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On two-point boundary value problems in multi-ion electrodiffusion

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Abstract

The solvability is established of certain two-point boundary value problems for nonlinear equations that arise in multi-ion electrodiffusion. Topological methods are adduced to prove the existence of solutions under appropriate conditions on the physical parameters.

1. Introduction

The theory of electrodiffusion provides a macroscopic description of the migration of charged particles through material barriers. Its origin resides in the liquid-junction theory of Nernst and Planck [1] and it has subsequently applied in the modelling of biological membranes [2–7]. The theory is also of importance in electrochemistry [8]. Schlögl [9] demonstrated that it is convenient to partition the ions present in the electrodiffusion model into classes which have the same charge q_i. The distinct species that pertain to a given charge are indexed by j. In steady, one-dimensional régimes, the model may then be reduced to the form [10]

$$dn_i/dx = v_i p n_i - c_i, \quad i = 1, 2, \ldots, m,$$

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$$dp/dx = \sum_{i=1}^{m} \nu_i n_i, \tag{1.1}$$

where, if N_{ij} denotes the number density of the ion labeled ij then

$$n_j = N_0^{-1} \sum_{j=1}^{k_i} N_{ij} \tag{1.2}$$

with N_0 a unit of ion density; p denotes the electric field E appropriately scaled. The total number of ion species is $\sum_{i=1}^{m} k_i$. The quantity

$$\nu_i = q_i/q_0, \tag{1.3}$$

where q_0 is a unit of charge is the signed valency of the ion. In general terms, the complexity of the nonlinear coupled system (1.1) depends on the number of distinct charges present and for m charges it leads to an mth order nonlinear differential equation in p [10]. The case $m = 2$ produces

$$p'' - (\nu_1 + \nu_2)pp' + \frac{1}{2}\nu_1\nu_2 p^3 - \nu_1\nu_2 cxp + \nu_1 c_1 + \nu_2 c_2 = 0. \tag{1.4}$$

The case of one positive and one negative ion was considered by Bruner [11] while that for ions of equal and opposite charges so that $\nu_1 + \nu_2 = 0$ by Bass [12] and by Cohen and Cooley [13]. In the latter instance, a Painlevé II reduction is obtained. Two point boundary value problems with Dirichlet and periodic side conditions for Painlevé II have recently been investigated in [14]. The case $m = 3$ yields [10]

$$\begin{aligned} &pp''' - p'p'' - (\nu_1 + \nu_2 + \nu_3)p^2 p'' + (\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)p^3 p' \\ &\quad - (\nu_1 c_1 + \nu_2 c_2 + \nu_3 c_3)p' - \frac{1}{2}\nu_1\nu_2\nu_3 p^5 + \nu_1\nu_2\nu_3(c_1 + c_2 + c_3)xp^3 \\ &\quad - \big[(\nu_2 + \nu_3)\nu_1 c_1 + (\nu_1 + \nu_3)\nu_2 c_2 + (\nu_1 + \nu_2)\nu_3 c_3\big]p^2 = 0. \end{aligned} \tag{1.5}$$

Here, attention is restricted to the cases $m = 2$ and $m = 3$. The existence of solutions to the two-point boundary problem for (1.4) with Dirichlet and periodic side conditions is investigated in the general case $\nu_1 + \nu_2 \neq 0$. For $m = 3$, conditions are set down for the existence of a solution of a two-point boundary value problem for (1.5).

2. The two-charge case

Let us consider the boundary value problem consisting of

$$p'' - (\nu_1 + \nu_2)pp' + \frac{1}{2}\nu_1\nu_2 p^3 - \nu_1\nu_2 cxp + \nu_1 c_1 + \nu_2 c_2 = 0, \quad x \in (0, T), \tag{2.1}$$

subject, in turn, to either Dirichlet or periodic boundary conditions, namely

$$p(0) = p_0, \quad p(T) = p_T, \quad \mathbb{D}, \tag{2.2}$$

and

$$p(0) = p(T), \quad p'(0) = p'(T), \quad \mathbb{P}, \tag{2.3}$$

where $c = c_1 + c_2$ while p_0, p_T are known constants. Note that if $\nu_1 + \nu_2 = 0$ then (2.1) may be reduced to Painlevé II. The existence of solutions of these boundary value problems in this case has been established by Mariani et al. [14]. Here, we deal with the generic case when $\nu_1 + \nu_2 \neq 0$. To establish the existence of solutions to the boundary value problems under discussion we shall apply the method of upper and lower solutions. This method which relies on maximum principles was developed by, notably, Scorza-Dragoni [15], Nagumo [16] and Jackson [17].

