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Yosida–Moreau Regularization of Sweeping Processes with Unbounded Variation

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Let $t \mapsto C(t)$ be a Hausdorff-continuous multifunction with closed convex values in a Hilbert space H such that $C(t)$ has nonempty interior for all t. We show that the Yosida–Moreau regularizations of the sweeping process with moving set $C(t)$, i.e., the solutions of

$$\frac{du_\lambda}{dt}(t) + \frac{1}{\lambda}\left[u_\lambda(t) - \operatorname{proj}(u_\lambda(t), C(t))\right] = 0 \qquad \text{a.e.} \quad \text{on } [0, T], \qquad u_\lambda(0) = \xi_0,$$

are strongly pointwisely convergent as $\lambda \to 0^+$ to the solution of the corresponding sweeping process, formally written as

$$-du \in N_{C(t)}(u(t)), \qquad u(t) \in C(t), \quad u(0) = \xi_0.$$

1. INTRODUCTION

The sweeping process by a convex moving set was introduced by J. J. Moreau in the early 1970s with a strong motivation from mechanics, such as elastoplasticity or quasistatics. From these early days, the general technique of Yosida regularization for evolution problems governed by monotone operators was also sucessfully applied to the sweeping process, at least in the case that the convex set depends on t in an absolutely continuous manner; cf., e.g., [6]. Later, it was shown in [3] that if the convex set has only right-continuous bounded variation (rcbv), the Yosida–Moreau approximations still converge—in the sense of graphs—to the

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corresponding rcbv solution of the sweeping process. These results have as a common feature the existence of an a priori measure, induced by the bv moving set, and this a priori measure is of course heavily used in the proofs.

In this paper we intend to study this problem in the case of an only Hausdorff-continuous moving set $t \mapsto C(t)$ with non-empty interior $\operatorname{int} C(t) \neq \emptyset$. Although the set may now have unbounded variation, there is a continuous bv solution of the corresponding sweeping process, cf. [2], so that it makes sense to study the convergence of Yosida–Moreau approximations to this solution. We will show that this convergence is in fact even pointwise strongly here, and our present study has some remarkable new features distinguishing it from the earlier related results.

First, to obtain a uniform bound for the variations of the approximations, we cannot use the existence of an a priori measure. To apply a result on consecutive projections, cf. Lemma 1 below, we now let "the set come to the solution" instead of "the solution approach the set," cf. Lemma 6. Moreover, since in general $\dim H = \infty$ and we have $\operatorname{int} C(t) \neq \emptyset$, it makes no sense to impose an additional compactness condition for $C(t)$, as was done in [3]. This results in the problem that a priori we can assume only *weak* convergence of the approximations to some bv function u. To show that this u is the solution of the sweeping process and that the convergence is in fact strong, we were led to overcome the resulting difficulties by working provisionally with $\bar{u} = (u^+ + u^-)/2$ instead of u itself.

We remark that there is no a priori reason that the convergence of the Yosida–Moreau approximations to the solution is not uniform, since the solution is continuous. But despite the fact that we have no counter-example, we believe that there should be one.

This paper is organized as follows: after stating some preliminary results in Section 2, we will derive a uniform bound for the variations of the Yosida-Moreau approximations in Section 3. Finally, in Section 4 the convergence of the approximations is shown.

We shall start by introducing some notation. Let H be a Hilbert space (of arbitrary dimension) with scalar product $\langle x, y \rangle$ or $x \cdot y$ and norm $|x|$. For $A, B \subset H$, the Hausdorff-distance d_H between A and B is defined as

$$d_H(A, B) = \max\{\sup_{x \in B} \operatorname{dist}(x, A), \sup_{x \in A} \operatorname{dist}(x, B)\}$$

$$\text{with} \quad \operatorname{dist}(x, A) = \inf\{|x - y| : y \in A\}.$$

We fix $T_* > 0$ and assume that the multifunction $C: [0, T_*] \to 2^H \setminus \{\emptyset\}$ be d_H-continuous. Furthermore, we suppose that $C(t) \subset H$ is closed and convex for every $t \in [0, T_*]$.

For $\xi_0 \in C(0)$ and every $\lambda > 0$ let u_λ denote the corresponding Yosida–Moreau-approximation of the sweeping process with moving set $C(\cdot)$, i.e., the absolutely continuous unique solution of

$$\frac{du_\lambda}{dt}(t) + \frac{1}{\lambda}\left[u_\lambda(t) - \mathrm{proj}(u_\lambda(t), C(t))\right] = 0 \quad \text{a.e. on } [0, T_*], \quad u_\lambda(0) = \xi_0. \tag{1}$$

Here $\mathrm{proj}(x, A)$ is the projection of $x \in H$ onto a closed and convex $A \subset H$, i.e., $y = \mathrm{proj}(x, A)$ if and only if $y \in A$ as well as $\langle x - y, y - a \rangle \geqslant 0$ for all $a \in A$. Moreover, for $x \in H$ and closed $A \subset H$ the normal cone to A at x is $N_A(x) = \emptyset$ for $x \notin A$ and

$$\begin{aligned} N_A(x) &= \{ y \in H : \langle y, x - a \rangle \geqslant 0 \text{ for all } a \in A \} \\ &= \{ y \in H : \langle y, x \rangle = \delta^*(y, A) = \sup_{a \in A} \langle y, a \rangle \} \end{aligned}$$

for $x \in A$.

