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Complexes of nonpositive curvature for extensions of F_2 by Z

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Abstract

For any extension G of F_2 by Z, which is not virtually a direct product of the two factors, we construct a 2-complex of nonpositive curvature with fundamental group G. As a corollary we obtain a new proof of the fact that any such extension has an automatic structure.

Keywords: Automatic structure; Nonpositive curvature

AMS (MOS) Subj. Class.: 20F

1. Introduction

In [5, Ch. 4], Thurston constructs a spine for the figure-8 knot complement. This 2-complex consists of two 2-cells, four 1-cells and two 0-cells. If the 2-complex is given a metric in which each 2-cell is isometric to a regular Euclidean hexagon, it is easy to check that the 2-complex has nonpositive curvature. Furthermore, the complement is the mapping torus of a homeomorphism of a once punctured torus. Hence the fundamental group of the complement is an extension of the rank-2 free group F_2 by Z, which acts cocompactly on a 2-complex of nonpositive curvature. The purpose of this paper is to construct similar 2-complexes for general extensions of F_2 by Z. Specifically, if G is any extension of F_2 by Z, which is not virtually a direct product of the two factors, we will construct a 2-complex of nonpositive curvature with fundamental group G. We would like to thank Noel Brady and Allen Hatcher for several helpful discussions and the referee for suggestions.

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2. The 3-manifold M

The following construction of a 3-manifold with boundary is attributed to Jørgensen in [3]. Take a collection of n tetrahedra whose vertices have been deleted. Label the tetrahedra 1 through n and orient the edges as shown in Fig. 1. We will refer to the edges as horizontal, vertical, front diagonal and back diagonal as shown. Similarly, we will refer to the faces as back left, back right, front left and front right. We identify the vertical pair of edges and the horizontal pair of edges on each tetrahedron to form a fattened copy of a once punctured torus. Next we identify the two front faces of tetrahedron k with the two back faces of tetrahedron $k+1$ (mod n) according to one of the patterns shown in Fig. 2. The resulting 3-manifold M is homeomorphic to the mapping torus of a homeomorphism of the once punctured torus. Thus $G=\pi_1(M)$ is an extension of F_2 by Z, where Z acts on F_2 by an automorphism ϕ. It follows from the construction that ϕ has a special form, namely, it can be expressed as a product in which each factor is one of the automorphisms $\lambda:(x, y)\rightarrow(xy, y)$ or $\rho:(x, y)\rightarrow(x, yx)$. Note that no factor can be an inverse of one of these, so that ϕ lies in the semigroup generated by λ and ρ.

3. The spine X

We will be interested in a certain retraction of the 3-manifold above, onto a 2-dimensional spine which is dual to the cell structure given by the ideal tetrahedra. For the moment we ignore the case where the automorphism ϕ is given by λ^n or ρ^n, i.e., the cases where the abelianization of ϕ is parabolic. So X will be a

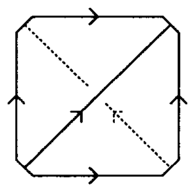

Fig. 1.

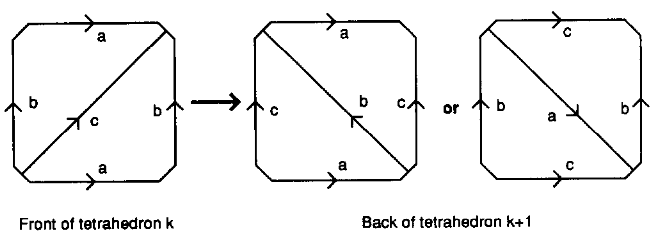

Fig. 2.

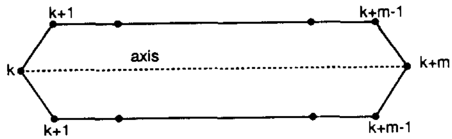

Fig. 3.

