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Several parameters of generalized Mycielskians☆

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Abstract

The generalized Mycielskians (also known as cones over graphs) are the natural generalization of the Mycielski graphs (which were first introduced by Mycielski in 1955). Given a graph G and any integer $m \geqslant 0$, one can transform G into a new graph $\mu_m(G)$, the generalized Mycielskian of G. This paper investigates circular clique number, total domination number, open packing number, fractional open packing number, vertex cover number, determinant, spectrum, and biclique partition number of $\mu_m(G)$.

Keywords: Generalized Mycielskian; Circular clique number; Fractional total domination number; Vertex cover number; Biclique partition number

1. Introduction

In 1955, Mycielski [13] introduced an interesting graph transformation which transforms a graph G into a new graph $\mu(G)$, we now call the Mycielskian of G. Using this construction, he created triangle-free graphs with large chromatic numbers. For a graph G with vertex set $V(G) = V$ and edge set $E(G) = E$, the *Mycielskian* of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$, and edge set $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. Recently, Mycielskians of graphs have been studied extensively, see [1,3–9,11].

Mycielskians have many interesting properties concerning various kinds of graph parameters. It was shown by Mycielski [13] that $\chi(\mu(G)) = \chi(G) + 1$ for any graph G and $\omega(\mu(G)) = \omega(G)$ for any graph G with at least one edge. Fisher et al. [7] investigated Hamiltonicity, diameter, domination, packing, and biclique partitions of Mycielskians. The same authors in [6] also obtained results for parameters related to the biclique partition number. Chang et al. [3] proved that if G has no isolated vertices, then $\kappa(\mu(G)) \geqslant \kappa(G) + 1$, where $\kappa(G)$ is the connectivity of G. In [11], Larsen et al. showed that $\chi_f(\mu(G)) = \chi_f(G) + 1/\chi_f(G)$ for any graph G, where $\chi_f(G)$ is the fractional chromatic number of G. This result was then used by Fisher [5] to construct examples of optimal fractional colorings that have large denominators.

The circular chromatic number $\chi_c(G)$ is a natural generalization of the ordinary chromatic number of a graph G, which was first introduced by Vince [16] under the name 'star chromatic number' of a graph. Readers are referred to [2,17] for surveys on circular chromatic number. The circular chromatic numbers of Mycielskians have been studied in several papers [3,4,9,8]. Huang and Chang [9] proved that $\chi_c(\mu(G_k^d)) = \chi(\mu(G_k^d)) = \lceil k/d \rceil + 1$, where G_k^d is the

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graph with vertex set $\{0, 1, \ldots, k-1\}$ in which i is adjacent to j if and only if $d \leqslant |i-j| \leqslant k-d$. Chang et al. [3] investigated the circular chromatic number of $\mu^m(G) = \mu(\mu^{m-1}(G))$ for $m \geqslant 2$.

The *generalized Mycielskians* are natural generalization of Mycielski graphs, which were also called by Tardif [15] *cones over graphs*. Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and edge set E^0. Given an integer $m \geqslant 1$ the *m-Mycielskian* of G, denoted by $\mu_m(G)$, is the graph $\mu_m(G)$ with vertex set

$$V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^m \cup \{u\},$$

where $V^i = \{v_j^i : v_j^0 \in V^0\}$ is the ith distinct copy of V^0 for $i = 1, 2, \ldots m$, and edge set

$$E^0 \cup \left(\bigcup_{i=0}^{m-1} \{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\} \right) \cup \{v_j^m u : v_j^m \in V^m\}.$$

We define $\mu_0(G)$ to be the graph obtained from G by adding an universal vertex u. It is evident that the so-called Mycielskian of a graph G is simply $\mu_1(G)$.

Given two graphs G and H. A *homomorphism* from G to H is a mapping f from $V(G)$ to $V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. Homomorphism of graphs are studied as a generalization of graph colorings. If there exists a homomorphism from G to H, we say G is *homomorphic* to H. From the definition of $\mu_m(G)$, it is easy to see that, for any two nonnegative integers i and j with $i \leqslant j$, $\mu_j(G)$ is homomorphic to $\mu_i(G)$ but not vice versa. And for any two graphs G and H, if G is homomorphic to H then $\mu_m(G)$ is homomorphic to $\mu_m(H)$ $(m \geqslant 0)$. There are many parameters of $\mu_m(G)$ which vary according to m, as the impact of the vertex u on G becomes weaker and weaker when m gets large.

An excellent generalization of the main result in [11] was made by Tardif in [15]. He proved that, for any graph G, $\chi_f(\mu_m(G)) = \chi_f(G) + 1/\sum_{k=0}^{m}(\chi_f(G)-1)^k$. Using different method, Stiebitz [14] and Tardif [15] proved that $\chi(\mu_m(C_{2k+1})) = 4$ for any integer $m \geqslant 0$ and any $k \geqslant 1$. Recently, it was proved in [12] that $\chi_c(\mu_m(C_{2k+1})) = 4$ for any integers $m \geqslant 0$ and $k \geqslant 1$. The circular chromatic numbers of the generalized Mycielskians of complete graphs were completely determined in [10]. They showed that $\chi_c(\mu_m(K_n)) = \chi(\mu_m(K_n)) = n+1$ for any odd integer $n \geqslant 3$ and any integer $m \geqslant 0$. And $\chi_c(\mu_m(K_n)) = n + 1/(\lfloor 2m/n \rfloor + 1)$ for any positive even number n and any nonnegative integer m.