In this paper we shall prove the following:

THEOREM 1. *Let H be a Hilbert space and $T_* > 0$. Moreover, let $C: [0, T_*] \to 2^H \setminus \{\emptyset\}$ be a d_H-continuous multifunction such that $C(t)$ is closed convex and has nonempty interior for all $t \in [0, T_*]$. If $\xi_0 \in C(0)$ then, as $\lambda \to 0^+$, the Yosida–Moreau-approximations* (1) *are pointwise strongly convergent on $[0, T_*]$ to the unique solution u of the corresponding sweeping process*

$$-du \in N_{C(t)}(u(t)), \qquad u(t) \in C(t), \qquad u(0) = \xi_0. \tag{2}$$

Here we call $u: [0, T_*] \to H$ a solution of (2) if u is continuous and of bounded variation (cbv) on $[0, T_*]$ such that $u(0) = \xi_0$, $u(t) \in C(t)$ for $t \in [0, T_*]$ and

$$-u'(t) \in N_{C(t)}(u(t)) \qquad \text{for} \quad |du| - \text{almost all } t \in [0, T_*], \tag{3}$$

where u' is a Radon–Nikodým density of du against $|du|$, cf. [4, Chap. 0.2.1] for details. From [4, Theorem 2.2.1] we know that under the given hypotheses such a unique solution to the sweeping process always exists, and we may assume that

$$\bar{B}_r(a) := \{ x \in H : |x - a| \leqslant r \} \subset \bigcap_{t \in [0, T_*]} C(t) \tag{4}$$

for some $a \in H$ and $r > 0$ by the arguments given in [4, Lemma 2.2.3], cf. also Lemma 5 below and [4, Lemma 2.3.2(a)].

Before going on to the proof of Theorem 1 we shall collect some (more or less known) results which will be needed in the remaining part of the paper.

2. PRELIMINARY RESULTS

We start with a lemma on consecutive projections.

LEMMA 1. *Let* $C_1, ..., C_n \subset H$ *be closed convex sets such that* $\bar{B}_r(a) \subset \bigcap_{i=1}^n C_i$ *for some* $a \in H$ *and* $r > 0$. *If* $x_0 \in H$ *and if* $x_i = \operatorname{proj}(x_{i-1}, C_i)$, *then*

$$|x_0 - a| \geqslant |x_1 - a| \geqslant \cdots \geqslant |x_n - a|,$$

$$\sum_{i=1}^{n} |x_i - x_{i-1}| \leqslant \frac{1}{2r}(|x_0 - a|^2 - |x_n - a|^2) \leqslant \frac{1}{2r}|x_0 - a|^2.$$

Proof. Cf. [9], [10], or [4, Lemma 0.4.4]. ∎

Lemma 1 shows that, to bound the variation of a polygonal path, it is important to identify certain sequences as sequences of consecutive projections. In this respect the following geometrical lemma will turn out to be very useful.

LEMMA 2. *Let* $C \subset H$ *be closed convex,* $x \in H$ *and* $y = \operatorname{proj}(x, C)$. *For* $z \in \{\lambda y + (1-\lambda) x : \lambda \in [0, 1]\} =: [x, y]$ *define* $C_z = \overline{\operatorname{co}}(C \cup \{z\})$, *where* $\overline{\operatorname{co}}\, A$ *denotes the closed convex hull of* $A \subset H$. *Then* $z = \operatorname{proj}(x, C_z)$.

Proof. We fix $z = \lambda y + (1-\lambda)x$ with some $\lambda \in [0, 1]$ and define $M = \{\xi \in H: \langle x - y, \xi \rangle \leqslant \langle x - y, z \rangle\}$. Since $\langle x - y, y \rangle \leqslant \langle x - y, x \rangle$ and $C \subset \{\xi \in H: \langle x - y, \xi \rangle \leqslant \langle x - y, y \rangle\}$, we clearly obtain $C \cup \{z\} \subset M$, and hence $C_z \subset M$, because M is a closed half-space. Thus, $x - y$ (and hence also $x - z$, which is colinear) is a normal to C_z at z and $z = \operatorname{proj}(x, C_z)$. ∎

To prepare for the next result we have to recall some notions. Fix $T_1 > 0$ and let $\mathscr{M}^b([0, T_1], H)$ denote the set of H-valued vector measures of bounded variation on $\mathscr{B}([0, T_1])$, the σ-algebra of Borelian subsets of $[0, T_1]$. If $m \in \mathscr{M}^b([0, T_1], H)$, then $|m|$ will be the nonnegative variation measure corresponding to m, and a Radon–Nikodým density of m against $|m|$ is denoted by m'; cf. [4, Ch. 0.0.1] for further information. A sequence $(m_n)_{n \in \mathbb{N}} \in [\mathscr{M}^b([0, T_1], H)]^{\mathbb{N}}$ converges vaguely to some $m \in \mathscr{M}^b([0, T_1], H)$ if for every continuous $\phi: [0, T_1] \to H$

$$\lim_{n \to \infty} \int_{[0, T_1]} \phi \cdot dm_n = \int_{[0, T_1]} \phi \cdot dm. \tag{5}$$

