_\infty$  contractible in  $J_{c_2+\varepsilon}$ , where  $c_2 = (S_n)^{4/n} \bar{c}^{(4-n)/n}$ . Indeed, from  $(A_4)$ , it follows that there exists a contraction  $h : [0, 1] \times X \rightarrow K^{\bar{c}}$ ,  $h$  continuous, such that for any  $a \in X$ ,  $h(0, a) = a$  and  $h(1, a) = a_0 \in X$ . Such a contraction gives rise to the following contraction  $\tilde{h} : [0, 1] \times X_\infty \rightarrow V_\eta(\Sigma^+)$  defined by:

$$[0, 1] \times X \times [A, +\infty[ \ni (t, a, \lambda) \mapsto P\delta_{(h(t,a), \lambda)} + \bar{v} \in V_\eta(\Sigma^+).$$

In fact,  $\tilde{h}$  is continuous and it satisfies  $\tilde{h}(0, a, \lambda) = P\delta_{(a, \lambda)} + \bar{v} \in X_\infty$  and  $\tilde{h}(1, a, \lambda) = P\delta_{(a_0, \lambda)} + \bar{v}$ .

Now, using Proposition 3.1, we deduce that

$$J(P\delta_{(h(t,a), \lambda)} + \bar{v}) = (S_n)^{4/n} (K(h(t, a)))^{(4-n)/n} (1 + O(A^{-2})),$$

where  $K(h(t, a)) \geq \bar{c}$  by construction.

Therefore such a contraction is performed under  $c_2 + \varepsilon$ , for  $A$  large enough, so  $X_\infty$  is contractible in  $J_{c_2+\varepsilon}$ .

In addition, choosing  $c_0$  small enough, we see that there is no critical point at infinity for  $J$  between the levels  $c_2 + \varepsilon$  and  $c_1$ , thus  $J_{c_2+\varepsilon}$  retracts by deformation on  $J_{c_1}$ , which retracts by deformation on  $X_\infty$ , therefore  $X_\infty$  is contractible leading to the contractibility of  $X$ , which is in contradiction with assumption  $(A_3)$ . Hence  $J$  has a critical point in  $V_\eta(\Sigma^+)$ . Using Proposition 4.1, we derive that such a critical point is positive. Therefore our theorem follows.  $\square$

Now, we give the proof of Corollary 1.2.

**Proof of Corollary 1.2.** Arguing by contradiction, we may assume that the Morse index of the solution provided by Theorem 1.1 is  $\leq m - 1$ .

Perturbing, if necessary  $J$ , we may assume that all the critical points of  $J$  are non-degenerate and have their Morse index  $\leq m - 1$ . Such critical points do not change the homological group in dimension  $m$  of level sets of  $J$ .

Since  $X_\infty$  defines a homological class in dimension  $m$  which is nontrivial in  $J_{c_1}$ , but trivial in  $J_{c_2+\varepsilon}$ , our result follows.

### 5. Proof of Theorem 1.3

Arguing by contradiction, we suppose that  $J$  has no critical points in  $V_\eta(\Sigma^+)$  defined by (4.9). We denote by  $z_1, \dots, z_r$  the critical points of  $K$  among of  $y_i$  ( $1 \leq i \leq l$ ), where

$$-\Delta K(z_j) \leq 0 \quad (1 \leq j \leq r).$$

The idea of the proof of Theorem 1.3 is to perturb the function  $K$  in the  $C^1$  sense in some neighborhoods of  $z_1, \dots, z_r$  such that the new function  $\tilde{K}$  has the same critical points with the same Morse indices but satisfying  $-\Delta \tilde{K}(z_j) > 0$  for  $1 \leq j \leq r$ . Notice that the new  $\tilde{X}$

corresponding to  $\tilde{K}$ , defined in assumption  $(A_3)$ , is also not contractible and its homology group in dimension  $m$  is nontrivial.

Under the level  $2^{4/n} S_n^{4/n} (K(y_0))^{(4-n)/n}$ , the associated functional  $\tilde{J}$  is close to the functional  $J$  in the  $C^1$  sense. Under the level  $c_2 + \varepsilon$ , where  $c_2$  is defined in the proof of Theorem 1.1, the functional  $\tilde{J}$  may have other critical points, however a careful choice of  $\tilde{K}$  ensures that all these critical points have Morse indices less than  $m - 2$  (see Proposition 5.1 below), and so they do not change the homology in dimension  $m$ , therefore the arguments used in the proof of Theorem 1.1 lead to a contradiction. It follows that Theorem 1.3 will be a corollary of the following proposition:

**Proposition 5.1.** *Assume that  $J$  has no critical points in  $V_\eta(\Sigma^+)$ . We can choose  $\tilde{K}$  close to  $K$  in the  $C^1$  sense such that  $\tilde{K}$  has the same critical points with the same Morse indices and such that*

- (i)  $-\Delta \tilde{K}(z_j) > 0$  for  $1 \leq j \leq r$ ,
- (ii)  $-\Delta \tilde{K}(y) > 0$  for  $y \in \{y_0, \dots, y_l\} \setminus \{z_1, \dots, z_r\}$ ,
- (iii)  $-\Delta \tilde{K}(y_i) < 0$  for  $l + 1 \leq i \leq s$ ,
- (iv) if  $\tilde{J}$  has critical points under the level  $c_2 + \varepsilon$ , then their Morse indices are less than  $m - 2$ , where  $m$  is defined in assumption  $(A_3)$ ,
- (v) the new  $\tilde{X}$  corresponding to  $\tilde{K}$ , defined in assumption  $(A_3)$ , is also not contractible and its homology group in dimension  $m$  is nontrivial.