2-complex which has a 0-cell for each tetrahedron, a 1-cell for each face common to a pair of tetrahedra and a 2-cell for each edge in the identification space. We label the 0-cells of X by the integers 1 through n also. Note that X has n 0-cells and $2n$ 1-cells. To compute the number of 2-cells, we observe that for a given tetrahedron, say the kth, the front diagonal edge becomes either vertical or horizontal in the $(k+1)$st tetrahedron. We say this front diagonal edge is "born at the kth stage". Similarly the back diagonal edge "dies at the kth stage", and the vertical and horizontal edges "live through the kth stage". Since each edge is born at some stage and two edges cannot be born at once, there are n 2-cells also. A closer look at a 2-cell reveals that it has an axis of symmetry and a preferred direction along that axis. This is because the vertices on the boundary of the 2-cell are labeled by the stages or tetrahedra. There is a stage at which the edge is born and a stage where the edge dies. For an intermediate stage the edge lives through and two 0-cells with that label appear on the boundary of the 2-cell opposite each other across the axis. See Fig. 3.

We now put a metric on X which will make X a piecewise Euclidean 2-complex. This is achieved by assigning unit length to each of the 1-cells and assigning an angle at each 0-cell on the boundary of a 2-cell. There are two cases depending on the lifespan of the corresponding edge. Suppose the edge is born at the kth stage and dies at the $(k+m)$th stage. If $m>2$ then we assign the angle $2\pi/3$ to the 0-cell at the kth stage, to both 0-cells at the $(k+1)$st stage, to both 0-cells at the $(k+m-1)$st stage and to the 0-cell at the $(k+m)$th stage. The 0-cells at all other stages are assigned the angle π. If $m=2$ then we assign the angle $2\pi/3$ to the 0-cell at the kth stage and to the 0-cell at the $(k+2)$nd stage while the two 0-cells at the $(k+1)$st stage are assigned the angle $\pi/3$. See Fig. 4.

Recall from [4] that a piecewise Euclidean 2-complex has nonpositive curvature if every nontrivial circuit in the link of every vertex has total angular measure at least 2π.

Theorem 3.1. *X has nonpositive curvature.*

Proof. We consider the length of loops in the links of 0-cells. The link of the 0-cell corresponding to the kth stage has a vertex for each face of tetrahedron k and an edge joining a pair of vertices if the corresponding faces share an edge of the tetrahedron. Thus the link is the graph shown in Fig. 5. There are six corners of 2-cells of X incident at a given 0-cell. These correspond to the edge e_b, which is

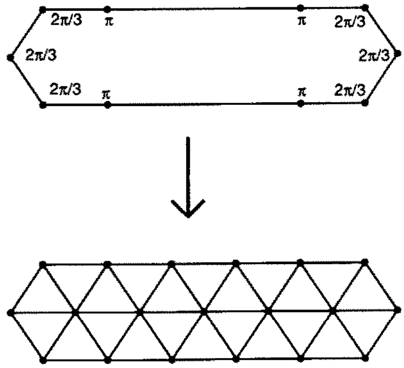

Fig. 4.

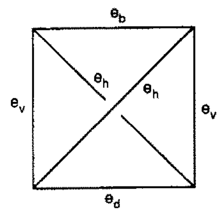

Fig. 5.

born at this stage, the edge e_d, which dies at this stage and the edges e_v and e_h, the vertical and horizontal edges respectively, which live through this stage. The edges e_v and e_h each contribute two corners of 2-cells, these corners are opposite in the link and have the same angle size. Let a_b, a_d, a_v, a_h be the sizes of the angle at the corner corresponding to the edges e_b, e_d, e_v, e_h, respectively. We know that the values of e_b and e_d are both $2\pi/3$. To calculate the possible values of a_v and a_h, we observe that, at stage $k-1$, either the edge e_v or the edge e_h was born and that, at stage $k+1$, either the edge e_v or the edge e_h will die. This gives four possibilities which fall into two equivalent sets. We consider one of each; the other two are obtained by interchanging the roles of e_h and e_v.

(a) The edge e_h is born at stage $k-1$ and dies at stage $k+1$. So $a_h = \pi/3$. However, this means that e_v must live through each of the stages $k-1$, k and $k+1$. So $a_v = \pi$. In this case, we see that the link has no loops with angle sum less than 2π. See Fig. 6.

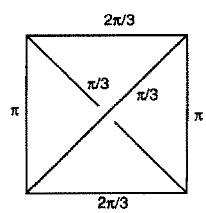

Fig. 6.