LEMMA 3. *Let* $(m_n)_{n\in\mathbb{N}} \in [\mathcal{M}^b([0, T_1], H)]^{\mathbb{N}}$ *converge vaguely to some* $m \in \mathcal{M}^b([0, T_1], H)$ *and let* $C: [0, T_1] \to 2^H \setminus \{\varnothing\}$ *be a* d_H*-continuous multifunction with closed convex values. Then*

$$\liminf_{n\to\infty} \int_{[0, T_1]} \delta^*(-m_n'(t), C(t))\, d\,|m_n|\,(t) \geqslant \int_{[0, T_1]} \delta^*(-m'(t), C(t))\, d\,|m|\,(t).$$

Proof. This is a very special case of [1, Theorem 2.1]. ∎

One main ingredient in the proof of Theorem 1 will be the following well-known result on the weak compactness of bv functions. Recall that for $T > 0$ and a bv function $u: [0, T] \to H$ the left limits resp. the right limits

$$u^+(t) := \lim_{s\to t^+} u(t) \qquad \text{resp.} \quad u^-(t) := \lim_{s\to t^-} u(t)$$

always exist for $t \in [0, T[$ resp. $t \in]0, T]$, and $u^+(T) := u(T)$ resp. $u^-(0) := u(0)$ by convention. We let also $|u|_\infty = \sup_{t\in[0, T]} |u(t)|$ denote the uniform norm of a uniformly bounded function $u: [0, T] \to H$.

LEMMA 4. *Let* $T > 0$ *and* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of functions* $u_n: [0, T] \to H$ *which is uniformly bounded in norm and variation, i.e.*

$$\sup_{n\in\mathbb{N}} |u_n|_\infty \leqslant K_1 \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \operatorname{var}(u_n) \leqslant K_2$$

for some constants $K_1, K_2 > 0$. *Then there exists a subsequence* $(u_{n_k})_{k\in\mathbb{N}}$ *of* $(u_n)_{n\in\mathbb{N}}$ *and a bv function* $u: [0, T] \to H$ *such that*

$$\operatorname{var}(u) \leqslant K_2 \quad \text{and} \quad u_{n_k}(t) \to u(t) \quad \text{weakly as } k\to\infty \quad \text{for all } t\in[0, T].$$

If all u_n *are right-continuous, then for every* $T_1 \in]0, T]$ *and every continuous* $\phi: [0, T_1] \to H$

$$\lim_{k\to\infty} \int_{]0, T_1]} \phi \cdot du_{n_k} = \int_{]0, T_1]} \phi \cdot du + \phi(0)\cdot[u^+(0) - u(0)] - \phi(T_1)\cdot[u^+(T_1) - u(T_1)]. \tag{6}$$

Proof. This follows from [4, Theorem 0.2.2] with $s = 0$ and $t = T_1$. ∎

3. UNIFORM BOUNDS FOR THE VARIATIONS

Let $T>0$ and a d_H-continuous multifunction $C\colon [0, T] \to 2^H \setminus \{\emptyset\}$ with closed convex values be fixed. The purpose of this section is to prove the following Lemma 5. If we let u_λ denote the solution of (1) on $[0, T]$, then this lemma in particular implies $\sup_{\lambda>0} \operatorname{var}(u_\lambda) \leqslant K$ for some constant $K>0$.

LEMMA 5. *For every $\lambda>0$ and $T_0, T_1 \in [0, T]$ with $T_0 < T_1$ and*

$$\bar{B}_r(a) \subset \bigcap_{t \in [T_0, T_1]} C(t) \tag{7}$$

for some $a \in H$ and $r>0$ we have

$$\operatorname{var}(u_\lambda; [T_0, T_1]) \leqslant \frac{1}{2r} |u_\lambda(T_0) - a|^2. \tag{8}$$

In particular, if

$$\bar{B}_r(a) \subset \bigcap_{t \in [0, T]} C(t), \tag{9}$$

then

$$\sup_{\lambda>0} \operatorname{var}(u_\lambda; [0, T]) \leqslant \frac{1}{2r} |\xi_0 - a|^2, \textit{ and hence } \sup_{\lambda>0} |u_\lambda|_\infty \leqslant \frac{1}{2r} |\xi_0 - a|^2 + |\xi_0|. \tag{10}$$

Proof of Lemma 5. We keep some $\lambda>0$ fixed throughout this section and approximate the corresponding u_λ by means of a suitable sequence $(v_n)_{n \in \mathbb{N}}$ of rcbv functions $v_n\colon [T_0, T_1] \to H$. This sequence is obtained by discretizing $t \mapsto C(t)$. (To simplify notations, we suppress the dependence of the v_n on λ.) Then we show $\sup_{n \in \mathbb{N}} \operatorname{var}(v_n) \leqslant |u_\lambda(T_0) - a|^2/2r$ in Lemma 6 below, and afterwards for $t \in [T_0, T_1]$ the pointwise convergence $v_n(t) \to u_\lambda(t)$ as $n \to \infty$ in Lemma 8. (In fact, we obtain even uniform convergence.) By definition of the variation $\operatorname{var}(u_\lambda)$, this implies (8).