This paper investigates serval parameters of generalized Mycielskans. In Section 2, we show that if m is large enough then the circular clique number of $\mu_m(G)$ equals that of G. Section 3 deals with domination and packing of $\mu_m(G)$. We determine exact values of the total domination number, open packing number, and fractional total domination number of $\mu_m(G)$ in terms of m and that of G. Upper and lower bounds (both are sharp) for the vertex cover number of $\mu_m(G)$ in terms of m and that of G are given in Section 4. Section 5 studies the determinant, spectrum and biclique partition number of $\mu_m(G)$.

2. Circular clique number

Suppose G is a graph. If H is an induced subgraph of G which is isomorphic to some G_p^q, then we call H a *p/q-circular clique* of G. Clearly, G_p^q itself is a p/q-circular clique. The *circular clique number* $\omega_c(G)$ of G is

$$\omega_c(G) = \sup \left\{ \frac{k}{d} : G_k^d \text{ is homomorphic to } G \right\}.$$

This notion of circular clique number of a graph was first introduced by Zhu in [18]. From the definition we can see that for two graphs G and H, if G is homomorphic to H, then $\omega_c(G) \leqslant \omega_c(H)$. Particularly, if G is a subgraph of H, then $\omega_c(G) \leqslant \omega_c(H)$. The following two lemmas can be found in [18].

Lemma 2.1. *For any graph G, $\omega(G) \leqslant \omega_c(G) < \omega(G) + 1$.*

Lemma 2.2. *If G is a finite graph then $\omega_c(G) = k/d$ for some integers $d \leqslant k \leqslant |V(G)|$. Moreover, if $\omega_c(G) = k/d$ and $(k, d) = 1$, then G_k^d is an induced subgraph of G.*

Theorem 2.3. *Let G be an nonempty graph. If* $\omega_c(G)=2$, *then* $\omega_c(\mu_m(G)) > \omega_c(G)$ *for any integer* $m \geqslant 0$. *If* $\omega_c(G) \geqslant 3$, *then* $\omega_c(\mu_m(G)) = \omega_c(G)$ *for any integer* $m \geqslant 1$. *For* $2 + 1/k \leqslant \omega_c(G) < 2 + 1/(k-1)$, $(k \geqslant 2)$, *if* $m \leqslant k-2$ *then* $\omega_c(\mu_m(G)) > \omega_c(G)$; *and if* $m \geqslant k$ *then* $\omega_c(\mu_m(G)) = \omega_c(G)$.

Proof. Since G is an induced subgraph of $\mu_m(G)$, $\omega_c(\mu_m(G)) \geqslant \omega_c(G)$. For any graph G with at least one edge we have C_{2m+3} as a subgraph of $\mu_m(G)$. So $\omega_c(\mu_m(G)) \geqslant \omega_c(C_{2m+3}) = 2 + 1/(m+1)$. If $\omega_c(G) = 2$, then clearly $\omega_c(\mu_m(G)) > \omega_c(G)$ for any integer $m \geqslant 0$. And for $2 + 1/k \leqslant \omega_c(G) < 2 + 1/(k-1)$, $(k \geqslant 2)$, if $m \leqslant k-2$ then $\omega_c(\mu_m(G)) > \omega_c(G)$. For other two cases we only need to show that $\omega_c(\mu_m(G)) \leqslant \omega_c(G)$.

Let $\omega_c(\mu_m(G)) = p/q$ and G_p^q is a p/q-circular-clique of $\mu_m(G)$. By Lemma 2.2, G_p^q is an induced subgraph of $\mu_m(G)$. Since any subgraph of $\mu_m(G) - u$ is homomorphic to G, we conclude that if u is not in $V(G_p^q)$ then $\omega_c(\mu_m(G)) \leqslant \omega_c(G)$. So we may assume that u is in $V(G_p^q)$. Let $V(G_p^q) = \{0, 1, 2, \ldots, p-1\}$. Without loss of generality, we may assume $u = 0$. The neighborhood of u in G_p^q is $[q, p-q]_p$ (where $[a, b]_p = \{a, a+1, \ldots, b\}$, additions are carried out modulo p). By the definition of $\mu_m(G)$, $I_1 = [q, p-q]_p$ is contained in V^m and so is an independent set of order $p - 2q + 1$. This implies that $|I_1| = (p-2q) + 1 \leqslant q$. Thus if $\omega_c(G) \geqslant 3$, then $\omega_c(\mu_m(G)) = \omega_c(G)$ for any integer $m \geqslant 1$. So we now assume that $2 + 1/k \leqslant \omega_c(G) < 2 + 1/(k-1)$ for some $k \geqslant 2$. The neighborhoods of q and $p-q$ in G_p^q are $[2q, 0]_p$ and $[0, p-2q]_p$, respectively, both containing in $V^{m-1} \cup \{u\}$. Thus, we have $I_2 = [2q, 0]_p \cup [0, p-2q]_p = [2q, p-2q]_p$ is contained in $V^{m-1} \cup \{u\}$ and hence is independent, which implies that $|I_2| = 2(p-2q) + 1 \leqslant q$. Now consider the two ends $p - 2q$ and $2q$ of the interval I_2. Their neighborhoods in G_p^q are $[p-q, 2p-3q]_p$ and $[3q-p, q]_p]_p$, respectively, both containing in $V^m \cup V^{m-2}$. So $I_3 = [p-q, 2p-3q]_p \cup I_1 \cup [3q-p, q]_p = [3q-p, 2p-3q]_p$ is contained in $V^m \cup V^{m-2}$ and is independent with $3(p-2q)+1$ vertices. Suppose $I_1, I_2, \ldots, I_j$ $(k-1 \geqslant j \geqslant 2)$ have been obtained. Then I_{j+1} is the union of I_{j-1} and the neighborhoods of the two ends of I_j. We have, for $i = 1, 2, \ldots, \lfloor k/2 \rfloor$,

$$I_{2i} = [2iq - (i-1)p, ip - 2iq]_p \subseteq \{u\} \cup \left(\bigcup_{t=1}^{i} V^{k-2t+1} \right),$$

and for $i = 0, 1, \ldots, \lceil k/2 \rceil - 1$,

$$I_{2i+1} = [(2i+1)q - ip, (i+1)p - (2i+1)q]_p \subseteq \left(\bigcup_{t=0}^{i} V^{k-2t} \right).$$

