JOURNAL OF NUMBER THEORY 52, 145–156 (1995)

Irrationality Criteria for Mahler's Numbers

J. W. SANDER

Institut für Mathematik, Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

Communicated by Alan C. Woods

Received March 24, 1993; revised June 30, 1993

For positive integers m and $h \geqslant 2$, let $(m)_h$ denote the finite sequence of digits of m written in h-ary notation. It is known that the real number

$$a_h(g) = 0 \cdot (g^{n_1})_h (g^{n_2})_h (g^{n_3})_h \cdots$$

with $g \geqslant 2$, $h \geqslant 2$ is irrational, if the sequence (n_i) of non-negative integers is unbounded. We study the case where (n_i) is bounded, and prove several irrationality criteria.

1. INTRODUCTION

It is well known that the real number

$$\alpha = 0.12345678910111213\ldots \tag{1}$$

is irrational. This follows at once from the fact that the sequence of digits of α contains arbitrarily long finite sequences of zeros. Obviously, the reasoning is not restricted to the decimal representation of integers. Going beyond examples of the above kind, Mahler showed some 15 years ago that the real number

$$\beta = 0.1248163264128\ldots$$

is irrational. In order to formulate Mahler's more general result, let $m \geqslant 1$, $h \geqslant 2$ be integers, and

$$m = m_1 h^{r-1} + m_2 h^{r-2} + \cdots + m_r$$

for some integers $r > 0$ and $0 \leqslant m_i < h$ $(1 \leqslant i \leqslant r)$ with $m_1 \neq 0$. Then we define

$$(m)_h = m_1 m_2 \cdots m_r,$$

i.e., the sequence of digits of m written in h-ary notation.

0022-314X/95 $6.00

In 1981, Mahler [2] proved that for any $g \geq 2$, the number

$$a(g) := 0 \cdot (g^0)_{10} (g^1)_{10} (g^2)_{10} \cdots,$$

obtained by concatenation of the digits of $(g^0)_{10}$, $(g^1)_{10}$, $(g^2)_{10}$, ..., is irrational. In 1984, Bundschuh [1] generalized this to arbitrary bases by showing that for any $g \geq 2$ and $h \geq 2$,

$$a_h(g) := 0 \cdot (g^0)_h (g^1)_h (g^2)_h \cdots$$

is irrational. In 1986, Niederreiter [3] gave a simpler proof of Bundschuh's result. Yet another proof using Kronecker's theorem was published by Shan [4] in 1987.

More generally, one studied numbers

$$a_h(g) := a_h^{(n_i)}(g) := 0 \cdot (g^{n_0})_h (g^{n_1})_h (g^{n_2})_h \cdots$$

for given $g \geq 2$, $h \geq 2$ and arbitrary sequences $(n_i)_{i \in \mathbb{N}}$ of non-negative integers. Results in this direction were obtained by Yang [7] and Yu [8] in 1988. One year later, Shan and Wang [5] showed by use of Thue's well known theorem on the finite number of solutions of the Diophantine equation

$$by^n - ax^n = c \qquad (a > 0, b > 0, n \geq 3, c \neq 0),$$

that $a_h(g)$ is irrational, if $g \geq 2$, $h \geq 2$ and (n_i) is strictly increasing. However, quite similar to the proof of the irrationality of α in (1), their argument needs only the fact that the sequence (g^{n_i}) contains integers with arbitrarily many digits in h-ary expansion. Therefore, the state of art is summarized by

THEOREM 1 [5]. *Let $g \geq 2$, $h \geq 2$, and let (n_i) be an unbounded sequence of non-negative integers. Then $a_h(g)$ is irrational.*

The remaining problem is to consider $a_h(g)$ for bounded sequences (n_i), which means, in particular, sequences with a finite number of limit points. Clearly, the corresponding $a_h(g)$ may be rational, for instance for cyclic (n_i), or irrational, for instance for $g = 2$, $h = 10$, $n_i \in \{0, 1\}$ and (n_i) not ultimately periodic (a sequence $(n_i)_{i \in \mathbb{N}}$ is called ultimately periodic, if $(n_i)_{i \geq i_0}$ is periodic for some i_0). For sequences (n_i) with exactly one limit point, i.e., convergent sequences, $a_h(g)$ is obviously rational. The first interesting case, namely sequences (n_i) with exactly two limit points, turns out to be special among bounded sequences.