So we fix $\lambda>0$, choose $\delta_n \to 0$, and for every $n \in \mathbb{N}$ a partition $T_0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = T_1$ with $|t_{i+1}^n - t_i^n| \leqslant \delta_n$. Let the step multifunction C_n be defined through $C_n(t) = C(t_i^n)$ for $t \in I_i^n = [t_i^n, t_{i+1}^n[$ and $C_n(T_1) = C(T_1)$. Moreover, let v_n be the solution of

$$\frac{dv_n}{dt}(t) + \lambda^{-1}[v_n(t) - \operatorname{proj}(v_n(t), C_n(t))] = 0 \text{ a.e. in } [T_0, T_1],$$

$$v_n(T_0) = u_\lambda(T_0). \tag{11}$$

Then we obtain

$$v_n(t) = e^{-(t-T_0)/\lambda} u_\lambda(T_0) + \lambda^{-1} \int_{T_0}^{t} e^{-(t-s)/\lambda} \operatorname{proj}(v_n(s), C_n(s))\, ds \tag{12}$$

for $t \in [T_0, T_1]$. The v_n may be also calculated more explicitly, cf. [4, p. 31/32]: if we let $x_i^n = v_n(t_i^n)$ and $y_i^n = \operatorname{proj}(x_i^n, C(t_i^n))$ for $i = 0, ..., k_n - 1$, then

$$v_n(t) = y_i^n + e^{-(t-t_i^n)/\lambda}(x_i^n - y_i^n) \qquad \text{for} \quad t \in I_i^n.$$

In particular, this yields

$$v_n(t) \in [x_i^n, y_i^n] = \{\mu y_i^n + (1-\mu)\, x_i^n : \mu \in [0, 1]\} \qquad \text{for} \quad t \in I_i^n. \tag{13}$$

LEMMA 6. *We have*

$$\sup_{n \in \mathbb{N}} \operatorname{var}(v_n; [T_0, T_1]) \leqslant \frac{1}{2r} |u_\lambda(T_0) - a|^2.$$

Proof. Let $D_i^n = \overline{\text{co}}(C(t_i^n) \cup \{x_{i+1}^n\})$ for $i \in \{0, ..., k_n - 1\}$. Then (13) and the continuity of v_n imply $x_{i+1}^n \in [x_i^n, y_i^n]$. Hence Lemma 2 may be used to give $x_{i+1}^n = \operatorname{proj}(x_i^n, D_i^n)$. Moreover, as a consequence of (7) we have

$$\bar{B}_r(a) \subset D_i^n \qquad \text{for} \quad n \in \mathbb{N}, \quad i \in \{0, ..., k_n - 1\}.$$

Therefore, by means of Lemma 1 we obtain

$$\sum_{i=1}^{k_n} |v_n(t_i^n) - v_n(t_{i-1}^n)| = \sum_{i=1}^{k_n} |x_i^n - x_{i-1}^n| \leqslant \frac{1}{2r} |x_0^n - a|^2 = \frac{1}{2r} |u_\lambda(T_0) - a|^2. \tag{14}$$

Since the left-hand side of (14) is the variation of v_n in $[T_0, T_1]$, the claim follows. ∎

This also yields

LEMMA 7. *We have*

$$\sup\{|v_n(t)| : t \in [T_0, T_1], n \in \mathbb{N}\} \leqslant \frac{1}{2r} |u_\lambda(T_0) - a|^2 + |u_\lambda(T_0)|, \tag{15}$$

and with $K_1 = (1/2r^3)\, |u_\lambda(T_0) - a|^4 + (2/r)\, |u_\lambda(T_0)|^2 + (1/r)\, |a|^2$

$$\sup\{\operatorname{dist}(v_n(t), C_n(t)) : t \in [T_0, T_1], n \in \mathbb{N}\} \leqslant K_1. \tag{16}$$

Proof. Since $|v_n(t)| \leqslant \mathrm{var}(v_n; [T_0, T_1]) + |v_n(T_0)|$, the first inequality holds. To prove the second one, it follows from (7) and the case $n=1$ in Lemma 1 that

$$\begin{aligned}\mathrm{dist}(v_n(t), C_n(t)) &= |v_n(t) - \mathrm{proj}(v_n(t), C_n(t))| \\ &\leqslant \frac{1}{2r}\left(|v_n(t)-a|^2 - |\mathrm{proj}(v_n(t), C_n(t)) - a|^2\right) \\ &\leqslant \frac{1}{2r}|v_n(t)-a|^2.\end{aligned}$$

By (15) and by twice using $(x+y)^2 \leqslant 2(x^2+y^2)$ for $x, y \in \mathbb{R}$, we also obtain (16). ∎

The following lemma shows in particular that $|u_\lambda(t) - v_n(t)| \to 0$ as $n \to \infty$ for every $t \in [T_0, T_1]$, and hence completes the proof of Lemma 5. We let

$$\omega(\delta) := \sup\{d_H(C(s), C(t)) \colon s, t \in [0, T], |s-t| \leqslant \delta\}$$

denote the modulus of continuity of $C(\cdot)$. Since $C(\cdot)$ is uniformly d_H-continuous, ω is continuously nondecreasing with $\omega(0)=0$.