Let us recall that (α, β) is deemed to be an ordered couple of a lower and an upper solution for the problem if $\alpha, \beta \in C^2([0,T])$ with $\alpha(x) \leqslant \beta(x)$ for $x \in [0,T]$, and if $\forall x \in [0,T]$ we have

$$\alpha'' - (\nu_1 + \nu_2)\alpha\alpha' + \frac{1}{2}\nu_1\nu_2\alpha^3 - \nu_1\nu_2 c x\alpha + \nu_1 c_1 + \nu_2 c_2 \geqslant 0,$$
$$\beta'' - (\nu_1 + \nu_2)\beta\beta' + \frac{1}{2}\nu_1\nu_2\beta^3 - \nu_1\nu_2 c x\beta + \nu_1 c_1 + \nu_2 c_2 \leqslant 0,$$

and

$$\alpha(0) \leqslant p_0 \leqslant \beta(0), \quad \alpha(T) \leqslant p_T \leqslant \beta(T), \quad (\mathbb{D}),$$
$$\alpha(0) = \alpha(T), \quad \alpha'(0) = \alpha'(T), \quad \beta(0) = \beta(T), \quad \beta'(0) = \beta'(T), \quad (\mathbb{P}).$$

Then we have the following result:

Theorem 1. *Let (α, β) be an ordered couple consisting of a lower and an upper solution for side conditions $\mathbb{D}$ or $\mathbb{P}$. Then the respective boundary value problems admit at least one solution p with $\alpha \leqslant p \leqslant \beta$.*

Proof. Let us consider the function $g : [0,T] \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x,y,z) = -(\nu_1 + \nu_2)\delta(x,y)z + \frac{1}{2}\nu_1\nu_2\big[\delta(x,y)\big]^3 - \nu_1\nu_2 c x\delta(x,y) + \nu_1 c_1 + \nu_2 c_2,$$

where

$$\delta(x,y) = \begin{cases} y & \text{if } \alpha(x) \leqslant y \leqslant \beta(x), \\ \alpha(x) & \text{if } \alpha(x) > y, \\ \beta(x) & \text{if } y > \beta(x). \end{cases}$$

The problem of finding a solution p with $\alpha \leqslant p \leqslant \beta$ is equivalent to solving

$$p'' + g(x,p,p') = 0, \quad \alpha \leqslant p \leqslant \beta,$$

under the respective boundary conditions $\mathbb{D}$ or $\mathbb{P}$. Set

$$R = |\nu_1 + \nu_2|M + |\nu_1\nu_2|\left(|c|T + \frac{3}{2}M^2\right), \tag{2.4}$$

where

$$M = \max\big\{\|\alpha\|_{C^1}, \|\beta\|_{C^1}\big\}$$

and choose $\lambda \geqslant R$. By standard results, for $\bar{p} \in C([0,T])$ the linear problem

$$p'' + g\big(x, \bar{p}(x), p'\big) - \lambda p = -\lambda\bar{p}(x), \quad x \in (0,T)$$

(under the respective boundary conditions), admits a unique solution $p \in C^2([0,T])$, and the mapping $\mathcal{K}: C([0,T]) \to C([0,T])$ defined by $\mathcal{K}(\bar{p}) = p$ is compact. Moreover, for $\alpha \leqslant \bar{p} \leqslant \beta$ it is seen that

$$p'' + g(x, \bar{p}, p') - \lambda p + R\bar{p} = (R-\lambda)\bar{p} \geqslant (R-\lambda)\beta + \beta'' + g(x, \beta, \beta').$$

Hence, if $u = p - \beta$ we obtain

$$u'' - (\nu_1 + \nu_2)\bar{p}u' - \lambda u \geqslant g(x, \beta, \beta') + R\beta - \left[g(x, \bar{p}, \beta') + R\bar{p}\right].$$