Since $P/q < q/(k-1)$, $q > (k-1)(p-2q)$, which means $p \geqslant (2k-1)(p-2q) + 2$. It follows that I_k and I_{k-1} are disjoint. For each $j = 1, 2, \ldots, k$, since $m \geqslant k$, I_j is an independent set with $j(p-2q)+1$ vertices. Particularly $|I_k| = k(p-2q)+1$. Since I_k is independent, $k(p-2q)+1 \leqslant q$. On the other hand, the assumption $p/q \geqslant 2 + 1/k$ implies that $k(p-2q) \geqslant q$, this is a contradiction. The theorem follows. □

Corollary 2.4. *For any graph G with* $\omega_c(G) \geqslant 3$, *if* $m \geqslant 1$ *then* $\omega_c(\mu^m(G)) = \omega_c(G)$, *where* $\mu^i(G) = \mu(\mu^{i-1}(G))$ *and* $\mu^1(G) = \mu(G)$.

Remark. For $2 + 1/k \leqslant \omega_c(G) < 2 + 1/(k-1)$, $(k \geqslant 2)$, when $m = k-1$ we do not know whether $\omega_c(\mu_m(G)) = \omega_c(G)$.

3. Domination and packing

A *closed neighborhood* of a vertex consists of the vertex and all the vertices adjacent to it. An *open neighborhood* of a vertex consists of all the vertices adjacent to it (but not the vertex itself). A *domination* of a graph G is a set of vertices whose closed neighborhoods include every vertex of G. The *domination number* $\gamma(G)$ is the minimum size of a domination of G. A *total domination* of a graph G is a set of vertices whose open neighborhoods include every vertex of G. The *total domination number* $\gamma^t(G)$ is the minimum size of a total domination of a graph G without isolated vertices (a graph with isolated vertices has no total domination).

Let S be a subset of $V(\mu_m(G))$. Denote by S_i the vertex set $S \cap V^i$ for $i = 0, 1, \ldots, m$. And denote by $S(j)$ the set of jth twin vertices of all vertices in $S \backslash \{u\}$. For example, $S_i(j)$ is the set of jth twin vertices of all vertices in $S \cap V^i$.

Theorem 3.1. *For any graph G without isolated vertices*,

$$\gamma^t(\mu_m(G)) = \begin{cases} \dfrac{m}{2}\gamma^t(G)+2 & \text{if } m \text{ is even},\\ \dfrac{(m+1)}{2}\gamma^t(G)+1 & \text{if } m \text{ is odd}. \end{cases}$$

Proof. Suppose S is an minimum total domination of G. View S as a subset of V^0 in $\mu_m(G)$. Let

$$S' = \begin{cases} \left(\bigcup_{t=0}^{k-1}(S(4t+1)\cup S(4t+2))\right)\cup\{v_1^m, u\} & \text{if } m=4k,\\ S(0)\cup\left(\bigcup_{t=0}^{k-1}(S(4t+3)\cup S(4t+4))\right)\cup\{v_1^m, u\} & \text{if } m=4k+2,\\ \left(\bigcup_{t=0}^{k-1}(S(4t+1)\cup S(4t+2))\right)\cup S(4k+1)\cup\{u\} & \text{if } m=4k+1,\\ S(0)\cup\left(\bigcup_{t=0}^{k-1}(S(4t+3)\cup S(4t+4))\right)\cup S(4k+3)\cup\{u\} & \text{if } m=4k+3. \end{cases}$$

Where if $k=0$ then $\bigcup_{t=0}^{k-1}(\cdots)=\emptyset$. Note that $S(0)$ can dominate $V^0\cup V^1$ and that $S(i)$ can dominate $V^{i-1}\cup V^{i+1}$ for $1\leqslant i\leqslant m-1$, it is not difficult to see that S' is a total domination of $\mu_m(G)$ and that S' has the order given in the right hand of the equality in the theorem. Thus the "less than or equal to" inequality of the theorem holds. Following we shall prove the opposite inequality.

Suppose S is any total domination of $\mu_m(G)$, we shall show that $|S|\geqslant(m/2)\gamma^t(G)+2$ for m even and $|S|\geqslant((m+1)/2)\gamma^t(G)+1$ for m odd. If $m=0$, then clearly $|S|\geqslant 2$. So we assume $m\geqslant 1$.

Since for each j, v_j^0 and v_j^1 have the same neighborhood in V^0, it follows that $S_0\cup S_1(0)$ is a total domination of G. Hence, $|S_0\cup S_1(0)|\geqslant\gamma^t(G)$. This implies that $|S_0|+|S_1|\geqslant\gamma^t(G)$. For $1\leqslant i\leqslant m-1$, since V^i is dominated by $S_{i-1}\cup S_{i+1}$, we conclude that $(S_{i-1}\cup S_{i+1})(0)$ dominates V^0, which implies that $|(S_{i-1}\cup S_{i+1})(0)|\geqslant\gamma^t(G)$. Thus, $|S_{i-1}|+|S_{i+1}|\geqslant\gamma^t(G)$ for $1\leqslant i\leqslant m-1$. By summing up these inequalities together with $|S_0|+|S_1|\geqslant\gamma^t(G)$, we obtain

$$2\sum_{i=0}^{m}|S_i|\geqslant m\gamma^t(G)+|S_{m-1}|+|S_m|.$$

That is,

$$|S\backslash\{u\}|=\sum_{i=0}^{m}|S_i|\geqslant\frac{m}{2}\gamma^t(G)+\frac{|S_{m-1}|+|S_m|}{2}.$$

We first deal with the case when $m(\geqslant 2)$ is even. Since the vertex u can only be dominated by S_m, $|S_m|\geqslant 1$. It follows that $|S\backslash\{u\}|\geqslant(m/2)\gamma^t(G)+1$. If $u\in S$, then clearly $|S|\geqslant(m/2)\gamma^t(G)+2$. If $u\notin S$, then to dominate all vertices of V^m, there must be at least two vertices in S_{m-1}. Again we have $|S|\geqslant(m/2)\gamma^t(G)+2$.