LEMMA 8. *We have*

$$\sup_{t \in [T_0, T_1]} |u_\lambda(t) - v_n(t)| \to 0 \qquad \textit{as} \quad n \to \infty.$$

Proof. From the definition of ω and the step-multi C_n it follows that $d_H(C(t), C_n(t)) \leqslant \omega(\delta_n)$ for all $t \in [T_0, T_1]$. Let $\phi_n(t) = |u_\lambda(t) - v_n(t)|$ on $[T_0, T_1]$ and

$$K_2 = 2\,[K_1 + \sup_{t \in [T_0, T_1]} \mathrm{dist}(u_\lambda(t), C(t))].$$

Then we find from (16) and from [5, (2.17)], cf. also [4, Prop. 0.4.7], that for $t \in [T_0, T_1]$

$$\begin{aligned}&|\mathrm{proj}(u_\lambda(t), C(t)) - \mathrm{proj}(v_n(t), C_n(t))|^2 \\ &\quad\leqslant \phi_n^2(t) + 2\, d_H(C(t), C_n(t))[\mathrm{dist}(v_n(t), C_n(t)) + \mathrm{dist}(u_\lambda(t), C(t))] \\ &\quad\leqslant \phi_n^2(t) + K_2\,\omega(\delta_n).\end{aligned}$$

Since (1) for u_λ may also be written in integrated form like (12) follows from (11) for v_n, we obtain with $\varepsilon_n = \sqrt{K_2 \omega(\delta_n)}$

$$\begin{aligned}\phi_n(t) &= |u_\lambda(t) - v_n(t)| \\ &\leqslant \lambda^{-1} \int_{T_0}^{t} e^{-(t-s)/\lambda} \, |\mathrm{proj}(u_\lambda(s), C(s)) - \mathrm{proj}(v_n(s), C_n(s))| \, ds \\ &\leqslant \lambda^{-1} \int_{T_0}^{t} e^{-(t-s)/\lambda} [\phi_n(s) + \varepsilon_n] \, ds \\ &\leqslant \lambda^{-1} \int_{T_0}^{t} e^{-(t-s)/\lambda} \phi_n(s) \, ds + \varepsilon_n.\end{aligned}$$

Because K_2 depends only on the fixed $\lambda > 0$ (but not on $n \in \mathbb{N}$) we have $\varepsilon_n \to 0$ as $n \to \infty$. This yields the claimed uniform convergence by differentiating $\varphi_n(t) = \lambda^{-1} \int_{T_0}^{t} e^{-(t-s)/\lambda} \phi_n(s) \, ds$, cf. [4, p. 33/34]. Hence the proof of Lemma 5 is also complete. ∎

4. PROOF OF THEOREM 1

Below we are going to show the following

LEMMA 9. *Let $T > 0$ and a d_H-continuous multifunction $C: [0, T] \to 2^H \setminus \{\emptyset\}$ with closed convex values and satisfying (9) be given. If $\xi_0 \in C(0)$, then, for every sequence $\lambda_n \to 0^+$, the corresponding sequence $(u_n)_{n \in \mathbb{N}} := (u_{\lambda_n})_{n \in \mathbb{N}}$ of Yosida–Moreau approximants on $[0, T]$ has a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ such that there exists a cbv function $u: [0, T] \to H$ with $u_{n_k}(t) \to u(t)$ strongly as $k \to \infty$ for every $t \in [0, T]$ and $u(t) \in C(t)$ for $t \in [0, T[$. Moreover, $u(0) = \xi_0$ and*

$$u'(t) \cdot [z - u(t)] \geqslant 0 \qquad \text{for} \quad |du|\text{-almost all } t \in [0, T[\text{ and all } z \in C(t), \quad (17)$$

i.e. u is a solution to the sweeping process on $[0, T[$.

It is clear that Theorem 1 follows from Lemma 9, since it only remains to deal with the right-end point T_*. (Note that we do not know a priori $|du|(\{T_*\}) = 0$, i.e. the left-continuity of u at $t = T_*$.) This is done by extending the multifunction C to a larger interval $[0, \hat{T}] \supset [0, T_*]$ with $\hat{T} > T_*$ by letting

$$\hat{C}(t) := C(t) \quad \text{for } t \in [0, T_*] \quad \text{and} \quad \hat{C}(t) := C(T_*) \quad \text{for } t \in [T_*, \hat{T}].$$

Because (9) holds for $\hat{C}$, and $\hat{C}$ is d_H-continuous on $[0, \hat{T}]$, we may apply Lemma 9 with $\hat{C}$ and $T = \hat{T}$ to obtain a pointwise strongly convergent (sub-) sequence of $(\hat{u}_n)_{n \in \mathbb{N}}$, the Yosida–Moreau approximants on $[0, \hat{T}]$ with initial value $\xi_0 \in C(0) = \hat{C}(0)$ corresponding to $\hat{C}$. Therefore $u_n(t) = \hat{u}_n(t)$ on $[0, T_*]$ obviously yields the claim of Theorem 1.

Hence it remains to show that Lemma 9 holds. Note first that by (4), Lemma 5 and Lemma 4 we may assume (by indexing subsequences again with $n \in \mathbb{N}$) that there is a bv function u: $[0, T] \to H$ with

$$u_n(t) \to u(t) \text{ weakly} \qquad \text{for every} \quad t \in [0, T] \tag{18}$$

and such that (6) is satisfied.

