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Finiteness conditions on the Ext-algebra of a cycle algebra

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Abstract

Let A be a finite-dimensional algebra given by quiver and monomial relations. In [E.L. Green, D. Zacharia, Manuscripta Math. 85 (1994) 11–23] we see that the Ext-algebra of A is finitely generated only if all the Ext-algebras of certain cycle algebras overlying A are finitely generated. Here a cycle algebra Λ is a finite-dimensional algebra given by quiver and monomial relations where the quiver is an oriented cycle. The main result of this paper gives necessary and sufficient conditions for the Ext-algebra of such a Λ to be finitely generated; this is achieved by defining a computable invariant of Λ, the smo-tube. We also give necessary and sufficient conditions for the Ext-algebra of Λ to be Noetherian.

Keywords: Ext-algebra; Finite generation; Noetherian; Monomial algebra; Cycle algebra

Introduction

Let $\mathcal{Q}$ be an oriented cycle with n vertices and n arrows. Label the vertices with the natural ordering $1, \dots, n$ so that there is one arrow from i to $i+1$ for $1 \leqslant i < n$ and one arrow from n to 1. Let $\Bbbk$ be an algebraically closed field, $\mathcal{I}$ an admissible ideal of the path

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algebra $\Bbbk\mathcal{Q}$ with a minimal generating set ρ of m monomial relations, which we fix, such that $\Bbbk\mathcal{Q}/\mathcal{I}$ is a finite-dimensional algebra. We say that a path in $\Bbbk\mathcal{Q}$ is a *relation* if and only if it is in the set ρ. Hereinafter we let $\Lambda = \Bbbk\mathcal{Q}/\mathcal{I}$, reading paths from left to right. Let $\underline{r}$ denote the Jacobson radical of Λ, and let $\bar{\Lambda} = \Lambda/\underline{r}$. Then the Ext-algebra of Λ, denoted $E(\Lambda)$, is the graded $\Bbbk$-algebra $\bigoplus_{i=0}^{\infty} \mathrm{Ext}^i(\bar{\Lambda}, \bar{\Lambda})$, with multiplication the Yoneda product.

The main result of this paper, Theorem 4.14, gives necessary and sufficient conditions for $E(\Lambda)$ to be finitely generated when $\mathcal{Q}$ is an oriented cycle. We also present a fast method for determining if these conditions hold. This is motivated by [4, Proposition 1.6], which states that the Ext-algebra of an arbitrary monomial algebra given by quiver and relations is finitely generated only if the Ext-algebras of certain cycle algebras are all finitely generated. Cycle algebras have an oriented cycle as underlying quiver and therefore are exactly those we study here. Using the same machinery that we use for finite generation of $E(\Lambda)$, we also give necessary and sufficient conditions for the Ext-algebra of Λ to be Noetherian.

The paper is structured as follows. In Section 1 we present a convenient way of representing the basis elements of $E(\Lambda)$: the smo-tube $\mathcal{T}_\Lambda$. This is improved in Section 2, where Theorems 2.3, 2.5, and 2.11 give conditions on the smo-tube that speed-up its calculation. Section 3 demonstrates the existence of natural constraints on $\mathcal{T}_\Lambda$, culminating in Theorem 3.10, which serves to reduce computational work even further. This section paves the way for Section 4 where our main result, Theorem 4.14, gives necessary and sufficient conditions for the finite generation of $E(\Lambda)$. We then give some special cases of this result. In Section 5 we give necessary and sufficient conditions for $E(\Lambda)$ to be a Noetherian ring.

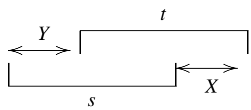
Background and preliminaries

We now give some further notation that will be required. Let p be a path in $\Bbbk\mathcal{Q}$. We denote by $\ell(p)$ the length of p and by $\mathfrak{o}(p)$ and $\mathfrak{t}(p)$ the start and end vertices of p, respectively. A path q is an *initial* subpath of p if $p = qs$ for some path $s \in \Bbbk\mathcal{Q}$. A path q is a *terminal* subpath of p if $p = rq$ for some path $r \in \Bbbk\mathcal{Q}$.

Define a cycle algebra Λ to be a quotient of a path algebra that has an oriented cycle $\mathcal{Q}$ for a quiver, and that has monomial relations. Then we have a very nice description of the Yoneda product for $E(\Lambda)$ via maximal overlaps of these relations. We recall the basic definitions from [3] and recursively define certain sets of paths denoted $\mathcal{Q}_{z+1}$. Let $\mathcal{Q}_0$ be the set of trivial paths of $\mathcal{Q}$, $\mathcal{Q}_1$ the set of arrows and set $\mathcal{Q}_2 = \rho$. Let B be the usual basis of $\Bbbk\mathcal{Q}$ consisting of all finite paths, and let $\mathcal{M} = \{b \in B\colon$ no subpath of b lies in $\rho\}$. For $z \geqslant 1$, a path p in $\Bbbk\mathcal{Q}$ is a *z-prechain* if $p = qwu$, where $q \in \mathcal{Q}_{z-1}$, $qw \in \mathcal{Q}_z$, $u \in \mathcal{M} - \mathcal{Q}_0$, and wu has a subpath in $\mathcal{Q}_2$. Call a z-prechain a *z-chain* if no proper initial subpath is a z-prechain. Then $\mathcal{Q}_{z+1}$ is defined as the set of all z-chains, $z \geqslant 1$.

These z-chains correspond to the paths of maximal overlap sequences of [2], [4] and also [1]. The terminology is understood in the following way. Let $s, t \in \rho$. The relation

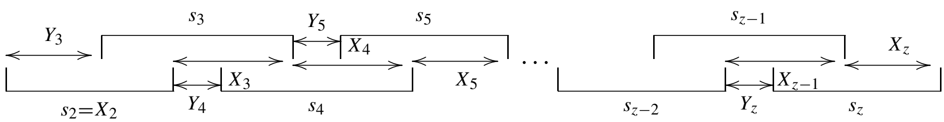
t is said to *overlap* the relation s if there are paths X, Y in $\Bbbk\mathcal{Q}$ such that $Yt = sX$, with $1 \leqslant \ell(Y) < \ell(s)$ and $1 \leqslant \ell(X) < \ell(t)$. This is illustrated in the following diagram:



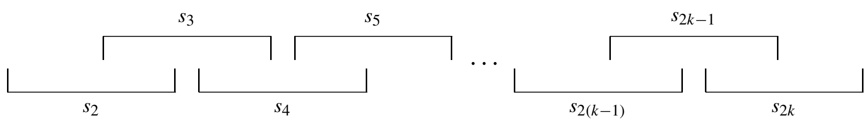
It is then clear that if t overlaps s, the path sX is a 2-prechain. Moreover, we say t *maximally overlaps* s if t overlaps s and further, there are only 2 subpaths of Yt that are relations: the proper initial subpath s and the proper terminal subpath t. If t maximally overlaps s, then (s, t) is a maximal overlap sequence and sX is the underlying path of (s, t). In this case the path sX is a 2-chain and so $sX \in \mathcal{Q}_3$. More generally, for $z \geqslant 3$ and $s_2, s_3, \ldots, s_z \in \rho$, $(s_2, s_3, \ldots, s_z)$ is a maximal left overlap sequence with underlying path $X_2X_3 \cdots X_z$ if

(i) $X_2 = s_2$;
(ii) $1 \leqslant \ell(X_i) < \ell(s_i)$ for $i = 3, \ldots, z$;
(iii) there exist paths $Y_3, \ldots, Y_z$ with $X_iX_{i+1} = Y_{i+1}s_{i+1}$ for $i = 2, \ldots, z-1$;
(iv) there are only 2 subpaths of Y_3s_3 that are relations: the proper initial subpath s_2 and the proper terminal subpath s_3;
(v) for $i = 4, \ldots, z$, there is only 1 subpath of Y_is_i that is a relation: the proper terminal subpath s_i.

The above conditions are visualised, thus:



The path $X_2X_3 \cdots X_z$ is a $(z-1)$-chain and is thus an element of $\mathcal{Q}_z$. The *degree* of the maximal left overlap sequence $(s_2, s_3, \ldots, s_z)$ is z. Throughout the paper a maximal left overlap sequence will be used interchangeably with its underlying path and will usually be illustrated, thus:



Note that if for some $z \geqslant 3$, $P^z = (s_2, s_3, \ldots, s_z)$ is a maximal left overlap sequence with underlying path $X_2X_3 \cdots X_z$, then so is $P^{z-1} = (s_2, s_3, \ldots, s_{z-1})$ with underlying path $X_2X_3 \cdots X_{z-1}$. We call X_z the *path of unoverlapped arrows* of P^z. Note that $P^z = P^{z-1}X_z$.

There is an analogous concept of a maximal right overlap sequence. In fact, from [1] we know that the underlying path of a maximal left overlap sequence of degree z is also the underlying path of a maximal right overlap sequence of degree z and vice versa. Henceforth, we refer to such a path as the underlying path of a maximal overlap sequence; however the construction will always be considered as from the left. We comment further in Section 5.

These maximal overlap sequences are of major importance, since they describe a minimal projective resolution of $\bar{\Lambda}$. Following our notation, for $z \geqslant 0$ the zth projective $\mathcal{P}_z$ in such a resolution is given in [2] as

$$\mathcal{P}_z = \bigoplus_{p \in \mathcal{Q}_z} e_p \Lambda,$$

where e_p is the trivial path at $\mathfrak{t}(p)$. For each $p \in \mathcal{Q}_z$ there is a corresponding element ε_p^z in $\mathrm{Ext}^z_\Lambda(\bar{\Lambda}, \bar{\Lambda})$. This element ε_p^z is represented by the Λ-homomorphism $h_p^z \in \mathrm{Hom}(\mathcal{P}_z, \bar{\Lambda})$ given by

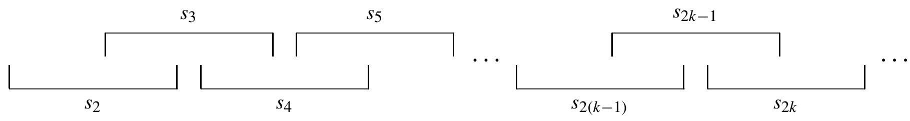
$$h_p^z(e_q \lambda) = \begin{cases} 0 & \text{if } p \neq q \text{ in } \Bbbk\mathcal{Q}, \\ e_p \bar{\lambda} & \text{if } p = q \text{ in } \Bbbk\mathcal{Q}, \end{cases}$$

where $\bar{\lambda}$ is the image of λ under the canonical surjection $\Lambda \to \bar{\Lambda}$. Each set $\mathcal{Q}_z$ is identified with a $\Bbbk$-basis of $\mathrm{Ext}^z_\Lambda(\bar{\Lambda}, \bar{\Lambda})$ in the obvious way by taking p in $\mathcal{Q}_z$ to ε_p^z in $\mathrm{Ext}^z_\Lambda(\bar{\Lambda}, \bar{\Lambda})$.

The set $\mathcal{G}_z := \{\varepsilon_p^z\colon\ p \in \mathcal{Q}_z\}$ is a basis for $\mathrm{Ext}^z_\Lambda(\bar{\Lambda}, \bar{\Lambda})$ and from [4] we have that the union of all the $\mathcal{G}_z$, for $z \geqslant 0$, forms a multiplicative basis for $E(\Lambda)$. This means that for $\varepsilon_p^z \in \mathcal{G}_z$ and $\varepsilon_q^w \in \mathcal{G}_w$, either $\varepsilon_p^z \varepsilon_q^w = 0$, or $\varepsilon_p^z \varepsilon_q^w \in \mathcal{G}_{z+w}$. The one-to-one correspondence between $\mathcal{Q}_z$ and $\mathcal{G}_z$, for each $z \geqslant 0$, given in [4], means that for the remainder of the paper we may deal with maximal overlap sequences as if they themselves form the multiplicative basis of $E(\Lambda)$. With this identification, a maximal overlap sequence of degree 0 is a trivial path, and a maximal overlap sequence of degree 1 is an arrow. If P^{z_1} and Q^{z_2} are maximal overlap sequences of degree z_1 and z_2, respectively, then $P^{z_1} Q^{z_2}$ represents a non-zero element of $E(\Lambda)$ if and only if the product of paths $P^{z_1} Q^{z_2}$ in $\Bbbk\mathcal{Q}$ is the underlying path of a maximal overlap sequence of degree $z_1 + z_2$. In this case $P^{z_1} Q^{z_2}$ represents an element in $E(\Lambda)$ of degree $z_1 + z_2$.

In particular, with this description we can avoid the lifting of maps usually associated to the Yoneda product. Therefore, maximal overlap sequences are fundamental to the results in this paper.

The definitions of the above paragraphs work for a general quiver $\mathcal{Q}$ with monomial relations, but in this paper we restrict $\mathcal{Q}$ to an oriented cycle. This is a special case where, given a $(z-1)$-chain p, there is at most one z-chain of the form pr. We can thus form a sequence of z-chains defined as follows. Let v be the start vertex of some relation r, and for $z \geqslant 2$ let A_v^z be the unique $(z-1)$-chain, if it exists, starting at v. Note that $A_v^2 = r$. Then we say the sequence $A_v := (A_v^z)_{z \geqslant 2}$ is the *extending sequence* of Λ starting at v. We define $\mathfrak{o}(A_v)$ as being the vertex v. The suffix will often be omitted if the start vertex itself is clear from the context or if it is unspecified. Thus A_v is formed from the sequence of maximal overlaps



with $\mathfrak{o}(s_2) = v$ and A_v^z being the maximal overlap sequence $(s_2, s_3, \dots, s_z)$. We also define the *lower half* of A_v; this is the sequence of relations $(s_{2j})_{j \geqslant 1}$, where each s_{2j} is the unique relation such that $\mathfrak{t}(s_{2j}) = \mathfrak{t}(A_v^{2j})$. The *upper half* of A_v is the sequence of relations $(s_{2j+1})_{j \geqslant 1}$, where each s_{2j+1} is the unique relation such that $\mathfrak{t}(s_{2j+1}) = \mathfrak{t}(A_v^{2j+1})$. We may also define the *maximum degree* attained by A_v, denoted $\operatorname{maxdeg} A_v$. If the sequence A_v terminates at some degree $z \geqslant 2$ (that is A_v^z is a $(z-1)$-chain but A_v^{z+1} does not exist), then $\operatorname{maxdeg} A_v = z$; it is defined as ∞ otherwise. Note that $\operatorname{maxdeg} A_v$ must always be at least 2.

Monomial algebras and cycle algebras

Here we review the results that show the fundamentality of cycle algebras in the study of the Ext-algebra of a monomial algebra. We draw our material from [4] and from discussion with the authors of [4].

Definition 0.1 *(*[4]*).* Let B be a monomial algebra and let $\mathcal{Q}$ be a quiver consisting of a single oriented cycle in the quiver Γ of B. Note that we allow $\mathcal{Q}$ to go through the same vertex or arrow of Γ more than once, so certain vertices or arrows of Γ may have several copies appearing in $\mathcal{Q}$. Let $f : \mathcal{Q} \to \Gamma$ be a map of quivers, that is, f sends vertices to vertices and arrows to arrows, so that if η is an arrow from i to j then $f(\eta)$ is an arrow from $f(i)$ to $f(j)$. We take the relations on $\mathcal{Q}$ by pulling back the relations on Γ, that is, a path in $\mathcal{Q}$ is a relation if its image in Γ is a relation in B. Let $Z_{\mathcal{Q}}$ be the algebra with quiver $\mathcal{Q}$ and with the above relations on $\mathcal{Q}$. Then $Z_{\mathcal{Q}}$ is said to be a *cycle algebra overlying B*.

For oriented cycles $\mathcal{Q}$ and $\mathcal{Q}'$, $Z_{\mathcal{Q}'}$ is a *finite covering* of $Z_{\mathcal{Q}}$ if there is a surjective quiver map $\mathcal{Q}' \to \mathcal{Q}$ that takes relations to relations and each relation in $\mathcal{Q}$ lifts to one in $\mathcal{Q}'$. Then we say that a cycle algebra $Z_{\mathcal{Q}'}$ overlying B is a *minimal* cycle algebra overlying B if, for all cycle algebras $Z_{\mathcal{Q}}$, whenever $Z_{\mathcal{Q}'}$ is a finite covering of $Z_{\mathcal{Q}}$ then $Z_{\mathcal{Q}} \cong Z_{\mathcal{Q}'}$.

The reason we consider minimal cycle algebras is due to the following result from [4]. For completeness we provide a more detailed proof.

Proposition 0.2 *(*[4]*). Let $\mathcal{Q}$ be the quiver of a minimal cycle algebra $Z_{\mathcal{Q}}$ and let $\mathcal{Q}'$ be the quiver of a cycle algebra $Z_{\mathcal{Q}'}$ overlying $Z_{\mathcal{Q}}$. Then $E(Z_{\mathcal{Q}'})$ is finitely generated if and only if $E(Z_{\mathcal{Q}})$ is finitely generated.*

Proof. Let $\mathcal{Q}$ have q vertices and q arrows. Then there exists a positive integer d such that $\mathcal{Q}'$ has dq vertices and dq arrows.

Let $E(Z_{\mathcal{Q}})$ be finitely generated, with $b_1, \ldots, b_l$ a complete set of generators from our usual basis described in the previous section. Each b_i in $E(Z_{\mathcal{Q}})$ lifts to one of d different paths in $\mathcal{Q}'$. Since the relations on $\mathcal{Q}'$ are taken from $\mathcal{Q}$, we get that each b_i corresponds to a set of d basis elements of $E(Z_{\mathcal{Q}'})$, denoted $b_i' = \{b_{i,1}', \ldots, b_{i,d}'\}$. We will now show that $E(Z_{\mathcal{Q}'})$ is finitely generated with generating set $\bigcup_{i=1}^{l} b_i'$.

Let a' be an element from the usual basis of $E(Z_{\mathcal{Q}'})$. The underlying path in $\mathcal{Q}'$ of a' corresponds to a path a in $\mathcal{Q}$. Since the relations on $\mathcal{Q}'$ are taken from $\mathcal{Q}$ we have that a is a basis element of $E(Z_{\mathcal{Q}})$. Write a as a finite product of the b_i's. For each copy of a generator that appears in this product there is a single natural choice $1 \leqslant j \leqslant d$ so that a' is written as a non-zero product of the $b_{i,j}'$'s. We can be certain of obtaining a non-zero product because the relations of $\mathcal{Q}'$ are taken from $\mathcal{Q}$. Since a' was arbitrary, $E(Z_{\mathcal{Q}'})$ is finitely generated.