Before we can state the criterion for this situation, we have to introduce some notation. Let $h \geq 2$ and $1 \leq m_1 \leq m_2$ be integers with

$(m_1)_h = m_{11} \cdots m_{1t}$, say. We define: m_1 h-divides m_2 (or m_2 is h-divisible by m_1, or $m_1 \mid_h m_2$) if and only if

$$\frac{m_2}{m_1} = \frac{h^{tL} - 1}{h^t - 1}$$

for some integer $L \geqslant 1$, i.e., $(m_1)_h$ is a period of $(m_2)_h$. Furthermore, m_1 and m_2 are called h-dependent if and only if some positive integer a h-divides m_1 as well as m_2, i.e., $(m_1)_h$ and $(m_2)_h$ have a common period. Otherwise, m_1 and m_2 are said to be h-independent.

THEOREM 2. *Let $g \geqslant 2$, $h \geqslant 2$, and let (n_i) be a bounded sequence of non-negative integers, which is not ultimately periodic and has exactly two limit points $N_1 < N_2$. Then $a_h(g)$ is irrational if and only if*

$$g^{N_1} \nmid_h g^{N_2}. \tag{2}$$

For ultimately periodic sequences (n_i), the corresponding $a_h(g)$ is clearly rational, which makes the condition of non-periodicity indispensable. The restriction to sequences (n_i) with exactly two limit points is also necessary. In order to see this, we first mention that for $N_1 < N_2$, the assertion (2) is equivalent to the h-independence of g^{N_1} and g^{N_2} (as demonstrated in the proof of Theorem 2). Now suppose we have a sequence (n_i) with three limit points N_1, N_2 and N_3. If we assume that g^{N_1}, g^{N_2} and g^{N_3} are pairwise h-independent, we still cannot conclude that $a_h(g)$ is irrational. Take for instance $g = 2$, $h = 5$; then

$$(g^1)_h = 2, \qquad (g^2)_h = 4, \qquad (g^6)_h = 224$$

are pairwise h-independent. However, it is easy to define a sequence (n_i), which is not ultimately periodic, with $n_i \in \{1, 2, 6\}$ having the limit points 1, 2 and 6 such that

$$a_5(2) = 0.\overline{224} \in \mathbb{Q}.$$

By use of a theorem of Shorey and Tijdeman [6], which in turn is an application of Baker's results for linear forms in logarithms, we can prove that for sequences as defined in Theorem 2, the number $a_h(g)$ is irrational except for finitely many pairs $N_1 < N_2$.

THEOREM 3. *Let $g \geqslant 2$ have a fixed prime factor or $h \geqslant 2$ be a fixed integer. Then there is an effectively computable constant $c(g)$ resp. $c(h)$ only depending on g resp. h and, moreover, a finite set V of pairs of integers with*

$$\text{card } V < c(g),$$

resp.

$$\text{card } V < c(h),$$

such that for any not ultimately periodic sequence (n_i) *with exactly two limit points* $N_1 < N_2$, $(N_1, N_2) \notin V$, $a_h(g)$ *is irrational.*

The proof of Theorem 3 will show that the constant $c(g)$, hence the cardinality of the exceptional set V is effectively computable. The method does, however, not yield a bound for the size of the largest pair in V, since Lemma 2 is ineffective in this sense.

For sequences (n_i) with more than two limit points, we show the following irrationality criterion.