Next, we are going to prove Proposition 5.1. For this purpose, we need the following lemmas:

**Lemma 5.2.** *Let  $z_0$  be a point of  $\Omega$  such that  $d(z_0, \partial\Omega) \geq c_0 > 0$  and let  $\pi$  be the orthogonal projection (with respect to the scalar inner  $(u, v)_2 = \int_\Omega \Delta u \Delta v$ ) onto*

$$E^\perp = \text{Vect}(P\delta_{(z_0, \lambda)}, \lambda^{-1} \partial P\delta_{(z_0, \lambda)} / \partial z, \lambda \partial P\delta_{(z_0, \lambda)} / \partial \lambda).$$

Then, we have the following estimates:

$$(i) \|J'(P\delta_{(z_0, \lambda)})\| = O\left(\frac{1}{\lambda}\right); \quad (ii) \left\| \frac{\partial \pi}{\partial z} \right\| = O(\lambda); \quad (iii) \left\| \frac{\partial^2 \pi}{\partial z^2} \right\| = O(\lambda^2).$$

**Proof.** The proof of claim (i) is easy, so we will omit it. Now, we prove claim (ii). Let  $\varphi \in \{P\delta_{(z_0, \lambda)}, \lambda^{-1} \partial P\delta_{(z_0, \lambda)} / \partial z, \lambda \partial P\delta_{(z_0, \lambda)} / \partial \lambda\}$ . We then have  $\pi \varphi = \varphi$ , therefore,

$$\frac{\partial \pi}{\partial z}(\varphi) = \frac{\partial \varphi}{\partial z} - \pi \frac{\partial \varphi}{\partial z},$$

thus  $\left\| \frac{\partial \pi}{\partial z}(\varphi) \right\| = O(\lambda)$ .

Now, for  $v \in E$ , we have  $\pi v = 0$ , thus

$$\frac{\partial \pi}{\partial z} v = -\pi \frac{\partial v}{\partial z} = \sum_{i=1}^3 a_i \varphi_i,$$

where  $\varphi_1 = P\delta_{(z_0, \lambda)}$ ,  $\varphi_2 = \lambda^{-1} \partial P\delta_{(z_0, \lambda)} / \partial z$ ,  $\varphi_3 = \lambda \partial P\delta_{(z_0, \lambda)} / \partial \lambda$ .

But, we have:

$$a_i \|\varphi_i\|^2 = \left( \frac{\partial v}{\partial z}, \varphi_i \right)_2 = - \left( v, \frac{\partial \varphi_i}{\partial z} \right)_2 = O(\lambda \|v\|).$$

Thus claim (ii) follows.

In the same way, claim (iii) follows and hence the proof of our lemma is completed.  $\square$

**Lemma 5.3.** *Let  $z_0$  be a point of  $\Omega$  close to a critical point of  $K$  such that  $d(z_0, \partial\Omega) \geq c_0 > 0$ . Let  $\bar{v} = \bar{v}(z_0, \alpha, \lambda) \in E$  defined in Proposition 3.2. Then, we have the following estimates:*

$$(i) \|\bar{v}\| = o\left(\frac{1}{\lambda}\right), \quad (ii) \left\| \frac{\partial \bar{v}}{\partial z} \right\| = o(1).$$

**Proof.** We notice that claim (i) follows from Proposition 3.2. Then, we need only to show that claim (ii) is true. We know that  $\bar{v}$  satisfies,

$$A\bar{v} = f + O(\|\bar{v}\|^{(n+4)/(n-4)}) \quad \text{and} \quad \frac{\partial A}{\partial z} \bar{v} + A \frac{\partial \bar{v}}{\partial z} = \frac{\partial f}{\partial z} + O\left(\|\bar{v}\|^{8/(n-4)} \left| \frac{\partial \bar{v}}{\partial z} \right| \right),$$

where  $A$  is the operator associated to the quadratic form  $Q$  defined on  $E$  ( $Q$  and  $f$  are defined in Proposition 3.1).

Then, we have:

$$A \left( \frac{\partial \bar{v}}{\partial z} - \pi \left( \frac{\partial \bar{v}}{\partial z} \right) \right) = \frac{\partial f}{\partial z} - \frac{\partial A}{\partial z} \bar{v} - A \pi \left( \frac{\partial \bar{v}}{\partial z} \right) + O\left(\|\bar{v}\|^{8/(n-4)} \left| \frac{\partial \bar{v}}{\partial z} \right| \right).$$

Since  $Q$  is a positive quadratic form on  $E$  (see [9]), we then derive,

$$\left\| \frac{\partial \bar{v}}{\partial z} - \pi \left( \frac{\partial \bar{v}}{\partial z} \right) \right\| \leq C \left( \left\| \frac{\partial f}{\partial z} \right\| + \left\| \frac{\partial A}{\partial z} \right\| \|\bar{v}\| + \left\| \pi \left( \frac{\partial \bar{v}}{\partial z} \right) \right\| + \|\bar{v}\|^{8/(n-4)} \left\| \frac{\partial \bar{v}}{\partial z} \right\| \right).$$

Now, we estimate each term of the right-hand side in the above estimate. First, it is easy to see  $\left\| \frac{\partial A}{\partial z} \right\| = O(\lambda)$ . Therefore, using (i), we obtain  $\left\| \frac{\partial A}{\partial z} \right\| \|\bar{v}\| = o(1)$ . Secondly, we have:

$$\begin{aligned}
 \left(\frac{\partial f}{\partial z}, v\right)_2 &= c \int K P \delta_{(z_0, \lambda)}^{8/(n-4)} \frac{\partial P \delta}{\partial z} v = c \nabla K(z_0) \int d(z_0, x) \delta^{8/(n-4)} \frac{\partial \delta}{\partial z} v \\
 &\quad + O\left(\int d^2(x, z_0) \delta^{(n+4)/(n-4)} \lambda |v|\right) \\
 &\quad + O\left(\int_{\Omega} \delta^{8/(n-4)} \varphi |v| + \int_{\Omega} \delta^{8/(n-4)} \left|\frac{\partial \varphi}{\partial z}\right| |v|\right) \\
 &\leq c \|v\| \left( |\nabla K(z_0)| + \frac{1}{\lambda} \right), \tag{5.1}
 \end{aligned}$$

where  $\varphi = \delta - P\delta$ .

Since  $z_0$  is close to a critical point of  $K$ , we derive that  $\|\frac{\partial f}{\partial z}\| = o(1)$ .