From (2.4), for fixed x, the function $\phi(V) := g(x, V, \beta'(x)) + RV$ is nondecreasing when $\alpha(x) \leqslant V \leqslant \beta(x)$. It follows that

$$u'' - (\nu_1 + \nu_2)\bar{p}(x)u' - \lambda u \geqslant 0,$$

and by the maximum principle we conclude that $u \leqslant 0$, i.e., $p \leqslant \beta$. In the same way we obtain that $p \geqslant \alpha$ and the result follows from the Schauder fixed point theorem. □

Remark. (i) If $\nu_1 c_1 + \nu_2 c_2 \geqslant 0$ then $\alpha \equiv 0$ is a lower solution for $\mathbb{P}$, and also for $\mathbb{D}$ if $p_0, p_T \geqslant 0$.

(ii) If $\nu_1 c_1 + \nu_2 c_2 \leqslant 0$ then $\beta \equiv 0$ is an upper solution for $\mathbb{P}$, and also for $\mathbb{D}$ if $p_0, p_T \leqslant 0$.

As a simple consequence of the preceding theorem we have the following

Corollary 1. *If $\nu_1 \nu_2 < 0$ then the boundary value problems consisting of the nonlinear equation* (2.1) *supplemented by the Dirichlet conditions $\mathbb{D}$ or periodic conditions $\mathbb{P}$ are solvable.*

Proof. It suffices to consider $\alpha \leqslant \beta$ to be constants such that

$$\frac{1}{2}\nu_1\nu_2\alpha^3 - \nu_1\nu_2 cx\alpha + \nu_1 c_1 + \nu_2 c_2 \geqslant 0 \geqslant \frac{1}{2}\nu_1\nu_2\beta^3 - \nu_1\nu_2 cx\beta + \nu_1 c_1 + \nu_2 c_2$$

(with $\alpha \leqslant p_0$, $p_T \leqslant \beta$ for the Dirichlet case). □

Comments. (i) Corollary 1 holds, in particular, when $\nu_1 + \nu_2 = 0$. Thus, the present result may be considered as an extension of the existence results in [14]. Note that, in this case, if $c \leqslant 0$, the respective solutions are unique; however, uniqueness does not necessarily hold if we replace $(0, T)$ by an arbitrary bounded interval, although Corollary 1 is still valid. This may be illustrated by the boundary value problem

$$Y'' = 2Y^3 + xY + 1,$$
$$Y(-2\pi) = Y(-\pi), \qquad Y'(-2\pi) = Y'(-\pi)$$

which admits at least two solutions: indeed, it suffices to take

$$\alpha_1 \equiv K_1 \ll 0, \qquad \beta_1 \equiv 0,$$
$$\alpha_2 \equiv \sqrt{\frac{\pi}{6}}, \qquad \beta_2 \equiv K_2 \gg 0.$$

(ii) For $\nu_1\nu_2 < 0$, if $c \leqslant 0$ then the following "minimum principle" holds for solutions of (2.1):

$$\text{if } \nu_1 c_1 + \nu_2 c_2 \geqslant 0 \text{ and } p \text{ is periodic or } p_0, p_T \geqslant 0 \text{ then } p \geqslant 0.$$

Indeed, if $p(x_0) < 0$ then we may assume that x_0 is a minimum so that $p'(x_0) = 0$ and

$$p''(x_0) = -\frac{1}{2}\nu_1\nu_2 p^3(x_0) + \nu_1\nu_2 c x_0 p(x_0) - (\nu_1 c_1 + \nu_2 c_2) < 0,$$

so we obtain a contradiction. In the same way, it is readily proved that

$$\text{if } \nu_1 c_1 + \nu_2 c_2 \leqslant 0 \text{ and } p \text{ is periodic or } p_0, p_T \leqslant 0 \text{ then } p \leqslant 0.$$

In particular, as $c = c_1 + c_2$, if $0 < \nu_1 = -\nu_2$ then the previous minimum and maximum principles read as

$$\text{if } |c_1| + c_2 \leqslant 0 \text{ and } p \text{ is periodic or } p_0, p_T \geqslant 0 \text{ then } p \geqslant 0,$$
$$\text{if } c_1 + |c_2| \leqslant 0 \text{ and } p \text{ is periodic or } p_0, p_T \leqslant 0 \text{ then } p \leqslant 0.$$