THEOREM 4. *Let* $g \geqslant 2$, $h \geqslant 2$, *and let* (n_i) *be a bounded, not ultimately periodic sequence of non-negative integers having exactly* J *limit points* $N_1 < \cdots < N_J$. *If*

$$g^{N_i} \not\equiv g^{N_j} \bmod h \tag{3}$$

for all $1 \leqslant i < j \leqslant J$, *then* $a_h(g)$ *is irrational.*

The theorem says that $a_h(g)$ is irrational whenever the last digits of the integers g^{N_i} $(1 \leqslant i \leqslant J)$ are pairwise distinct. As the proof will show the conclusion also holds if the first digits of the g^{N_i} are pairwise distinct.

For $J = 1$, the theorem is obviously true, since then (n_i) is ultimately periodic. For $J = 2$, Theorem 4 is an immediate consequence of Theorem 2.

Condition (3) implies pairwise h-independence of the g^{N_i}. The example following Theorem 2 demonstrates, however, that pairwise independence is not sufficient for the irrationality of $a_h(g)$ for $J > 2$.

2. SEQUENCES WITH EXACTLY TWO LIMIT POINTS: PROOF OF THEOREM 2

In this section, we shall write

$$\underline{a} = a_1 \cdots a_r$$

for the finite sequence of non-negative integers $a_1, \ldots, a_r$. Such an $\underline{a}$ will be called self-overlapping if for some j, $1 \leqslant j < r$, we have $a_{i+j} = a_i$ for all $1 \leqslant i \leqslant r$, where $a_k := a_i$ for $k \equiv i \bmod r$ and $1 \leqslant i \leqslant r$. The following lemma reveals that a sequence is self-overlapping if and only if it has a proper subperiod.

LEMMA 1. *Let* $\underline{a} = a_1 \cdots a_r$ *be given. If* $\underline{a}$ *is self-overlapping, then* $\underline{a}$ *is periodic and contains a period of length* $d < r$, $d \mid r$.

Proof. Since $\underline{a}$ is self-overlapping, there is some j, $1 \leqslant j < r$, such that for all $1 \leqslant i \leqslant r$,

$$a_i = a_{i+j} = a_{i+2j} = \cdots = a_{i+(r_0-1)j}, \tag{4}$$

where

$$r_0 := \frac{r}{(r, j)}.$$

The subscripts in (4) are pairwise incongruent mod r, but pairwise congruent mod (r, j). Hence $a_{i_1} = a_{i_2}$ for all $i_1 \equiv i_2 \bmod (r, j)$, which proves the lemma with $d = (r, j)$.

PROPOSITION. *Let $g \geqslant 2$, $h \geqslant 2$, m and n be non-negative integers such that g^m and g^n are h-independent. Let (n_i) be a sequence with $n_i \in \{m, n\}$, and define $\underline{g_i} = (g^{n_i})_h$. If*

$$a_h(g) = 0 \cdot \underline{g_1}\underline{g_2}\underline{g_3} \cdots$$

is rational, then (n_i) is ultimately periodic.

Proof. Without loss of generality , m and n both occur infinitely often in (n_i), because otherwise (n_i) is obviously finally constant, thus ultimately periodic.

We assume that (n_i) is not ultimately periodic. Let

$$\underline{a} := a_1 \cdots a_r := (g^n)_h,$$

and

$$\underline{b} := b_1 \cdots b_s := (g^m)_h,$$

say.

Suppose $\underline{a}$ is periodic and has a period

$$\underline{a}' := a_1 \cdots a_u$$

with $u < r$, and minimal u, hence

$$\underline{a} = (\underline{a}')^{r/u},$$

where $(\underline{a})^1 := \underline{a}$ and $(\underline{a})^{n+1} := (\underline{a})^n \underline{a}$ for $n \geqslant 1$. By definition, we have for all i

$$\underline{g_i} = \begin{cases} \underline{a} & \text{for} \quad n_i = n, \\ \underline{b} & \text{for} \quad n_i = m. \end{cases}$$