Similarly, if we assume that $E(Z_{\mathcal{Q}'})$ is finitely generated, then an arbitrary basis element a in $E(Z_{\mathcal{Q}})$ corresponds to some a' in $E(Z_{\mathcal{Q}'})$, and thus it is shown that $E(Z_{\mathcal{Q}})$ is finitely generated. $\square$

We now consider the following claim from [4].

Claim *([4, Proposition 1.6]). Let B be a monomial algebra. Then the Ext-algebra $E(B)$ is finitely generated if and only if the $\Bbbk$-algebras $E(Z_{\mathcal{Q}})$ are finitely generated for all minimal cycle algebras $Z_{\mathcal{Q}}$ overlying B.*

We give a proof for one direction stated in the following proposition.

Proposition 0.3. *Let B be a monomial algebra and let the Ext-algebra $E(B)$ be finitely generated. Then the $\Bbbk$-algebras $E(Z_{\mathcal{Q}})$ are finitely generated for all minimal cycle algebras $Z_{\mathcal{Q}}$ overlying B.*

Proof. Let B be a monomial algebra with quiver Γ and let $E(B)$ be finitely generated with (basis) generators $b_1, \ldots, b_l$. Also let $Z_{\mathcal{Q}}$ be an overlying minimal cycle algebra of B. Reordering if necessary, let $b_1, \ldots, b_k$ be precisely those generators of $E(B)$ whose underlying paths lie on the closed path in Γ that is the image of $\mathcal{Q}$. Since the relations of $Z_{\mathcal{Q}}$ are lifted from B, we have that $b_1^*, \ldots, b_k^*$ are corresponding basis elements of $E(Z_{\mathcal{Q}})$.

Now let a^* be a basis element of $E(Z_{\mathcal{Q}})$. Then we have a as the corresponding basis element of $E(B)$, so a can be written as a product of $b_1, \ldots, b_l$. However, if b_i is a subpath of a, then b_i^* is a path in $\mathcal{Q}$, thus a is a product of $b_1, \ldots, b_k$ and so a^* is a product of $b_1^*, \ldots, b_k^*$.

Since a^* was arbitrary, we get that $E(Z_{\mathcal{Q}})$ is finitely generated. $\square$

The details of the proof of this proposition are not given in [4]. Following discussion with the authors of [4], we state here that the reverse implication of the claim above is false. They have provided a counter-example which we give as Example 11 in Section 4, where we can treat it more fully.

However, it is the direction proved in Proposition 0.3 above that is most useful to us. With it, it is clear that finding just one overlying cycle algebra with infinitely generated Ext-algebra gives us the Ext-algebra of B infinitely generated. Hence, studying the Ext-algebras of cycle algebras is fundamental. Of course, B has finite global-dimension if its quiver has no oriented cycles: then $E(B)$ is trivially finitely generated.

We note here that a similar result exists for the respective Ext-algebras being Noetherian. In Section 5 we look at when the Ext-algebra of a cycle algebra is Noetherian, and Proposition 5.5 is the Noetherian analogue to Proposition 0.3.

1. The smo-tube

We now turn exclusively to the study of the Ext-algebra of a monomial cycle algebra Λ, that is $\Lambda = \Bbbk\mathcal{Q}/\mathcal{I}$, where $\mathcal{Q}$ is an oriented cycle and $\mathcal{I}$ has monomial generators. In this section we give the basic definition of the smo-tube and show how it relates to the extending sequences of Λ. We begin by giving some notation that will be used throughout the paper.

If v and w are vertices let $v \to w$ denote the path in $\Bbbk\mathcal{Q}$ from v to w with length in the range 0 to $n-1$ inclusive. The arrow will only ever be used with this precise meaning. Let v be a vertex, z some integer. Then $v+z$ is a vertex where the addition is integer addition modulo n. We identify the vertex v with the trivial path e_v in $\Bbbk\mathcal{Q}$ of length 0 at v. Let p be a path in $\Bbbk\mathcal{Q}$. If $p \in \Bbbk\mathcal{Q}x\Bbbk\mathcal{Q}$ for some path x of length 0 (or 1), then we say that x is a *vertex in* p (respectively *arrow in* p). Likewise if $p \in JxJ$, where J is the 2-sided ideal of $\Bbbk\mathcal{Q}$ generated by the arrows, we say that x is a *vertex* (respectively *arrow*) *strictly in* p.

Label the relations $r_1, \dots, r_m$ such that the concatenation of m paths $H_1 \to H_2 \to \cdots \to H_m \to H_1$ has length n, where $H_i = \mathfrak{o}(r_i)$. Likewise let $T_i = \mathfrak{t}(r_i)$. Define $m+1=1$. We call H_i the *head vertex* or *head*, and T_i the *tail vertex* or *tail* of the relation r_i. An arrow is called a *head arrow* of r_i if its start vertex is H_i. Let α_i be the head arrow of r_i, so $\mathfrak{o}(\alpha_i) = H_i$, and let ω_i be the arrow that ends at T_i, so $\mathfrak{t}(\omega_i) = T_i$. A head vertex H is said to *follow* a tail vertex T if the path $T \to H$ is of minimal length among all paths starting at T and ending at a head vertex. If a tail vertex T is also a head vertex H then we also say H follows T.

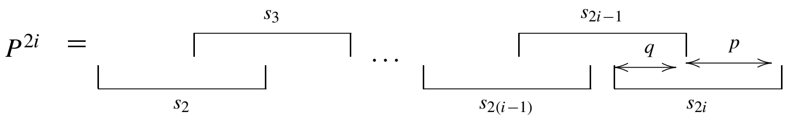
Our first results give us some control over the behaviour of maximal overlap sequences. Proofs of the first two results use induction and are left to the reader.

Proposition 1.1. *Let Λ have m relations $r_1, \dots, r_m$ with the path $H_1 \to \cdots \to H_m \to H_1$ of length n. Then the path $T_1 \to \cdots \to T_m \to T_1$ is also of length n.*

Proposition 1.2. *The path of unoverlapped arrows at the end of an odd-degree maximal overlap sequence always has length less than or equal to n.*

Lemma 1.3. *Let P^{2i} be a maximal overlap sequence of degree $2i$ with last relation s_{2i}. If $\ell(s_{2i}) > n$, then one can always overlap P^{2i} on the right with another relation.*

Proof. If $i = 1$, the result follows since a relation of length greater than n has to overlap itself. For $i > 1$, P^{2i} takes the form:



If there are n or more unoverlapped arrows at the end of P^{2i}, then we are done, so set p equal to the path of unoverlapped arrows and suppose $\ell(p) < n$; let q be such that $s_{2i} = qp$. Now, since $\ell(s_{2i}) > n$, we must have at least two copies of the head arrow of s_{2i} appearing in s_{2i}. By Proposition 1.2, only one copy can appear in q and so we must have at least one copy in p. $\square$

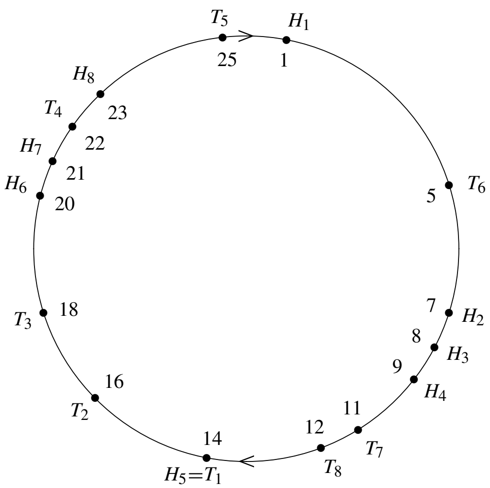
We will use the following example throughout the paper to illustrate our method.

Example 1. Let $\mathcal{Q}$ be an oriented cycle with 25 vertices labelled $1, \ldots, 25$. Label an arrow η_i if it starts at vertex i. Let $\mathcal{I} = \langle r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8 \rangle$, where

$$r_1 = \eta_1 \cdots \eta_{13}, \qquad r_2 = \eta_7 \cdots \eta_{15}, \qquad r_3 = \eta_8 \cdots \eta_{17}, \qquad r_4 = \eta_9 \cdots \eta_{21},$$
$$r_5 = \eta_{14} \cdots \eta_{24}, \qquad r_6 = \eta_{20} \cdots \eta_4, \qquad r_7 = \eta_{21} \cdots \eta_{10}, \qquad r_8 = \eta_{23} \cdots \eta_{11},$$

and let $\Lambda = \mathbb{k}\mathcal{Q}/\mathcal{I}$.

We thus have head vertices $H_1 = 1$, $H_2 = 7$, $H_3 = 8$, etc., and tail vertices $T_1 = 14$, $T_2 = 16$, $T_3 = 18$, etc. The following diagram illustrates which vertices are head or tail vertices in our example:



Note that H_2 is the head vertex that follows T_6, and H_5 follows T_7, T_8, and T_1.

Definition 1.4. A *semi-maximal-overlap-sequence* or *smo-sequence* of a tail vertex T_i is a sequence of indices $(a_1, a_2, \dots)$ from the set $X_m = \{1, \dots, m\}$. It is defined inductively in the following way:

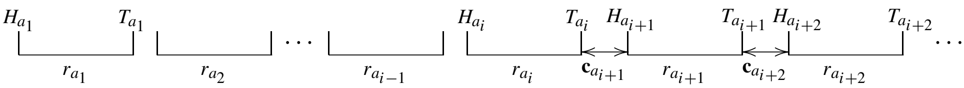
(i) $a_1 = i$.
(ii) For $k \geqslant 1$, $a_{k+1} = l$ where H_l is the head vertex that follows T_{a_k}.

The *smo-function* of the algebra Λ, $f_\Lambda : X_m \to X_m$, is defined as $f_\Lambda(a_k) = a_{k+1}$.

Since entries in an smo-sequence are taken from a finite set and an entry is dependent only on its direct predecessor, we have that after a certain stage the sequence will repeat.

Definition 1.5. Let (a_l) be an smo-sequence. Then a subsequence $(a_i, a_{i+1}, \dots, a_{i+j-1})$ is a *repetition in Λ* if $j \geqslant 1$ is minimal such that $a_i = a_{i+j}$. The *order* of this repetition is j. If a_k is a component of a repetition, i.e., $i \leqslant k \leqslant i + j - 1$ as above, call a_k a *repetition index* and r_{a_k} a *repetition relation*. We say two repetitions are equal if they share a common component, that is they have respective components a_k and $b_{k'}$ such that $a_k = b_{k'}$. It is clear that if two repetitions share one component in this way, they share all components. The *connective path* for repetition relations r_{a_k} and $r_{a_{k+1}}$ is the path $T_{a_k} \to H_{a_{k+1}}$, denoted $\mathbf{c}_{a_{k+1}}$. The connective paths of a repetition $R = (a_i, a_{i+1}, \dots, a_{i+j-1})$ are the paths $\mathbf{c}_{a_{i+1}}, \mathbf{c}_{a_{i+2}}, \dots, \mathbf{c}_{a_{i+j}}$ and the connective paths of Λ are those of all the repetitions of Λ. If x is a component of R, then the head vertex H_x is said to be *R-indexed*.

The lower half of an extending sequence starting at the vertex H_{a_1} may be illustrated as below, with a_i the first repetition index in the sequence (a_k):

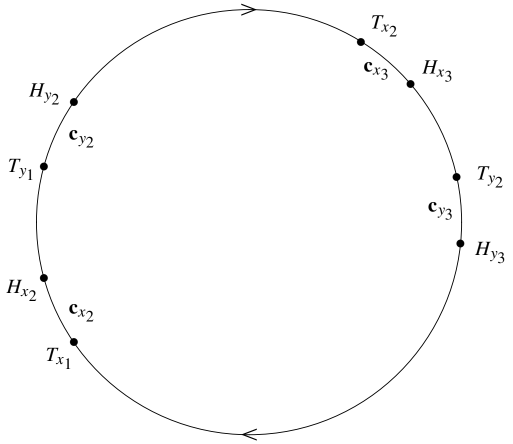


Note that no vertex of $\mathcal{Q}$ can be in two distinct connective paths of Λ.

Example 2. In Example 1 we have two distinct repetitions, each of order 2: $R_1 = (1, 5)$ and $R_2 = (2, 6)$. We have four connective paths, η_{25} and e_{14} associated with R_1, $\eta_5\eta_6$ and $\eta_{16}\eta_{17}\eta_{18}\eta_{19}$ associated with R_2. The smo-sequences of T_1 and T_2 are $(1, 5, 1, 5, \dots)$ and $(2, 6, 2, 6, \dots)$, respectively.

Lemma 1.6. *All repetitions of Λ are of the same order.*

Proof. If Λ has precisely one repetition, and this is of order 1, we are immediately done. We treat all other cases together. Thus let R_x be a repetition of order k with connective paths $\mathbf{c}_{x_2}, \mathbf{c}_{x_3}, \dots, \mathbf{c}_{x_{k+1}}$, let R_y be a repetition of order l with connective paths $\mathbf{c}_{y_2}, \mathbf{c}_{y_3}, \dots, \mathbf{c}_{y_{l+1}}$. Recall that $\mathbf{c}_{x_2}$ is the path $T_{x_1} \to H_{x_2}$ and $\mathbf{c}_{y_2}$ is the path $T_{y_1} \to H_{y_2}$. Relabelling if necessary, suppose that $\mathbf{c}_{x_2}$ and $\mathbf{c}_{y_2}$ are adjacent on the quiver, that is they are the only distinct connective paths of R_x and R_y, respectively, that are subpaths of the path $T_{x_1} \to H_{y_2}$. We will show that $\mathbf{c}_{x_3}$ and $\mathbf{c}_{y_3}$ are the only connective paths of R_x and R_y, respectively, that are subpaths of the path $T_{x_2} \to H_{y_3}$:



Let $\sigma \subseteq \rho$ be the set of relations whose indices appear in R_x or R_y. In particular, the head vertex of each of these relations is the end vertex of a connective path of either R_x or R_y. By hypothesis, we have no head vertex indexed by a relation in σ in the path $H_{x_2} \to H_{y_2}$ except the start and end vertices. Thus by Proposition 1.1, there is no tail vertex indexed by a relation in σ in the path $T_{x_2} \to T_{y_2}$ except the start and end vertices. Hence $\mathbf{c}_{x_3}$ and $\mathbf{c}_{y_3}$ are the only connective paths of R_x or R_y that are subpaths of the path $T_{x_2} \to H_{y_3}$. Inductively it follows that $k+1 = l+1$, giving equality of the orders of R_x and R_y. $\square$

Remark. The above proof gives us some insight not present in the statement of the lemma: we will thus refer to the proof itself later in the paper. In particular, we note here a consequence. Suppose that Λ has two or more repetitions, and let $\mathbf{c}_{x_2}$ and $\mathbf{c}_{y_2}$ be any two distinct connective paths, with no other connective paths in the path $T_{x_1} \to H_{y_2}$. Then $\mathbf{c}_{x_2}$ and $\mathbf{c}_{y_2}$ are in different repetitions.

The reader may like to note that diagrams of the sort in the proof above can be drawn to illustrate most of the proofs in this paper.

We now define the smo-tube, a combinatorial description of the maximal overlap sequences of Λ.

Definition 1.7.

(1) A *degeneration path* is a path $T_p \to H_q$, denoted $\mathbf{d}_q$, with no head or tail vertices strictly in the path $\mathbf{d}_q$ and such that, unless $\mathbf{d}_q$ is of zero length, $\mathfrak{o}(\mathbf{d}_q)$ is not a head vertex and $\mathfrak{t}(\mathbf{d}_q)$ is not a tail vertex. Then H_q follows T_p and we call T_p and H_q respectively *degeneration tail* and *head* vertices. Notice that every connective path has exactly one degeneration path as a terminal subpath: this means

that every repetition index is also the index of a degeneration head vertex. Let D be the set of degeneration paths of Λ, with elements labelled $\mathbf{d}_{q_1}, \mathbf{d}_{q_2}, \ldots, \mathbf{d}_{q_{|D|}}$ so that $p_1 < p_2 < \cdots < p_{|D|}$ with respect to the ordering of the relations of Λ, where $T_{p_i} = \mathfrak{o}(\mathbf{d}_{q_i})$ for $1 \leqslant i \leqslant |D|$.

(2) Place the smo-sequence of T_{p_i} in the ith row of an array where row 1 is at the bottom. We call this array the *smo-array*.

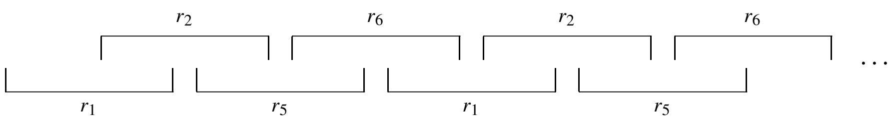
In practice one need only write down the first L columns, for $L = M + \lambda + 1$, where the Mth column is the first to contain only repetition indices and λ is the order of the repetitions. We can bound the size of L as follows. Consider any row i in the smo-array that has entry $(i, M-1)$ not a repetition index. Since there are m relations, at least λ of which are repetition relations, we have $M - 1 \leqslant m - \lambda$. This yields a bound of $L = M + \lambda + 1 \leqslant m + 2$. Since $|D| \leqslant m$, we can have no more than $m(m+2)$ entries in the first L columns of the smo-array.

Fix the above definitions of λ, M, and D for the remainder of the paper. We consider the top row of the above array to be joined to the bottom, and so, once the flags of the next definition have been placed, we will call this array the *smo-tube* and denote it $\mathcal{T}_\Lambda$. In this spirit we will refer to the jth column as *band* j. Henceforth *entry* (i, j) refers to the entry of the smo-tube (or array) in row i, band j. Thus, if (a_k) is an smo-sequence such that $a_s = (i, j)$, for some s, i, j, then $a_{s+1} = (i, j+1)$.