For the case when $m(\geqslant 1)$ is odd, we first show that $|S\backslash\{u\}|\geqslant((m+1)/2)\gamma^t(G)$. Since $|S_{i-1}|+|S_{i+1}|\geqslant\gamma^t(G)$ for $1\leqslant i\leqslant m-1$, we have $\sum_{j=i}^{i+3}|S_j|\geqslant 2\gamma^t(G)$ for $0\leqslant i\leqslant m-3$. Note that $|S_0|+S_1|\geqslant\gamma^t(G)$ and that $m+1$ is even, it is easy to see that $|S\backslash\{u\}|=\sum_{i=0}^{m}|S_i|\geqslant((m+1)/2)\gamma^t(G)$. If $|S_{m-1}|+|S_m|\geqslant\gamma^t(G)+1$, then by the inequality above, $|S|\geqslant|S\backslash\{u\}|>((m+1)/2)\gamma^t(G)$. So we assume $|S_{m-1}|+|S_m|\leqslant\gamma^t(G)$. As $S_m\neq\emptyset$ (otherwise S cannot dominate the vertex u), $|S_{m-1}|<\gamma^t(G)$.This implies that S_{m-1} cannot dominate all vertices of V^m. It follows that $u\in S$ and hence $|S|\geqslant((m+1)/2)\gamma^t(G)+1$. $\square$

A *packing* is a set of vertices whose closed neighborhoods are disjoint. The *packing number* $\eta(G)$ of a graph G is the maximum order of a packing of G. An *open packing* is a set of vertices whose open neighborhoods are disjoint. The *open packing number* $\eta^o(G)$ of a graph G is the maximum order of an open packing of G.

Theorem 3.2. *For any graph G without isolated vertices*,

$$\eta^o(\mu_m(G)) = \begin{cases} \dfrac{m}{2}\,\eta^o(G)+1 & \text{if } m \text{ is even},\\ \dfrac{(m+1)}{2}\,\eta^o(G) & \text{if } m \text{ is odd and } \eta^o(G)\geqslant 2,\\ \dfrac{(m+1)}{2}\,\eta^o(G)+1 & \text{if } m \text{ is odd and } \eta^o(G)=1. \end{cases}$$

Proof. Suppose S is a maximum open packing of G. Let

$$S' = \begin{cases} \left(\bigcup_{t=0}^{k-1}(S(4t+1)\cup S(4t+2))\right)\cup\{u\} & \text{if } m=4k,\\ S(0)\cup\left(\bigcup_{t=0}^{k-1}(S(4t+3)\cup S(4t+4))\right)\cup\{u\} & \text{if } m=4k+2,\\ S(0)\left(\bigcup_{t=0}^{k-1}(S(4t+3)\cup S(4t+4))\right) & \text{if } m=4k+1 \text{ and } \eta^o(G)\geqslant 2,\\ \left(\bigcup_{t=0}^{k-1}(S(4t+1)\cup S(4t+2))\right) & \text{if } m=4k+3 \text{ and } \eta^o(G)\geqslant 2,\\ \left(\bigcup_{t=0}^{k-1}(S(4t+1)\cup S(4t+2))\right)\cup S(4k+1)\cup\{u\} & \text{if } m=4k+1 \text{ and } \eta^o(G)=1,\\ S(0)\left(\bigcup_{t=0}^{k-1}(S(4t+3)\cup S(4t+4))\right)\cup S(4k+3)\cup\{u\} & \text{if } m=4k+3 \text{ and } \eta^o(G)=1. \end{cases}$$

It is straightforward to check that S' is an open packing of $\mu_m(G)$ with order given in the right hand of the equality in the theorem. Following we shall prove the opposite direction of the theorem.

Let S be any open packing of $\mu_m(G)$. We prove that $|S|$ is less than or equal to left item of the equality in the theorem. If $m=0$, then clearly $|S|\leqslant 1=(m/2)\eta^o(G)+1$. So we assume $m\geqslant 1$. Since for each j, v_j^0 and v_j^1 have same neighborhood in V^0, we conclude that $S_0\cap S_1(0)=\emptyset$ and that $S_0\cup S_1(0)$ is an open packing of G. Thus, $|S_0|+|S_1|=|S_0\cup S_1(0)|\leqslant\eta^o(G)$. For $1\leqslant i\leqslant m-1$, we can similarly prove that $|S_{i-1}|+|S_{i+1}|\leqslant\eta^o(G)$. By summing up these inequalities, we have

$$|S\backslash\{u\}|=\sum_{i=0}^{m}|S_i|\leqslant\frac{m}{2}\eta^o(G)+\frac{|S_{m-1}|+|S_m|}{2}.$$