We define the sequence $(\underline{g}_i')$ by replacing the terms $\underline{g}_i = \underline{a}$ in the sequence $(\underline{g}_i)$ by r/u terms $\underline{a}'$; the terms $\underline{g}_i = \underline{b}$ remain unchanged (apart from their index). One might think of this procedure as stretching the old sequence. We clearly have

$$0 \cdot \underline{g}_1' \underline{g}_2' \underline{g}_3' \cdots = a_h(g).$$

We claim that the sequence $(\underline{g}_i')$ is not ultimately periodic. To prove this we proceed as follows. Since (n_i) is not ultimately periodic by assumption, the same holds for $(\underline{g}_i)$. Hence the terms $\underline{a}$ and $\underline{b}$ both occur infinitely often in $(\underline{g}_i)$, thus $\underline{a}'$ and $\underline{b}$ both occur infinitely often in $(\underline{g}_i')$. Suppose $(\underline{g}_i')$ were ultimately periodic. Without loss of generality we then may assume that $(\underline{g}_i')$ is periodic,

$$\underline{g}_1', \underline{g}_2', \underline{g}_3', \ldots = \overline{\underline{g}_1', \ldots, \underline{g}_t'}$$

for some t, say. Therefore, we have for $1 \leqslant j \leqslant t$

$$\underline{g}_1', \underline{g}_2', \underline{g}_3', \ldots = \underline{g}_1', \ldots, \underline{g}_{j-1}', \overline{\underline{g}_j', \ldots, \underline{g}_t', \underline{g}_1', \ldots, \underline{g}_{j-1}'}.$$

We choose the least $j = j_0$ such that $\underline{g}_{j_0}' = \underline{b}$ and $\underline{g}_{j_0+1}' = \underline{a}'$. By definition, $\underline{a}'$ occurs in $(\underline{g}_j')$ always in blocks of lengths r/u. By choice of j_0, each of these blocks

$$(\underline{a}')^{kr/u} \qquad (k \in \mathbb{N}) \tag{5}$$

lies completely in a period

$$\underline{g}_{j_0+lt}', \ldots, \underline{g}_{(l+1)t}', \underline{g}_{(l+1)t+1}', \ldots, \underline{g}_{(l+1)t+j_0-1}' \tag{6}$$

for some $l \geqslant 0$, since the next period starts with $\underline{b}$. By replacing each block (5) in (6) by a block $(\underline{a})^k$, we have reconstructed our old sequence $(\underline{g}_i)$, which is now obviously periodic with the period corresponding to (6). This contradicts the initial assumption, hence $(\underline{g}_i')$ cannot be ultimately periodic.

After renaming, we have two finite sequences of digits

$$\underline{a} = a_1 \cdots a_r, \qquad \underline{b} = b_1 \cdots b_s,$$

with $r \leqslant s$, where $\underline{a}$ has no subperiod and is itself not a period of $\underline{b}$ (by h-independence). Moreover, there is a not ultimately periodic sequence (n_i) over $\{m, n\}$, such that for

$$\underline{g}_i = \begin{cases} \underline{a} & \text{for } n_i = n, \\ \underline{b} & \text{for } n_i = m, \end{cases}$$

the number

$$0 \cdot \underline{g_1 g_2 g_3} \cdots = 0 \cdot \overline{c_1 \cdots c_l},$$

say, is rational.

For $1 \leqslant j \leqslant l$, let

$$I_j := \{nl + j : n \geqslant 0\}.$$

Since $\underline{a}$ appears infinitely often in $(\underline{g_i})$, the box principle shows that for some j, $1 \leqslant j \leqslant l$, there is an infinite set $I^* \subseteq I_j$, such that for all $k \in I^*$ the digit c_k is the first digit of $\underline{a}$, i.e., c_k is the first digit of a $\underline{g_i}$ with $n_i = n$. Let $k_1 < k_2$ be two elements of I^*. Clearly,

$$k_1 \equiv k_2 \bmod l. \tag{7}$$

Let c_{k_1} resp. c_{k_2} be the first digits of $\underline{g_{i_1}}$ resp. $\underline{g_{i_2}}$ with $n_{i_1} = n_{i_2} = n$. If $\underline{g_{i_1+i}} = \underline{g_{i_2+i}}$ for all $i \geqslant 0$, then $n_{i_1+i} = n_{i_2+i}$ for all $i \geqslant 0$, and (n_i) would be ultimately periodic. Thus there is a least $i_0 \geqslant 1$ satisfying $\underline{g_{i_1+i_0}} \neq \underline{g_{i_2+i_0}}$, i.e.,

$$\underline{g_{i_1+i_0}} = \underline{a}, \qquad \underline{g_{i_2+i_0}} = \underline{b}$$