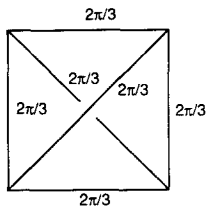

Fig. 7.

(b) The edge e_h is born at stage $k-1$ while the edge e_v dies at stage $k+1$. So $a_h = 2\pi/3$ and $a_v = 2\pi/3$. In this case also we see that the link has no loops with angle sum less than 2π. See Fig. 7. □

Example 3.2. Consider the case where the automorphism is given by

$$\phi(x, y) = (xyyxyxyyxyxy,\ yxyxy).$$

This can be expressed as a composition as:

$$\begin{aligned}(x, y) &\to (x(y), y) = (xy, x)\\ &\to (xy, y(xy)(xy)) = (xy, yxyxy)\\ &\to (xy(yxyxy)(yxyxy), yxyxy) = (xyyxyxyyxyxy, yxyxy).\end{aligned}$$

Thus $\phi = \lambda\lambda\rho\rho\lambda$ and the tetrahedra and identifications are as shown in Fig. 8. There are five 2-cells in the 2-complex X. Fig. 9 shows how the lifts of the 2-cells fit together to form a flat plane in the universal cover of X. This flat plane is the universal cover of the torus boundary in the 3-manifold M. In the figure, each

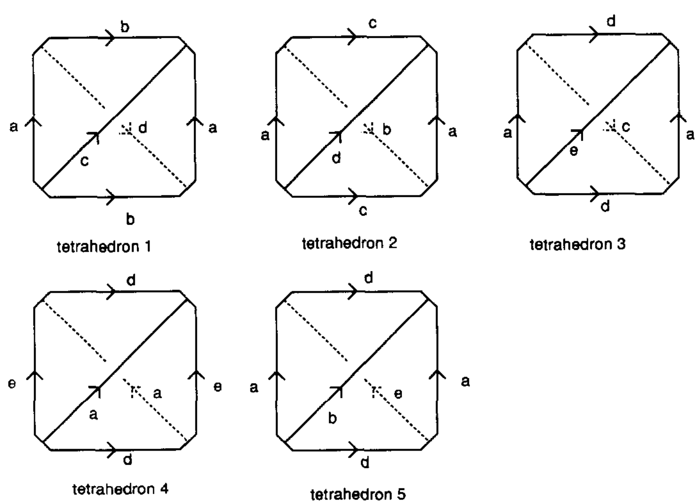

Fig. 8.

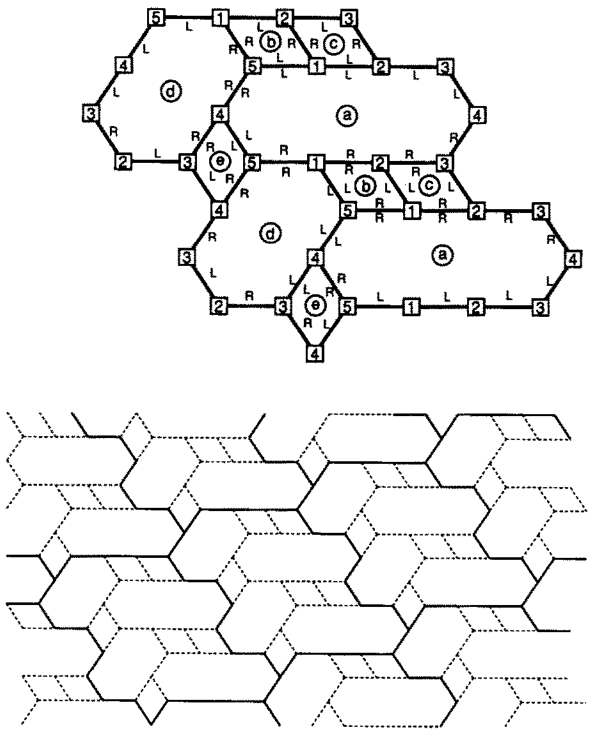

Fig. 9.

1-cell joins a 0-cell labeled k to a 0-cell labeled $k+1$, and is labeled R or L according as the corresponding front face of tetrahedron k is right or left.