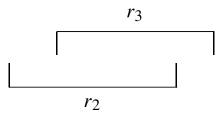
Example 3. In Example 1 we have 5 degeneration paths: η_{25}, $\eta_5\eta_6$, e_{14}, $\eta_{18}\eta_{19}$, and η_{22}. We thus have the smo-array:

	1	**2**	**3**	**4**	**5**	**6**	······
5	6	2	6	2	6	2	······
4	5	1	5	1	5	1	······
3	4	8	5	1	5	1	······
2	3	6	2	6	2	6	······
1	1	5	1	5	1	5	······

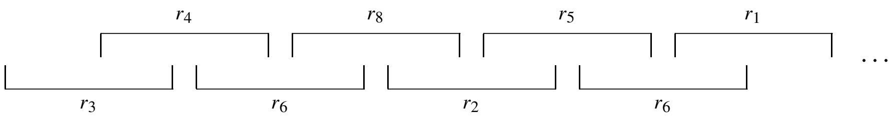
Here we have $M = 3$ and $\lambda = 2$, giving $L = 6$. Let us look again at extending sequences to see how they relate to the smo-array. The extending sequence A_{H_1} has lower half $(r_1, r_5, r_1, r_5, \ldots)$ and upper half $(r_2, r_6, r_2, r_6, \ldots)$, yielding the overlaps



However, the extending sequence A_{H_2} terminates almost immediately, that is at degree 3, with the maximal overlap sequence

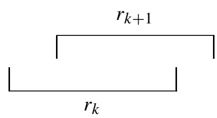


It is clear that the reason for this degree 3 termination is that T_2 is not the start of a degeneration path. We also view the overlaps from which A_{H_3} is formed:



Note that in all but position 1, the upper half of A_{H_1} is the same as the lower half of A_{H_3}. This occurs because the same head vertex, H_6, follows T_2 and T_3.

The above example well illustrates what happens for general Λ, inasmuch as two things go. Firstly, if T_k is not a degeneration tail vertex then A_{H_k} terminates at degree 3. This is clear since if T_k is not a degeneration tail vertex then it is immediate that we have no head arrows in the path $T_k \to T_{k+1}$. By Proposition 1.2, this path is equal to the path of unoverlapped arrows in the maximal overlap sequence



This maximal overlap sequence therefore cannot be overlapped by a relation on the right. Note that r_k can always be overlapped by r_{k+1} if T_k is not a degeneration tail vertex. This is why we include only degeneration tail vertices in the first band of the smo-array: all other tail vertices give rise to extending sequences that terminate at degree 3.

Secondly, suppose that T_a and T_b are degeneration tail vertices. If Λ has only one degeneration path take $T_a = T_b$, otherwise take T_a and T_b such that the path $T_a \to T_b$ is of positive length (so $T_a \neq T_b$) and contains no other degeneration tail vertices. We consider the upper-half of A_{H_a}, $(r_{a+1}, r_{f_\Lambda(a+1)}, r_{f_\Lambda^2(a+1)}, \dots)$, and the lower-half of A_{H_b}, $(r_b, r_{f_\Lambda(b)}, r_{f_\Lambda^2(b)}, \dots)$. Let us show that $r_{f_\Lambda(a+1)} = r_{f_\Lambda(b)}$. If Λ has only one degeneration path, then this is immediate since $H_{f_\Lambda(b)}$ is the head vertex that follows all tail vertices, and thus follows T_{a+1}. So suppose T_b is different from T_a, as above, and consider T_{a+1}. By hypothesis there cannot be a head vertex in the path $T_{a+1} \to T_b$, other than possibly T_b itself, else we would have a degeneration tail vertex strictly in the path $T_a \to T_b$. Hence we have that $H_{f_\Lambda(b)}$ is the head vertex that follows both T_{a+1} and T_b, giving $r_{f_\Lambda(a+1)} = r_{f_\Lambda(b)}$. It follows immediately that $r_{f_\Lambda^k(a+1)} = r_{f_\Lambda^k(b)}$ for all $k \geqslant 1$, and hence that the upper-half of A_{H_a} is identical to the lower-half of A_{H_b} in all places but the first. For this reason we include only the lower halves of extending sequences A_{H_k}, where T_k is a degeneration tail vertex, as rows in the smo-array. We then get the upper halves automatically from the row above. This simplifies matters a great deal, as long as we keep track of what is really happening in band 1.

The above reasoning has shown that the smo-array contains all the information needed to build the infinite extending sequences of Λ. However, with the smo-array as it stands, we are unable to tell which, if any, of the extending sequences terminate. The problem is that smo-sequences are infinite whilst extending sequences can be finite, terminating in a maximal overlap sequence that cannot be overlapped on the right by a relation. We solve this problem by introducing *flags* to the smo-array. The position of a flag in some row i of $\mathcal{T}_\Lambda$ indicates that the extending sequence with lower half associated to row i terminates; the exact point of termination depends on the type of flag. For an extending sequence A_v of Λ, Theorem 1.10 gives the exact values of $\operatorname{maxdeg}(A_v)$ for each position and type of flag. The following definition gives the rules for marking the smo-array with the different types of flag. Recall that α_k is the first arrow of the relation r_k.

Definition 1.8.

(0) *Flags of type 0.* For $1 \leqslant i \leqslant |D|$, entry (i, j) is marked flag0 if and only if $j = 1$, $\ell(r_{(i,1)}) \leqslant n$ and $r_{(i,1)}$ contains no head arrow other than $\alpha_{(i,1)}$.

(1) *Flags of type 1.* If $|D| = 1$, then no entry is marked flag1.
If $|D| > 1$, then for $1 \leqslant i \leqslant |D|$ and $2 \leqslant j \leqslant L$, entry (i, j) of the smo-tube is marked flag1 if and only if $(i, j) \neq (i + 1, j)$ and no head arrow lies in the path $T_{(i,j)} \to T_{(i+1,j)}$. Note that if $i = |D|$ then $i + 1 = 1$.

(2) *Flags of type 2.* If $|D| = 1$, take $(2, 1) = (1, 1) + 1$, and $(2, j) = (1, j)$ for $j \geqslant 2$. If $|D| > 1$ and $i = |D|$, then take $i + 1 = 1$. Let $N(i, j)$ be the number of occurrences of $\alpha_{(1,1)}$ in $r_{(i,j)}$.
   (i) For $2 \leqslant i \leqslant |D|$, $2 \leqslant j \leqslant L$, entry (i, j) is marked flag2 if and only if $\sum_{k=1}^{j-1} N(i + 1, k) = \sum_{k=1}^{j} N(i, k)$ and
      * for $j = 2$ we have $(i, 1) + 1 \neq (i, 2)$ and no head arrow lies in the path $T_{(i,1)+1} \to T_{(i,2)}$;
      * for $3 \leqslant j \leqslant L$ we have $(i, j) \neq (i + 1, j - 1)$ and no head arrow lies in the path $T_{(i+1,j-1)} \to T_{(i,j)}$.
   (ii) Entry $(1, j)$, for $2 \leqslant j \leqslant L$, is marked flag2 if and only if $1 + \sum_{k=1}^{j-1} N(2, k) = \sum_{k=1}^{j} N(1, k)$ and
      * for $j = 2$ we have $(1, 1) + 1 \neq (1, 2)$ and no head arrow lies in the path $T_{(1,1)+1} \to T_{(1,2)}$;
      * for $3 \leqslant j \leqslant L$ we have $(1, j) \neq (2, j - 1)$ and no head arrow lies in the path $T_{(2,j-1)} \to T_{(1,j)}$.

Example 4. In Example 1 the only entry to be marked with a flag is entry $(3, 2)$: it is a flag1. We return to this example in Section 2 with more detail.

The above definition is a very slow way to calculate flags and so Section 2 will present three theorems which speed up this calculation. However, we do have the following corollary of Lemma 1.3.

Corollary 1.9. *If all relations r are such that $\ell(r) > n$, then no flag0s or flag2s can be placed in the smo-tube.*

We complete this section by proving that the flags defined in Definition 1.8 really do give us precisely the termination points of the finite extending sequences.

Theorem 1.10. *Let A be an extending sequence with $\mathfrak{o}(A) = H_a$ and let $\mathcal{T}_\Lambda$ be the smo-tube of Λ. Then*

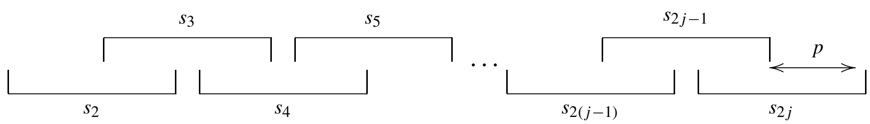
(1) $\operatorname{maxdeg} A = 2$ *if and only if $a = (i, 1)$ for some row i of $\mathcal{T}_\Lambda$, and $(i, 1)$ is marked flag0*;
(2) $\operatorname{maxdeg} A = 3$ *if and only if $a \neq (i, 1)$ for all rows i*;
(3) $\operatorname{maxdeg} A = 2j$, *some $j \geqslant 2$, if and only if $a = (i, 1)$ for some row i, and (i, j) is marked flag2 and is the first flagged entry in row i*;
(4) $\operatorname{maxdeg} A = 2j + 1$, *some $j \geqslant 2$, if and only if $a = (i, 1)$ for some row i, and (i, j) is marked flag1 and is the first flagged entry in row i*;
(5) $\operatorname{maxdeg} A = \infty$ *if and only if $a = (i, 1)$ for some unflagged row i.*

Proof. Let us look in turn at the different ways in which an extending sequence might terminate.

(1) Let A be an extending sequence; by definition A attains at least degree 2. The degree 2 maximal overlap sequence is just a single relation, r_k say. Thus by previous reasoning, A terminating at degree 2 is equivalent to k equalling $(i, 1)$, for some row i in $\mathcal{T}_\Lambda$, and r_k not having any relation overlapping it. This in turn is equivalent to k equalling $(i, 1)$, for some row i in $\mathcal{T}_\Lambda$, and r_k containing no head arrows other than α_k once at the start: exactly the condition needed to mark entry $(i, 1)$ with a flag0.

(2) We have seen already how $\mathcal{T}_\Lambda$ excludes precisely those extending sequences that terminate at degree 3.

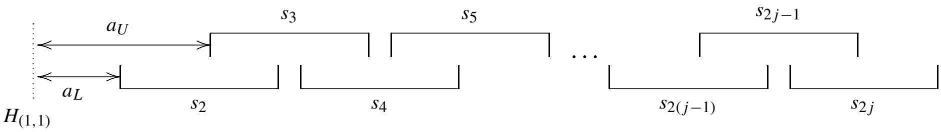
(3) Whether the extending sequence A terminates at degree $2j$, for some $j \geqslant 2$, is, a priori, more difficult to determine. We know already that this condition is equivalent to the degree $2j$ maximal overlap sequence of A,

$A^{2j} =$ 

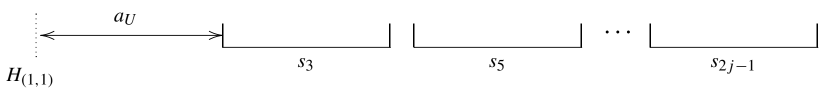
having no head arrows in the path of unoverlapped arrows p. The problem is that p may have length less, greater or equal to n. If $\ell(p) \geqslant n$, then there will be a head arrow in p, and we can conclude that A does not terminate at degree $2j$. If $\ell(p) < n$, then p equals the path $\mathfrak{t}(s_{2j-1}) \to \mathfrak{t}(s_{2j})$. We then just need to check the path $\mathfrak{t}(s_{2j-1}) \to \mathfrak{t}(s_{2j})$ for head arrows to determine whether or not A terminates at degree $2j$. We calculate whether $\ell(p) < n$

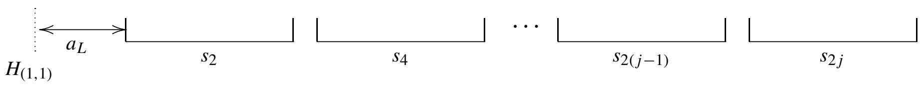
with a counting argument. Let A^{2j} be the maximal overlap sequence shown above, with $\mathfrak{o}(A^{2j}) = \mathfrak{o}(s_2)$.

Suppose first that the lower half of A^{2j} is not in row 1 of $\mathcal{T}_A$. Consider the two paths in $\Bbbk\mathcal{Q}$ that consist of A^{2j-1} and A^{2j} with the path a_L appended to the start of each, where $\ell(a_L) < n$, $\mathfrak{o}(a_L) = H_{(1,1)}$, $\mathfrak{t}(a_L) = \mathfrak{o}(s_2)$ and a_U is the path as shown below:



Let $U = a_L A^{2j-1}$ and $L = a_L A^{2j}$. We visualise the paths U and L respectively in the following natural way:





Write $U = c^k q$, where c is the cycle such that $\ell(c) = n$, $\mathfrak{o}(c) = \mathfrak{t}(c) = H_{(1,1)}$ and q is such that $\ell(q) < n$, $\mathfrak{o}(q) = H_{(1,1)}$, $\mathfrak{t}(q) = \mathfrak{t}(s_{2j-1})$. Then $L = Up = c^k qp$. We will count the number of occurrences of $\alpha_{(1,1)}$ in U and in L. Then, if $\alpha_{(1,1)}$ occurs more often in L than in U, we can conclude that at least one $\alpha_{(1,1)}$ is in p. Since $\alpha_{(1,1)}$ is a head arrow we would know that A does not terminate at degree $2j$. Otherwise, if $\alpha_{(1,1)}$ occurs the same number of times in U as it does in L, then we know it cannot occur in p. Thus $\ell(p) < n$ and so we check the path $\mathfrak{t}(s_{2j-1}) \to \mathfrak{t}(s_{2j})$ for head arrows as detailed above.

For any finite path $\nu \in \Bbbk\mathcal{Q}$ let $N(\nu)$ be the number of occurrences of $\alpha_{(1,1)}$ in ν. Then

$$N(L) = 1 + \sum_{z=1}^{j} N(s_{2z}) \quad \text{and} \quad N(U) = 1 + \sum_{z=2}^{j} N(s_{2z-1}).$$

For firstly the "gaps" between the relations (the paths Y_z, $4 \leqslant z \leqslant 2j$, in the Preliminaries section) contain no head arrows by maximality of the construction of A^{2j}, and secondly, since $\mathfrak{o}(s_2) \neq H_{(1,1)}$ the paths a_U and a_L contain exactly one copy of $\alpha_{(1,1)}$. Thus

$$\sum_{z=1}^{j} N(s_{2z}) = \sum_{z=2}^{j} N(s_{2z-1}) \iff N(L) = N(U) \implies \ell(p) < n; \quad \text{and}$$

$$\sum_{z=1}^{j} N(s_{2z}) > \sum_{z=2}^{j} N(s_{2z-1}) \quad \Longleftrightarrow \quad N(L) > N(U) \quad \Longrightarrow \quad p$$

contains a head arrow. Note that the function N here is an extension to all paths of the function of the same name in Definition 1.8. We leave it to the reader to verify that $N(s_3) = N(i+1, 1)$, where $s_2 = r_{(i,1)}$. This shows that marking entry (i, j) with a flag2 via Definition 1.8, part (2i), is equivalent to the corresponding maximal overlap sequence terminating at degree $2j$.

The case where $\mathfrak{o}(s_2) = H_{(1,1)}$ is almost identical to above; the only change is that now $\ell(a_L) = 0$ and we have $N(a_L) = 0$, $N(a_U) = 1$. This is left to the reader.

(4) This part follows part (3) above, but without the necessity for the counting argument, since by Proposition 1.2 the path of unoverlapped arrows p is always such that $\ell(p) \leqslant n$.

(5) This part is also left in the hands of the reader. □

2. Calculating flags

In this section we present three theorems that speed up calculation of the smo-tube with its flags: for this reason the section becomes rather technical. Note that certain of the auxiliary results presented here will be drawn upon throughout the remainder of the paper. We will illustrate calculation of the smo-tube with two examples.

Lemma 2.1. *For any entry (i, j) in an smo-tube, we have that the concatenation of paths $H_{(i,j)} \to H_{(i+1,j)} \to \cdots \to H_{(|D|,j)} \to H_{(1,j)} \to \cdots \to H_{(i-1,j)} \to H_{(i,j)}$ is of length n.*

Proof. Using Proposition 1.1 and the discussion in Definition 1.7, the result follows by induction on j. □

Lemma 2.2. *If x is some repetition index, then for all j greater or equal to 2, there exists some row i in $\mathcal{T}_\Lambda$ such that $x = (i, j)$.*

Proof. Recall that band 1 contains the indices of all degeneration tail vertices. By definition, the next entry in the smo-sequence of a such a tail vertex is the index of a degeneration head. We get all degeneration head indices this way; these appear in band 2. As remarked in Definition 1.7, each repetition index is also a degeneration head index, and so each repetition index appears in band 2. Immediately we get that each repetition index appears in band j, for all $j \geqslant 2$. □

Theorem 2.3. *If an entry (i, j) of $\mathcal{T}_\Lambda$ is assigned a flag1, then $(i, j) \neq (i+1, j)$ and $(i, j+1) = (i+1, j+1)$.*

Moreover, if Λ has more than one repetition index, then (i, j) is assigned a flag1 if and only if $(i, j) \neq (i+1, j)$ and $(i, j+1) = (i+1, j+1)$.

Proof. If entry (i, j) is marked with a flag1, then by definition $(i+1, j) \neq (i, j)$ and there is no head arrow in the path $T_{(i,j)} \to T_{(i+1,j)}$. This means that $H_{(i,j+1)}$ follows both $T_{(i,j)}$ and $T_{(i+1,j)}$ giving $H_{(i,j+1)} = H_{(i+1,j+1)}$ and so $(i, j+1) = (i+1, j+1)$.