(or the other way round, which is dealt with analogously). Since $r \leqslant s$, we conclude by (7) and the minimality of i_0

$$\underline{b} = \underline{a}\, b_{r+1} \cdots b_s \tag{8}$$

for $s > r$; for $s = r$, we have $\underline{a} = \underline{b}$. Clearly, $\underline{g_{i_1+i_0+1}} = \underline{a}$ or $\underline{g_{i_1+i_0+1}} = \underline{b}$. By (8), the sequence of digits of $\underline{g_{i_1+i_0+1}}$ begins in any case with $\underline{a}$, hence

$$\underline{b} = \underline{a}\,\underline{a}\, b_{2r+1} \cdots b_s$$

for $s > 2r$, and

$$\underline{b} = \underline{a}\, a_1 \cdots a_{s-r}$$

for $r < s \leqslant 2r$. Writing $s = qr + d$ with $q \geqslant 1$ and $0 \leqslant d < r$, we obtain by induction

$$\underline{b} = (\underline{a})^q\, a_1 \cdots a_d. \tag{9}$$

For $d = 0$, we have

$$\underline{b} = (\underline{a})^q,$$

but this contradicts the conditions on $\underline{a}$ and $\underline{b}$. Thus $0 < d < r$. By (9), the first $(q+1)r + d$ digits of

$$\underline{g_{i_2+i_0}}\,\underline{g_{i_2+i_0+1}} \cdots$$

are

$$\underline{b}\underline{a} = (\underline{a})^q a_1 \cdots a_d \underline{a}. \tag{10}$$

Since $\underline{g_{i_1+i_0}} = \underline{a}$, the first $(q+1)r + d$ digits of

$$\underline{g_{i_1+i_0} g_{i_1+i_0+1}} \cdots$$

are

$$(\underline{a})^{q+1} a_1 \cdots a_d. \tag{11}$$

By (7), the sequences (10) and (11) are equal, in other words, $\underline{a}$ is self-overlapping. By Lemma 1, $\underline{a}$ contains a smaller subperiod, which contradicts the properties of $\underline{a}$. Hence the initial assumption on (n_i) cannot hold. This proves the proposition.

Proof of Theorem 2. First suppose that $g^{N_1} |_h g^{N_2}$. Then

$$(g^{N_1})_h = a_1 \cdots a_t,$$

say, is a period of $(g^{N_2})_h$, and

$$\alpha_h(g) = 0 \cdot b_1 \cdots b_l \overline{a_1 \cdots a_t}$$

for some integers $b_i (1 \leqslant i \leqslant l)$, thus $\alpha_h(g)$ is rational.

On the contrary, let

$$g^{N_1} \nmid_h g^{N_2}. \tag{12}$$

We assume g^{N_1} and g^{N_2} are h-dependent, namely

$$(g^{N_1})_h = (\underline{a})^{s_1}$$

and

$$(g^{N_2})_h = (\underline{a})^{s_2}$$

with $\underline{a} = a_1 \cdots a_t$, $s_1 \leqslant s_2$, say. Let $s_2 = qs_1 + r$ for some $q \geqslant 1$ and $0 \leqslant r < s_1$.

