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The binomial transform and the analysis of skip lists☆

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Abstract

To any sequence of real numbers $\langle a_n \rangle_{n \geqslant 0}$, we can associate another sequence $\langle \hat{a}_s \rangle_{s \geqslant 0}$, which Knuth calls its *binomial transform*. This transform is defined through the rule

$$\hat{a}_s = \mathbf{B}_s a_n = \sum_n (-1)^n \binom{s}{n} a_n.$$

We study the properties of this transform, obtaining rules for its manipulation and a table of transforms, that allow us to invert many transforms by inspection.

We use these methods to perform a detailed analysis of *skip lists*, a probabilistic data structure introduced by Pugh as an alternative to balanced trees. In particular, we obtain the mean and variance for the cost of searching for the first or the last element in the list (confirming results obtained previously by other methods), and also for the cost of searching for a random element (whose variance was not known).

We obtain exact solutions, although not always in closed form. From them we are able to find the corresponding asymptotic expressions.

Keywords: Binomial transform; Skip lists; Analysis of algorithms

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1. Introduction

The inversion formula

$$a_n = \sum_k (-1)^k \binom{n}{k} b_k \iff b_n = \sum_k (-1)^k \binom{n}{k} a_k \tag{1}$$

plays an important rôle in the analysis of some algorithms and data structures, and in the solution of many combinatorial problems [19,8].

In [11] Knuth used this relation to define a *transform*, mapping sequences of real numbers onto sequences of real numbers. Even though the experience with other similar transforms indicates that they can be powerful tools, in this case the concept does not seem to have been developed, and the literature is lacking in tables of binomial transforms, rules for their manipulation, inversion, etc.

In this paper, we develop the theory of the binomial transform, and show how it can be applied to analyze the performance of *skip lists*, a probabilistic data structure introduced by Pugh [18,14].

Using the binomial transform, we give alternative derivations for some known results and solve an open problem. Our results give exact expressions for the relevant performance measures, as functions of n (the number of keys in the data structure). This is useful, because it allows us to check the consistency of our solutions, by comparing them to the values obtained by numerical evaluation of the corresponding recurrence equations for small values of n. From these exact expressions it is easy to obtain asymptotic expressions. As a possible drawback, the manipulations required for the analysis may become complicated, and often require the use of a symbolic algebra system.

2. Notation

We generally follow the notational conventions of [8]. In particular, for any Boolean expression C, $[C]$ denotes a function that has the value 1 if C is true, and 0 otherwise. If $\Gamma(x)$ is the Gamma function, $\psi(x)$ is its logarithmic derivative ($\psi(x) = \Gamma'(x)/\Gamma(x)$). The harmonic numbers are defined as $H_n = \sum_{1 \leqslant k \leqslant n} \frac{1}{k}$. Equivalently, $H_n = \psi(n+1)+\gamma$, where $\gamma = 0.5772\ldots$ is Euler's constant. Asymptotically, $H_n = \ln n + \gamma + \mathrm{O}(\frac{1}{n})$. The harmonic numbers can be generalized to noninteger arguments by the formula $H_x = \sum_{k \geqslant 1} (\frac{1}{k} - \frac{1}{k+x})$, and it can be shown that $H_x = \psi(x+1) + \gamma$. The second-order harmonic numbers $H_n^{(2)}$ are defined as $H_n^{(2)} = \sum_{1 \leqslant k \leqslant n} \frac{1}{k^2}$, and their limiting value is $H_\infty^{(2)} = \frac{\pi^2}{6}$. The rising and falling factorial powers are defined as $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ and $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$, respectively. They can be extended to negative exponents by means of the relation $x^{\overline{-n}} = 1/(x-1)^{\underline{n}}$. In addition to the standard binomial coefficients, we also use a notation for the symmetric binomial coefficients $(i, j) = \binom{i+j}{i} = \binom{i+j}{j}$. The equivalent formula $(x, n) = (x+1)^{\overline{n}}/n!$ extends this to the case of one noninteger argument. The notation $Q_r(m, n)$ denotes the Q functions [11], defined as $Q_r(m, n) = \sum_{0 \leqslant j \leqslant n} (j, r) \frac{n^{\underline{j}}}{m^j}$, for $r \geqslant 0$. The notation $B(x, y)$ denotes the Beta function ($B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$). The operator $\mathbf{D}_x$ denotes a partial derivative with respect to x. Whenever the variables p and q are used, it will be assumed that $p + q = 1$. Finally, throughout the paper, each occurrence of the symbol $\chi(\cdot)$ will denote a (possibly different) periodic function, usually of very small amplitude and mean 0 (see Section 3.8).