Conversely suppose that (i, j) is such that $(i, j) \neq (i+1, j)$ and $(i, j+1) = (i+1, j+1)$; this means that either the path $T_{(i,j)} \to T_{(i+1,j)}$ or the path $T_{(i+1,j)} \to T_{(i,j)}$ is free from head arrows. Suppose also that Λ has more than one repetition index. By Lemma 2.2, all repetition indices occur in band $j+1$ so, since Λ has more than one repetition index, there must be a third row in $\mathcal{T}_\Lambda$, row k say, such that $(i+1, j+1) \neq (k, j+1)$. This gives us $(i+1, j) \neq (k, j)$ and $(i, j) \neq (k, j)$. Thus we have that $T_{(i,j)} \to T_{(i+1,j)} \to T_{(k,j)} \to T_{(i,j)}$ is a path of length n by Lemma 2.1 and Proposition 1.1. Since a different head vertex follows $T_{(k,j)}$ than follows $T_{(i,j)}$ and $T_{(i+1,j)}$, there is a head vertex, namely $H_{(k,j+1)}$, in the path $T_{(i+1,j)} \to T_{(i,j)}$. Thus there must no head arrow in the path $T_{(i,j)} \to T_{(i+1,j)}$, and so (i, j) will be assigned a flag1. □

Proposition 2.4. *The number of rows in $\mathcal{T}_\Lambda$ that do not have a flag1 is equal to the number of distinct repetition indices of Λ.*

Proof. Recall that the Mth band of $\mathcal{T}_\Lambda$ is the first to contain only repetition indices. If Λ has $\mu \geqslant 2$ repetition indices, then by Lemma 2.2 there must be some row i such that $(i, M+1) \neq (i+1, M+1)$. By Lemma 1.6, we get $(i, j) \neq (i+1, j)$, $\forall j \geqslant 1$. Thus by Theorem 2.3, row i will never be marked flag1. Lemmas 2.1 and 2.2 give us exactly μ rows i in $\mathcal{T}_\Lambda$ such that $(i, M+1) \neq (i+1, M+1)$, which gives us at least μ rows without a flag1 by above. By Theorem 2.3, any row i with $(i, M+1) = (i+1, M+1)$ has a flag1. This gives us precisely the same number of unflagged rows as we have repetition indices.

Suppose then that Λ has only one repetition index. We get our result immediately if $|D| = 1$, so suppose $|D| \geqslant 2$. All entries of band M are equal to the same repetition index, x say. From the proof of Lemma 2.2, band 2 contains at least 2 distinct indices and so we have $M \geqslant 3$. Now, all entries of band $M-1$ index tail vertices that are followed by H_x, so by Lemma 2.1 and Proposition 1.1 there exists exactly one row, i say, such that the path $T_{(i,M-1)} \to T_{(i+1,M-1)}$ contains a head arrow. It follows from Theorem 2.3 that row i will not get a flag1. Let row k be different to row i. Then there exists j, with $1 \leqslant j \leqslant M-1$, such that $(k, j) \neq (k+1, j)$ and $(k, j+1) = (k+1, j+1)$. We will show (k, j) is marked flag1. To seek a contradiction suppose it is not. Then by definition, $T_{(k,j)} \to T_{(k+1,j)}$ contains a head arrow. Since both $T_{(k,j)}$ and $T_{(k+1,j)}$ are followed by $H_{(k,j+1)}$, this means the path $T_{(k+1,j)} \to T_{(k,j)}$ must contain no head arrows. Hence by Lemma 2.1, we must have $j = M-1$, but since $k \neq i$, we get a contradiction. Thus entry (k, j) is marked flag1 for all $k \neq i$. □

Theorem 2.5. *If entry $(i, 1)$ of $\mathcal{T}_\Lambda$ is marked flag0, then $(i, 1) + 1 = (i, 2)$ modulo m.*

Moreover, if $\ell(r_{(i,1)}) \leqslant n$, entry $(i, 1)$ is marked flag0 if and only if $(i, 1) + 1 = (i, 2)$ modulo m.

Proof. Suppose $(i, 1)$ is marked flag0. This means no head arrow lies in $r_{(i,1)}$ except $\alpha_{(i,1)}$ once at the start. Therefore $H_{(i,1)+1}$ is the head vertex that follows $T_{(i,1)}$, and hence $(i, 1) + 1 = (i, 2)$.

Conversely, suppose $(i,1)+1=(i,2)$ and that $\ell(r_{(i,1)}) \leqslant n$. We have that $H_{(i,1)+1}$ is the head vertex that follows $T_{(i,1)}$ and so the path $H_{(i,1)} \to T_{(i,1)} \to H_{(i,1)+1}$ must have length less than or equal to n. Suppose $r_{(i,1)}$ is not the only relation, else we are done. This means $(i,1) \neq (i,1)+1$ and so $\alpha_{(i,1)}$ is the only head arrow in the path $H_{(i,1)} \to H_{(i,1)+1}$. Since $\ell(r_{(i,1)}) \leqslant n$, $r_{(i,1)}$ must be an initial subpath of $H_{(i,1)} \to H_{(i,1)+1}$. Thus $r_{(i,1)}$ contains no head arrow other than $\alpha_{(i,1)}$. Hence $(i,1)$ gets marked with a flag0. $\square$

To prove a similar theorem concerning flag2s, we need access to a few more results. Recall that λ is the order of the repetitions.

Proposition 2.6. *If $\lambda \geqslant 2$, and r_{x_1} and r_{x_2} are two distinct relations, then the path $H_{x_1} \to T_{x_1} \to T_{x_2} \to H_{x_2} \to H_{x_1}$ has length greater than n.*

Proof. We proceed by contradiction. Assume that r_{x_1} and r_{x_2} are distinct relations such that the path $H_{x_1} \to T_{x_1} \to T_{x_2} \to H_{x_2} \to H_{x_1}$ is of length n. Since $\lambda \geqslant 2$, H_{x_2} cannot follow T_{x_2}. We therefore have a third relation, r_{x_3} say, distinct from r_{x_1} and r_{x_2}, such that H_{x_3} is in the path $T_{x_2} \to H_{x_2} - 1$. By Proposition 1.1, T_{x_3} must lie in the path $T_{x_1} \to T_{x_2}$.

It is clear that each time this argument is applied to r_{x_1} and r_{x_i}, for some $i \geqslant 2$, we get a new relation $r_{x_{i+1}}$ distinct from all the others. Since $\mathcal{I}$ has a fixed finite generating set, we get our contradiction. $\square$

Corollary 2.7. *Suppose $\lambda \geqslant 2$ and let r_k be a relation such that $cn < \ell(r_k) \leqslant (c+1)n$, for some non-negative integer c. Then $cn < \ell(r) \leqslant (c+1)n$ for all relations r.*

Proof. Let r_k be a relation such that $cn < \ell(r_k) \leqslant (c+1)n$, for some positive integer c and let r_l be a relation such that $(c-1)n < \ell(r_l) \leqslant cn$. To ensure that r_l is not a subpath of r_k it is clear that the path $H_k \to T_k \to T_l \to H_l \to H_k$ must have length n. This contradicts the above proposition. $\square$

Definition 2.8. Let $r_{y_1}, \dots, r_{y_l}$ be the repetition relations of Λ, ordered such that the concatenation of l paths $H_{y_1} \to H_{y_2} \to \cdots \to H_{y_l} \to H_{y_1}$ is of length n. Then we call the path $H_{y_i} \to H_{y_{i+1}}$ the *repetition path* $\mathbf{b}_{y_{i+1}}$, where $l+1=1$. Clearly every arrow in $\Bbbk\mathcal{Q}$ is in precisely one repetition path.

Example 5. In Example 1 the repetition relations are r_1, r_2, r_5, and r_6; the repetition paths are $\eta_1 \cdots \eta_6$, $\eta_7 \cdots \eta_{13}$, $\eta_{14} \cdots \eta_{19}$, and $\eta_{20} \cdots \eta_{25}$.

Lemma 2.9. *Suppose $\lambda \geqslant 2$. Let H_l be a degeneration head vertex with r_l not a repetition relation, and let H_l and T_l lie in the repetition path $\mathbf{b}_a$, with $T_l \neq \mathfrak{o}(\mathbf{b}_a)$. Then $\mathfrak{t}(\mathbf{b}_a) = H_a$ is the head vertex that follows T_l.*

Proof. Let the setup be as above and let $H_k = \mathfrak{o}(\mathbf{b}_a)$, so that $\mathbf{b}_a$ is the path $H_k \to H_a$. To seek a contradiction suppose H_a does not follow T_l. Firstly, T_l lies in the path $H_k + 1 \to H_l - 1$, otherwise the path $H_l \to T_l \to T_k \to H_k \to H_l$ would have length n, since T_k is a repetition tail vertex. This would contradict Proposition 2.6. Now $\lambda \geqslant 2$, so H_a may not follow T_a and so T_a, since it is a repetition tail vertex, is in the path $H_a \to H_k$. By the ordering imposed by Proposition 1.1, we get that the path $H_a \to T_a \to T_k \to H_k \to H_a$ is of length n. This contradicts Proposition 2.6 and we get our result. $\square$

Lemma 2.10. *Suppose $\lambda \geqslant 2$ and let (i, j) be an entry of $\mathcal{T}_\Lambda$. Then*

(1) *if $j = 2$, the path $H_{(i,1)} \to H_{(i,1)+1} \to H_{(i,2)} \to H_{(i+1,2)} \to H_{(i,1)}$ has length n;*
(2) *if $j \geqslant 3$, the path $H_{(i,j-1)} \to H_{(i+1,j-1)} \to H_{(i,j)} \to H_{(i+1,j)} \to H_{(i,j-1)}$ has length n.*

Proof. (1) Suppose that $j = 2$. Since $H_{(i,2)}$ and $H_{(i+1,2)}$ are both degeneration head vertices, the path $T_{(i,1)} \to H_{(i,2)} \to T_{(i,1)+1} \to H_{(i+1,2)} \to T_{(i,1)}$ is of length n. Also $T_{(i+1,2)}$ must be in the path $H_{(i+1,2)} + 1 \to T_{(i,1)}$ since $\lambda \geqslant 2$. Now, $T_{(i,2)}$ must be in the path $T_{(i,1)} \to T_{(i+1,2)}$ else we contradict Proposition 2.6 with $r_{(i+1,2)}$ and $r_{(i,2)}$. Since $\lambda \geqslant 2$, $T_{(i,2)} \neq T_{(i,1)}$ so in fact $T_{(i,2)}$ is in the path $T_{(i,1)+1} \to T_{(i+1,2)}$. Thus the path $T_{(i,1)} \to T_{(i,1)+1} \to T_{(i,2)} \to T_{(i+1,2)} \to T_{(i,1)}$ has length n and Proposition 1.1 yields our result.

(2) Suppose now that $j \geqslant 3$. We show that the path $H_{(i,j-1)} \to H_{(i+1,j-1)} \to H_{(i,j)} \to H_{(i+1,j)} \to H_{(i,j-1)}$ has length n. Note that if $(i, j-1) = (i+1, j-1)$, then $(i, j) = (i+1, j)$ and we immediately get our result. Thus we assume $(i, j-1) \neq (i+1, j-1)$. Consider the two vertices $H_{(i,j-1)}$ and $H_{(i+1,j-1)}$, the terminating vertices of the degeneration paths $\mathbf{d}_{(i,j-1)}$ and $\mathbf{d}_{(i+1,j-1)}$, respectively.

We first wish to place $H_{(i,j)}$ in the path $H_{(i+1,j-1)} \to H_{(i,j-1)}$. We assume $(i, j) \neq (i+1, j-1)$ and $(i, j) \neq (i, j-1)$. To seek a contradiction suppose $H_{(i,j)}$ is in the path $H_{(i,j-1)} + 1 \to H_{(i+1,j-1)} - 1$. Then by Lemmas 2.1 and 2.2, $\mathbf{d}_{(i,j)}$, and therefore $\mathbf{d}_{(i,j-1)}$, cannot be a repetition degeneration path. By Lemma 2.9 this means $H_{(i,j-1)}$ and $T_{(i,j-1)}$ cannot be in the same repetition path. Thus there must be a repetition head vertex in the path $H_{(i,j-1)} + 1 \to H_{(i,j)} - 1$, contradicting Lemmas 2.1 and 2.2 regarding band $j - 1$. Hence $H_{(i,j)}$ must lie in the path $H_{(i+1,j-1)} \to H_{(i,j-1)} - 1$.

It remains only to locate $H_{(i+1,j)}$ in the path $H_{(i,j)} \to H_{(i,j-1)}$. Assume $(i+1, j) \neq (i, j)$ and $(i+1, j) \neq (i, j-1)$. There are two cases to consider:

(i) If $H_{(i+1,j)}$ lies in the path $H_{(i,j-1)} \to H_{(i+1,j-1)}$, then so does $T_{(i+1,j-1)}$. Two subcases arise. If $H_{(i+1,j-1)}$ is a repetition head, then so is $H_{(i+1,j)}$. This is contradicted by Lemmas 2.1 and 2.2. If $H_{(i+1,j-1)}$ is not a repetition head, then we must have a repetition head in the path $H_{(i+1,j)} \to H_{(i+1,j-1)}$ or Lemma 2.9 will be contradicted. However, the existence of this repetition head again contradicts Lemmas 2.1 and 2.2.
(ii) If $H_{(i+1,j)}$ lies in the path $H_{(i+1,j-1)} + 1 \to H_{(i,j)} - 1$, then so does $T_{(i+1,j-1)}$. Proposition 2.6 on relations $r_{(i,j-1)}$ and $r_{(i+1,j-1)}$ provides the contradiction.

Hence $H_{(i+1,j)}$ must lie in the path $H_{(i,j)} \to H_{(i,j-1)}$. This completes the proof. $\square$

We can at last prove our final theorem of this section.

Theorem 2.11. *If an entry (i, j) of $\mathcal{T}_\Lambda$ is assigned a flag2, then*

(1) *for $j = 2$ we have $(i, 1) + 1 \neq (i, 2)$ and $(i + 1, 2) = (i, 3)$;*
(2) *for $j \geqslant 3$ we have $(i + 1, j - 1) \neq (i, j)$ and $(i + 1, j) = (i, j + 1)$.*

Moreover, if $\lambda \geqslant 2$ and $\ell(r) \leqslant n$ for all relations r, then (i, j) is assigned a flag2 if and only if the appropriate condition (1) *or* (2) *holds.*

Proof. If (i, j) is assigned a flag2, then showing the appropriate condition (1) or (2) is easy.

For the reverse direction, suppose $\lambda \geqslant 2$ and $\ell(r) \leqslant n$ for all relations r. Suppose also that $j \geqslant 3$ and the conditions from (2) hold. For a contradiction assume that no flag2 is assigned to entry (i, j). As $\ell(r) \leqslant n$ for all relations r, and by Theorem 1.10, no flag2 assigned to entry (i, j) is equivalent to the existence of a head arrow in the path of unoverlapped arrows $T_{(i+1,j-1)} \to T_{(i,j)}$. However, since $T_{(i+1,j-1)}$ and $T_{(i,j)}$ are both followed by the same head vertex, we have the path $T_{(i,j)} \to T_{(i+1,j-1)}$ free from head arrows. As $\lambda \geqslant 2$ we must have $T_{(i+1,j)}$ in the path $T_{(i+1,j-1)} + 1 \to T_{(i,j)} - 1$. This means the path $T_{(i+1,j-1)} \to T_{(i,j)} \to T_{(i+1,j)}$ has length greater than n, contradicting Lemma 2.10.

The case of $j = 2$ is similar. $\square$

Example 6. For our algebra of Example 1, we have a single flag1 in the smo-tube. There are no flag0s or flag2s. The position of the flag1 is indicated by the square box:

$$\begin{array}{c|cccccccc}
 & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \cdots\cdots \\
\mathbf{5} & 6 & 2 & 6 & 2 & 6 & 2 & \cdots\cdots \\
\mathbf{4} & 5 & 1 & 5 & 1 & 5 & 1 & \cdots\cdots \\
\mathbf{3} & 4 & \boxed{8} & 5 & 1 & 5 & 1 & \cdots\cdots \\
\mathbf{2} & 3 & 6 & 2 & 6 & 2 & 6 & \cdots\cdots \\
\mathbf{1} & 1 & 5 & 1 & 5 & 1 & 5 & \cdots\cdots
\end{array}$$

Theorem 2.3 was used to mark the flag1: notice above that $(3, 2) \neq (4, 2)$ but that $(3, 3) = (4, 3)$. Since here $\ell(r) \leqslant n$, for all relations r, and $\lambda \geqslant 2$, we may use the full equivalences of Theorems 2.5 and 2.11 to conclude that there are no flag0s or flag2s present in $\mathcal{T}_\Lambda$.

Example 7. Let us consider a different example. We keep the same quiver of 25 vertices and 25 arrows, but this time put on 6 relations:

$$r_1 = \eta_1 \cdots \eta_6, \qquad r_2 = \eta_2 \cdots \eta_{10}, \qquad r_3 = \eta_4 \cdots \eta_{13},$$
$$r_4 = \eta_{10} \cdots \eta_{15}, \qquad r_5 = \eta_{15} \cdots \eta_{22}, \qquad r_6 = \eta_{18} \cdots \eta_{24}.$$

We get a different smo-tube; this time all rows have a flag. The position of a flag0 or flag2 is indicated by a circle; the flag1 by the square:

$$\begin{array}{c|ccccccc c}
 & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \cdots\cdots \\
\mathbf{4} & 6 & 1 & 4 & 6 & 1 & 4 & 6 & \cdots\cdots \\
\mathbf{3} & 4 & 6 & 1 & 4 & 6 & 1 & 4 & \cdots\cdots \\
\mathbf{2} & 3 & \boxed{5} & 1 & 4 & 6 & 1 & 4 & \cdots\cdots \\
\mathbf{1} & 1 & 4 & 6 & 1 & 4 & 6 & 1 & \cdots\cdots
\end{array}$$

Notice in each example that the number of rows without a flag1 is equal to the number of repetition indices, as stated in Proposition 2.4. It is no fluke that all the rows have flags in the second example above. The next section, while introducing the notion of shifts, shows that if an smo-tube has any flag0s or flag2s at all, then all rows have a flag.

3. Repetition shift

The aim of this section is to present some further structure of the smo-tube, the so-called repetition shift, and we see how important the repetition shift is in governing the placement of flags. This importance is shown in Theorem 3.10, which gives exact conditions for when and how the different types of flag will be present on the smo-tube. We begin with a definition. Recall that the Mth band of $\mathcal{T}_\Lambda$ is the first to contain only repetition indices and that f_Λ is the smo-function of Λ.

Definition 3.1. Let a and b be rows in $\mathcal{T}_\Lambda$ that share the same repetition. Then row a is said to have a *b-shift of N* if $(a, M+N) = (b, M)$ and $N \geqslant 0$ is minimal with this property.