It is evident that $|S_m|\leqslant 1$. If $u\in S$ then $S_{m-1}=\emptyset$. In this case, $|S|=|S\backslash\{u\}|+1\leqslant(m/2)\eta^o(G)+\frac{3}{2}$. And the theorem holds clearly. Now assume $u\notin S$. Since $|S_{m-1}|\leqslant\eta^o(G)$ and $|S_m|\leqslant 1$, if m is odd then $|S|=|S\backslash\{u\}|\leqslant((m+1)/2)\eta^o(G)+\frac{1}{2}$. That is the theorem holds for odd m. If m is even and $m=4k+2$, then

$$|S|=|S_0|+|S_1|+\sum_{t=0}^{k-1}(|S_{4t+2}|+|S_{4t+4}|)+\sum_{t=0}^{k-1}(|S_{4t+3}|+|S_{4t+5}|)+|S_{4k+2}|\leqslant\frac{m}{2}\,\eta^o(G)+1.$$

If $m=4k$, then

$$|S|=\sum_{t=0}^{k-1}(|S_{4t}|+|S_{4t+2}|)+\sum_{t=0}^{k-1}(|S_{4t+1}|+|S_{4t+3}|)+|S_{4k}|\leqslant\frac{m}{2}\,\eta^o(G)+1.$$

This completes the proof. □

A *fractional total domination* puts nonnegative weights on vertices so the weights in any open neighborhood sum to at least one. The *fractional total domination number* $\gamma_f^t(G)$ is the minimum sum of weights in a fractional total domination of a graph G without isolated vertices (a graph with isolated vertices has no fractional total dominations). A *fractional open packing* puts nonnegative weights on vertices so the weights in any open neighborhood sum to at most one. The *fractional open packing number* $\eta^o_f(G)$ is the maximum sum of weights in a fractional open packing of a graph G without isolated vertices (a fractional open packing can have arbitrarily large weights on isolated vertices). A linear programming formulation that finds $\gamma^t_f(G)$ is the dual of that finds $\eta^o_f(G)$. Thus, for all graphs G without isolated vertices, we have

$$\eta^o(G) \leqslant \eta^o_f(G) = \gamma^t_f(G) \leqslant \gamma^t(G).$$

Theorem 3.3. *For any graph G without isolated vertices,*

$$\gamma^t_f(\mu_m(G)) = \begin{cases} \dfrac{m}{2}\gamma^t_f(G) + 2 - \dfrac{1}{\gamma^t_f(G)} & \text{if } m \text{ is even,} \\ \dfrac{(m+1)}{2}\gamma^t_f(G) + \dfrac{1}{\gamma^t_f(G)} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let $a = 1/\gamma^t_f(G)$. For $j = 1, 2, \ldots, n$, let $w(v^0_j)$ be the weight on vertex v^0_j in a minimal fractional total domination of G. We assign weights w' to the vertices of $\mu_m(G)$. If $m = 4k$ $(k \geqslant 0)$, then let

$$\begin{cases} w'(u) = 1 - a, \ w'(v^0_j) = aw(v^0_j), & \\ w'(v^i_j) = (1-a)w(v^0_j) & \text{for } i = 4t+1, \ 4t+2, \ 0 \leqslant t \leqslant k-1 \text{ if } k \geqslant 1, \\ w'(v^i_j) = aw(v^0_j) & \text{for } i = 4t+3, \ 4t+4, \ 0 \leqslant t \leqslant k-1 \text{ if } k \geqslant 1. \end{cases}$$

If $m = 4k+1$ $(k \geqslant 0)$, then let

$$\begin{cases} w'(u) = a, \ w'(v^0_j) = (1-a)w(v^0_j), & \\ w'(v^i_j) = aw(v^0_j) & \text{for } i = 4t+1, 4t+2, \ 0 \leqslant t \leqslant k-1 \text{ if } k \geqslant 1, \\ w'(v^i_j) = (1-a)w(v^0_j) & \text{for } i = 4t+3, 4t+4, \ 0 \leqslant t \leqslant k-1 \text{ if } k \geqslant 1. \\ w'(v^{4k+1}_j) = aw(v^0_j) & . \end{cases}$$

If $m = 4k+2$ $(k \geqslant 0)$, then let

$$\begin{cases} w'(u) = 1 - a, \ w'(v^0_j) = (1-a)w(v^0_j), & \\ w'(v^i_j) = aw(v^0_j) & \text{for } i = 4t+1, \ 4t+2, 0 \leqslant t \leqslant k, \\ w'(v^i_j) = (1-a)w(v^0_j) & \text{for } i = 4t+3, \ 4t+4, 0 \leqslant t \leqslant k-1 \text{ if } k \geqslant 1. \end{cases}$$

If $m = 4k+3$ $(k \geqslant 0)$, then let

$$\begin{cases} w'(u) = a, \ w'(v^0_j) = aw(v^0_j), & \\ w'(v^i_j) = (1-a)w(v^0_j) & \text{for } i = 4t+1, \ 4t+2, \ 0 \leqslant t \leqslant k, \\ w'(v^i_j) = aw(v^0_j) & \text{for } i = 4t+3, \ 4t+4, \ 0 \leqslant t \leqslant k-1 \text{ if } k \geqslant 1. \\ w'(v^{4k+3}_j) = aw(v^0_j) & . \end{cases}$$

It is straightforward to check that $w'(v^0_j) + w'(v^1_j) = w(v^0_j)$, $w'(v^{i-1}_j) + w'(v^{i+1}_j) = w(v^0_j)$ for $i = 1, 2, \ldots, m-1$. Hence, for each vertex v^0_j of V^0, the weights in its open neighborhood sum to $\sum_{v^0_{j'} v^0_j \in E^0}(w'(v^0_{j'}) + w'(v^1_{j'})) = \sum_{v^0_{j'} v^0_j \in E^0} w(v^0_{j'}) \geqslant 1$. And for each vertex v^i_j of V^i, the weights in its open neighborhood sum to $\sum_{v^0_{j'} v^0_j \in E^0}(w'(v^{i-1}_{j'}) + w'(v^{i+1}_{j'})) = \sum_{v^0_{j'} v^0_j \in E^0} w(v^0_{j'}) \geqslant 1$. For each vertex v^m_j of V^m, the weights in its open neighborhood sum to $\sum_{v^0_{j'} v^0_j \in E^0} w'(v^{m-1}_{j'}) + w'(u) \geqslant (1-a) + a$ (or $a + (1-a)$). Finally, for the vertex u, the weights in its open neighborhood sum to $\sum_{j=1}^n w'(v^m_j) = a\sum_{j=1}^n w(v^0_j) = a\gamma^t_f(G) = 1$. We conclude that w' is a fractional total domination of $\mu_m(G)$. And it is evident that the sum of weights in this fractional total domination is the right item in the equality of the theorem. This proves one direction of the theorem.