Note that for Painlevé II,

$$Y'' = 2Y^3 + xY + C,$$

we have

$$\text{if } C \leqslant 0 \text{ and } Y \text{ is periodic or } Y(0), Y(T) \geqslant 0 \text{ then } Y \geqslant 0,$$
$$\text{if } C \geqslant 0 \text{ and } Y \text{ is periodic or } Y(0), Y(T) \leqslant 0 \text{ then } Y \leqslant 0.$$

For the case $\nu_1\nu_2 > 0$, the method ensures the existence of solutions of the problem with Dirichlet boundary conditions $\mathbb{D}$ when T is sufficiently small.

Corollary 2. *Assume that $\nu_1\nu_2 > 0$. Then there exists a positive constant T^* such that the boundary value problem with Dirichlet conditions $\mathbb{D}$ admits at least one solution for any $T < T^*$.*

Proof. If $\nu_1, \nu_2 > 0$, consider $\beta(x) = rx + s$. Then β is an upper solution if and only if

$$s \geqslant p_0, \qquad rT + s \geqslant p_T$$

and

$$-(\nu_1 + \nu_2)(rx + s)r + \frac{1}{2}(rx + s)^3 - \nu_1\nu_2 cx(rx + s) + \nu_1 c_1 + \nu_2 c_2 \leqslant 0. \tag{2.5}$$

For $x = 0$ the l.h.s of (2.5) reads as

$$-(\nu_1 + \nu_2)sr + \frac{1}{2}s^3 + \nu_1 c_1 + \nu_2 c_2.$$

Thus, if $s > 0$, p_0, p_T and $r > 0$ is large enough then (2.5) holds on $[0, T]$ for small values of T. On the other hand, we may consider $\alpha(x) = \bar{r}x + \bar{s}$, taking $\bar{s} < 0$, p_0, p_T and $\bar{r} > 0$ large enough such that

$$-(\nu_1 + \nu_2)\bar{s}\bar{r} + \frac{1}{2}\bar{s}^3 + \nu_1 c_1 + \nu_2 c_2 > 0.$$

Moreover, if T is small we also have that $\bar{r}T + \bar{s} \leqslant p_T$, and the result follows. The proof is analogous if $\nu_1, \nu_2 < 0$. $\square$

3. Convergent iterative sequences ($m = 2$)

In this section we introduce iterative sequences that converge to solutions of the Dirichlet and periodic boundary value problems in the case $m = 2$ under the conditions of Theorem 1. We shall need the following:

Lemma 1. *Assume that (α, β) is an ordered couple consisting of a lower and an upper solution for the respective boundary conditions $\mathbb{D}$ and $\mathbb{P}$ and $\lambda > K^2$, where*

$$K = \frac{|\nu_1 + \nu_2|}{2} \max\bigl\{\|\alpha\|_C, \|\beta\|_C\bigr\}.$$

Then there exists a constant M such that for any $\bar{p} \in C([0, T])$ with $\alpha \leqslant \bar{p} \leqslant \beta$, if

$$p'' + g\bigl(x, \bar{p}(x), p'\bigr) - \lambda p = -\lambda \bar{p}(x), \quad x \in (0, T),$$

with p satisfying periodic or Dirichlet conditions, then

$$\|p\|_{C^1} \leqslant M.$$

Proof. Consider the operator given by

$$Sp = p'' + g(\cdot, \bar{p}, p') - \lambda p$$

and $P = p - \varphi$, where $\varphi(x) = (p_T - p_0)x/T + p_0$ for the Dirichlet conditions, and $\varphi \equiv 0$ for the periodic conditions. Then

$$\|Sp - S\varphi\|_{L^2} \|P\|_{L^2} \geqslant -\int_0^T (Sp - S\varphi)P$$

$$= \|P'\|_{L^2}^2 + \lambda \|P\|_{L^2}^2 - \int_0^T \bigl[g(\cdot, \bar{p}, p) - g(\cdot, \bar{p}, \varphi)\bigr].P.$$