Lemma 3.2. *Let R be a repetition of Λ with order $\lambda \geqslant 2$. Let $x_1, \ldots, x_\lambda$ be the repetition indices of R with the path $H_{x_1} \to \cdots \to H_{x_\lambda} \to H_{x_1}$ of length n. If $N \geqslant 0$ is minimal such that $f_\Lambda^N(x_1) = x_2$, then N is minimal such that $f_\Lambda^N(x_i) = x_{i+1}$, for all $i = 1, \ldots, \lambda$. Note that if $i = \lambda$ we take $i + 1 = 1$.*

Proof. Pick $2 \leqslant k \leqslant \lambda$. Let l be minimal such that $f_\Lambda^l(x_1) = x_k$. Thus $f_\Lambda^{l+N}(x_1) = f_\Lambda^N(x_k)$ giving $f_\Lambda^l(x_2) = f_\Lambda^N(x_k)$. Now the proof of Lemma 1.6, applied l times, gives us no R-indexed head vertex in the path $H_{f_\Lambda^l(x_1)} \to H_{f_\Lambda^l(x_2)}$ except the start and end vertices. Hence there is no R-indexed head vertex in the path $H_{x_k} \to H_{f_\Lambda^l(x_2)}$ except the start and end vertices. Thus $f_\Lambda^l(x_2) = x_{k+1}$, and so $f_\Lambda^N(x_k) = x_{k+1}$. Minimality of N follows since x_1 was arbitrary. $\square$

Motivated by this result, we now make the following definition.

Definition 3.3. Let R be a repetition of Λ, with x_1 and x_2 repetition indices of R such that no R-indexed head vertex lies in the path $H_{x_1} \to H_{x_2}$ except the start and end vertices. The *repetition shift* of R is the least positive integer N such that $f_\Lambda^N(x_1) = x_2$.

Example 8. We take the usual oriented cycle $\mathcal{Q}$ with 25 vertices and 25 arrows, and let $\Lambda = \Bbbk\mathcal{Q}/\mathcal{I}$, where $\mathcal{I}$ is generated by the 8 relations:

$$r_1 = \eta_1 \cdots \eta_{16}, \qquad r_2 = \eta_3 \cdots \eta_{19}, \qquad r_3 = \eta_5 \cdots \eta_{21}, \qquad r_4 = \eta_8 \cdots \eta_{22},$$
$$r_5 = \eta_{12} \cdots \eta_{24}, \qquad r_6 = \eta_{14} \cdots \eta_4, \qquad r_7 = \eta_{19} \cdots \eta_{11}, \qquad r_8 = \eta_{21} \cdots \eta_{14}.$$

The only flags on this smo-tube are flag1s:

$$\begin{array}{c|cccccccc}
 & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \cdots\cdots \\
\mathbf{5} & 7 & 5 & 1 & 7 & 5 & 1 & 7 & \cdots\cdots \\
\mathbf{4} & 6 & \boxed{3} & 1 & 7 & 5 & 1 & 7 & \cdots\cdots \\
\mathbf{3} & 5 & 1 & 7 & 5 & 1 & 7 & 5 & \cdots\cdots \\
\mathbf{2} & 2 & \boxed{8} & 7 & 5 & 1 & 7 & 5 & \cdots\cdots \\
\mathbf{1} & 1 & 7 & 5 & 1 & 7 & 5 & 1 & \cdots\cdots
\end{array}$$

There is only one repetition here, $R = (1, 7, 5)$. The repetition shift N of R is equal to 2.

The repetition shift can be observed in the smo-tube by taking a flag1-free row containing R: the b-shift of that row gives the repetition shift, where row b is the next flag1-free row up that contains R.

Example 9. The algebra in Example 1 has a repetition shift of $N = 1$.

We use the proof of Lemma 1.6 to get the following result.

Lemma 3.4. *Let Λ have a repetition with repetition shift N. Then all repetitions of Λ have repetition shift N.*

For the remainder of the paper, fix N as the repetition shift of all the repetitions of Λ.

Proposition 3.5. *Suppose Λ has only one repetition R, and this is of order $\lambda \geqslant 2$. Then for each i, row i of the smo-tube of Λ has $(i+1)$-shift equal to 0 or N.*

Proof. If we have only 2 degeneration paths then, since $\lambda \geqslant 2$, the indices of both paths must be repetition indices. So clearly R has a repetition shift of $N = 1$. Now, since there are only 2 rows in $\mathcal{T}_\Lambda$, we get immediately from Lemma 2.2 that each row i has $(i+1)$-shift of 1.

Thus assume Λ has at least 3 degeneration paths. Let $x_1, \ldots, x_\lambda$ be the repetition indices of R, with the path $H_{x_1} \to \cdots \to H_{x_\lambda} \to H_{x_1}$ of length n. Let y and z be distinct degeneration head indices such that no degeneration head vertices lie in the path $H_y \to H_z$ other than the start and end vertices. Then for some $k \in \{1, \ldots, \lambda\}$ (with $\lambda + 1 = 1$) we have that H_y and H_z lie in the path $H_{x_k} \to H_{x_{k+1}}$. We will show that either $f_\Lambda^M(y) = f_\Lambda^M(z)$ or $f_\Lambda^{M+N}(y) = f_\Lambda^M(z)$.

By applying Proposition 1.1 M times, we get that the path

$$H_{f_\Lambda^M(x_k)} \to H_{f_\Lambda^M(y)} \to H_{f_\Lambda^M(z)} \to H_{f_\Lambda^M(x_{k+1})}$$

has length less than or equal to n. By M applications of the proof of Lemma 1.6 we have no repetition head vertices in this path except the start and end vertices. Since by the definition of M, $f_\Lambda^M(y)$ and $f_\Lambda^M(z)$ are repetition indices, three possibilities occur:

(1) $f_\Lambda^M(y) = f_\Lambda^M(z) = f_\Lambda^M(x_k)$,
(2) $f_\Lambda^M(y) = f_\Lambda^M(z) = f_\Lambda^M(x_{k+1})$,
(3) $f_\Lambda^M(y) = f_\Lambda^M(x_k)$, $f_\Lambda^M(z) = f_\Lambda^M(x_{k+1})$.

Let i be such that $y = (i, 2)$, and so by hypothesis $z = (i+1, 2)$. If either possibility (1) or (2) occurs, then row i has an $(i+1)$-shift of 0, since $(i, M+2) = f_\Lambda^M(y) = f_\Lambda^M(z) = (i+1, M+2)$ and hence $(i, M+\lambda) = (i+1, M+\lambda)$, so $(i, M) = (i+1, M)$. If possibility (3) occurs, then $f_\Lambda^{M+N}(y) = f_\Lambda^{M+N}(x_k) = f_\Lambda^M(x_{k+1}) = f_\Lambda^M(z)$, and by a similar argument $(i, M+N) = (i+1, M)$; so row i has an $(i+1)$-shift of N. $\square$

We can bring the above results together to form the next proposition, which builds upon Proposition 2.4. First though, as a consequence of the proof of Lemmas 1.6 and 2.1 we have the following result.

Lemma 3.6. *If $\lambda \geqslant 2$ then there is an ordering on the repetitions of Λ: $R_1, \ldots, R_l$ such that whenever $(i, j) \neq (i+1, j)$, for some $j \geqslant M$, we have $(i, j) \in R_k$ and $(i+1, j) \in R_{k+1}$, some $k \in \{1, \ldots, l\}$ with $l + 1 = 1$.*

Remark. Notice that in the case of Example 8 we have only one repetition. This renders Lemma 3.6 somewhat trivial in that $l = 1$, giving an ordered list of one element. Thus $R_{k+1} = R_k$ and the lemma is then obvious in this case.

Proposition 3.7. *Suppose $\lambda \geqslant 2$. If the following conditions all occur*:

(1) *Λ has only one repetition R,*
(2) *R has repetition shift $N = 1$,*
(3) *$\ell(r) \leqslant n$ for all relations r,*

then all rows in $\mathcal{T}_\Lambda$ are flagged. Otherwise the number of unflagged rows is equal to the number of distinct repetition indices.

Proof. If conditions (1) and (2) hold, then by Proposition 3.5, for each i, row i has $(i+1)$-shift either 0 or 1. If this shift is 0 then, since $\lambda \geqslant 2$, we can use Theorem 2.3 to get a flag1 in row i. If the $(i+1)$-shift is 1 then by condition (3) we can use Theorems 2.5 and 2.11 to get a flag0 or flag2 in row i. Hence all the rows are flagged.

Otherwise Proposition 2.4 states there are the same number of rows without a flag1 as there are repetition indices. Thus we need to show there are no flag0s or flag2s in $\mathcal{T}_\Lambda$ whenever one of the three conditions above fails.

If condition (1) fails, we have 2 or more repetitions. Let (i, j) be an entry of $\mathcal{T}_\Lambda$, with $j \geqslant 3$. Suppose for a contradiction that $(i, j) = (i+1, j-1)$; then we have $(i, j+M) = (i+1, j+M-1)$ and so, since $\lambda \geqslant 2$, this gives $(i, j+M) \neq (i+1, j+M)$ with $(i, j+M)$ and $(i+1, j+M)$ in the same repetition. This is prohibited by Lemma 3.6. We can thus assume that $(i, j) \neq (i+1, j-1)$ for all i, j, and so by Theorem 2.11 we have no flag2s. A similar argument in the case $j = 2$ shows that $(i, 2) \neq (i, 1) + 1$ and hence by Theorem 2.5 that there are no flag0s.

If condition (2) fails, then for all i and for all $j \geqslant 3$ we have $(i, j) \neq (i+1, j-1)$ and $(i, 2) \neq (i+1, 1) + 1$. Theorems 2.5 and 2.11 then give us no flag0s or flag2s.

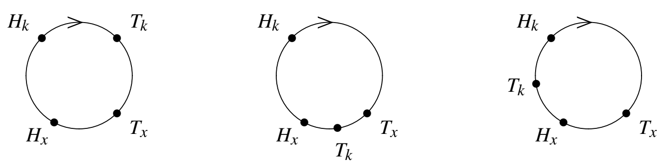
If condition (3) fails, we get our result by Corollaries 2.7 and 1.9. □

We now focus our attention to the case where the order of the repetitions is 1.

Lemma 3.8. *Let $\lambda = 1$ and let r_x be a repetition relation. If $(a-1)n < \ell(r_x) \leqslant an$ for some positive integer a, then $(a-1)n < \ell(r_k) \leqslant (a+1)n$ for all relations r_k.*

Proof. Let $a \in \mathbb{Z}$ be such that $(a-1)n < \ell(r_x) \leqslant an$ and let r_k be some relation. Clearly we must have $\ell(r_k) \leqslant (a+1)n$ else r_x would be a subpath of r_k.

Now, since H_x must follow T_x, we must have H_k in the path $H_x \to T_x$. The following three diagrams show the possible relative positions of T_k: note that we allow $T_x = H_x$ and $T_k = H_x$ or H_k where appropriate. By looking at each diagram in turn it is not hard to see that we must have $\ell(r_k) > (a-1)n$ to prevent r_k being a subpath of r_x.



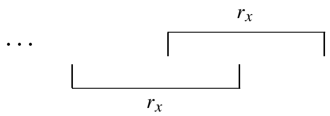
□

Proposition 3.9. *Suppose $\lambda = 1$. If the following conditions both occur:*

(1) *Λ has only one repetition relation r_x,*
(2) *$\ell(r_x) \leqslant n$,*

then all rows in $\mathcal{T}_\Lambda$ are flagged. Otherwise the number of unflagged rows is equal to the number of distinct repetition indices.

Proof. Suppose both conditions hold and yet we have an unflagged row in the smo-tube of Λ. This gives rise to a degree $2M+1$ maximal overlap sequence ending with



which contradicts condition (2), since the head arrow of r_x appears here in r_x twice.

For the converse we will show that if either condition (1) or (2) fails, then the number of unflagged rows in $\mathcal{T}_\Lambda$ is equal to the number of repetition indices.

Suppose (1) fails. Then we have $k \geqslant 2$ repetition relations. We know from Proposition 2.4 that there are k rows in $\mathcal{T}_\Lambda$ with no flag1: let row i be one of these. To seek a contradiction suppose that entry (i, j) has a flag2 and that $j \geqslant 3$. By Theorem 2.11, we have $(i, j) \neq (i+1, j-1)$ and $(i, j+1) = (i+1, j)$. Thus $(i, j+M+1) = (i+1, j+M)$ and so, since $\lambda = 1$, $(i, j+M+1) = (i+1, j+M+1)$. By Theorem 2.3, row i has a flag1, contradicting our hypothesis. The cases where $j = 1$ or 2 are similar to the above, with the case $j = 1$ prohibiting flag0s.

Finally, if (2) fails then Lemma 3.8 and Corollary 1.9 show that there are no flag0s or flag2s in $\mathcal{T}_\Lambda$. Thus by Proposition 2.4, the number of unflagged rows is equal to the number of distinct repetition indices. $\square$

Putting the last two propositions together gives us the theorem of this section. Using Theorem 1.10, we follow with a useful corollary.

Theorem 3.10. *The smo-tube $\mathcal{T}_\Lambda$ has every row flagged if and only if one of the following occurs*:

(1) $\lambda \geqslant 2$, *there is only* 1 *repetition,* $N = 1$, *and* $\ell(r) \leqslant n$ *for all relations* r.
(2) $\lambda = 1$, *there is only* 1 *repetition relation* r_x, *and* $\ell(r_x) \leqslant n$.

Otherwise the smo-tube has the same number of unflagged rows as it does distinct repetition indices.

Corollary 3.11. *If Λ has an infinite extending sequence, then $\mathcal{T}_\Lambda$ has no flag0s or flag2s.* $\square$

4. Finite generation of the Ext-algebra

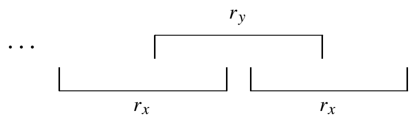
The previous sections have given us a method to identify exactly those extending sequences that are infinite in extent. In this section we bring together these results to determine precisely when $E(\Lambda)$ is finitely generated: Theorem 4.14, our main result, does this for us. Toward the end of this section there are a number of propositions each of which, for different conditions on $E(\Lambda)$, give an explicit finite generating set. As noted in the introduction, we will freely interchange a maximal overlap sequence and its underlying path.

However, when we talk about products of maximal overlap sequences, we refer to the corresponding product as basis elements in $E(\Lambda)$. Thus the product of two maximal overlap sequences may be zero in $E(\Lambda)$, whilst the product of their underlying paths may be non-zero in $\Bbbk\mathcal{Q}$. We recall here that, for Λ a cycle algebra, λ is the order of the repetitions, N is the repetition shift, and M is the first band of $\mathcal{T}_\Lambda$ to contain only repetition indices.

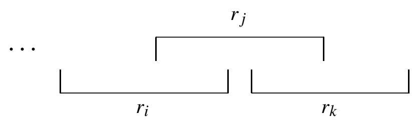
Our first three results, for certain conditions on Λ, give us restrictions on the behaviour of the maximal overlap sequences. These results will aid us when determining finite generation of $E(\Lambda)$.

Lemma 4.1.

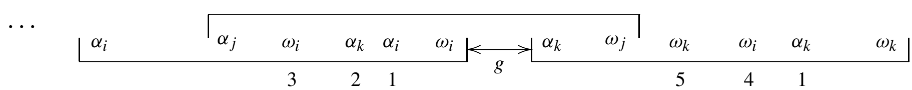
(1) *If $\lambda \geqslant 2$ and $\ell(r) > n$ for all relations r, then there are always n or more unoverlapped arrows at the end of any even-degree maximal overlap sequence.*
(2) *If $\lambda = 1$, r_x and r_y are repetition relations, and $\ell(r) > n$ for all repetition relations r, then there are always n or more unoverlapped arrows at the end of any even-degree maximal overlap sequence which ends*



Proof. (1) If the degree of the maximal overlap sequence is 2, then it is a relation and there is nothing to prove. Thus suppose the degree is $\geqslant 4$. If $\ell(r) > 2n$ for all relations r, the result is immediate using Proposition 1.2. So by Corollary 2.7, we may suppose $n < \ell(r) \leqslant 2n$, for all relations r. Let



be the end of the even-degree maximal overlap sequence. Note that $r_i \neq r_k$ by hypothesis. If $r_j = r_k$, then the result is immediate, so assume $r_j \neq r_k$. By Theorems 2.3 and 1.10, we may assume $r_i \neq r_j$. The diagram below is a copy of the one above, but with the relative positions of certain arrows marked; the order in which an arrow has been marked is indicated below that arrow. Recall that for a relation r_l we have α_l as the start arrow and ω_l as the end arrow:



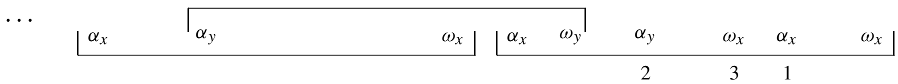
Once the start and end arrows of each relation have been marked on the diagram above, we know we can mark the other arrows for the following reasons, given by order of marking.

Note that from Lemma 2.10 and Proposition 1.1, the paths $H_i \to H_j \to H_k \to H_i$ and $T_i \to T_j \to T_k \to T_i$ each have length n.

(1) As $\ell(r_i) > n$, r_i contains two copies of α_i. One copy must lie in the overlapped part of r_i and r_j, by Proposition 1.2 if the degree of the maximal overlap sequence is greater than or equal to 6, or trivially if the degree is 4. By the same reasoning, a copy of α_k must lie in the unoverlapped part of r_k. We remark that $\alpha_i \neq \omega_i$ since a copy of α_j must lie between α_i and α_k, but cannot lie in the path g. At this stage we allow the possibility that $\alpha_k = \omega_k$.
(2) By the note above, a copy of α_k lies in r_i between α_j and the copy of α_i placed in 1.
(3) H_k is the head vertex that follows T_i by maximality of the overlap sequence, so ω_i sits as marked in the overlapped part of r_i and r_j.
(4) Three copies of ω_i cannot lie in r_j, but a copy of ω_i must lie in r_k, since $n < \ell(r_j), \ell(r_k) \leqslant 2n$. Thus there is a copy of ω_i as shown.
(5) By the note above, a copy of ω_k lies in r_k between ω_j and ω_i.

The presence of two copies of ω_k in the unoverlapped part of r_k yields our result.