For the term  $\|\pi(\frac{\partial \bar{v}}{\partial z})\|$ , we have, since  $\bar{v} \in E$ ,

$$\begin{aligned}
 \left(\frac{\partial \bar{v}}{\partial z}, \delta_{(z_0, \lambda)}\right)_2 &= -\left(\bar{v}, \frac{\partial \delta_{(z_0, \lambda)}}{\partial z}\right)_2 = 0, \\
 \left(\frac{\partial \bar{v}}{\partial z}, \lambda \frac{\partial \delta_{(z_0, \lambda)}}{\partial \lambda}\right)_2 &= -\left(\bar{v}, \lambda \frac{\partial^2 \delta_{(z_0, \lambda)}}{\partial \lambda \partial z}\right)_2 = O(\lambda \|\bar{v}\|) = o(1).
 \end{aligned}$$

In the same way, we have:

$$\left(\frac{\partial \bar{v}}{\partial z}, \frac{1}{\lambda} \frac{\partial P \delta}{\partial z}\right)_2 = o(1).$$

Therefore  $\|\pi(\frac{\partial \bar{v}}{\partial z})\| = o(1)$ . Now, using the following inequality:

$$\left\| \frac{\partial \bar{v}}{\partial z} \right\| \leq \left\| \frac{\partial \bar{v}}{\partial z} - \pi\left(\frac{\partial \bar{v}}{\partial z}\right) \right\| + \left\| \pi\left(\frac{\partial \bar{v}}{\partial z}\right) \right\|,$$

we easily derive our claim and our lemma follows.  $\square$

We are now able to prove Proposition 5.1.

**Proof of Proposition 5.1.** We suppose that  $J$  has no critical points in  $V_\eta(\Sigma^+)$  and we perturb the function  $K$  only in some neighborhoods of  $z_1, \dots, z_r$ , therefore claims (ii) and (iii) follow from assumption  $(A'_2)$ . We observe that under the level  $c_2 + \varepsilon$  and outside  $V(1, \varepsilon_0)$ , we have  $|\partial J| > c > 0$ . If  $\tilde{K}$  is close to  $K$  in the  $C^1$ -sense, then  $\tilde{J}$  is close to  $J$  in the  $C^1$ -sense, and therefore  $|\partial \tilde{J}| > c/2$  in this region. Thus, a critical point  $u_0$  of  $\tilde{J}$  under the level  $c_2 + \varepsilon$  has to be in  $V(1, \varepsilon_0)$ . Therefore, we can write  $u_0 = P\delta_{(z_0, \lambda)} + \bar{v}$ .

Next we will prove the following claim:

**Claim.**  $z_0$  has to be near a critical point  $z_i$  of  $K$ ,  $1 \leq i \leq r$  (recall that  $z_i$ 's satisfy  $\Delta K(z_i) \geq 0$ ).

To prove our claim, we will prove in the first step that  $d_{z_0} := d(z_0, \partial\Omega) \geq c_0 > 0$ . For this fact, arguing by contradiction, we assume that  $d_{z_0} \rightarrow 0$ . Thus, we have:

$$\frac{\partial K}{\partial v}(z_0) < -c < 0 \quad \text{and} \quad \frac{\partial H}{\partial v}(z_0, z_0) \sim \frac{c}{d_{z_0}^{n-3}}. \quad (5.2)$$

(The proof of the last fact is similar to the corresponding statement for the Laplacian operator in [29].)

Using Propositions 3.2 and 3.4, we obtain:

$$0 = \left( \partial \tilde{J}(u_0), \frac{1}{\lambda} \frac{\partial P\delta}{\partial z} \right)_2 \cdot v > \frac{c}{\lambda} + \frac{c}{(\lambda d_{z_0})^{n-3}} > 0.$$

Thus, we derive a contradiction and therefore  $z_0$  has to satisfy  $d_{z_0} \geq c_0 > 0$ .

Now, also using Propositions 3.2 and 3.4, we derive that

$$0 = \left( \partial \tilde{J}(u_0), \frac{1}{\lambda} \frac{\partial P\delta}{\partial z} \right)_2 = c \frac{\nabla \tilde{K}(z_0)}{\lambda} + o\left(\frac{1}{\lambda}\right),$$

thus  $z_0$  has to be close to  $y_i$ , where  $i \in \{0, \dots, s\}$ .

We also have, by Propositions 3.2 and 3.4:

$$0 = \left( \partial \tilde{J}(u_0), \lambda \frac{\partial P\delta}{\partial \lambda} \right)_2 = c \frac{\Delta \tilde{K}(z_0)}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right). \quad (5.3)$$

In the neighborhood of  $y_i$  with  $i \in \{k \mid -\Delta K(y_k) > 0\} \cup \{l+1, \dots, s\}$ ,  $\tilde{K} \equiv K$  and therefore  $|\Delta \tilde{K}| > c > 0$  in this neighborhood. Thus (5.3) implies that  $z_0$  has to be near  $z_i$  with  $1 \leq i \leq r$ . Thus our claim is proved.

In the sequel, we assume that  $\delta = \delta_{(z_0, \lambda)}$  satisfies  $\|\delta\| = 1$ , and thus  $\Delta^2 \delta = S_n^{4/(n-4)} \times \delta^{(n+4)/(n-4)}$ . We also assume that  $|D^2 \tilde{K}| \leq c(1 + |D^2 K|)$ , where  $c$  is a fixed positive constant.

Let  $u_0 = P\delta_{(z_0, \lambda)} + \bar{v}$  be a critical point of  $\tilde{J}$ . In order to compute the Morse index of  $\tilde{J}$  at  $u_0$ , we need to compute  $\frac{\partial^2}{\partial z^2} \tilde{J}(P\delta_{(z, \lambda)} + \bar{v})|_{z=z_0}$ . We observe that

$$\begin{aligned} \frac{\partial}{\partial z} \tilde{J}(P\delta_{(z, \lambda)} + \bar{v}) &= \tilde{J}'(P\delta_{(z, \lambda)} + \bar{v}) \frac{\partial}{\partial z} (P\delta_{(z, \lambda)} + \bar{v}) \\ &= \tilde{J}'(P\delta_{(z, \lambda)} + \bar{v}) \pi \left( \frac{\partial}{\partial z} (P\delta_{(z, \lambda)} + \bar{v}) \right) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \tilde{J}(P\delta_{(z, \lambda)} + \bar{v}) &= \tilde{J}''(P\delta_{(z, \lambda)} + \bar{v}) \frac{\partial}{\partial z} (P\delta_{(z, \lambda)} + \bar{v}) \pi \left( \frac{\partial}{\partial z} (P\delta_{(z, \lambda)} + \bar{v}) \right) \\ &\quad + \tilde{J}'(P\delta_{(z, \lambda)} + \bar{v}) \frac{\partial}{\partial z} \left( \pi \left( \frac{\partial}{\partial z} (P\delta_{(z, \lambda)} + \bar{v}) \right) \right). \end{aligned} \quad (5.4)$$