Hence

$$\bigl(\|P'\|_{L^2} - K\|P\|_{L^2}\bigr)^2 + (\lambda - K^2)\|P\|_{L^2}^2 \leqslant \|\lambda \bar{p} + S\varphi\|_{L^2} \|P\|_{L^2}$$

and it follows that

$$\|p - \varphi\|_{H^1} \leqslant M_0$$

for some constant M_0 independent of $\bar{p}$. Further, as $Sp = -\lambda \bar{p}$ we obtain that $\|p''\|_{L^2} \leqslant M_1$ for a constant M_1 independent of $\bar{p}$, and the proof follows from the imbedding $H^2(0, T) \hookrightarrow C^1([0, T])$. $\square$

Corollary 3. *Let K and M be as in Lemma* 1*, with $M \geqslant \|\alpha\|_{C^1}$, $M \geqslant \|\beta\|_{C^1}$, and let also $\lambda > K^2, R$, where R is given by* (2.4)*. Define the sequences $p_n^{\pm}$ given by*

$$p_0^- \equiv \alpha, \qquad p_0^+ \equiv \beta,$$

and

$$\left(p_{n+1}^{\pm}\right)'' + g\left(x, p_n^{\pm}, \left(p_{n+1}^{\pm}\right)'\right) - \lambda p_{n+1}^{\pm} = -\lambda p_n^{\pm}$$

under the respective boundary conditions $\mathbb{D}$ and $\mathbb{P}$. Then $\{p_n^-\}$ ($\{p_n^+\}$) is nondecreasing (*nonincreasing*) *and converges to a solution of the problem.*

Proof. From the arguments of Theorem 1, we have that $\alpha \leqslant p_1^+ \leqslant \beta$. Assume as an inductive hypothesis that $\alpha \leqslant p_n^+ \leqslant p_{n-1}^+ \leqslant \beta$, then $p_{n+1}^+ \geqslant \alpha$. Moreover,

$$\begin{aligned}
&\left(p_{n+1}^+ - p_n^+\right)'' + g\left(x, p_n^+, \left(p_{n+1}^+\right)'\right) - g\left(x, p_n^+, \left(p_n^+\right)'\right) - \lambda\left(p_{n+1}^+ - p_n^+\right) \\
&\quad = -\left[(\lambda - R)\left(p_n^+ - p_{n-1}^+\right) + g\left(x, p_n^+, \left(p_n^+\right)'\right) + Rp_n^+ \right.\\
&\qquad \left. - \left[g\left(x, p_{n-1}^+, \left(p_n^+\right)'\right) + Rp_{n-1}^+\right]\right).
\end{aligned}$$

Thus,

$$\left(p_{n+1}^+ - p_n^+\right)'' - (\nu_1 + \nu_2)p_n^+\left(p_{n+1}^+ - p_n^+\right)'' - \lambda\left(p_{n+1}^+ - p_n^+\right) \geqslant 0$$

and it follows that $p_{n+1}^+ \leqslant p_n^+$. As p_n^+ is nonincreasing and bounded, it converges pointwise to a function p^+. Furthermore, from the proof of Lemma 1 it is seen that $\{p_n^+\}$ is bounded for the H^2-norm, and from the compactness of the imbedding $H^2(0,T) \hookrightarrow C^1([0,T])$ it follows that $p_n^+ \to p^+$ in $C^1([0,T])$. From the definition of $\{p_n^+\}$ it is immediate that p^+ is a solution of the problem. The proof is analogous for $\{p_n^-\}$. □

4. The three-charge case

Let us consider the two-point boundary value problem

$$\begin{cases}
pp''' - p'p'' - (\nu_1 + \nu_2 + \nu_3)p^2p'' + (\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)p^3p' \\
\quad - (\nu_1c_1 + \nu_2c_2 + \nu_3c_3)p' - \frac{1}{2}\nu_1\nu_2\nu_3p^5 + \nu_1\nu_2\nu_3(c_1 + c_2 + c_3)xp^3 \\
\quad - [(\nu_2 + \nu_3)\nu_1c_1 + (\nu_1 + \nu_3)\nu_2c_2 + (\nu_1 + \nu_2)\nu_3c_3]p^2 = 0, \\
p(0) = p_0, \quad p(T) = p_T, \quad p''(0) = r_0.
\end{cases} \tag{4.1}$$