(2) is proved in a similar way with the diagram:

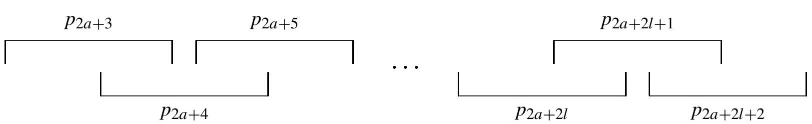


□

Proposition 4.2. *A maximal overlap sequence P^{2k}, $k \geqslant 2$, cannot be written as a product of maximal overlap sequences $P^{2a+1}F^{2l+1}Q^{2b}$, some $a \geqslant 0$, $l \geqslant M+1$, $b \geqslant 0$, if one of the following occurs*:

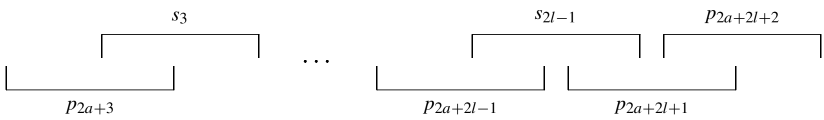
(1) $\lambda \geqslant 2$, *Λ has $\geqslant 3$ repetitions,*
(2) $\lambda \geqslant 2$, *Λ has 2 repetitions and $N \neq 1$,*
(3) $\lambda \geqslant 2$, *Λ has only 1 repetition and $2N \not\equiv 1 \pmod{\lambda}$,*
(4) $\lambda \geqslant 2$, *$\ell(r) > n$ for all relations r,*
(5) $\lambda = 1$, *Λ has 2 repetition relations r_x and r_y, and $\ell(r_x), \ell(r_y) > n$,*
(6) $\lambda = 1$, *Λ has $\geqslant 3$ repetitions.*

Proof. (1) If P^{2k} is to be written as such a product, we at least need the product in $\Bbbk Q$ of the three underlying paths to be non-zero: thus we assume this now. Suppose for a contradiction that P^{2k} can be written as the above product. Consider the underlying path of F^{2l+1}, represented thus



where the relations are from P^{2k}, in the same positions that they appear in the corresponding part of the underlying path of P^{2k}.

Let us also construct F^{2l+1} as a maximal overlap sequence, starting at $\mathfrak{o}(p_{2a+3})$:



Now look at $\mathcal{T}_\Lambda$. If $p_{2a+2l+1} = p_{2a+2l+2}$, then two cases arise. Either $s_3 = p_{2a+3}$, in which case p_{2a+3} is the only relation of Λ by maximality of the overlap sequence, or $s_3 \neq p_{2a+3}$, in which case by Theorems 2.3 and 1.10 there is no such maximal overlap sequence of degree $2l + 1$. Both cases give a contradiction to the hypothesis.

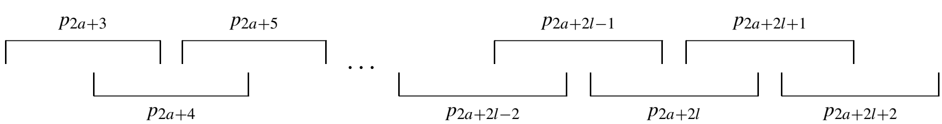
So assume $p_{2a+2l+1} \neq p_{2a+2l+2}$. Let f, g and h be integers such that $1 \leqslant f, g, h \leqslant m$ and $r_f = p_{2a+2l}$, $r_g = p_{2a+2l+1}$ and $r_h = p_{2a+2l+2}$. By the remark following Lemma 1.6, the two indices g and h are in different repetitions, respectively R_t and R_s say. By Lemma 3.6, we have a special ordering on the repetitions, which says that R_s follows R_t. However, since the index f is in R_s we have that R_t follows R_s. Since there are more than two repetitions this contradicts Lemma 3.6. Hence P^{2k} cannot be written as such a product.

The cases (2)–(6) are proved in a similar way, with (4) and (5) using Lemma 4.1. $\square$

Lemma 4.3. *Let Λ be such that $\ell(r) \leqslant n$ for all repetition relations r and suppose one of the following occurs*:

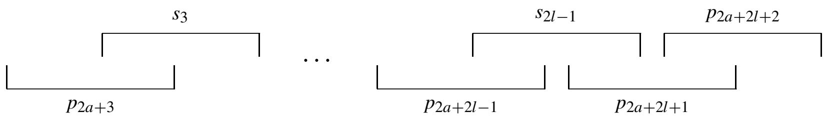
(1) $\lambda \geqslant 2$, *Λ has precisely* 2 *repetitions,* $N = 1$,
(2) $\lambda \geqslant 2$, *Λ has only* 1 *repetition,* $2N \not\equiv 1 \pmod{\lambda}$,
(3) $\lambda = 1$, *Λ has precisely* 2 *repetition relations.*

Let P^k be a maximal overlap sequence and let S be the subpath (*but not necessarily a maximal overlap sequence*):



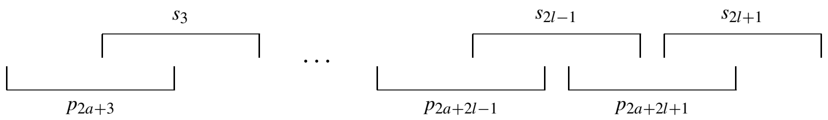
for some $l \geqslant M$. The relations above are from P^k, in the same positions that they appear in the corresponding part of the underlying path of P^k. Then the path S can be constructed as a maximal overlap sequence if and only if a maximal overlap sequence exists starting at $\mathfrak{o}(p_{2a+3})$ of degree $2l + 1$.

Proof. If the path S can be formed as a maximal overlap sequence, then it must take the form:



which is a maximal overlap sequence of degree $2l+1$.

We prove the converse in the case where condition (2) holds. Suppose there exists a maximal overlap sequence of degree $2l+1$ starting at $\mathfrak{o}(p_{2a+3})$. It will take the form:



Since $\ell(r) \leqslant n$ for all relations r, it is enough to show that $s_{2l+1} = p_{2a+2l+2}$. It is clear that $s_{2l+1} \neq p_{2a+2l+1}$ and $p_{2a+2l+2} \neq p_{2a+2l+1}$. Now in $\mathcal{T}_\Lambda$, $2N \not\equiv 1 \pmod{\lambda}$ is equivalent to having $(i, j) = (i+2+k(i), j-1)$ for every unflagged row i and for $j \geqslant M$, where $k(i)$ is the number of flagged rows counting up from row i to the next but one unflagged row. Consider band $a+l$ of $\mathcal{T}_\Lambda$. From the maximal overlap sequence P^k, and using Lemma 2.1, we can see this means that the index of $p_{2a+2l+2}$ is the next different one in band $a+l$ up from the index of $p_{2a+2l+1}$. Since $s_{2l+1} \neq p_{2a+2l+1}$ we thus get that $s_{2l+1} = p_{2a+2l+2}$.

Similar arguments may be used to prove the converse where either condition (1) or (3) holds. $\square$

The following definition and lemma are fundamental to the finite generation of $E(\Lambda)$.

Definition 4.4. Let e_v be a zero-length connective path of some repetition R. For ease of notation, we write $R = (a_1, a_2, \ldots, a_\lambda)$ and $\mathbf{c}_{a_1} = e_v$, with $\mathfrak{o}(r_{a_1}) = \mathfrak{t}(r_{a_\lambda}) = v$ and $f^i_\Lambda(a_1) = a_{i+1}$, for all $0 \leqslant i < \lambda$. The *multiplication path* of $\mathbf{c}_{a_1}$ is the path in $\Bbbk\mathcal{Q}$,

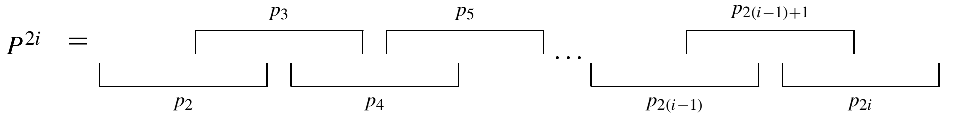
$$g_v = r_{a_1}\mathbf{c}_{a_2}r_{a_2}\mathbf{c}_{a_3}\cdots r_{a_{\lambda-1}}\mathbf{c}_{a_\lambda}r_{a_\lambda}.$$

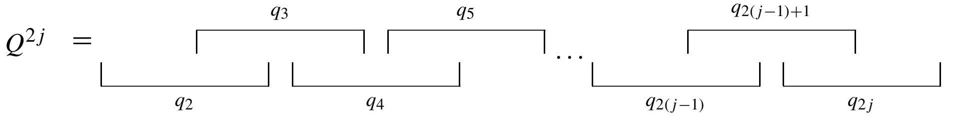
If a multiplication path can be formed as a maximal overlap sequence then it is called a *generative multiplication path*. Note that if this is the case, $\deg(g_v) = 2\lambda$.

Remark. If $g_v = r_{a_1}\mathbf{c}_{a_2}r_{a_2}\mathbf{c}_{a_3}\cdots r_{a_{\lambda-1}}\mathbf{c}_{a_\lambda}r_{a_\lambda}$ is a generative multiplication path, then $r_{a_1}\mathbf{c}_{a_2}r_{a_2}\mathbf{c}_{a_3}\cdots r_{a_{i-1}}\mathbf{c}_{a_i}r_{a_i}$ is also a maximal overlap sequence, for any $i \leqslant \lambda$.

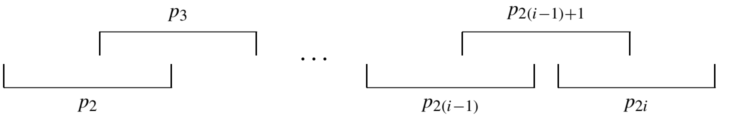
Proposition 4.5. *Let Λ be such that no flag0s or flag2s are present in $\mathcal{T}_\Lambda$ and let P^{2i} and Q^{2j} be even-degree maximal overlap sequences* (*of degree $2i$ and $2j$, respectively*) *such that the product of underlying paths $P^{2i}Q^{2j}$ is non-zero in $\Bbbk\mathcal{Q}$. Then the path $P^{2i}Q^{2j}$ is also a maximal overlap sequence, of degree $2i+2j$.*

Proof. Consider two maximal overlap sequences P^{2i} and Q^{2j}:

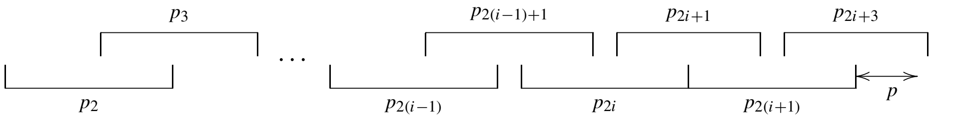




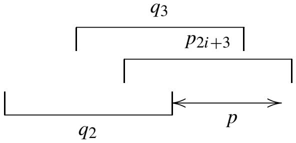
Suppose that $P^{2i}Q^{2j}$ is non-zero as a path in $\Bbbk\mathcal{Q}$; then $\mathfrak{t}(P^{2i}) = \mathfrak{o}(Q^{2j})$. We need to build the path $P^{2i}Q^{2j}$ as a maximal overlap sequence. We start with P^{2i} as above:



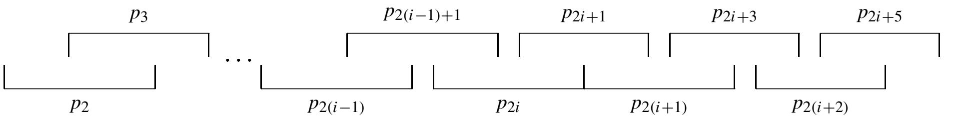
By hypothesis and Theorem 1.10, we know we may overlap with another relation, p_{2i+1} say. Since $\mathfrak{t}(p_{2i}) = \mathfrak{o}(q_2)$, and using the hypothesis and Theorem 1.10 again, we have the maximal overlap sequence:



where $p_{2(i+1)} = q_2$ and p is the path of unoverlapped arrows. In the following diagram we can see the path p within Q^{2j}:



Thus p has as an initial subpath the path of unoverlapped arrows of Q^3 and hence we may overlap our maximal overlap sequence above with the relation q_4. Using the hypothesis and Theorem 1.10, we get the maximal overlap sequence



where $p_{2(i+2)} := q_4$. We continue inductively setting $p_{2(i+k)} = q_{2k}$ for $1 \leqslant k \leqslant j$. $\square$

Corollary 4.6. *Let Λ be such that no flag0s or flag2s are present in $\mathcal{T}_\Lambda$. If g_v is a generative multiplication path, then $g_v^l \in E(\Lambda)$ is non-zero for all $l \geqslant 1$.*

Once we have stated the following definition, we will be in a position to start deciding if $E(\Lambda)$ is or is not finitely generated.

Definition 4.7. If A_w is an extending sequence of $E(\Lambda)$ starting at the vertex w then, along with w and the arrow starting at w, it naturally corresponds to $E_w := \mathrm{Ext}^*_\Lambda(S_w, \bar{\Lambda})$, where S_w is the simple Λ-module occurring at w. Then E_w is a (possibly non-unital) subring of $E(\Lambda)$. We also let $E_w^{\mathrm{ev}} := \mathrm{Ext}^{2*}_\Lambda(S_w, \bar{\Lambda})$ be the (possibly non-unital) subring of E_w consisting of the even-degree elements.

Say that a non-unital subring $\mathcal{R}$ of $E(\Lambda)$ has a *finite generating set* if there is a finite subset $\mathcal{S}$ of $E(\Lambda)$ in which every element of $\mathcal{R}$ may be expressed as a finite product of elements from $\mathcal{S}$.

A maximal overlap sequence a of degree z is said to be *in* an extending sequence A if $a = A^z$ for some $z \geqslant 2$. A generative multiplication path g_v is *in the lower half* (respectively *upper half*) of an infinite extending sequence A if there is some degree $z \geqslant 2$ and some even-degree (respectively odd-degree) maximal overlap sequence p in A such that $A^z = pg_v$, with the product in $E(\Lambda)$.

Using this definition, we get that $E(\Lambda)$ is finitely generated as a $\Bbbk$-algebra if and only if E_w has a finite generating set for all w such that A_w is an infinite extending sequence.

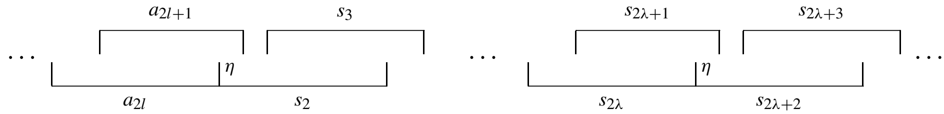
Remark. Once and for all we take care of the basis elements of $E(\Lambda)$ of degree 0 and 1; from the introduction we know these correspond respectively to the vertices and arrows of Λ. The question we resolve here is that of whether, to have a finite generating set $\mathcal{S}$ for $\mathrm{Ext}^{\geqslant 2}_\Lambda(\bar{\Lambda}, \bar{\Lambda})$, we need $\mathcal{S}$ to include elements of $E(\Lambda)$ of degree 0 or 1. The answer is that it does not, as the next proposition shows, when used with subsequent propositions.

Proposition 4.8. *Let A_w be an infinite extending sequence and let a be a maximal overlap sequence in A_w of degree greater than or equal to $2M$. Let η be an arrow in $\Bbbk\mathcal{Q}$; then η corresponds to a basis element of $E(\Lambda)$ of degree 1 and we have the following*:

(1) *If* $\deg(a)$ *is even and $a\eta \in \Bbbk\mathcal{Q}$ is non-zero in $E(\Lambda)$, then A_w has a generative multiplication path in its lower half.*
(2) *If* $\deg(a)$ *is odd and $a\eta \in \Bbbk\mathcal{Q}$ is non-zero in $E(\Lambda)$, then A_w has a generative multiplication path in its upper half.*

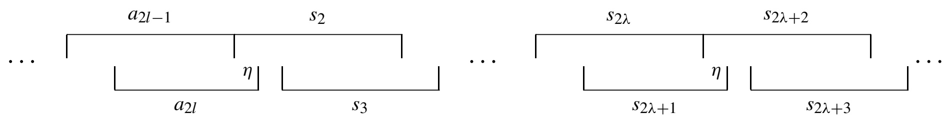
Proof. Let A_w, a and η be as above.

(1) Let $\deg(a)$ be even and $a\eta$ be non-zero in $E(\Lambda)$; we let a_{2l} and a_{2l+1} be the last relations of a and $a\eta$, respectively. Then since η is a path of length 1 and $\deg(a) \geqslant 2M$, A_w must take the form



with $s_{2\lambda+2} = s_2$ and $s_{2\lambda+3} = s_3$. It is immediate to see that the multiplication path starting at $\mathfrak{o}(\eta)$ is a maximal overlap sequence since clearly s_3 is the relation that maximally overlaps s_2: if η were a longer path this need not be true. We continue building the maximal overlap sequence up to degree 2λ in the obvious way using the relations from A_w. Thus A_w has a generative multiplication path in its lower half.

(2) Now let $\deg(a)$ be odd with $a\eta$ non-zero in $E(\Lambda)$ and let a_{2l-1} and a_{2l} be the last relations of a and $a\eta$, respectively. Then since η is a path of length 1 and $\deg(a) > 2M$, A_w must take the form



with $s_{2\lambda+2} = s_2$ and $s_{2\lambda+3} = s_3$. As before, it is immediate that the multiplication path starting at $\mathfrak{o}(\eta) = \mathfrak{o}(s_2)$ is a maximal overlap sequence and hence a generative multiplication path in the upper half of A_w. $\square$

Remark. If Λ satisfies one of the six conditions in Proposition 4.2, then the second diagram in the proof above yields the existence of an arbitrarily long odd-degree maximal overlap sequence in the position of the F^{2l+1} from Proposition 4.2. This contradicts that proposition and so we may conclude that condition (2) of Proposition 4.8 never occurs under any of the conditions from Proposition 4.2.

In Proposition 4.2 we showed that, given one of six conditions, a maximal overlap sequence could not be written as a product with a second odd-degree factor of degree greater than or equal to $2M + 3$ (called F^{2l+1} in Proposition 4.2). Non-existence of this factor is used as a hypothesis in part of the next proposition. The reason for this is that we want to examine the cases where we cannot use arbitrary powers of a generative multiplication path found in the upper half of an extending sequence, to get a finite generating set.

Proposition 4.9. *Let A_w be an infinite extending sequence of Λ.*

If there is a generative multiplication path g_v in the lower half of A_w, then E_w^{ev} has a finite generating set.