For  $z = z_0$ , we have  $\tilde{J}'(P\delta_{(z,\lambda)} + \bar{v}) = 0$ . We will estimate each term of the right-hand side of (5.4). First, we have by Lemma 5.3,

$$\tilde{J}''(P\delta_{(z,\lambda)} + \bar{v}) \frac{\partial \bar{v}}{\partial z} \pi \left( \frac{\partial \bar{v}}{\partial z} \right) = o(1).$$

Secondly, we compute:

$$\begin{aligned} T &= \tilde{J}''(P\delta_{(z,\lambda)} + \bar{v}) \frac{\partial P\delta}{\partial z} \pi \frac{\partial \bar{v}}{\partial z} \\ &= c \left[ \left( \frac{\partial P\delta}{\partial z}, \pi \frac{\partial \bar{v}}{\partial z} \right) - \frac{n+4}{n-4} \tilde{J}(u_0)^{n/(n-4)} \int \tilde{K}(P\delta + \bar{v})^{8/(n-4)} \frac{\partial P\delta}{\partial z} \pi \frac{\partial \bar{v}}{\partial z} \right] \end{aligned}$$

According to Proposition 3.1, we have:

$$\tilde{J}(P\delta + \bar{v}) = \frac{S_n^{4/n}}{\tilde{K}(z)^{(n-4)/n}} + O\left(\frac{\|\bar{v}\|}{\lambda} + \frac{1}{\lambda^2}\right). \tag{5.5}$$

Thus,

$$\begin{aligned} T &= c \left[ \left( \frac{\partial P\delta}{\partial z}, \pi \frac{\partial \bar{v}}{\partial z} \right)_2 - \frac{n+4}{n-4} S_n^{4/(n-4)} \int \frac{\tilde{K}}{\tilde{K}(z)} P\delta^{8/(n-4)} \frac{\partial P\delta}{\partial z} \pi \frac{\partial \bar{v}}{\partial z} \right] \\ &\quad + O\left( \int (\delta^{(12-n)/(n-4)} |\bar{v}| + |\bar{v}|^{8/(n-4)} \chi_{P\delta \leq |\bar{v}|}) \left\| \frac{\partial P\delta}{\partial z} \right\| \left\| \pi \frac{\partial \bar{v}}{\partial z} \right\| \right) + o(1) \\ &= c \frac{n+4}{n-4} S_n^{4/(n-4)} \int \left( 1 - \frac{\tilde{K}}{\tilde{K}(z)} \right) \delta^{8/(n-4)} \frac{\partial \delta}{\partial z} \pi \left( \frac{\partial \bar{v}}{\partial z} \right) \\ &\quad + O\left( \lambda \|\bar{v}\| \left\| \frac{\partial \bar{v}}{\partial z} \right\| + \lambda \|\bar{v}\|^{(n+4)/(n-4)} \left\| \frac{\partial \bar{v}}{\partial z} \right\| \right) + o(1) \\ &= o(1). \end{aligned}$$

Thus (5.4) becomes:

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \tilde{J}(P\delta_{(z,\lambda)} + \bar{v}) &= \tilde{J}''(P\delta_{(z,\lambda)} + \bar{v}) \frac{\partial P\delta}{\partial z} \left( \frac{\partial P\delta}{\partial z} + \frac{\partial \bar{v}}{\partial z} \right) + \tilde{J}'(P\delta_{(z,\lambda)} + \bar{v}) \frac{\partial^2 P\delta}{\partial z^2} + o(1) \\ &= 2\tilde{J}(u_0) \left[ \left( \frac{\partial P\delta}{\partial z} + \frac{\partial \bar{v}}{\partial z}, \frac{\partial P\delta}{\partial z} \right)_2 + \left( P\delta + \bar{v}, \frac{\partial^2 P\delta}{\partial z^2} \right)_2 \right. \\ &\quad \left. - \tilde{J}(u_0)^{n/(n-4)} \frac{n+4}{n-4} \left( \int K(P\delta + \bar{v})^{8/(n-4)} \left( \frac{\partial P\delta}{\partial z} \right)^2 \right. \right. \\ &\quad \left. \left. + \int K(P\delta + \bar{v})^{8/(n-4)} \frac{\partial P\delta}{\partial z} \frac{\partial \bar{v}}{\partial z} \right) \right] \end{aligned}$$

$$\begin{aligned}
& - \tilde{J}(u_0)^{n/(n-4)} \int K(P\delta + \bar{v})^{(n+4)/(n-4)} \frac{\partial^2 P\delta}{\partial z^2} \Big] + o(1) \\
= & 2\tilde{J}(u_0) \left[ \left( \frac{\partial P\delta}{\partial z} + \frac{\partial \bar{v}}{\partial z}, \frac{\partial P\delta}{\partial z} \right)_2 + \left( P\delta + \bar{v}, \frac{\partial^2 P\delta}{\partial z^2} \right)_2 \right. \\
& - \frac{n+4}{n-4} \tilde{J}(u_0)^{n/(n-4)} \left( \int K P\delta^{8/(n-4)} \left( \frac{\partial P\delta}{\partial z} \right)^2 \right. \\
& + \frac{8}{n-4} \int K P\delta^{(12-n)/(n-4)} \bar{v} \left( \frac{\partial P\delta}{\partial z} \right)^2 \\
& + \int K P\delta^{8/(n-4)} \frac{\partial P\delta}{\partial z} \frac{\partial \bar{v}}{\partial z} + \frac{n-4}{n+4} \int K P\delta^{(n+4)/(n-4)} \frac{\partial^2 P\delta}{\partial z^2} \\
& \left. \left. + \int K P\delta^{8/(n-4)} \bar{v} \frac{\partial^2 P\delta}{\partial z^2} \right) \right] + o(1).
\end{aligned}$$