Let $w(v_j^0)$ be the weight on vertex v_j^0 in a maximum fractional open packing of G. By defining a fractional open packing of $\mu_m(G)$ as we do in defining the fractional total domination w' above, we can prove the other direction of the theorem. $\square$

Theorem 3.3 generalizes Theorem 13 in [7] which asserts that $\gamma_f^t(\mu(G)) = \gamma_f^t(G) + 1/\gamma_f^t(G)$ for any graph G without isolated vertices.

4. Vertex cover number

A set of vertices S is called a *vertex cover* of a graph G if each edge of G has at least one end in S. The *vertex cover number* $\tau(G)$ is the minimum order of a vertex cover of G. This section investigates the vertex cover number of $\mu_m(G)$. We find that $\tau(\mu_m(G))$ is not a function of $\tau(G)$ and m. However, we can give upper and lower bounds (both are sharp) of $\tau(\mu_m(G))$ in terms of $\tau(G)$ and m.

Theorem 4.1. *For any graph G, $\tau(\mu_m(G)) \leqslant (m+1)\tau(G) + 1$.*

Proof. Suppose S is an minimum vertex cover of G. Then it is easy to verify that $S' = (\sum_{i=0}^{m} S(i)) \cup \{u\}$ is a vertex cover of $\mu_m(G)$. Thus $\tau(\mu_m(G)) \leqslant (m+1)\tau(G) + 1$. $\square$

This upper bound is sharp since we have graphs G with $\tau(\mu_m(G)) = (m+1)\tau(G) + 1$. For example, consider the star graph $K_{1,r}$, clearly $\tau(K_{1,r}) = 1$ and it is not difficult to verify that $\tau(\mu_m(K_{1,r})) = m + 2$.

Theorem 4.2. *For any graph G,*

$$\tau(\mu_m(G)) \geqslant \begin{cases} \dfrac{(m+1)}{2}(\tau(G)+1)+1 & \text{if } m \text{ is odd,} \\ \dfrac{(m+2)}{2}(\tau(G)+1) & \text{if } m \text{ is even.} \end{cases}$$

Proof. Let S be any vertex cover of $\mu_m(G)$. We shall show that $|S|$ is great than or equal to the lower bound given in the theorem. Clearly $|S_0| \geqslant \tau(G)$. For each $i = 0, 1, \ldots, m-1$, by identifying the vertex v_j^i and v_j^{i+1} for all $j = 1, 2, \ldots, n$ in the subgraph induced by $V^i \cup V^{i+1}$, we obtain a copy of G, denote this copy of G by H_i. Since the edges between V^i and V^{i+1} can only be covered by $S_i \cup S_{i+1}$, it follows that $S_i \cup S_{i+1}(i)$ is a vertex cover of H_i. Thus, $|S_i \cup S_{i+1}(i)| \geqslant \tau(G)$. Therefore, $|S_i| + |S_{i+1}| \geqslant \tau(G)$ for $i = 0, 1, \ldots, m-1$. Furthermore, we now verify that $|S_i| + |S_{i+1}| \geqslant \tau(G) + 1$ for $i = 0, 1, \ldots, m-1$. We need the following simple observation.

For any minimum vertex cover of G, since $\tau(G) \leqslant |V(G)| - 1$ (this can be seen from the definition of $\tau(G)$), there is at least one edge of G covered exactly once by the given minimum vertex cover.

If for some i, $|S_i| + |S_{i+1}| = \tau(G)$, then clearly $S_i \cap S_{i+1}(i) = \emptyset$ and $S_i \cup S_{i+1}(i)$ is an minimum vertex cover of H_i. From the above observation, suppose $e = v_1^i v_j^i$ is an edge of H_i such that v_1^i is in $S_i \cup S_{i+1}(i)$ but v_j^i is not in. Now it is easy to see that $v_1^i v_j^{i+1}$ or $v_1^{i+1} v_j^i$ cannot be covered by $S_i \cup S_{i+1}$ in $\mu_m(G)$. This contradicts the assumption that S is an vertex cover of $\mu_m(G)$. Hence $|S_i| + |S_{i+1}| \geqslant \tau(G) + 1$ for $i = 0, 1, \ldots, m-1$.