Let us assume first that $p_0 \neq 0$. It proves convenient to set $u = p''/p$ so that the above nonlinear problem becomes

$$\begin{aligned}
&p^2u' - (\nu_1 + \nu_2 + \nu_3)p^3u + (\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)p^3p' - (\nu_1c_1 + \nu_2c_2 + \nu_3c_3)p' \\
&\quad - \frac{1}{2}\nu_1\nu_2\nu_3p^5 + \nu_1\nu_2\nu_3(c_1 + c_2 + c_3)xp^3 \\
&\quad = \left[(\nu_2 + \nu_3)\nu_1c_1 + (\nu_1 + \nu_3)\nu_2c_2 + (\nu_1 + \nu_2)\nu_3c_3\right]p^2, \\
&p'' = pu, \\
&p(0) = p_0, \quad p(T) = p_T, \quad u(0) = \frac{r_0}{p_0}.
\end{aligned} \tag{4.2}$$

Here, we proceed with the constraint

$$\nu_1 c_1 + \nu_2 c_2 + \nu_3 c_3 = 0 \tag{4.3}$$

on the parameters. Let $\varphi(x) = (p_T - p_0)x/T + p_0$ and define a fixed point operator $\mathcal{K}: \varphi + H_0^1(0,T) \to \varphi + H_0^1(0,T)$ as follows: for each $p \in \varphi + H_0^1(0,T)$ let u be the unique solution of the problem

$$\begin{aligned} &u' - (\nu_1+\nu_2+\nu_3)pu \\ &\quad = -(\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3)pp' + \frac{1}{2}\nu_1\nu_2\nu_3 p^3 \\ &\qquad - \nu_1\nu_2\nu_3(c_1+c_2+c_3)xp + \big[(\nu_2+\nu_3)\nu_1 c_1 + (\nu_1+\nu_3)\nu_2 c_2 + (\nu_1+\nu_2)\nu_3 c_3\big] \\ &\quad := \phi_p(x) \end{aligned} \tag{4.4}$$

with $u(0) = r_0/p_0$. Next, define $\mathcal{K}p = \tilde{p}$ as the unique solution of the problem

$$\tilde{p}'' = pu, \quad \tilde{p}(0) = p_0, \quad \tilde{p}(T) = p_T.$$

We obtain

$$u(x) = \frac{r_0}{p_0} e^{(\nu_1+\nu_2+\nu_3)\int_0^x p} + \int_0^x \phi_p(s) e^{(\nu_1+\nu_2+\nu_3)\int_s^x p}\, ds. \tag{4.5}$$

Since

$$\begin{aligned} &\int_0^x p(s)p'(s) e^{(\nu_1+\nu_2+\nu_3)\int_s^x p}\, ds \\ &\quad = \frac{1}{2}\left[p^2(x) - p_0^2 e^{(\nu_1+\nu_2+\nu_3)\int_0^x p} + (\nu_1+\nu_2+\nu_3)\int_0^x p^3(s) e^{(\nu_1+\nu_2+\nu_3)\int_s^x p}\, ds\right], \end{aligned}$$

we deduce that

$$\big|u(x)\big| \leqslant \mathcal{P}\big(\|p\|_{L^2}, \|p\|_C\big) e^{|\nu_1+\nu_2+\nu_3| T^{1/2}\|p\|_{L^2}},$$

where $\mathcal{P}$ is the polynomial given by $\mathcal{P}(A,B) = C_0 + C_1 A + C_2 B^2 + C_3 A^2 B$ with

$$\begin{aligned} C_0 &= \left|\frac{r_0}{p_0}\right| + T\big|(\nu_2+\nu_3)\nu_1 c_1 + (\nu_1+\nu_3)\nu_2 c_2 + (\nu_1+\nu_2)\nu_3 c_3\big|, \\ C_1 &= \frac{T}{\sqrt{2}}\big|\nu_1\nu_2\nu_3(c_1+c_2+c_3)\big|, \\ C_2 &= \frac{1}{2}\big|\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3\big|, \\ C_3 &= \frac{1}{2}\big(|\nu_1\nu_2\nu_3| + |\nu_1+\nu_2+\nu_3|.|\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3|\big). \end{aligned}$$