3. The binomial transform

3.1. Definition

For any sequence of real numbers $\langle a_n \rangle_{n \geqslant 0}$ we define its *binomial transform* as the sequence $\langle \hat{a}_s \rangle_{s \geqslant 0}$, where

$$\hat{a}_s = \mathbf{B}_s a_n = \sum_n (-1)^n \binom{s}{n} a_n. \tag{2}$$

Note that, since s is a nonnegative integer, the binomial coefficients vanish for $n < 0$ or $n > s$. Therefore the actual range of the summation is really $0 \leqslant n \leqslant s$, but we prefer to write it as an unconstrained sum to simplify notation.

Furthermore, we will always assume that $a_n = 0$ for all $n < 0$, and $\hat{a}_s = 0$ for all $s < 0$. For example, when we write "$a_n = 1$" we actually mean "$a_n = [n \geqslant 0]$."

Another interesting formulation can be given if we rewrite the definition of the transform using the operators $\mathbf{Z}a_n = a_0$, $\mathbf{E}a_n = a_{n+1}$, $\mathbf{I}a_n = a_n$ and $\mathbf{\Delta}a_n = (\mathbf{E} - \mathbf{I})a_n = a_{n+1} - a_n$. In terms of these operators, we have

$$\mathbf{B}_s \equiv \mathbf{Z}\sum_k (-1)^k \binom{s}{k} \mathbf{E}^k \equiv \mathbf{Z}(\mathbf{I} - \mathbf{E})^s \equiv \mathbf{Z}(-\mathbf{\Delta})^s.$$

3.2. Inverse transform

Because of the inversion formula (1), it is easy to see that we can recover a_n from $\hat{a}_s$ by using the same formula (2), with the rôles of n and s reversed. Thus,

$$a_n = \mathbf{B}_n \hat{a}_s = \sum_s (-1)^s \binom{n}{s} \hat{a}_s. \tag{3}$$

3.3. Relation with other transforms

For any sequence of real numbers $\langle a_n \rangle_{n \geqslant 0}$, its Poisson Transform [7,15] can be defined as

$$\mathbf{P}_x a_n = \mathrm{e}^{-x} \sum_{n \geqslant 0} a_n \frac{x^n}{n!}$$

(this corresponds to the case $m = 1$ in [7,15]). For any function $f(x)$, its Mellin Transform [12,4,6] is defined as

$$\mathbf{M}_s f(x) = \int_0^\infty f(x) x^{s-1}\, \mathrm{d}x.$$

It is not hard to prove that the following formal identity holds between these two transforms and the binomial transform:

$$\mathbf{B}_s a_n = \frac{1}{\Gamma(-s)} \mathbf{M}_{-s} \mathbf{P}_x a_n.$$

The binomial transform is also related to the exponential generating function. If $A(z)$ is the egf of the sequence a_n, then $\mathrm{e}^z A(-z)$ is the egf of the transformed sequence.

3.4. Rules for the manipulation of binomial transforms

The following is a list of useful rules for the manipulation of binomial transforms. For the sake of clarity and ease of reference, the proofs for these properties have been collected in Appendix A.

R1. $\mathbf{B}_s(\alpha a_n + \beta b_n) = \alpha \mathbf{B}_s a_n + \beta \mathbf{B}_s b_n$ (Linearity).

R2. $\mathbf{B}_s \frac{a_{n+1}}{n+1} = -\frac{\hat{a}_{s+1}}{s+1}$.