When m is odd, $|S \backslash \{u\}| = \sum_{i=0}^{m} |S_i| = \sum_{t=0}^{(m-1)/2} (|S_{2t}| + |S_{2t+1}|) \geqslant ((m+1)/2)(\tau(G)+1)$. If $u \in S$, then $|S| \geqslant ((m+1)/2)(\tau(G)+1)+1$. If $u \notin S$, then we must have $S_m = V^m$. It follows that $|S_m| \geqslant \tau(G)+1$. And $|S| = \sum_{i=0}^{m} |S_i| = \sum_{t=0}^{(m-1)/2-1} (|S_{2t+1}| + |S_{2t+2}|) + |S_0| + |S_m| \geqslant ((m+1)/2)(\tau(G)+1)+1$. When m is even, $|S \backslash \{u\}| = \sum_{i=0}^{m} |S_i| = \sum_{t=0}^{m/2-1} (|S_{2t+1}| + |S_{2t+2}|) + |S_0| \geqslant ((m+2)/2)(\tau(G)+1) - 1$. If $u \in S$, then $|S| \geqslant ((m+2)/2)(\tau(G)+1)$. If $u \notin S$, then $|S_m| \geqslant \tau(G)+1$ and $|S| = \sum_{i=0}^{m} |S_i| = \sum_{t=0}^{m/2-1} (|S_{2t}| + |S_{2t+1}|) + |S_m| \geqslant ((m+2)/2)(\tau(G)+1)$. $\square$

This lower bound for $\tau(\mu_m(G))$ is also sharp. Consider the complete graph K_n. Clearly, $\tau(K_n) = n - 1$ for $n \geqslant 2$. It is straightforward to check that $\bigcup_{t=0}^{m/2} V^{2t}$ is a vertex cover of $\mu_m(K_n)$ of order $(m+2)n/2$ if m is even, and $(\bigcup_{t=0}^{(m-1)/2} V^{2t}) \cup \{u\}$ is a vertex cover of $\mu_m(K_n)$ of order $(m+1)n/2 + 1$ if m is odd.

5. Determinant, spectrum and biclique partition

Given a graph G with vertices $\{v_1, v_2, \ldots, v_n\}$, the adjacency matrix $A(G)$ is the $n \times n$ matrix with $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$ otherwise. Let $1 = (1, 1, \ldots, 1)^{\mathrm{T}}$, $\theta = (0, 0, \ldots, 0)^{\mathrm{T}}$, and I_k the identity matrix of order k. Let 0 be a 0-matrix with its order varying according to the situation. If we arrange the vertices of $\mu_m(G)$ in the order $V^0, V^1, \ldots, V^m, u$, then it is easy to see that

$$A(\mu_0(G)) = \begin{pmatrix} A(G) & 1 \\ 1^T & 0 \end{pmatrix}, \quad A(\mu_1(G)) = \begin{pmatrix} A(G) & A(G) & \theta \\ A(G) & 0 & 1 \\ \theta^T & 1^T & 0 \end{pmatrix}.$$

And in general,

$$A(\mu_m(G)) = \begin{pmatrix} A(G) & A(G) & 0 & \cdots & 0 & 0 & \theta \\ A(G) & 0 & A(G) & \cdots & 0 & 0 & \theta \\ 0 & A(G) & 0 & \cdots & 0 & 0 & \theta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A(G) & \theta \\ 0 & 0 & 0 & \cdots & A(G) & 0 & 1 \\ \theta^T & \theta^T & \theta^T & \cdots & \theta^T & 1^T & 0 \end{pmatrix}.$$

For $i \geqslant 1$, let

$$H_i = \begin{pmatrix} I_n & 0 & 0 & \cdots & 0 & 0 & \theta \\ I_n & (-1)^1 I_n & 0 & \cdots & 0 & 0 & \theta \\ 0 & 0 & (-1)^2 I_n & \cdots & 0 & 0 & \theta \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^{i-1} I_n & 0 & \theta \\ 0 & 0 & 0 & \cdots & 0 & (-1)^i I_n & \theta \\ \theta^T & \theta^T & \theta^T & \cdots & \theta^T & \theta^T & (-1)^{i+1} \end{pmatrix}.$$

It is not too difficult to check that

$$A(\mu_i(G)) = H_i \begin{pmatrix} A(G) & 0 \\ 0 & -A(\mu_{i-1}(G)) \end{pmatrix} H_i^{\mathrm{T}}.$$

For $1 \leqslant i \leqslant m$, let

$$H_i' = \begin{pmatrix} I_{(m-i)n} & 0 \\ 0 & H_i \end{pmatrix},$$

and let

$$A_m = \begin{pmatrix} A(G) & 0 & 0 & \cdots & 0 & 0 \\ 0 & (-1)^1 A(G) & 0 & \cdots & 0 & 0 \\ 0 & 0 & (-1)^2 A(G) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^m A(G) & 0 \\ 0 & 0 & 0 & \cdots & 0 & (-1)^m A(\mu_0(G)) \end{pmatrix}.$$

Then

$$A(\mu_m(G)) = (H_m' \cdots H_1') \cdot A_m \cdot (H_m' \cdots H_1')^{\mathrm{T}}.$$

That is $A(\mu_m(G))$ is congruent to A_m $(m \geqslant 1)$.

Let $\xi_+(A)$, $\xi_-(A)$, and $\xi_0(A)$ be the number of positive, negative, and zero eigenvalues (counting multiplicities) of a symmetric matrix A. And let $r(A)$ be the rank of a matrix A.

We then get the following theorem which is a generalization of Theorem 4 in [7].