4. The A_2 complex Y

Next we define Y to be the 2-complex obtained from X by subdividing each 2-cell into two rows of equilateral triangles as shown in Fig. 4. Since each 2-cell of Y is now an equilateral triangle, this makes Y a 2-complex of type A_2, using the terminology of [4]. Furthermore, since the links of the new 0-cells have no short loops and the lengths of loops in the links of the old 0-cells are unchanged, Y also has nonpositive curvature. Thus the results of [4] apply to give:

Corollary 4.1. *$\pi_1(Y)$ has an automatic structure.*

Note 4.2. This corollary also follows from [2, Ch. 12]. This is because the existence of an automatic structure can be deduced from the structure of the 3-manifold M as a circle bundle over a once punctured torus.

5. Variations on the construction

The above construction can be modified in various ways to give different manifolds. For each of the variations the links of 0-cells in Y are the same as in Section 4 above, so the properties of having nonpositive curvature and being A_2 are maintained.

(a) *The parabolic case*: Here $\phi = \lambda^n$ or ρ^n, the construction is the same except that one edge lives through every stage, is never born and never dies. It contributes an annulus to X rather than a 2-cell. The angles assigned at vertices on the boundary of the annulus each have value π, and the annulus is subdivided into equilateral triangles in Y in the obvious way.

(b) *Rotation by* π: Here we rotate the front of the final tetrahedron by π before gluing to the back of the first tetrahedron. The effect is to change ϕ by composing with the automorphism τ of F_2, which inverts both generators:

$$\tau : (x, y) \to (x^{-1}, y^{-1}).$$

(c) *Reflection*: Here we reflect the front of the final tetrahedron across the front diagonal before gluing to the back of the first tetrahedron. The resulting manifold is nonorientable and the effect on ϕ is to compose with the automorphism σ of F_2, which interchanges the generators:

$$\sigma : (x, y) \to (y, x).$$

(d) *Combinations*: We can apply both processes (b) and (c) to an automorphism ϕ, or we can apply one or both processes to an automorphism of type (a).

6. Extensions of F_2 by Z

Now let G be an extension of F_2 by Z, which is not virtually a direct product of the two factors. In this section, we will show that G is isomorphic to the fundamental group of one of the complexes Y, constructed in Section 4 or one of the variations in Section 5 above. First note that G fits into the exact sequence

$$1 \to F_2 \to G \to Z \to 1,$$

which splits since Z is free. Thus G can be expressed as the semidirect product of F_2 by Z where Z acts on F by some element ϕ of $\mathrm{Aut}(F_2)$. Since G has a presentation given by

$$G_\phi = \langle x, y, t \mid t^{-1}xt = \phi(x),\ t^{-1}yt = \phi(y) \rangle,$$

it follows that the isomorphism class of G will depend only on the conjugacy class of ϕ in $\mathrm{Aut}(F_2)$. Next the group $\mathrm{Aut}(F_2)$ fits into the exact sequence

$$1 \to \mathrm{Inn}(F_2) \to \mathrm{Aut}(F_2) \to \mathrm{GL}(2, Z) \to 1,$$

so that the isomorphism class of G depends only on the image of ϕ in $\mathrm{GL}(2, Z)$ under the map induced by abelianization. For, if ϕ and ψ have the same abelianization then $\phi = \psi I_z$ where I_z is an inner automorphism. Thus the extensions determined by ϕ and ψ are isomorphic.

So let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in GL(2, Z). In what follows we will need to refer to the specific 2×2 integral matrices:

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and}$$
$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that L, R, $-I$ and S are the images of λ, ρ, τ and σ, respectively under the map induced by abelianization. The matrix A has determinant $\det = ad - bc$ and trace $\mathrm{tr} = a + d$. We will consider six cases corresponding to the possible values of det and tr:

Case 1: $\det = 1$, $\mathrm{tr} = -1, 0, 1$ *or* $\det = -1$, $\mathrm{tr} = 0$. In each of these cases A is of finite order and G_ϕ contains the direct product $F_2 \times Z$ as a subgroup of finite index. We will omit these cases.