R3. $\mathbf{B}_s \sum_{0 \leqslant k < n} a_k = -\hat{a}_{s-1}$, $\mathbf{B}_s \sum_{0 \leqslant k \leqslant n} a_k = \hat{a}_s - \hat{a}_{s-1}$,
$\mathbf{B}_s a_{n-1} = -\sum_{0 \leqslant t < s} \hat{a}_t$, $\mathbf{B}_s(a_n - a_{n-1}) = \sum_{0 \leqslant t \leqslant s} \hat{a}_t$.

R4. $\mathbf{B}_s \frac{1}{n+1} \sum_{0 \leqslant k \leqslant n} a_k = \frac{\hat{a}_s}{s+1}$, $\mathbf{B}_s \frac{a_n}{n+1} = \frac{1}{s+1} \sum_{0 \leqslant t \leqslant s} \hat{a}_t$.

R5. $\mathbf{B}_s(a_{n+1} - a_n) = -\hat{a}_{s+1}$, $\mathbf{B}_s a_{n+1} = \hat{a}_s - \hat{a}_{s+1}$.

R6. $\mathbf{B}_s \binom{n}{k} a_{n-k} = (-1)^k \binom{s}{k} \hat{a}_{s-k}$ (or equivalently, $\mathbf{B}_s n^{\underline{k}} a_{n-k} = (-1)^k s^{\underline{k}} \hat{a}_{s-k}$).

R7. $\mathbf{B}_s n a_n = s(\hat{a}_s - \hat{a}_{s-1})$, $\mathbf{B}_s n(a_n - a_{n-1}) = s\hat{a}_s$.

For the following three properties, recall our convention that $p + q = 1$:

R8. $\mathbf{B}_s \sum_k \binom{n}{k} p^k q^{n-k} a_k = p^s \hat{a}_s$.

R9. $\mathbf{B}_s \sum_k \binom{n}{k} p^k q^{n-k} a_k b_{n-k} = \sum_k \binom{s}{k} p^k q^{s-k} \hat{a}_k \hat{b}_{s-k}$.

Table 1
Pairs of binomial transforms

	a_n	$\rightleftharpoons$	b_n
T1.	1	$\rightleftharpoons$	$[n=0]$
T2.	$[n>0]$	$\rightleftharpoons$	$-[n>0]$
T3.	p^n	$\rightleftharpoons$	q^n
T4.	$\binom{n}{k}$	$\rightleftharpoons$	$(-1)^k[n=k]$
T5.	$\binom{n}{k}p^{n-k}$	$\rightleftharpoons$	$(-1)^k\binom{n}{k}q^{n-k}$
T6.	$\frac{1}{n+1}$	$\rightleftharpoons$	$\frac{1}{n+1}$
T7.	H_n	$\rightleftharpoons$	$-\frac{[n>0]}{n}$
T8.	$\frac{1}{2}(H_n^2+H_n^{(2)})$	$\rightleftharpoons$	$-\frac{[n>0]}{n^2}$
T9.	$\frac{1}{n+x}$	$\rightleftharpoons$	$\frac{n!}{\overline{x^{n+1}}}=B(x,n+1)$
T10.	$\frac{1}{(n+x)^2}$	$\rightleftharpoons$	$B(x,n+1)(\psi(x+n+1)-\psi(x))$ $=B(x,n+1)(H_{x+n}-H_{x-1})$
T11.	$n(H_n-1)$	$\rightleftharpoons$	$-\frac{[n>1]}{n-1}$
T12.	$Q_r(m,n)$	$\rightleftharpoons$	$(-1)^n(n,r)\frac{n!}{m^n}$

R10. $\mathbf{B}_s\sum_k\binom{n+1}{k+1}p^kq^{n-k}a_k=p^s\hat{a}_s-p^{s-1}q\sum_{0\leqslant t<s}\hat{a}_t=p^{s-1}\hat{a}_s-p^{s-1}q\sum_{0\leqslant t\leqslant s}\hat{a}_t.$
R11. $\mathbf{B}_s(a_n-a_0)=\mathbf{B}_s(a_n-a_0)[n>0]=\hat{a}_s[s>0].$

3.5. Table of transforms

Because the binomial transform and the inverse binomial transform are defined by the same summation, with the rôles of n and s interchanged, we use the following format for our table of transforms: Table 1 lists pairs of functions $a_n \rightleftharpoons b_n$ that are related by the transform; when applying the binomial transform, the table can be used to find $a_s=\mathbf{B}_s b_n$ and $b_s=\mathbf{B}_s a_n$, and when inverting the transform, $a_n=\mathbf{B}_n b_s$ and $b_n=\mathbf{B}_n a_s$. The derivations of these pairs of transforms have been collected in Appendix A.