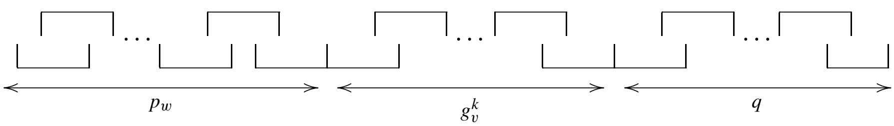
Moreover, if no even-degree maximal overlap sequence A_w^{2k} in A_w may be written as a product of maximal overlap sequences $A_w^{2a+1} F^{2l+1} Q^{2b}$, for any $a \geqslant 0$, $l \geqslant M + 1$, $b \geqslant 0$, *then there is a generative multiplication path g_v in the lower half of A_w if and only if E_w^{ev} has a finite generating set.*

Proof. Let A_w be an infinite extending sequence with a generative multiplication path g_v in its lower half. Let G_v be the infinite extending sequence starting at v, so that g_v is in G_v. Define p_w as the maximal overlap sequence of least even-degree in A_w such that $\deg(p_w) \geqslant 2M$ and $\mathfrak{t}(p_w) = v$. We take as our generating set for E_w^{ev}:

(1) The trivial path e_v.
(2) All even-degree maximal overlap sequences in A_w with degree less than or equal to $\deg(p_w)$.
(3) The maximal overlap sequence g_v of the hypothesis.
(4) All even-degree maximal overlap sequences in G_v that have degree less than the degree of g_v.

Let a be an even-degree maximal overlap sequence in A_w. We show how to get a from the above set by considering the degree of a:

- If $\deg(a) \leqslant \deg(p_w)$, then a is in the chosen generating set.
- If $\deg(a) > \deg(p_w)$, then, using the remark following Definition 4.4, we may write $a = p_w g_v^k q$, for some $k \geqslant 0$, and where q is a maximal overlap sequence in G_v of even-degree less than or equal to $\deg(g_v)$. We show this below:



The above product is non-zero in $E(\Lambda)$ by Proposition 4.5.

Conversely, suppose that we do have some finite generating set $\mathcal{S}$ for E_w^{ev} and that no maximal overlap sequence A_w^{2k} in A_w may be written as a product of maximal overlap sequences $A_w^{2a+1} F^{2l+1} Q^{2b}$, for any $a \geqslant 0$, $l \geqslant M + 1$, $b \geqslant 0$. We consider a maximal overlap sequence in A_w of sufficiently high even-degree such that, in any expression of it as a product of elements of $\mathcal{S}$, at least one element of $\mathcal{S}$ of degree $\geqslant 2$ occurs with multiplicity at least 2. Without loss of generality, we may choose a maximal overlap sequence a in A_w with $a = h_0 d h_1 d$, where $d \in \mathcal{S}$, $\deg(d) \geqslant 2$, each h_i is a product of generators, $i = 0, 1$, and $\deg(h_1) \geqslant 2M + 1$. Now, since $\deg(h_1 d) \geqslant 2M + 3$ and $\deg(a)$ is even, we have by hypothesis that $\deg(h_1 d)$ is even. Thus $\deg(h_0 d)$ is even. Therefore, since $\mathfrak{t}(d) = \mathfrak{o}(h_1)$, we have that the maximal overlap sequence $h_1 d$ is either a generative multiplication path, or some power (with multiplication in $E(\Lambda)$) of a generative multiplication path, in the lower half of A_w. $\square$

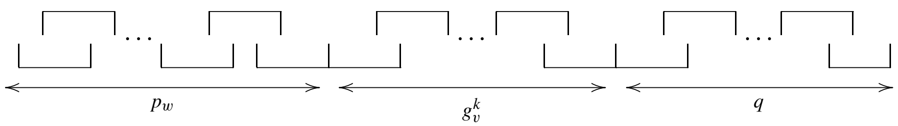
Theorem 4.10. *If each infinite extending sequence of $E(\Lambda)$ contains a generative multiplication path in its lower half, then $E(\Lambda)$ is finitely generated as a $\Bbbk$-algebra.*

Proof. Let A_w be an infinite extending sequence with g_v, p_w, and G_v as in Proposition 4.9. Then by Corollary 3.11, $\mathcal{T}_\Lambda$ has no flag0s or flag2s. Note also that $\deg(g_v) = 2\lambda$ and $2M \leqslant \deg(p_w) \leqslant 2M + 2\lambda - 2$. From the first part of Proposition 4.9 we get all the even-degree maximal overlap sequences in A_w with the finite generating set for E_w^{ev} as given there. Now augment that generating set by including the following elements:

(5) The arrow starting at w.
(6) All odd-degree maximal overlap sequences s in A_w such that $\deg(s) \leqslant \deg(p_w) + 2M + 1$.
(7) All odd-degree maximal overlap sequences t in G_v such that $2M + 1 \leqslant \deg(t) < 2M + 2\lambda + 1$.

With this set we now describe how to get any odd-degree maximal overlap sequence b in A_w. Note that the degrees are larger here than in the generating set of the last proposition for the following reason. Let q be a maximal overlap sequence in G_v of odd-degree less than $\deg(g_v)$. Then the underlying path of q is an initial subpath of that of g_v. Now, if b is of high-degree, its last relation must be a repetition relation. However, the last relation of q need not be. We must therefore choose the right-most factor of b, denoted t below, to be of sufficiently high degree to end with a repetition relation. We then use Lemma 2.1 and Theorem 2.3 to give us the correct end relation for the product (this is trivial if Λ has only 1 repetition relation):

- If $\deg(b) \leqslant \deg(p_w) + 2M + 1$, then b is in the chosen generating set.
- If $\deg(b) > \deg(p_w) + 2M + 1$, then $b = p_w g_v^k t$, for $k \geqslant 0$ and where t is some maximal overlap sequence of G_v as in (7) above, so that $\deg(t) = \deg(b) - \deg(p_w) - k\deg(g_v)$. We illustrate $p_w g_v^k t$ below and then prove that such a maximal overlap sequence b may indeed be expressed in this way:



Firstly, $p_w g_v^k t'$ is non-zero in $E(\Lambda)$ by Proposition 4.5, where t' is the even-degree maximal overlap sequence in G_v of degree $\deg(t) - 1$. By maximality of our overlap sequences, and using the remark following Definition 4.4, we have that the last relation of t' is the same as that of b', where b' is the even-degree maximal overlap sequence in A_w of degree $\deg(b) - 1$. By Proposition 1.2, it is enough to show that the last relation of t is the same as the last relation of b. If Λ has only 1 repetition relation then this is immediate, so suppose Λ has more than 1 repetition relation. Then, since G_v and A_w are infinite extending sequences, Theorem 2.3 says that $\mathfrak{t}(t) \neq \mathfrak{t}(t')$ and $\mathfrak{t}(b) \neq \mathfrak{t}(b')$. Moreover, since $\deg(t) \geqslant 2M + 1$, we have that the last relation of t is a repetition relation. By Lemmas 2.1 and 2.2, we have that the last relation of t is equal to that of b. Thus $p_w g_v^k t$ is non-zero in $E(\Lambda)$ and is equal to b.

Since there are only finitely many such A_w, taking the union of the above sets over each infinite A_w, along with all other trivial paths, arrows and maximal overlap sequences in finite extending sequences, gives us a finite generating set for $E(\Lambda)$. $\square$

The next two propositions examine the cases where arbitrary powers of a generative multiplication path in the upper half of an extending sequence can also be used to get a finite generating set.

Proposition 4.11. *Let Λ be such that $\ell(r) \leqslant n$ for all repetition relations r. Suppose one of the following occurs*:

(1) $\lambda \geqslant 2$, *Λ has precisely* 2 *repetitions*, $N = 1$,
(2) $\lambda \geqslant 2$, *Λ has only* 1 *repetition*, $2N \equiv 1 \pmod{\lambda}$,
(3) $\lambda = 1$, *Λ has precisely* 2 *repetition relations r_x and r_y.*

Let A_w be an infinite extending sequence. Then E_w has a finite generating set if and only if there is a generative multiplication path g_v in the lower or upper half of A_w.

Proof. Suppose first that E_w has a finite generating set $\mathcal{S}$, which we fix. We consider a maximal overlap sequence in A_w of sufficiently high even-degree such that, in any expression of it as a product of elements of $\mathcal{S}$, at least one element of $\mathcal{S}$ of degree $\geqslant 2$ occurs with multiplicity at least 3. Without loss of generality, we may choose a maximal overlap sequence a in A_w with $a = h_0 d h_1 d h_2 d$, where $d \in \mathcal{S}$, $\deg(d) \geqslant 2$, each h_i is a product of generators, $i = 0, 1, 2$, and $\deg(h_1) \geqslant 2M + 1$.

We have that $\mathfrak{t}(h_i) = \mathfrak{o}(d)$, for $i = 0, 1$ and 2. Thus $\mathfrak{o}(d)$ is a zero-length connective path. We need to show that $\mathfrak{o}(d)$ has a generative multiplication path. Since at least one out of dh_1, dh_2, and dh_1dh_2 is of even degree, one of the three will be a generative multiplication path or a power of one. Thus A_w has a generative multiplication path in its lower or upper half.

Conversely, suppose Λ has a zero-length connective path e_v that has a generative multiplication path g_v. If g_v is in the lower half of A_w, then by the proof of Theorem 4.10 we get a finite generating set for E_w. Thus suppose g_v is in the upper half of A_w. Note that for condition (2), since we have only 1 repetition, g_v being in the upper half of A_w is equivalent to g_v being in the lower half of A_w. Thus we get our result immediately for condition (2). Define s_w as the maximal overlap sequence of least odd-degree in A_w, such that $\deg(s_w) \geqslant 2M + 1$ and $\mathfrak{t}(s_w) = v$. Let G_v be the extending sequence starting at v, a sequence infinite by Corollary 3.11 and Proposition 4.5. For condition (1) we take as a finite generating set for E_w:

(1) The trivial path e_w and the arrow starting at w.
(2) All even-degree maximal overlap sequences in A_w with degree less than or equal to $\deg(s_w) + 2M - 1$.
(3) All odd-degree maximal overlap sequences in A_w with degree less than or equal to $\deg(s_w)$.

(4) All even-degree maximal overlap sequences in G_v with degree less than or equal to that of g_v.
(5) All odd-degree maximal overlap sequences t in G_v, such that $2M + 1 \leqslant \deg(t) < 2M + 2\lambda + 1$.

For condition (3) we take:

(1) The trivial path e_w and the arrow starting at w.
(2) All even-degree maximal overlap sequences in A_w with degree less than or equal to $\deg(s_w) + 2M - 1$.
(3) All odd-degree maximal overlap sequences in A_w with degree less than or equal to $\deg(s_w)$.
(4) The (degree 2) maximal overlap sequence g_v,
(5) The maximal overlap sequence in G_v of degree $2M + 1$.

It is left to the reader to verify that in each case we have a finite generating set for E_w. □

Proposition 4.12. *Let Λ be such that $\ell(r) \leqslant n$ for all repetition relations r. Suppose one of the following occurs*:

(1) $\lambda \geqslant 2$, *Λ has precisely* 2 *repetitions,* $N = 1$,
(2) $\lambda \geqslant 2$, *Λ has only* 1 *repetition,* $2N \equiv 1 \pmod{\lambda}$,
(3) $\lambda = 1$, *Λ has precisely* 2 *repetition relations r_x and r_y.*

Then $E(\Lambda)$ is finitely generated as a $\Bbbk$-algebra if and only if each infinite extending sequence has a generative multiplication path in either its lower or upper half.

Proof. This follows immediately from Proposition 4.11 since $E(\Lambda)$ is finitely generated as a $\Bbbk$-algebra if and only if for each infinite extending sequence A_w, we have a finite generating set for E_w. □

The last proposition in this section deals with a special case.

Proposition 4.13. *Let Λ have only one order* 1 *repetition relation, r_x, with $\ell(r_x) > n$, and let A_w be the single infinite extending sequence. Then $E(\Lambda)$ is finitely generated as a $\Bbbk$-algebra if and only if $w = \mathfrak{o}(r_x) = \mathfrak{t}(r_x)$.*

Proof. If r_x is the only repetition relation, then from Theorem 3.10 we have a single unflagged row in $\mathcal{T}_\Lambda$, which corresponds to A_w. Thus finite generation of $E(\Lambda)$ is equivalent to E_w having a finite generating set. If $w = \mathfrak{o}(r_x) = \mathfrak{t}(r_x)$, then we take as generating set:

(1) All odd-degree maximal overlap sequences in A up to degree $2M + 1$.
(2) The (degree 2) maximal overlap sequence r_x.

Clearly this is a finite generating set for E_w.

Conversely, suppose E_w has a finite generating set. As there is only one repetition relation, and by Proposition 4.8, we must have a generative multiplication path in the upper or lower half of A_w. Since r_x is the only repetition relation this means we must have $\mathfrak{o}(r_x) = \mathfrak{t}(r_x)$. Now, r_x is a maximal overlap sequence (of degree 2), and by hypothesis there are no flag0s or flag2s in $\mathcal{T}_\Lambda$. We may therefore apply Proposition 4.5 to get $\mathfrak{o}(r_x)$ the start of some infinite extending sequence G. By hypothesis $G = A_w$; hence $w = \mathfrak{o}(r_x)$. $\square$

The following theorem is our main result and provides the classification of the finite generation of $E(\Lambda)$ as a $\mathbb{k}$-algebra.

We put Theorem 3.10 with Corollary 2.7, and Propositions 4.12 and 4.13 together to form conditions (1)–(6) in Theorem 4.14 below. Any conditions other than those of (1)–(6) are captured by Proposition 4.2. Proposition 4.9 and Theorem 4.10 then yield the stated result for these.

It is remarked here that, for any of our algebras Λ, Theorem 4.10 gives sufficient conditions for $E(\Lambda)$ to be finitely generated.

Theorem 4.14. *Let $\Lambda = \mathbb{k}\mathcal{Q}/\mathcal{I}$ be a finite-dimensional algebra, with $\mathcal{Q}$ an oriented cycle and $\mathcal{I}$ an admissible ideal, with the notation of the previous section. Then $E(\Lambda)$ is finite-dimensional if and only if one of the following occurs*:

(1) $\lambda \geqslant 2$, *there is only* 1 *repetition,* $N = 1$ *and* $\ell(r) \leqslant n$ *for all repetition relations* r,
(2) $\lambda = 1$, *there is only* 1 *repetition relation* r_x *and* $\ell(r_x) \leqslant n$.

If $E(\Lambda)$ has infinite dimension and one of the following occurs:

(3) $\lambda \geqslant 2$, Λ *has precisely* 2 *repetitions,* $N = 1$ *and* $\ell(r) \leqslant n$ *for all repetition relations* r,
(4) $\lambda \geqslant 2$, Λ *has only* 1 *repetition,* $2N \equiv 1 \pmod{\lambda}$ *and* $\ell(r) \leqslant n$ *for all repetition relations* r,
(5) $\lambda = 1$, Λ *has precisely* 2 *repetition relations* r_x *and* r_y, *and* $\ell(r_x), \ell(r_y) \leqslant n$,
then $E(\Lambda)$ is finitely generated as a $\mathbb{k}$-algebra if and only if each infinite extending sequence has a generative multiplication path in either its lower or upper half.

If $E(\Lambda)$ has infinite dimension and the following occurs:

(6) $\lambda = 1$, Λ *has only* 1 *repetition relation* r_x *and* $\ell(r_x) > n$,
then $E(\Lambda)$ is finitely generated as a $\mathbb{k}$-algebra if and only if $\mathfrak{o}(A) = \mathfrak{o}(r_x) = \mathfrak{t}(r_x)$, where A is the single infinite extending sequence of Λ.

Otherwise, if $E(\Lambda)$ has infinite dimension, then $E(\Lambda)$ is finitely generated as a $\mathbb{k}$-algebra if and only if each infinite extending sequence has a generative multiplication path in its lower half.

We can now use the above theorem to yield an immediate result in some special cases; many of these algebras are considered in the literature. It should be noted that the bound on the size of the smo-tube given after Definition 1.7 does not grow too large next to the size of the algebra. Therefore any reasonably sized examples can easily be checked by hand.

Corollary 4.15. *Let Λ have only one relation r and let $\ell(r) = kn + c$, for $k \geqslant 0$, $0 \leqslant c < n$ and $\ell(r) \geqslant 2$.*

(1) *If $k = 0$, then $E(\Lambda)$ is finite-dimensional.*
(2) *If $c = 0$, then $E(\Lambda)$ is finitely generated as a $\Bbbk$-algebra (with $E(\Lambda)$ finite-dimensional if $k = 1$).*
(3) *If $k \neq 0$ and $c \neq 0$, then $E(\Lambda)$ is infinitely generated as a $\Bbbk$-algebra.*

Proof. If $k = 0$ or if $k = 1$ and $c = 0$ we are in condition (2) of the theorem. If $k \geqslant 2$ and $c = 0$ we have condition (6) in Theorem 4.14, with $\mathfrak{o}(r) = \mathfrak{t}(r)$. If $k \neq 0$ and $c \neq 0$ we have condition (6) again, but this time $\mathfrak{o}(r) \neq \mathfrak{t}(r)$. □

The algebras covered in the following corollary are those Λ which are self-injective.

Corollary 4.16. *Let J be the 2-sided ideal of $\Bbbk\mathcal{Q}$ generated by the arrows and let $\Lambda = \Bbbk\mathcal{Q}/J^l$, so that Λ has $m = n$ relations, each of length $l \geqslant 2$. Then $E(\Lambda)$ has infinite dimension but is finitely generated as a $\Bbbk$-algebra.*

Proof. Since the tail of any relation is the head of another, any relation is the start of some infinite extending sequence. This also means each extending sequence has a generative multiplication path in its lower (and upper) half. □

Corollary 4.17. *Let Λ have m relations, each of length $kn \geqslant 2$, for some fixed $k \geqslant 1$. Then $E(\Lambda)$ is finitely generated as a $\Bbbk$-algebra. Moreover, if $m = 1$ and $k = 1$, then $E(\Lambda)$ is finite-dimensional.*

Proof. If $m = 1$ and $k = 1$, then we are in case (2) of the theorem. If $m \neq 1$ or $k \neq 1$, then since $\mathfrak{t}(r) = \mathfrak{o}(r)$ for all relations r, every extending sequence is infinite so we cannot have any flags in $\mathcal{T}_\Lambda$. Every relation r is then the start of an infinite extending sequence that has r as a (degree 2) generative multiplication path in its lower half. □

Example 10. From its smo-tube, the algebra in Example 1 can be identified as having 2 repetitions of order $\lambda = 2$. The repetition shift is $N = 1$ and the length of all repetition relations is less than or equal to n. We thus have condition (3), so by Theorem 4.14 we need to find a zero-length connective path e_v that has a generative multiplication path g_v. From Example 2 and an inspection of the quiver we see that e_{14} is a zero-length connective path, with multiplication path $g_v = r_5\eta_{25}r_1$. In this case g_v is generative. Checking the smo-tube, we find g_v is in either the upper or lower half of each of the 4 infinite extending sequences. This gives $E(\Lambda)$ finitely generated as a $\Bbbk$-algebra.