Case 2: $\det = 1$, $\mathrm{tr} = 2$. Here A has a single eigenvalue equal to 1, and the corresponding eigenvector is $\begin{pmatrix} b \\ 1-a \end{pmatrix}$ or $\begin{pmatrix} 1-d \\ c \end{pmatrix}$ if $b = 0$. Thus A can be conjugated in GL(2, Z) to R^n or L^n. So ϕ can be conjugated in Aut(F_2) to ρ^n or λ^n and we are in the case of variation (a) of Section 5 above.

Case 3: $\det = 1$, $\mathrm{tr} = -2$. Now Case 2 can be applied to $-IA$ and since $-I$ is in the centre of GL(2, Z), A is conjugate to $-IR^n$ or $-IL^n$. So ϕ can be conjugated in Aut(F_2) to $\tau\rho^n$ or $\tau\lambda^n$ and we have a combination of variations (a) and (b).

Case 4: $\det = 1$, $\mathrm{tr} > 2$. The eigenvalues of A are given by

$$\lambda = \left\{ \mathrm{tr} \pm \sqrt{(\mathrm{tr})^2 - 4} \right\} / 2,$$

which in this case consists of a pair of irrational real numbers. The corresponding eigenvectors are

$$\begin{pmatrix} 2b \\ (d-a) \pm \sqrt{(\mathrm{tr})^2 - 4} \end{pmatrix}.$$

The action of A on the hyperbolic plane is as a hyperbolic translation along the axis with endpoints given by the slopes of these eigenvectors. Since the endpoints are irrational the axis crosses at least one geodesic whose endpoints are adjacent Farey fractions. This allows us to conjugate A, within GL(2, Z), to a matrix $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, the slopes of whose eigenvectors are separated by the line joining 0 to ∞. This means that the slopes of the eigenvectors have different signs, so that

$$\begin{aligned} 0 &> \left\{ (s-p) + \sqrt{(p+s)^2 - 4} \right\} \left\{ (s-p) - \sqrt{(p+s)^2 - 4} \right\} \\ &= (s-p)^2 - \left\{ (p+s)^2 - 4 \right\} \\ &= 4\{1 - ps\}. \end{aligned}$$

Thus p and s are nonzero and have the same sign. Since $p + s > 2$, p and s must both be positive. Now $qr = ps - 1 > 0$, so q and r are nonzero and have the same sign. If q and r are both negative, conjugation by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ will give a matrix with positive entries. Then, by [1], for example, this positive matrix with determinant 1 can be expressed as a product in R and L. This allows us to conjugate ϕ in $\mathrm{Aut}(F_2)$ to a product in ρ and λ of the form described in Section 2.

Case 5: det = 1, tr < −2. Now $-IA$ is of the form discussed in Case 4, so $-IA$ is conjugate to a product in R and L. Since $-I$ is in the centre of $\mathrm{GL}(2, Z)$, A is conjugate to a product in R and L premultiplied by $-I$ and variation (b) applies.

Case 6: det = −1, tr ≠ 0. Here the eigenvalues of A are

$$\lambda = \left\{\mathrm{tr} \pm \sqrt{(\mathrm{tr})^2 + 4}\right\}/2.$$

These are always real and irrational. As in Case 4, we can conjugate A to a matrix $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, the slopes of whose eigenvectors have different signs. So

$$\begin{aligned} 0 &> \left\{(s-p) + \sqrt{(p+s)^2 + 4}\right\}\left\{(s-p) - \sqrt{(p+s)^2 + 4}\right\} \\ &= (s-p)^2 - \left\{(p+s)^2 + 4\right\} \\ &= 4\{-1 - ps\}. \end{aligned}$$

Thus $ps > -1$ and $qr = ps + 1 > 0$. If p and s are both nonzero, we see that p and s have the same sign and q and r have the same sign. Thus either A or $-IA$ is conjugate to a positive matrix. Since det = −1, this matrix can be expressed as a product in R and L, premultiplied by the matrix T. Thus we are in the case of variation (c) from Section 5 above or a combination of variations (c) and (b). Finally, if one of p or s is zero, then $q = r = \pm 1$ and B can be expressed in a similar form, except that only one of the matrices R or L will occur in the product.

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