Suppose that $r > 0$. For

$$A := a_1 h^{t-1} + a_2 h^{t-2} + \cdots + a_t$$

and $H := h^t$, we have

$$g^{N_1} = A(H^{s_1-1} + \cdots + H + 1)$$

and

$$g^{N_2} = A(H^{s_2-1} + \cdots + H + 1),$$

therefore

$$\begin{aligned} g^{N_2-N_1} &= \frac{H^{s_2-1} + \cdots + H + 1}{H^{s_1-1} + \cdots + H + 1} \\ &= (H^{s_1-1} + \cdots + H + 1)^{-1} \left(\left(\sum_{j=1}^{q} \sum_{s=1}^{s_1} H^{js_1+r-s} \right) + \right. \\ &\quad \left. + H^{r-1} + \cdots + H + 1 \right) \\ &= \sum_{j=1}^{q} H^{(j-1)s_1+r} + \frac{H^{r-1} + \cdots + H + 1}{H^{s_1-1} + \cdots + H + 1}. \end{aligned}$$

Since $r < s_1$, the fraction is not integral, which is contradictive. Hence, $r = 0$, which means that $s_1 \mid s_2$. This in turn contradicts (12). For this reason, g^{N_1} and g^{N_2} have to be h-independent.

Since $(n_i)_{i \in \mathbb{N}}$ is bounded, having exactly two limit points $N_1 < N_2$, there is an i_0 such that $(n_i)_{i \geqslant i_0}$ and $n := N_1$, $m := N_2$ satisfy the conditions of the Proposition. Since (n_i) is not ultimately periodic, we conclude that $a_h(g)$ is irrational, as desired.

3. Sequences with Exactly Two Limit Points: Proof of Theorem 3

Lemma 2 [6]. (i) *Let $g \geqslant 2$ have a fixed prime factor. Then the equation*

$$\frac{h^{tL} - 1}{h^t - 1} = g^d \tag{13}$$

has only finitely many solutions in integers $h \geqslant 2$, $t \geqslant 1$, $L \geqslant 1$ and $d \geqslant 3$ (bounded by an effectively computable constant only depending on g).

(ii) *Let $h \geqslant 2$ be a fixed integer. Then eq. (13) has only finitely many solutions in integers $g \geqslant 2$, $t \geqslant 1$, $L \geqslant 1$ and $d \geqslant 3$ (again effectively bounded).*

Proof. By taking $a := b := 1$, $x := h^t$, $y := g$ and $g := d$, our lemma is an immediate consequence of [6], Theorem 5(iv) for even L and Theorem 5(vi) for odd L.

Proof of Theorem 3. If (n_i) is unbounded, $a_h(g)$ is irrational by Theorem 1. Therefore, we may assume that (n_i) is bounded. Then by Theorem 2, $a_h(g)$ is rational only if

$$g^{N_1} \mid_h g^{N_2}, \tag{14}$$

i.e., there are integers $t \geqslant 1$, $L \geqslant 2$ and digits $0 \leqslant a_i < h$ $(1 \leqslant i \leqslant t)$, such that

$$g^{N_1} = a_1 h^{t-1} + \cdots + a_t, \tag{15}$$

$$g^{N_2} = g^{N_1}(h^{(L-1)t} + \cdots + h^t + 1). \tag{16}$$

Set $d := N_2 - N_1$. Then

$$g^d = h^{(L-1)t} + \cdots + h^t + 1 = \frac{h^{tL} - 1}{h^t - 1}.$$

Assume that g has some fixed prime factor. By Lemma 2(i), d, h, L, and t are effectively bounded in terms of g. Since $a_i < h$, we conclude from (15) that g^{N_1}, hence N_1 is bounded by some constant only depending on g. Since $d = N_2 - N_1$, this implies that N_2 is also bounded which proves the theorem in this case.

The same argument yields the desired result, if h is fixed.

4. Bounded Sequences: Proof of Theorem 4

In order to prove Theorem 4, we make the assumption that $a_h(g)$ is rational, without loss of generality

$$a_h(g) = 0 \cdot \overline{a_1 \cdots a_l},$$

say. By the remark following Theorem 4, we may further assume that $J \geqslant 3$. It remains to show that for some $1 \leqslant i < j \leqslant J$,

$$g^{N_i} \equiv g^{N_j} \quad \bmod h. \tag{17}$$

(This is, of course, trivial for $J > h$ by a box principle argument.)