Using (5.5) and Proposition 2.1, we derive that

$$\begin{aligned}
\frac{\partial^2}{\partial z^2} \tilde{J}(P\delta_{(z,\lambda)} + \bar{v}) &= 2\tilde{J}(u_0) \left[ S_n^{4/(n-4)} \left( \int \frac{n+4}{n-4} \delta^{8/(n-4)} \left( \frac{\partial \delta}{\partial z} \right)^2 + \delta^{(n+4)/(n-4)} \frac{\partial^2 \delta}{\partial z^2} \right) \right. \\
& - \tilde{J}(u_0)^{n/(n-4)} \left( \frac{n+4}{n-4} \int K \delta^{8/(n-4)} \left( \frac{\partial \delta}{\partial z} \right)^2 \right. \\
& + \int K \delta^{(n+4)/(n-4)} \frac{\partial^2 \delta}{\partial z^2} \Big) \\
& + S_n^{4/(n-4)} \frac{n+4}{n-4} \left( \int \left( 1 - \frac{K}{K(z)} \right) \delta^{8/(n-4)} \frac{\partial \delta}{\partial z} \frac{\partial \bar{v}}{\partial z} \right. \\
& + \int \left( 1 - \frac{K}{K(z)} \right) \delta^{8/(n-4)} \frac{\partial^2 \delta}{\partial z^2} \bar{v} \\
& \left. \left. + \frac{8}{n-4} \int \left( 1 - \frac{K}{K(z)} \right) \delta^{(12-n)/(n-4)} \left( \frac{\partial \delta}{\partial z} \right)^2 \bar{v} \right) \right] + o(1) \\
= & 2\tilde{J}(u_0) \left[ S_n^{4/(n-4)} \frac{\partial}{\partial z} \left( \int_{R^n} \delta^{(n+4)/(n-4)} \frac{\partial \delta}{\partial z} \right) \right. \\
& \left. - \tilde{J}(u_0)^{n/(n-4)} \int_{\Omega} K \frac{\partial^2 \delta^{2n/(n-4)}}{\partial z^2} \right] + o(1).
\end{aligned}$$

Thus,

$$\frac{\partial^2}{\partial z^2} \tilde{J}(P\delta_{(z,\lambda)} + \bar{v})|_{z=z_0} = -cD^2K(z_0) + o(1),$$

where  $c$  is a positive constant.



Therefore, taking account of the  $\lambda$ -space, we derive that

$$\text{index}(\tilde{J}, u_0) \leq n - \text{index}(K, z_0) + 1 \leq m - 2.$$

Then claims (i) and (iv) of Proposition 5.1 follow.

On the other hand, according to assumption  $(A'_2)$  we have:

$$n - m + 3 \leq \text{index}(K, z_j) = \text{index}(\tilde{K}, z_j) \quad \text{for } 1 \leq j \leq r.$$

Thus, for any pseudogradient of  $\tilde{K}$ , the dimension of the stable manifold of  $z_j$  is less than  $m - 3$ . Note that our perturbation changes the pseudogradient  $Z$  to  $\tilde{Z}$ , but only in some neighborhoods of  $z_1, \dots, z_r$ . Therefore the stable manifolds of  $y_i$  for  $i \notin \{1, \dots, r\}$ , remain unchanged. Since the dimension of  $X$  is greater than  $m$  and its homology group in dimension  $m$  is nontrivial, we derive that the homology group of  $\tilde{X}$  in dimension  $m$  is also nontrivial. This completes the proof of Proposition 5.1.  $\square$

### 6. Proof of Theorems 1.5 and 1.6

In this section we assume that assumptions  $(A_0)$ ,  $(A_5)$  and  $(A_6)$  hold and we are going to prove Theorems 1.5 and 1.6. First, we start by proving the following main results:

**Proposition 6.1.** *Let  $n \geq 7$ . There exists a pseudogradient  $Y_2$  such that the following holds:*

(1) *There exists a constant  $c > 0$  independent of  $u = \sum_{i=1}^2 \alpha_i P\delta_{(a_i, \lambda_i)} \in V(2, \varepsilon)$  such that*

$$(-\partial J(u), Y_2)_2 \geq c \left( \varepsilon_{12}^{(n-3)/(n-4)} + \sum \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{(\lambda_i d_i)^{n-3}} \right);$$

(2) 
$$\left( -\partial J(u + \bar{v}), Y_2 + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(Y_2) \right)_2$$

$$\geq c \left( \varepsilon_{12}^{(n-3)/(n-4)} + \sum \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{(\lambda_i d_i)^{n-3}} \right);$$

(3)  *$Y_2$  is bounded and the only case where the maximum of the  $\lambda_i$ 's increases along  $Y_2$  is when the points  $a_i$ 's are close to two different critical points  $y_j$  and  $y_r$  of  $K$  with  $-\Delta K(y_l) > 0$  for  $l = j, r$ . Furthermore the least distance to the boundary only increases if it is small enough.*

**Proof.** We divide the set  $V(2, \varepsilon)$  into three sets  $A_1 \cup A_2 \cup A_3$  where, for  $u = \sum \alpha_i P\delta_{(a_i, \lambda_i)} \in V(2, \varepsilon)$ ,  $A_1 = \{u \mid d_1 \geq d_0 \text{ and } d_2 \geq d_0\}$ ,  $A_2 = \{u \mid d_1 \leq d_0 \text{ and } d_2 \geq 2d_0\}$ ,  $A_3 = \{u \mid d_1 \leq 2d_0 \text{ and } d_2 \leq 2d_0\}$ . We will build a vector field on each set and then,  $Y_2$  will be a convex combination of those vector fields.

- *1st set:* For  $u \in A_1$ . We can assume without loss of generality that  $\lambda_1 \leq \lambda_2$ . We introduce the following set  $T = \{i \mid |\nabla K(a_i)| \geq C_2/\lambda_i\}$  where  $C_2$  is a large constant. The set  $A_1$  will be divided into four subsets:

*1st subset:* The set of  $u$  such that  $\varepsilon_{12} \geq C_1/\lambda_2^2$  and  $(10\lambda_1 \geq \lambda_2$  or  $|\nabla K(a_1)| \geq C_2/\lambda_1)$ , where  $C_1$  is a large constant. In this case, we define  $W_1$  as

$$W_1 = -M\lambda_2 \frac{\partial P \delta_2}{\partial \lambda_2} + \sum_{i \in T} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|},$$

where  $M$  is a large constant. Using Propositions 3.3 and 3.4, we derive that

$$\begin{aligned} (-\partial J(u), W_1)_2 &\geq M \left( c\varepsilon_{12} + O\left(\frac{1}{\lambda_2^2}\right) \right) + \sum_{i \in T} \left( \frac{|\nabla K(a_i)|}{\lambda_i} + O\left(\frac{1}{\lambda_i^2} + \varepsilon_{12}\right) \right) \\ &\geq c \left( \varepsilon_{12} + \sum \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right). \end{aligned} \quad (6.1)$$