Moreover, from the equation $\tilde{p}'' = pu$ and the boundary conditions we have

$$\left\|\tilde{p}' - \frac{p_T - p_0}{T}\right\|_{L^2}^2 \leqslant \|pu\|_{L^2}\|\tilde{p} - \varphi\|_{L^2}.$$

By the Poincaré inequality we conclude that

$$\left\|\tilde{p}' - \frac{p_T - p_0}{T}\right\|_{L^2} \leqslant \frac{T}{\pi}\|pu\|_{L^2} \leqslant \frac{T}{\pi}\|p\|_{L^2}\mathcal{P}\big(\|p\|_{L^2}, \|p\|_C\big)e^{|\nu_1+\nu_2+\nu_3|T^{1/2}\|p\|_{L^2}}.$$

On the other hand, it is readily shown that

$$\|p\|_{L^2} \leqslant \left(\frac{T}{3}\big(p_0^2 + p_0 p_T + p_T^2\big)\right)^{1/2} + \frac{T}{\pi}\left\|p' - \frac{p_T - p_0}{T}\right\|_{L^2}$$

and

$$\|p\|_C \leqslant \max\big\{|p_0|, |p_T|\big\} + T^{1/2}\left\|p' - \frac{p_T - p_0}{T}\right\|_{L^2}.$$

Remark. Replacing r_0/p_0 by 0, it is clear that the previous computations also hold for the case $p_0 = r_0 = 0$.

Thus we have

Theorem 2. *Assume that* (4.3) *holds and let* $A, B, \theta : \mathbb{R}^+ \to \mathbb{R}^+$ *be defined by*

$$A(R) = \frac{T}{\pi}R + \left(\frac{T}{3}\big(p_0^2 + p_0 p_T + p_T^2\big)\right)^{1/2},$$
$$B(R) = \max\big\{|p_0|, |p_T|\big\} + T^{1/2}R,$$
$$\theta(R) = \frac{T}{\pi}A(R)\mathcal{P}\big(A(R), B(R)\big)e^{|\nu_1+\nu_2+\nu_3|T^{1/2}A(R)},$$

where $\mathcal{P}$ *is the polynomial introduced as above. Assume there exists* $R > 0$ *such that* $\theta(R) \leqslant R$. *Then the boundary value problem* (4.1) *admits at least one solution.*

Proof. From the previous computations, if

$$\left\|p' - \frac{p_T - p_0}{T}\right\|_{L^2} \leqslant R$$

then

$$\left\|\tilde{p}' - \frac{p_T - p_0}{T}\right\|_{L^2} \leqslant R,$$

and the result follows from the Schauder theorem. □

In particular, we deduce

Corollary 4. *Assume that* (4.3) *holds and*

$$\big|(\nu_2 + \nu_3)\nu_1 c_1 + (\nu_1 + \nu_3)\nu_2 c_2 + (\nu_1 + \nu_2)\nu_3 c_3\big| < \frac{\pi^2}{T^3}.$$

Then there exist $\delta_0, \delta_T, \delta > 0$ *such that the boundary value problem* (4.1) *admits a solution for any* p_0, p_T *with* $|p_0| < \delta_0$, $|p_T| < \delta_T$ *and* $|r_0| \leqslant \delta|p_0|$.

Proof. With the notation of Theorem 2, if δ is small enough we have that $|C_0| < \pi^2/T^2$. Moreover, it is seen that

$$A(R), B(R) \to 0 \quad \text{as } R, \delta_0, \delta_T \to 0$$

and hence, if δ_0, δ_T, δ and R are small enough,

$$\mathcal{P}\big(A(R), B(R)\big)e^{|\nu_1+\nu_2+\nu_3|T^{1/2}A(R)} < \frac{\pi^2}{(1+\varepsilon)T^2}$$

for some positive constant ε. Letting $\delta_0, \delta_T \to 0$ we may assume that

$$\left(\frac{T}{3}\big(p_0^2 + p_0 p_T + p_T^2\big)\right)^{1/2} < \frac{T}{\pi}\varepsilon R$$

and the result follows from Theorem 2. □

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