We define

$$\underline{g_i} := (g^{n_i})_h.$$

Hence

$$a_h(g) = 0 \cdot \underline{g_1}\,\underline{g_2}\,\underline{g_3} \cdots,$$

say. For $1 \leqslant j \leqslant l$, let

$$I_j := \{nl + j : n \geqslant 0\}.$$

Since N_1 is a limit point of (n_i), for at least one j, $1 \leqslant j \leqslant l$, there is an infinite set $I^* \subseteq I_j$, such that for all $k \in I^*$ the digit a_k is the first digit of some $\underline{g}_i$ with

$$\underline{g}_i = (g^{N_1})_h.$$

Let $1 < k_1 < k_2$ be two elements of I^*. We clearly have

$$k_1 \equiv k_2 \qquad \text{mod } l. \tag{18}$$

If a_{k_1} resp. a_{k_2} are the first digits of $\underline{g}_{i_1}$ resp. $\underline{g}_{i_2}$, say, with

$$\underline{g}_{i_1} = \underline{g}_{i_2} = (g^{N_1})_h,$$

then by (18), we have $a_{k_1-1} = a_{k_2-1}$, in other words, the last digits of $\underline{g}_{i_1-1}$ and $\underline{g}_{i_2-1}$ are equal. If $\underline{g}_{i_1-1} \neq \underline{g}_{i_2-1}$, (17) is proven. Otherwise, we look at $\underline{g}_{i_1-2}$ and $\underline{g}_{i_2-2}$ and proceed recursively.

A problem occurs only in case

$$\underline{g}_{i_1-i} = \underline{g}_{i_2-i} \tag{19}$$

for all $1 \leqslant i < i_1$, and this for all pairs $i_1 < i_2$ corresponding to some $k_1 < k_2$ in I^*. Then we fix k_1, $k_2 \in I^*$, $k_1 < k_2$, and pick some $k \in I^*$ satisfying $k - k_2 > k_2 - k_1$, where a_k is the first digit of $\underline{g}_i$, say. By (18) and (19),

$$\underline{g}_{i_1}, \underline{g}_{i_1+1}, ..., \underline{g}_{i_2-1}$$

and

$$\underline{g}_{i_1}, \underline{g}_{i_1+1}, ..., \underline{g}_{i-1}$$

share a common subperiod. Since this holds for any $k \in I^*$ and I^* is an infinite set, we conclude that the subperiod is a period of

$$\underline{g}_{i_1}, \underline{g}_{i_1+1}, \underline{g}_{i_1+2}, \cdots$$

This means that the sequence $(\underline{g}_i)$ is ultimately periodic. By definition of $\underline{g}_i$, this also holds for (n_i), contradicting the conditions of the theorem. Therefore, (19) does not occur, and this gives the result.

REFERENCES

1. P. BUNDSCHUH, Generalization of a recent irrationality result of Mahler, *J. Number Theory* **19** (1984), 248–253.
2. K. MAHLER, On some irrational decimal fractions, *J. Number Theory* **13** (1981), 268–269.
3. H. NIEDERREITER, On an irrationality theorem of Mahler and Bundschuh, *J. Number Theory* **24** (1986), 197–199.

4. Z. SHAN, A note on irrationality of some numbers, *J. Number Theory* **25** (1987), 211–212.
5. Z. SHAN AND E. T. H. WANG, Generalization of a theorem of Mahler, *J. Number Theory* **32** (1989), 111–113.
6. T. N. SHOREY AND R. TIJDEMAN, New applications of Diophantine approximations to Diophantine equations, *Math. Scand.* **39** (1976), 5–18.
7. S. YANG, Generalization of a theorem of Mahler, *Acta Univ. Guang Xi* (*Guang Xi Daxue Xuebao*), to appear.
8. H. YU, A note on a theorem of Mahler, *J. China Univ. Sci. Tech.* **18**, No. 3 (1988), 388–389.