*2nd subset:* The set of  $u$  such that  $\varepsilon_{12} \geq C_1/\lambda_2^2$ ,  $10\lambda_1 \leq \lambda_2$  and  $|\nabla K(a_1)| \leq C_2/\lambda_1$ . In this case, the point  $a_1$  is close to a critical point  $y$  of  $K$ . We define  $W_2$  as

$$W_2 = W_1 + \sqrt{M}\lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} (\text{sign}(-\Delta K(y))).$$

Using Propositions 3.3 and 3.4, we obtain:

$$\begin{aligned} (-\partial J(u), W_2)_2 &\geq M \left( c\varepsilon_{12} + O\left(\frac{1}{\lambda_2^2}\right) \right) + \sqrt{M} \left( \frac{c}{\lambda_1^2} + O(\varepsilon_{12}) \right) \\ &\quad + \sum_{i \in T} \left( \frac{|\nabla K(a_i)|}{\lambda_i} + O\left(\frac{1}{\lambda_i^2} + \varepsilon_{12}\right) \right) \\ &\geq c \left( \varepsilon_{12} + \sum \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right). \end{aligned} \quad (6.2)$$

*3rd subset:* The set of  $u$  such that  $\varepsilon_{12} \leq C_1/\lambda_2^2$  and  $(|\nabla K(a_1)| \geq C_2/\lambda_1$  or  $|\nabla K(a_2)| \geq C_2/\lambda_2)$ . In this case, the set  $T$  is not empty, thus we define:

$$W'_3 = \sum_{i \in T} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|}.$$

Using Proposition 3.4, we find:

$$(-\partial J(u), W'_3)_2 \geq c \sum_{i \in T} \left( \frac{|\nabla K(a_i)|}{\lambda_i} + O\left(\frac{1}{\lambda_i^2} + \varepsilon_{12}\right) \right). \quad (6.3)$$

If we assume that  $(|\nabla K(a_1)| \geq C_2/\lambda_1$  or  $10\lambda_1 \geq \lambda_2)$  and we choose  $C_1 \ll C_2$ , (6.3) implies the desired estimate. In the other situation, i.e.,  $(|\nabla K(a_1)| \leq C_2/\lambda_1$  and  $10\lambda_1 \leq \lambda_2)$ , the point  $a_1$  is close to a critical point  $y$  of  $K$ . As in the second case, we define  $W_3''$  as

$$W_3'' = \frac{1}{\lambda_2} \frac{\partial P \delta_2}{\partial a_2} \frac{\nabla K(a_2)}{|\nabla K(a_2)|} + \lambda_1 \frac{\partial P \delta_1}{\partial \lambda_1} (\text{sign}(-\Delta K(y))).$$

Using Propositions 3.3 and 3.4, we derive that

$$\begin{aligned} (-\partial J(u), W_3'')_2 &\geq c \left( \frac{|\nabla K(a_2)|}{\lambda_2} + O\left(\frac{1}{\lambda_2^2} + \varepsilon_{12}\right) \right) + c \left( \frac{1}{\lambda_1^2} + O(\varepsilon_{12}) \right) \\ &\geq c \left( \varepsilon_{12} + \sum \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right). \end{aligned} \tag{6.4}$$

$W_3$  will be a convex combination of  $W_3'$  and  $W_3''$ .

*4th subset:* The set of  $u$  such that  $\varepsilon_{12} \leq C_1/\lambda_2^2$  and  $|\nabla K(a_i)| \leq C_2/\lambda_i$  for  $i = 1, 2$ . In this case, the concentration points are near two critical points  $y_i$  and  $y_j$  of  $K$ . Two cases may occur: either  $y_i = y_j$  or  $y_i \neq y_j$ .

– If  $y_i = y_j = y$ . Since  $y$  is a nondegenerate critical point, we derive that  $\lambda_k |a_k - y| \leq c$  for  $k = 1, 2$  and therefore  $\lambda_1 |a_1 - a_2| \leq c$ . Thus we obtain  $\varepsilon_{12} \geq c(\lambda_1/\lambda_2)^{(n-4)/2}$  and therefore  $\varepsilon_{12} \leq C_1/\lambda_2^2 = o(1/\lambda_1^2)$ . In this case we define  $W_4' = \lambda_1 (\partial P \delta_1 / \partial \lambda_1) (\text{sign}(-\Delta K(y)))$ . Using Proposition 3.3, we derive that

$$(-\partial J(u), W_4')_2 \geq \frac{c}{\lambda_1^2} + O(\varepsilon_{12}) \geq c \left( \varepsilon_{12} + \sum \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right). \tag{6.5}$$

– If  $y_i \neq y_j$ . In this case we have  $\varepsilon_{12} = o(1/\lambda_k^2)$  for  $k = 1, 2$ . The vector field  $W_4''$  will depend on the sign of  $-\Delta K(y_k)$ ,  $k = i, j$ . If  $-\Delta K(y_i) < 0$  ( $y_i$  is near  $a_1$ ), we decrease  $\lambda_1$ . If  $-\Delta K(y_i) > 0$  and  $-\Delta K(y_j) < 0$ , we decrease  $\lambda_2$  in the case where  $10\lambda_1 \geq \lambda_2$  and we increase  $\lambda_1$  in the other case. If  $-\Delta K(y_k) > 0$  for  $k = i, j$ , we increase both  $\lambda_k$ 's. Thus we obtain:

$$(-\partial J(u), W_4'')_2 \geq c \left( \varepsilon_{12} + \sum \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right). \tag{6.6}$$

The vector field  $W_4$  will be a convex combination of  $W_4'$  and  $W_4''$ .

- *2nd set:* For  $u \in A_2$ , we have  $|a_1 - a_2| \geq d_0$ . Therefore  $\varepsilon_{12} = o(1/\lambda_1)$  and  $H(a_2, \cdot) \leq c$ . Let us define  $W_5 = (1/\lambda_1) (\partial P \delta_1 / \partial a_1) (-v_1)$ . Using Proposition 3.4, we find:

$$(-\partial J(u), W_5)_2 \geq \frac{c}{\lambda_1} + O(\varepsilon_{12}) + \frac{c}{(\lambda_1 d_1)^{n-3}} \geq \frac{c}{\lambda_1} + \frac{c}{(\lambda_1 d_1)^{n-3}}. \tag{6.7}$$

If  $\lambda_1 \leq 10\lambda_2$ , then, in the lower bound of (6.7), we can make appear  $1/\lambda_2$  and all the terms needed in (1). In the other case, i.e.,  $\lambda_1 \geq 10\lambda_2$ , we define  $W_6$  as  $W_6 = W_5 + Y_1(P\delta_2)$  and we obtain the desired estimate in this case also.