Theorem 5.1. *Let G be a graph on n vertices and $m(\geqslant 1)$ an integer. Then*

(a) $r(A(\mu_m(G))) = m \cdot r(A(G)) + r(A(\mu_0(G)))$,
(b) $\xi_0(A(\mu_m(G))) = m \cdot \xi_0(A(G)) + \xi_0(A(\mu_0(G)))$,

(c)

$$\xi_+(A(\mu_m(G))) = \begin{cases} \dfrac{m}{2}\,\xi_+(A(G)) + \dfrac{m}{2}\,\xi_-(A(G)) + \xi_+(A(\mu_0(G))) & \text{if } m \text{ is even},\\ \dfrac{m+1}{2}\,\xi_+(A(G)) + \dfrac{m-1}{2}\,\xi_-(A(G)) + \xi_-(A(\mu_0(G))) & \text{if } m \text{ is odd}.\end{cases}$$

(d)

$$\xi_-(A(\mu_m(G))) = \begin{cases} \dfrac{m}{2}\,\xi_-(A(G)) + \dfrac{m}{2}\,\xi_+(A(G)) + \xi_-(A(\mu_0(G))) & \text{if } m \text{ is even},\\ \dfrac{m+1}{2}\,\xi_-(A(G)) + \dfrac{m-1}{2}\,\xi_+(A(G)) + \xi_+(A(\mu_0(G))) & \text{if } m \text{ is odd}.\end{cases}$$

(e)

$$\det(A(\mu_m(G))) = \begin{cases} \det(A)^m \det(A(\mu_0(G))) & \text{if } m = 4k,\\ (-1)^{n+1}\det(A)^m \det(A(\mu_0(G))) & \text{if } m = 4k+1,\\ (-1)^{n}\det(A)^m \det(A(\mu_0(G))) & \text{if } m = 4k+2,\\ -\det(A)^m \det(A(\mu_0(G))) & \text{if } m = 4k+3.\end{cases}$$

As indicated by Fisher et al. in [7] that the invertibility of $A(G)$ and $A(\mu_0(G))$ are independent. In general, we have $r(A(G)) \leqslant r(A(\mu_0(G))) \leqslant r(A(G)) + 1$. The lower and the upper bound are both attainable. So according to the above theorem, to investigate many parameters of $A(\mu_m(G))$ it is necessary and sufficient to investigate those of $A(G)$ and $A(\mu_0(G))$.

A *biclique* of a graph G is a complete bipartite subgraph of G. A *biclique partition* of a graph G is a set of bicliques of G with each edge of G in exactly one biclique. The *biclique partition number* $bp(G)$ is the minimum size of a biclique partition of G. The following lemma can be found in [7].

Lemma 5.2. *For any simple graph G,*

$$bp(G) \geqslant \max\{\xi_+(G), \xi_-(G)\} \geqslant \tfrac{1}{2} r(A(G)).$$

Theorem 5.1 and Lemma 5.2 allow us to investigate $bp(\mu_m(G))$ for $m \geqslant 1$.

Theorem 5.3. *Let G be a graph on n vertices. If $A(G)$ and $A(\mu_0(G))$ are invertible, then $bp(\mu_m(G)) = (m+1)n/2 + 1$ if m is odd; and $mn/2 + (n+1)/2 \leqslant bp(\mu_m(G)) \leqslant mn/2 + 1 + bp(G)$ if m is even.*

Proof. By Lemma 5.2 and Theorem 5.1 (a), $bp(\mu_m(G)) \geqslant (m+1)n/2 + \frac{1}{2}$. If m is odd then the integrality gives that $bp(\mu_m(G)) \geqslant (m+1)n/2 + 1$. On the other hand, by placing a star around each vertex of $(\bigcup_{t=0}^{(m-1)/2} V^{2t}) \cup \{u\}$, we obtain a biclique partition of $\mu_m(G)$ with $(m+1)n/2 + 1$ bicliques. Thus, if m is odd then $bp(\mu_m(G)) = (m+1)n/2 + 1$.

If m is even, then the $(mn)/2 + 1$ stars around each vertex of $(\bigcup_{t=1}^{m/2} V^{2t-1}) \cup \{u\}$ together with $bp(G)$ bicliques in a biclique partition of G form a biclique partition of $\mu_m(G)$. This gives the upper bound for $bp(\mu_m(G))$ when m is even. □

Theorem 5 in [7] is a special case of Theorem 5.3 for $m = 1$.

Theorem 5.4. *For any graph G, $bp(\mu_m(G)) \leqslant (m+1)bp(G) + 1$.*

Proof. Suppose $bp(G) = q$ and let $[T_1, U_1], \ldots, [T_q, U_q]$ be a biclique partition of G with q bicliques. To prove the theorem it is sufficient to partition the edge set of $\mu_m(G)$ into $(m+1)bp(G) + 1$ bicliques. We distinguish two case according to the parity of m.

Case 1: m is even. Suppose $m = 2k$ for some k. Then

$$[T_1(0), U_1(0)], \ldots, [T_q(0), U_q(0)],$$

$$[T_1(2t+1), U_1(2t) \cup U_1(2t+2)], \ldots, [T_q(2t+1), U_q(2t) \cup U_q(2t+2)],$$

$$[T_1(2t) \cup T_1(2t+2), U_1(2t+1)], \ldots, [T_q(2t) \cup T_q(2t+2), U_q(2t+1)],$$

$$(t = 0, 1, \ldots, k-1.)$$

and $[V^m, \{u\}]$ is a biclique partition of $\mu_m(G)$ with $(m+1)bp(G)+1$ bicliques.

Case 2: m is odd. Suppose $m = 2k+1$ for some k. Then

$$[T_1(0), U_1(0) \cup U_1(1)], \ldots, [T_q(0), U_q(0) \cup U_q(1)],$$

$$[T_1(1), U_1(0)], \ldots, [T_q(1), U_q(0)],$$

$$[T_1(2t), U_1(2t-1) \cup U_1(2t+1)], \ldots, [T_q(2t), U_q(2t-1) \cup U_q(2t+1)],$$

$$[T_1(2t-1) \cup T_1(2t+1), U_1(2t)], \ldots, [T_q(2t-1) \cup T_q(2t+1), U_q(2t)],$$

$$(t = 1, 2, \ldots, k.)$$

and $[V^m, \{u\}]$ is a biclique partition of $\mu_m(G)$ with $(m+1)bp(G)+1$ bicliques. $\square$

When $bp(G) < n/2$, the upper bound in Theorem 5.4 is better than that in Theorem 5.3.

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