First, $u_n(0) = \xi_0$ for all $n \in \mathbb{N}$ in particular implies

$$u(0) = \xi_0. \tag{19}$$

Moreover, since $\operatorname{dist}(x, A) = |x - \operatorname{proj}(x, A)|$, by (1) and (10) we have

$$\int_0^T \operatorname{dist}(u_n(t), C(t))\, dt = \lambda_n \int_0^T \left| \frac{du_n}{dt}(t) \right| dt = \lambda_n \operatorname{var}(u_n) \to 0 \qquad \text{as} \quad n \to \infty,$$

i.e. $\operatorname{dist}(u_n(\cdot), C(\cdot)) \to 0$ in $L^1([0, T])$, and hence w.l.o.g.

$$\operatorname{dist}(u_n(t), C(t)) \to 0 \qquad \text{as} \quad n \to \infty \text{ a.e.} \qquad \text{in} \quad [0, T].$$

Because every $C(t)$ is closed and convex, it follows from the Hahn–Banach theorem and (18) that $u(t) \in C(t)$ a.e. in $[0, T]$, which yields in particular

$$u^-(t) \in C(t) \qquad \text{for} \quad t \in\,]0, T], \qquad u^+(t) \in C(t) \quad \text{for} \quad t \in [0, T[\tag{20}$$

$$\text{and} \qquad \bar{u}(t) = \tfrac{1}{2}(u^-(t) + u^+(t)) \in C(t) \qquad \text{for} \quad t \in\,]0, T[. \tag{21}$$

The following lemma will show that u is right-continuous at $t = 0$.

LEMMA 10. *We have* $u^+(0) = u(0) = \xi_0$.

Proof. For fixed $\varepsilon > 0$ we claim that there are $b \in H$ and $\eta, \delta > 0$ such that

$$\bar{B}_{\eta/2}(b) \subset C(t) \qquad \text{for} \quad t \in [0, \delta] \qquad \text{and} \qquad \frac{1}{\eta} |\xi_0 - b|^2 \leqslant \varepsilon. \tag{22}$$

Because $C(\cdot)$ is d_H-continuous, this is clear if $\xi_0 = a$ since we can choose $b = a$. In case that $|\xi_0 - a| > 0$ we let $\varepsilon_1 = \varepsilon\, r / |\xi_0 - a|$ as well as $\eta = r\, |\xi_0 - b| / |\xi_0 - a|$. Then we conclude from $\bar{B}_r(a) \subset C(0)$, the convexity of

$C(0)$ and from $\xi_0 \in C(0)$ that there is a $b \in C(0)$ with $0 < |\xi_0 - b| \leqslant \varepsilon_1$ and $\bar{B}_\eta(b) \subset C(0)$. Hence the d_H-continuity of $C(\cdot)$ implies the existence of a $\delta > 0$ such that $\bar{B}_{\eta/2}(b) \subset C(t)$ for $t \in [0, \delta]$, and therefore by the choice of ε_1 we have proven (22).

Consequently, (8) in Lemma 5 yields for all $n \in \mathbb{N}$

$$\operatorname{var}(u_n, [0, \delta]) \leqslant \frac{1}{\eta} |\xi_0 - b|^2 \leqslant \varepsilon,$$

and this in turn gives

$$|u_n(t) - \xi_0| \leqslant \varepsilon \qquad \text{for all} \quad n \in \mathbb{N} \qquad \text{and} \qquad t \in [0, \delta].$$

By taking $\liminf_{n \to \infty}$ and then $\lim_{t \to 0+}$ we finally obtain $u^+(0) = \xi_0$. ▮

Next, note that for $|du_n|$-a.e. $t \in [0, T]$ and all $z \in C(t)$ the properties of a projection and (1) give

$$\begin{aligned}
&[z - u_n(t)] \cdot \frac{du_n}{dt}(t) \\
&\quad = [z - \operatorname{proj}(u_n(t), C(t))] \cdot \frac{du_n}{dt}(t) + [\operatorname{proj}(u_n(t), C(t)) - u_n(t)] \cdot \frac{du_n}{dt}(t) \\
&\quad = \frac{1}{\lambda_n} [z - \operatorname{proj}(u_n(t), C(t))] \cdot [\operatorname{proj}(u_n(t), C(t)) - u_n(t)] + \lambda_n \left| \frac{du_n}{dt}(t) \right|^2 \\
&\quad \geqslant \lambda_n \left| \frac{du_n}{dt}(t) \right|^2 \geqslant 0.
\end{aligned}$$

Let u_n' denote the density of $du_n = (du_n/dt)\, dt$ w.r.t. $|du_n| = |du_n/dt|\, dt$. Since u_n' is colinear with du_n/dt Lebesgue-a.e., hence $|du_n|$-a.e., by the previous inequality we obtain

$$[z - u_n(t)] \cdot u_n'(t) \geqslant 0 \qquad \text{for} \quad n \in \mathbb{N}, \qquad |du_n| - \text{a.e. } t \in [0, T] \quad \text{and all} \quad z \in C(t). \tag{23}$$