- *3rd set:* For  $u \in A_3$ , i.e.,  $d_i \leq 2d_0$  for  $i = 1, 2$ . We have three cases.

*1st case:* If there exists  $i \in \{1, 2\}$  (we denote by  $j$  the other index) such that  $M_1 d_i \leq d_j$ , where  $M_1$  is a large constant. In this case we define:

$$W_7 = \sum \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} (-v_i). \quad (6.8)$$

Using Proposition 3.4, we derive that

$$\begin{aligned} (-\partial J(u), W_7) &\geq c \sum_k \left( \frac{1}{\lambda_k} + \frac{1}{(\lambda_k d_k)^{n-3}} \right) + o(\varepsilon_{12}^{(n-3)/(n-4)}) \\ &+ O \left( \sum_k \frac{1}{\lambda_k} \left| \frac{\partial \varepsilon_{12}}{\partial a_k} \right| + \frac{1}{(\lambda_1 \lambda_2)^{(n-4)/2}} \frac{1}{\lambda_k} \left| \frac{\partial H(a_1, a_2)}{\partial a_k} \right| \right. \\ &\left. + \lambda_k |a_1 - a_2| \varepsilon_{12}^{(n-1)/(n-4)} \right). \end{aligned} \quad (6.9)$$

Since  $M_1 d_i \leq d_j$ , then we have  $|a_1 - a_2| \geq d_j/2 \geq M_1 d_i/2$ . Thus we obtain:

$$\frac{1}{\lambda_k} \left| \frac{\partial \varepsilon_{12}}{\partial a_k} \right| + \frac{1}{(\lambda_1 \lambda_2)^{(n-4)/2}} \frac{1}{\lambda_k} \left| \frac{\partial H(a_1, a_2)}{\partial a_k} \right| + \varepsilon_{12}^{(n-3)/(n-4)} = o \left( \sum_{r=1}^2 \frac{1}{(\lambda_r d_r)^{n-3}} \right). \quad (6.10)$$

The same estimate holds for  $\lambda_k |a_1 - a_2| \varepsilon_{12}^{(n-1)/(n-3)}$ . Thus claim (1) follows in this case.

*2nd case:* If  $d_2/M_1 \leq d_1 \leq M_1 d_2$  and  $\lambda_2/M_2 \leq \lambda_1 \leq M_2 \lambda_2$ , where  $M_2$  is chosen large enough. In this case we define:

$$W_8 = \frac{1}{\lambda_2} \sum_i \frac{\partial P \delta_i}{\partial a_i} (-\alpha_i v_i). \quad (6.11)$$

Using Proposition 3.4 we derive that

$$\begin{aligned} (-\partial J(u), W_8)_2 &\geq \frac{c}{\lambda_2} \left( 1 + \sum_k \frac{1}{d_k (\lambda_k d_k)^{n-4}} + c \alpha_1 \alpha_2 \frac{\partial \varepsilon_{12}}{\partial a_1} (v_1 - v_2) \right. \\ &\left. + \frac{c \alpha_1 \alpha_2}{(\lambda_1 \lambda_2)^{(n-4)/2}} \sum_k \frac{\partial H(a_1, a_2)}{\partial a_k} v_k \right) + o(\varepsilon_{12}^{(n-3)/(n-4)}). \end{aligned} \quad (6.12)$$

Observe that  $|\partial \varepsilon_{12} / \partial a_1| |v_1 - v_2| = O(\varepsilon_{12}) = o(1)$  and using the fact that  $\partial H(a_1, a_2) / \partial v_i \geq o((d_1 d_2)^{(3-n)/2})$ . It remains to appear  $\varepsilon_{12}$  in the lower bound. For this, if there

exists  $i$  such that  $\varepsilon_{12} \leq m/(\lambda_i d_i)^{4-n}$ , where  $m$  is a fixed large positive constant, then we can make to appear  $\varepsilon_{12}$  in (6.12). In the other case, we decrease both  $\lambda_i$ 's and we define  $W_9 = -\sum \lambda_i \partial P \delta_i / \partial \lambda_i$ . Using Proposition 3.3, we obtain:

$$(-\partial J(u), W_9)_2 \geq c\varepsilon_{12} + \sum_i O\left(\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{n-4}}\right) \geq c\varepsilon_{12} + \sum_i O\left(\frac{1}{\lambda_i^2}\right). \tag{6.13}$$

Thus, in this case, we define the vector field as  $W_8 + W_9$ . Using (6.12) and (6.13), we obtain the desired estimate.

3rd case: If  $d_2/M_1 \leq d_1 \leq M_1 d_2$  and there exists  $i$  (we denote  $j$  the other index) such that  $\lambda_i \geq M_2 \lambda_j$ . In this case we increase  $\lambda_j$ , we decrease  $\lambda_i$  and we move the points along the inward normal vector. Then we define  $W_{10} = -2m\lambda_i \partial P \delta_i / \partial \lambda_i + m\lambda_j \partial P \delta_j / \partial \lambda_j + W_7$ , where  $m$  is a large constant. Using Propositions 3.3 and 3.4, we derive that

$$\begin{aligned} (-\partial J(u), W_{10}) &\geq m\left(c\varepsilon_{12} + \frac{c}{(\lambda_j d_j)^{n-4}} + O\left(\frac{1}{(\lambda_i d_i)^{n-4}}\right)\right) \\ &\quad + c\left(\sum \frac{1}{\lambda_k} + \frac{1}{(\lambda_k d_k)^{n-3}} + O(\varepsilon_{12})\right). \end{aligned} \tag{6.14}$$

Observe that, in this case, we have  $\lambda_j d_j = o(\lambda_i d_i)$  if we choose  $M_1/M_2$  so small. Thus the desired estimate follows.