We close this section with a remark on generalising to monomial path algebras. We also give the counter-example provided by the authors of [4] to their claim of the reverse implication of Proposition 0.3.

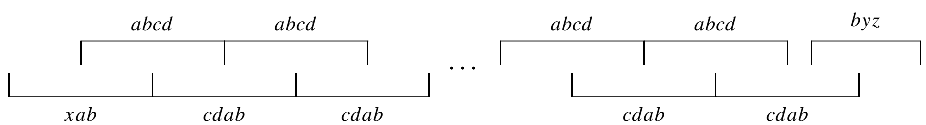
Remark. Let Γ be any finite quiver; we form the path algebra $\Bbbk\Gamma$. Recall that if $\mathcal{I}$ is an admissible ideal of $\Bbbk\Gamma$ generated by a finite set of paths such that $B := \Bbbk\Gamma/\mathcal{I}$ is finite-dimensional, then we say that B is a monomial algebra. From Proposition 0.3, to determine if $E(B)$ is infinitely generated it is enough to find one infinitely generated $E(\Lambda)$ for any minimal cycle algebra Λ overlying B. We can now use Theorem 4.14 on each overlying minimal cycle algebra Λ to determine whether or not $E(\Lambda)$ is infinitely generated. The bound on the size of the smo-tube means that for each cycle-algebra Λ this determination can be quickly made.

We now present an example of a monomial algebra with infinitely generated Ext-algebra, but with all overlying minimal cycle algebras having finitely generated Ext-algebra.

Example 11 *((E.L. Green, D. Zacharia))*. Let B be the $\Bbbk$-algebra with quiver

$$\begin{tikzcd}
 & & \bullet \arrow[dr, "b"] & & & \\
\bullet \arrow[r, "x"] & \bullet \arrow[ur, "a"] & & \bullet \arrow[dl, "c"] \arrow[r, "y"] & \bullet \arrow[r, "z"] & \bullet \\
 & & \bullet \arrow[ul, "d"] & & &
\end{tikzcd}$$

and relations $abcd$, $bcda$, $cdab$, $dabc$, xab, $daby$, and byz. This algebra has only one overlying minimal cycle algebra and by Theorem 4.14 this has finitely generated Ext-algebra. To show $E(B)$ is infinitely generated we consider basis elements in $E(B)$ with underlying path $x(abcd)^n abyz$, for $n \geqslant 1$. Such an element takes the form



Since multiplication relies on concatenation of paths, to non-trivially factor an element of the above form we need to split its underlying path in two at a vertex that is both the start of a relation and the end of one. So far we have many choices. However, it is clear that no matter where one chooses the split to be, the right hand path will not be a maximal overlap sequence. Thus maximal overlap sequences of the sort above cannot be non-trivially factored. Since we have infinitely many such maximal overlap sequences, $E(B)$ is infinitely generated.

5. Noetherian Ext-algebras

Now that we have determined precisely when $E(\Lambda)$ is finitely generated as a $\Bbbk$-algebra, we determine for which cycle algebras Λ the Ext-algebra is a Noetherian ring. In doing so we produce a class of examples for which the Ext-algebra is finitely generated but not Noetherian, and a further class that have Noetherian Ext-algebra with $\Lambda \ncong \Bbbk\mathcal{Q}/J^n$ for any $n \geqslant 2$, where J is the 2-sided ideal of $\Bbbk\mathcal{Q}$ generated by the arrows.

We may immediately state the main result of this section.

Theorem 5.1. *Let $\Lambda = \Bbbk\mathcal{Q}/\mathcal{I}$ be a finite-dimensional algebra, with $\mathcal{Q}$ an oriented cycle, $\mathcal{I}$ an admissible ideal. Suppose further that the Ext-algebra $E(\Lambda)$ has infinite dimension. Then $E(\Lambda)$ is a Noetherian ring if and only if every connective path of $E(\Lambda)$ is of zero length.*

Before we can prove this theorem we need the following.

Remark. So far in the paper we have talked of connective paths. In fact, just as a maximal overlap sequence is considered to be a left maximal overlap sequence if it is constructed from the left, so a left connective path and a left repetition come from an extending sequence constructed from the left. In the preceding sections, connective paths and repetitions have both been constructed from the left. This left construction of the extending sequences naturally shows the right $E(\Lambda)$-module structure of $E(\Lambda)$, which we exploit in Theorem 5.1. However, we also need to show how $E(\Lambda)$ behaves as a left $E(\Lambda)$-module: this is done by constructing right maximal overlap sequences. From [1] we know that left and right maximal overlap sequences have the same underlying path. In general however, the left and right repetitions need not be the same and so the left and right connective paths need not be the same. The following proposition gives us conditions under which the left and right repetitions do coincide.

Proposition 5.2. *Let Λ be such that $E(\Lambda)$ has infinite dimension and suppose all left connective paths are of zero length. Then the set of left repetition relations is equal to the set of right repetition relations. In particular, all right connective paths are also of zero length.*

Proof. Suppose $E(\Lambda)$ is of infinite dimension and that all left connective paths are of zero length. Let $r_{a_1}, r_{a_2}, \ldots, r_{a_\lambda}$ be the left repetition relations of $(a_1, a_2, \ldots, a_\lambda)$, one of the left repetitions of $E(\Lambda)$. Then $r_{a_1} r_{a_2} \cdots r_{a_\lambda}$ is a non-zero path in $\Bbbk\mathcal{Q}$. Since each r_{a_i} is a degree 2 left maximal overlap sequence, and $E(\Lambda)$ has infinite dimension, we may use Theorems 1.10 and 3.10 and Proposition 4.5 to conclude that the path $h := r_{a_1} r_{a_2} \cdots r_{a_\lambda}$ is also a left maximal overlap sequence. Similarly $r_{a_i} r_{a_{i+1}} \cdots r_{a_\lambda} h^k$ is a left maximal overlap sequence for all $1 \leqslant i \leqslant \lambda$ and $k \geqslant 0$. Hence we can construct a left maximal overlap sequence beginning at the vertex $\mathfrak{o}(r_{a_i})$ for all $1 \leqslant i \leqslant \lambda$, of degree greater than $2M$. From [1] we have that, as a path in $\Bbbk\mathcal{Q}$, each left maximal overlap sequence of degree l is also a right maximal overlap sequence of degree l. Thus the path $r_{a_i} r_{a_{i+1}} \cdots r_{a_\lambda} h^k$ is a right maximal overlap sequence for all $1 \leqslant i \leqslant \lambda$ and $k \geqslant 0$ and so r_{a_i} is a right repetition relation for all $1 \leqslant i \leqslant \lambda$. We thus have that all left repetition relations are also right repetition relations. By an identical argument we get that all right repetition relations are also left repetition relations. It follows immediately that all right connective paths are of zero length. $\square$

A dual argument yields the following corollary.

Corollary 5.3. *All left connective paths are of zero length if and only if all right connective paths are of zero length.*

We can now prove Theorem 5.1.

Proof of Theorem 5.1. Assume first that all left connective paths of $E(\Lambda)$ are of zero length. We will show that $E(\Lambda)$ is a Noetherian right $E(\Lambda)$-module. Let $r_{b_1}, r_{b_2}, \ldots, r_{b_\mu}$ be the left repetition relations of Λ. Since all left connective paths are of zero length, and $E(\Lambda)$ has infinite dimension, we have by Theorems 1.10 and 3.10 that every left repetition relation is the start of a generative multiplication path. Let g_{b_i} be the generative multiplication path such that $\mathfrak{o}(g_{b_i}) = \mathfrak{o}(r_{b_i})$. Set $\zeta = g_{b_1} + g_{b_2} + \cdots + g_{b_\mu} \in E(\Lambda)$. Then by Lemma 1.6, ζ is a homogeneous element of $E(\Lambda)$ in degree 2λ and has the property that $\zeta^l = g^l_{b_1} + g^l_{b_2} + \cdots + g^l_{b_\mu}$ for all $l \geqslant 1$. We thus have that 1 and ζ in $E(\Lambda)$ generate a graded subalgebra of $E(\Lambda)$ which is isomorphic to the polynomial ring in one variable, which we denote by $\Bbbk[\zeta]$.

Consider the usual basis of $E(\Lambda)$ consisting of trivial paths, arrows and maximal overlap sequences constructed from the left. Since $\Bbbk[\zeta]$ is a subring of $E(\Lambda)$ we consider $E(\Lambda)$ as a right $\Bbbk[\zeta]$-module. Let $\mathcal{S}' = \{A^z\colon\ A$ is an infinite extending sequence of $E(\Lambda),\ 2 \leqslant z \leqslant 2M + 2\lambda - 1\}$ and let $\mathcal{S} = \mathcal{S}' \cup \{$trivial paths and arrows of $\Lambda\}$. We will show that $\mathcal{S}$ is a (finite) generating set for $E(\Lambda)$ as a right $\Bbbk[\zeta]$-module.

Let A^y be a maximal overlap sequence of degree $y \geqslant 2M + 2\lambda$, in some infinite extending sequence A of $E(\Lambda)$. Then $\mathfrak{t}(A^y) = \mathfrak{t}(r_{b_i})$ for some $1 \leqslant i \leqslant \mu$. Write $y - 2M = c(2\lambda) + k'$, for some $0 \leqslant k' < 2\lambda$, $c \geqslant 1$, so that $y = c(2\lambda) + k$, for some $2M \leqslant k \leqslant 2\lambda + 2M - 1$, $c \geqslant 1$. Then $A^y = A^k g^c_{b_i} = A^k \zeta^c$, with $A^k \in \mathcal{S}$. Since A^y was arbitrary we get all maximal overlap sequences of degree greater or equal to $2M + 2\lambda$ in this way. Hence $E(\Lambda)$ is finitely generated as a right $\Bbbk[\zeta]$-module with generating set $\mathcal{S}$. As $\Bbbk[\zeta]$ is a Noetherian ring, we get that $E(\Lambda)$ is a Noetherian right $\Bbbk[\zeta]$-module. Hence $E(\Lambda)$ is a Noetherian right $E(\Lambda)$-module.

Now we must show that $E(\Lambda)$ is a Noetherian left $E(\Lambda)$-module. By Proposition 5.2 since the left connective paths of $E(\Lambda)$ are of zero length, so are the right ones. Also the right repetition relations are the same as the left. By a similar argument to that above, it follows that $E(\Lambda)$ is finitely generated as a left $\Bbbk[\zeta]$-module. That $E(\Lambda)$ is a Noetherian left $E(\Lambda)$-module then follows. Hence $E(\Lambda)$ is a Noetherian ring.

Conversely, assume now that there exists a connective path of $E(\Lambda)$ that has positive length. We can take this to be a left connective path by Corollary 5.3. Suppose this path starts at $\mathfrak{t}(r_{a_{i-1}})$ and ends at $\mathfrak{o}(r_{a_i})$, for $r_{a_{i-1}}$ and r_{a_i} repetition relations in some left repetition. The connective path is then denoted $\mathbf{c}_{a_i}$. By taking the basis of $E(\Lambda)$ of left maximal overlap sequences, we view $E(\Lambda)$ as a right $E(\Lambda)$-module. We now construct a strictly ascending chain of right submodules of $E(\Lambda)$ that is of infinite length. First consider some special basis elements of $E(\Lambda)$, namely those left maximal overlap sequences of degree greater than $2M$ that end at $\mathfrak{t}(r_{a_{i-1}})$. Since $r_{a_{i-1}}$ is a left repetition relation, there is some extending sequence A in which there are infinitely many of these maximal overlap sequences. Label these left maximal overlap sequences in A that end at the vertex $\mathfrak{t}(r_{a_{i-1}})$ by $\xi_1, \xi_2, \ldots$ in increasing order of degree.

Now let q be an element from our basis of $E(\Lambda)$: then q corresponds to a vertex, an arrow or a left maximal overlap sequence of degree $\geqslant 2$. Pick $j \geqslant 1$; then, since $\mathfrak{t}(\xi_j) = \mathfrak{o}(\mathbf{c}_{a_i})$ and $\ell(\mathbf{c}_{a_i}) > 0$, we get that the product $\xi_j q$ is zero in $E(\Lambda)$, for all $q \neq \mathfrak{t}(r_{a_{i-1}})$. Thus $\deg(\xi_j a) \leqslant \deg(\xi_j)$, for all $a \in E(\Lambda)$, $j \geqslant 1$. We now construct our chain of submodules. Let $I_0 = \{0\}$ and for $j \geqslant 1$ let $I_j = (\xi_1, \xi_2, \ldots, \xi_j)E(\Lambda)$. Then $I_0 \subset I_1 \subset I_2 \subset \cdots$ is clearly an infinite, strictly ascending chain of right $E(\Lambda)$-submodules of $E(\Lambda)$. Hence $E(\Lambda)$ is not right Noetherian and therefore not Noetherian. □

Recalling the definition of a minimal cycle algebra from Definition 0.1, we have the following corollary.

Corollary 5.4. *Let $Z_{\mathcal{Q}'}$ be a cycle algebra overlying a minimal cycle algebra $Z_{\mathcal{Q}}$. Then $E(Z_{\mathcal{Q}'})$ is Noetherian if and only $E(Z_{\mathcal{Q}})$ is Noetherian.*

Proof. The result is immediate since by definition all the connective paths in $E(Z_{\mathcal{Q}})$ are of zero length if and only if all the connective paths in $E(Z_{\mathcal{Q}'})$ are of zero length. □

Notice that Theorem 5.1 says nothing about the relations on Λ being of equal length if $E(\Lambda)$ is Noetherian. The following example shows they need not be. Note that in both examples below, $E(\Lambda)$ has infinite dimension.

Example 12. Let $\mathcal{Q}$ be an oriented cycle with 9 vertices labelled $1, \ldots, 9$. Let η_i be the arrow which starts at the vertex i. Let $\mathcal{I} = \langle r_1, r_2, r_3, r_4, r_5 \rangle$, where $r_1 = \eta_1\eta_2\eta_3$, $r_2 = \eta_2\eta_3\eta_4\eta_5$, $r_3 = \eta_3\eta_4\eta_5\eta_6$, $r_4 = \eta_4\eta_5\eta_6\eta_7\eta_8\eta_9$, $r_5 = \eta_6\eta_7\eta_8\eta_9\eta_1$. Let $\Lambda = \Bbbk\mathcal{Q}/\mathcal{I}$. Then the repetition relations of Λ are r_1, r_2, r_4, and r_5; the connective paths are the trivial paths e_1, e_2, e_4, and e_6. By Theorem 5.1, since $E(\Lambda)$ has infinite dimension, we get that $E(\Lambda)$ is Noetherian (and hence also finitely generated).

Example 13. Let Λ be as in Example 12, with the exception that here $r_4 = \eta_4\eta_5\eta_6\eta_7\eta_8$. We have the same left repetition relations as above, but now the left connective paths are η_9, e_2, e_4, and e_6. Thus from Theorem 5.1, $E(\Lambda)$ is not Noetherian. The positive length left connective path η_9 means that the ξ_j's from the proof of Theorem 5.1 arise, each ending at the vertex 9. However, from Theorem 4.14, we get that $E(\Lambda)$ is finitely generated.

Remark. It is clear that using Theorems 4.14 and 5.1, we can extend Examples 12 and 13 in both cases to a large class of examples with the same finiteness conditions on the Ext-algebra.

Lastly we return to our discussion on monomial algebras.

Proposition 5.5. *Let B be a monomial algebra and let the Ext-algebra $E(B)$ be Noetherian. Then the $\Bbbk$-algebras $E(Z_{\mathcal{Q}})$ are Noetherian for all minimal cycle algebras $Z_{\mathcal{Q}}$ overlying B.*

Proof. Let $Z_{\mathcal{Q}}$ be a minimal cycle algebra overlying B and let $E(Z_{\mathcal{Q}})$ be non-Noetherian. Let B have quiver Γ. We will show that $E(B)$ is non-Noetherian. By Theorem 5.1, we have a connective path of $E(Z_{\mathcal{Q}})$ of positive length, so in particular $E(Z_{\mathcal{Q}})$ is not right Noetherian. We thus have the infinite strictly ascending chain of right ideals of $E(Z_{\mathcal{Q}})$ constructed in the proof of Theorem 5.1. Recall that an ideal from this chain was written $I_j^{Z_{\mathcal{Q}}} = (\xi_1^*, \xi_2^*, \ldots, \xi_j^*)E(Z_{\mathcal{Q}})$. We use the $*$-notation to remain consistent with Proposition 0.3. As discussed in the proof of Proposition 0.3, each ξ_i^* in $E(Z_{\mathcal{Q}})$ corresponds to a maximal overlap sequence ξ_i in $E(B)$. We can thus form an infinite ascending chain of right ideals of $E(B)$: $I_0^B \subset I_1^B \subset I_2^B \subset \cdots$, where $I_0^B = \{0\}$ and $I_j^B = (\xi_1, \xi_2, \ldots, \xi_j)E(B)$. It remains to show that this chain is strictly ascending. Fix some $j \geqslant 0$ and consider the basis element ξ_{j+1}. To seek a contradiction suppose $I_j^B = I_{j+1}^B$. Then since ξ_{j+1} is one maximal overlap sequence (not a linear combination), $\xi_{j+1} = \xi_i b$, for some $1 \leqslant i \leqslant j$ and b some basis element of $E(B)$. Then the underlying path of b is a terminal subpath of ξ_{j+1} and so lies along the path in Γ that is covered by $\mathcal{Q}$. Thus b corresponds to a basis element b^* in $E(Z_{\mathcal{Q}})$. This gives us $\xi_{j+1}^* = \xi_i^* b^*$, and so $I_j^{Z_{\mathcal{Q}}} = I_{j+1}^{Z_{\mathcal{Q}}}$. This is a contradiction and therefore we conclude that $I_j^B \neq I_{j+1}^B$. Since j was arbitrary, we have that our chain of right ideals of $E(B)$ is strictly ascending. Hence $E(B)$ is not Noetherian. $\square$

As a counter-example to the reverse implication, Example 11 gives a monomial algebra with non-Noetherian Ext-algebra that has all its overlying minimal cycle algebras possessing Noetherian Ext-algebras.

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