In other words, for all $n \in \mathbb{N}$ and $|du_n|$-a.e. $t \in [0, T]$

$$0 \geqslant \delta^*(-u_n'(t), C(t)) + u_n'(t) \cdot u_n(t). \tag{24}$$

Let

$$I := \{t \in [0, T] : u^+(t) = u(t)\}$$

denote the points of right-continuity of u. Since u is bv, $[0, T]\backslash I$ is at most countable, hence I is dense in $[0, T]$. Therefore it is sufficient by uniqueness to prove that u is a solution to the sweeping process on every $[0, T_1]$ with $T_1 \in I$ and $T_1 < T$. In particular, it follows from Lemma 10 and (21) that

$$\bar{u}(t) \in C(t) \qquad \text{for all} \quad t \in [0, T_1]. \tag{25}$$

From (24) we obtain for all $n \in \mathbb{N}$

$$0 \geqslant \int_{[0, T_1]} \delta^*(-u_n'(t), C(t))\, |du_n|\,(t) + \int_{[0, T_1]} u_n'(t) \cdot u_n(t)\, |du_n|\,(t). \tag{26}$$

With $\bar{u}$ from (21) we intend to prove

$$u'(t) \cdot [z - \bar{u}(t)] \geqslant 0 \qquad \text{for} \quad |du| - \text{almost all } t \in [0, T_1]$$
$$\text{and all} \quad z \in C(t). \tag{27}$$

As a first step in this direction, we will take lim inf of (26). Afterwards, from the corresponding limiting equation, we will derive (27).

To take lim inf of (26), we first consider the second term on the right-hand side. Let us also use the notation u_n^2 for $|u_n|^2$. Because u_n is cbv, we have

$$\int_{[0, T_1]} u_n'(t) \cdot u_n(t)\, |du_n|\,(t) = \int_{[0, T_1]} u_n \cdot du_n = \tfrac{1}{2} \int_{[0, T_1]} d(u_n^2) = \tfrac{1}{2}(u_n^2(T_1) - u_0^2),$$

and consequently by (18), since $T_1 \in I$ and since $(u^2)^+ = (u^+)^2$ as well as $(u^2)^- = (u^-)^2$, it follows that

$$\begin{aligned} \liminf_{n \to \infty} \int_{[0, T_1]} u_n'(t) \cdot u_n(t) |du_n|(t) &\geqslant \tfrac{1}{2}(u^2(T_1) - u_0^2) \\ &= \tfrac{1}{2}((u^2)^+\,(T_1) - (u^2)^-\,(0)) \\ &= \tfrac{1}{2} \int_{[0, T_1]} du^2 \\ &= \int_{[0, T_1]} \bar{u} \cdot du \\ &= \int_{[0, T_1]} \bar{u} \cdot u'\, |du|, \end{aligned} \tag{28}$$

cf. [7] or [4, Chapter 0.0.1].

To take lim inf of the first term on the right-hand side of (26), we want to apply Lemma 3 with $m_n = du_n$ and $m = du$. For that, we have to show the vague convergence $du_n \to du$ in $\mathcal{M}^b([0, T_1], H)$. So let a continuous ϕ: $[0, T_1] \to H$ be given, and note that (6), Lemma 10, and $T_1 \in I$ imply

$$\lim_{n \to \infty} \int_{]0,\, T_1]} \phi \cdot du_n = \int_{]0,\, T_1]} \phi \cdot du. \tag{29}$$

Since $t = 0$ is neither an atom of du_n nor of du, we may replace $]0, T_1]$ by $[0, T_1]$ both on the left-hand side and on the right-hand side of (29), and this is just the vague convergence $du_n \to du$. Consequently, by Lemma 3,

$$\liminf_{n \to \infty} \int_{[0,\, T_1]} \delta^*(-u_n'(t), C(t))\, |du_n|\, (t) \geqslant \int_{[0,\, T_1]} \delta^*(-u'(t), C(t))\, |du|\, (t). \tag{30}$$

By taking lim inf of (26), we thus obtain from (30) and (28) that

$$\begin{aligned} 0 &\geqslant \int_{[0,\, T_1]} \delta^*(-u'(t), C(t))\, |du|\, (t) + \int_{[0,\, T_1]} \bar{u}(t) \cdot u'(t)\, |du|\, (t) \\ &=: \int_{[0,\, T_1]} \Delta(t)\, |du|\, (t) \end{aligned} \tag{31}$$

with

$$\Delta(t) = \delta^*(-u'(t), C(t)) + \bar{u}(t) \cdot u'(t) \geqslant 0 \qquad \text{for all} \quad t \in [0, T_1]$$

by (25). Hence (31) gives $\Delta(t) = 0$ for $|du|$-a.e. $t \in [0, T_1]$, and this in turn implies (27).

To conclude from (27) that u is continuous on $[0, T_1]$, let

$$J_1 := \{t \in [0, T_1] : \Delta(t) \neq 0\}$$

$$\text{and} \qquad J_2 := \left\{ t \in [0, T_1] : u'(t) = \lim_{\varepsilon \to 0^+} \frac{du([t, t+\varepsilon])}{|du|\, ([t, t+\varepsilon])} \right\}.$$

Then by the Moreau–Valadier extension of Jeffery's theorem (cf. [8] or [4, Theorem. 0.1.1]) we obtain $|du|\, (J_1 \cup ([0, T] \backslash J_2)) = 0$.