The proof of claim (1) is then completed. Claim (3) follows immediately from the construction of  $Y_2$ . Claim (2) follows from the estimate of  $\bar{v}$  as in [3] and [7].  $\square$

Now, arguing as in the proof of Proposition 4.3, we easily derive the following result:

**Corollary 6.2.** *Let  $n \geq 7$ . The only critical points at infinity in  $V(2, \varepsilon)$  correspond to  $P\delta_{(y_i, \infty)} + P\delta_{(y_j, \infty)}$ , where  $y_i$  and  $y_j$  are two different critical points of  $K$  satisfying  $-\Delta K(y_k) > 0$  for  $k = i, j$ . Such critical point has a Morse index equal to  $2n - \sum_{r=i, j} \text{index}(K, y_r) + 1$ .*

**Proposition 6.3.** *Let  $n \geq 7$  and assume that (P) has no solution. Then the following claims hold:*

- (i) *If  $X = \overline{\bigcup_{y \in B} W_s(y)}$ , where  $B = \{y \in \Omega \mid \nabla K(y) = 0, -\Delta K(y) > 0\}$ , then  $f_\lambda(C_{y_0}(X))$  retracts by deformation on  $\bigcup_{y_i \in X - \{y_0\}} W_u(y_0, y_i)_\infty \cup X_\infty$  where  $X_\infty = (\bigcup_{y_i \in X} W_u(y_i)_\infty)$ .*
- (ii) *If  $X = \overline{W_s(y_{i_0})}$ , where  $y_{i_0}$  satisfies:*

$$K(y_{i_0}) = \max\{K(y_i) \mid \text{index}(K, y_i) = n - k, -\Delta K(y_i) > 0\}$$

*and if assumption (A7) holds, then  $f_\lambda(C_{y_0}(X))$  retracts by deformation on  $\bigcup_{y_i \in X - \{y_0\}} W_u(y_0, y_i)_\infty \cup X_\infty \cup \sigma_1$ , where  $\sigma_1 \subset \bigcup_{y_i / \text{index}(K, y_i) \geq n-k} W_u(y_i)_\infty$ .*

**Proof.** Let us start by proving claim (i). Since  $J$  does not have any critical point, the manifold  $f_\lambda(C_{y_0}(X))$  retracts by deformation on the union of the unstable manifolds of the critical points at infinity dominated by  $f_\lambda(C_{y_0}(X))$  (see [6,25]). Proposition 4.3 and Corollary 6.2 allow us to characterize such critical points. Observe that we can modify the construction of the pseudogradient defined in Propositions 4.2 and 6.1 such that, when we move the point  $x$  it remains in  $X$ , i.e., we can use  $Z_K$  instead of  $\nabla K/|\nabla K|$  where  $Z_K$  is the pseudogradient for  $K$  which we use to build the manifold  $X$ .

For an initial condition  $u = (\alpha/K(y_0)^{(n-4)/8})P\delta_{(y_0,\lambda)} + ((1-\alpha)/K(x)^{(n-4)/8})P\delta_{(x,\lambda)}$  in  $f_\lambda(C_{y_0}(X))$ , the action of the pseudogradient (see Proposition 6.1) is essentially on  $\alpha$ . The action of bringing  $\alpha$  to zero or to 1 depends on whether  $\alpha < 1/2$  (in this case,  $u$  goes to  $X_\infty$ ) or  $\alpha > 1/2$  (in this case,  $u$  goes to  $\overline{W}_u((y_0)_\infty)$ ). On the other hand, we have another action on  $x \in X$ , when  $\alpha = 1 - \alpha = 1/2$ . Since only  $x$  can move, then  $y_0$  remains one of the concentration points of  $u$  and either  $x$  goes to  $W_s(y_j)$  where  $y_j$  is a critical point of  $K$  in  $X - \{y_0\}$  or  $x$  goes to a neighborhood of  $y_0$ . In the last case the flow has to exit from  $V(2, \varepsilon)$  (see the construction of  $Y_2$  in Proposition 6.1). The level of  $J$  in this situation is close to  $(2S_n)^{4/n}/K(y_0)^{(n-4)/n}$  and therefore it cannot dominate any critical point at infinity of two masses (since  $K(y_0) = \max K$ ). Thus the flow has to enter in  $V(1, \varepsilon)$  and it will dominate  $(y_i)_\infty$  for  $y_i \in X$ . Then  $u$  goes to

$$\left( \bigcup_{y_i \in X - \{y_0\}} W_u((y_0, y_i)_\infty) \right) \cup \left( \bigcup_{y_i \in X} W_u((y_i)_\infty) \right).$$

Then claim (i) follows. Now, using assumption (A<sub>7</sub>) and the same argument as in the proof of claim (i), we easily derive claim (ii). Thus our proposition follows.  $\square$

We now prove our theorems.

**Proof of Theorem 1.5.** Arguing by contradiction, we assume that  $(P)$  has no solution. Using Proposition 6.3 and the fact that  $\mu(y_{i_0}) = 0$ , we derive that  $f_\lambda(C_{y_0}(X))$  retracts by deformation on  $X_\infty \cup D$  where  $D \subset \sigma$  is a stratified set of dimension at most  $k$  (in the topological sense, that is,  $D \in \Sigma_j$ , the group of chains of dimension  $j$  with  $j \leq k$ ) and where  $\sigma = \bigcup_{y_i \in X - \{(y_{i_0}, y_0)\}} W_u((y_0, y_i)_\infty) \cup \bigcup_{y_i / \text{index}(K, y_i) \geq n-k} W_u((y_i)_\infty)$  is a manifold in dimension at most  $k$ .

As  $f_\lambda(C_{y_0}(X))$  is a contractible set, we then have  $H_*(X_\infty \cup D) = 0$ , for all  $* \in \mathbb{N}^*$ . Using the exact homology sequence of  $(X_\infty \cup D, X_\infty)$ , we derive  $H_k(X_\infty) = H_{k+1}(X_\infty \cup D, X_\infty) = 0$ . This yields a contradiction since  $X_\infty \equiv X \times [A, +\infty)$ , where  $A$  is a large positive constant. Therefore our theorem follows.  $\square$

**Proof of Theorem 1.6.** Assume that  $(P)$  has no solution. By the above arguments, if  $\mu(y_i) = 0$  for each  $y_i \in B_k$ , then  $f_\lambda(C_{y_0}(X))$  retracts by deformation on  $X_\infty \cup D$  where  $D \subset \sigma$  is a stratified set and where  $\sigma = \bigcup_{y_i \in X - (B_k \cup \{y_0\})} W_u((y_0, y_i)_\infty)$  is a manifold in dimension at most  $k$ .

As in the proof of Theorem 1.5, we derive that  $H_*(X_\infty \cup D) = 0$  for each  $*$ . Using the exact homology sequence of  $(X_\infty \cup D, X_\infty)$  we obtain  $H_k(X_\infty) = H_{k+1}(X_\infty \cup D, X_\infty) = 0$ , this yields a contradiction and therefore our result follows.  $\square$

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