Suppose that there is $t \in]0, T_1]$ with $u^+(t) \neq u^-(t)$. Then $du(\{t\}) = u^+(t) - u^-(t) \neq 0$, and consequently, since du is absolutely continuous w.r.t. $|du|$, it follows that $|du|(\{t\})>0$, hence $t\in([0,T]\backslash J_1)\cap J_2$. Because $z=u^-(t)\in C(t)$ by (20), (27) finally gives the contradiction

$$0\leqslant u'(t)\cdot(u^-(t)-\bar{u}(t))=\left\langle\frac{u^+(t)-u^-(t)}{|u^+(t)-u^-(t)|},\frac{1}{2}(u^-(t)-u^+(t))\right\rangle$$
$$=-\frac{1}{2}|u^+(t)-u^-(t)|.$$

Therefore we have $u^+(t)=u^-(t)=\bar{u}(t)$ for all $t\in[0,T_1]$, i.e. u^+ is continuous on $[0,T_1]$ since $(u^+)^+=u^+$ and $(u^+)^-=u^-=u^+$. Moreover $du=du^+$, and du does not have atoms in $[0,T_1]$. By (25) we may take $z=u^+(t)$ in (23); it follows that for $n\in\mathbb{N}$ and $t\in[0,T_1]$ we have

$$\int_{[0,t]}[u_n-u^+]\cdot du_n\leqslant 0.$$

Hence

$$\tfrac{1}{2}(|u_n(t)|^2-|u^+(t)|^2)=\tfrac{1}{2}\left(\int_{[0,t]}du_n^2-\int_{[0,t]}d(u^+)^2\right)$$
$$=\int_{[0,t]}u_n\cdot du_n-\int_{[0,t]}u^+\cdot du^+$$
$$\leqslant\int_{[0,t]}u^+\cdot du_n-\int_{[0,t]}u^+\cdot du^+$$

and so

$$|u_n(t)|^2\leqslant|u^+(t)|^2+2\int_{[0,t]}u^+\cdot du_n-2\int_{[0,t]}u^+\cdot du^+.$$

Because u^+ is continuous and neither du_n nor du has an atom at $t=0$ we conclude from the weak convergence $u_n(t)\to u(t)$, from Lemma 4 and $du=du^+$ that

$$|u(t)|^2\leqslant\liminf_{n\to\infty}|u_n(t)|^2\leqslant\limsup_{n\to\infty}|u_n(t)|^2$$
$$\leqslant|u^+(t)|^2+2\left(\int_{[0,t]}u^+\cdot du-u^+(t)\cdot[u^+(t)-u(t)]\right)$$
$$-2\int_{[0,t]}u^+\cdot du^+$$
$$=|u^+(t)|^2-2u^+(t)\cdot[u^+(t)-u(t)]=-|u^+(t)|^2+2u^+(t)\cdot u(t),\quad(32)$$

and consequently

$$|u(t)-u^{+}(t)|^{2}=|u(t)|^{2}+|u^{+}(t)|^{2}-2u^{+}(t)\cdot u(t)\leqslant 0.$$

Therefore $u(t)=u^{+}(t)$ on $[0, T_1]$, and hence u is continuous with $u(t)\in C(t)$ on $[0, T_1]$ by (25). Also, $u(t)=\bar{u}(t)$ in $[0, T_1]$, so that (27) is equivalent to (17). Moreover, the right-hand side of (32) equals $|u(t)|^2$, and this finally implies

$$|u(t)|=\lim_{n\to\infty}|u_n(t)| \qquad \text{for} \quad t\in[0, T_1].$$

Since we already know that $u_n(t)\to u(t)$ weakly in H we therefore obtain $u_n(t)\to u(t)$ strongly in H, and this completes the proof of Lemma 9. ∎

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REFERENCES

1. C. Castaing and V. Jalby, Epiconvergence of integral functionals on the space of vector measures, *C. R. Acad. Sci. Paris* **319** (1994), 669–674.
2. M. D. P. Monteiro Marques, Rafle par un convexe semicontinu d'intérieur non vide en dimension infinie, *Séminaire d'Analyse Convexe*, *USTL Montpellier*, No. 4 (1986).
3. M. D. P. Monteiro Marques, Regularization and graph approximation of a discontinuous evolution problem, *J. Differential Equations* **67** (1987), 145–164.
4. M. D. P. Monteiro Marques, "Differential Inclusions in Nonsmooth Mechanical Problems—Shocks and Dry Friction," Birkhäuser, Basel–Boston, 1993.
5 J. J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space, *J. Differential Equations* **26** (1977), 347–374.
6. J. J. Moreau, On unilateral constraints, friction and plasticity, *in* "New Variational Techniques in Mathematical Physics, C.I.M.E. ciclo, Bressanone, 1973" (G. Capriz and G. Stampacchia, Eds.), Edizione Cremonese, Rome, 1974.
7. J. J. Moreau, Bounded variation in time, *in* "Topics in Non-Smooth Mechanics" (J. J. Moreau, P. D. Panagiotopoulos, and G. Strang, Eds.), pp. 1–74, Birkhäuser, Basel–Boston, 1988.
8. J. J. Moreau and M. Valadier, A chain rule involving vector functions of bounded variation, *J. Funct. Anal.* **74** (1987), 333–345.
9. M. Valadier, Quelques résultats de base concernant le processus de rafle, *Séminaire d'Analyse Convexe*, *USTL Montpellier (3)* **18** (1988).
10. M. Valadier, Lipschitz approximation of the sweeping (or Moreau) process, *J. Differential Equations* **88** (1990), 248–264.