Transforms T6–T10 will be useful for inverting transforms expressed as partial fraction expansions. Transforms T11 and T12 are not used in the rest of the paper, but we include them for the sake of completeness, as the functions involved appear in similar analyses of other algorithms.

3.6. Two-dimensional binomial transforms

If a_{n_1,n_2} is a sequence indexed by two nonnegative integer variables, we can compute its binomial transform on the first variable ($\acute{a}_{s_1,n_2}$), on the second one ($\grave{a}_{n_1,s_2}$), or on both ($\hat{a}_{s_1,s_2}$). Note that in the latter case, the result is independent of the order in which the transforms are applied, as these transforms are finite summations.

All the known properties apply to each variable considered in isolation, but there are some interesting results about the two variables together. We will use the two following properties, whose proofs can be found in Appendix A.

P1. If a_{n_1,n_2} is a two-dimensional sequence, and b_n a sequence on one variable, then

$$a_{n_1,n_2}=b_{n_1+n_2}\quad\forall n_1,n_2\geqslant 0 \iff \hat{a}_{s_1,s_2}=\hat{b}_{s_1+s_2}\quad\forall s_1,s_2\geqslant 0.$$

P2. If a_{n_1,n_2} is a two-dimensional sequence, and b_n a sequence on one variable, then

$$b_n=\frac{1}{n+1}\sum_{0\leqslant k\leqslant n}a_{k,n-k}\quad\forall n\geqslant 0 \iff \hat{b}_s=\frac{1}{s+1}\sum_{0\leqslant t\leqslant s}\hat{a}_{t,s-t}\quad\forall s\geqslant 0.$$

3.7. Numerical computation of binomial transforms

In the course of solving a problem using binomial transforms, one is often performing complex manipulations, either manually or with the aid of a symbolic algebra system, or more likely with a mix of the two. As a consequence, one is likely to feel (or indeed to be) error prone, and it is very helpful to be able to evaluate some functions and their transforms numerically, as a consistency check. However, it may be cumbersome to use the definition of the transform for this computation, as it involves binomial coefficients that may become quite large. We will see that there is a simple recurrence that will let us simplify this process.

Using rule R5 and replacing s by $s-1$ and n by $n+k$, we have that $\mathbf{B}_s a_{n+k} = \mathbf{B}_{s-1} a_{n+k} - \mathbf{B}_{s-1} a_{n+k+1}$. Defining $A_{s,k} = \mathbf{B}_s a_{n+k}$, we have the recurrence $A_{s,k} = A_{s-1,k} - A_{s-1,k+1}$, with the boundary condition $A_{0,k} = a_k$. The transform $\hat{a}_s$ can be obtained as $\hat{a}_s = A_{s,0}$.

For example, the following table illustrates numerically the relation $\mathbf{B}_s \frac{[n>0]}{n} = -H_s$. Note that the input sequence $\langle a_n \rangle_{n \geqslant 0}$ goes in the first row, and the output sequence $\langle \hat{a}_s \rangle_{s \geqslant 0}$ appears in the first column.

s, n	0	1	2	3	4
0	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\cdots$
1	-1	$\frac{1}{2}$	$\frac{1}{6}$	$\cdots$	
2	$-\frac{3}{2}$	$\frac{1}{3}$	$\cdots$		
3	$-\frac{11}{6}$	$\cdots$			
4	$\cdots$				

If we want to compute the $\hat{a}_s$ sequentially, a single array suffices to implement this process. The following algorithm computes $\hat{a}_0, \hat{a}_1, \ldots$ given $a_0, a_1, \ldots$ using an array L, that satisfies the invariant $L[k] = A_{s-k,k}$:

```
for s = 0, 1, ... do
    L[s] ← a_s
    for k = s − 1, s − 2, ..., 0 do
        L[k] ← L[k] − L[k + 1]
    â_s ← L[0]
```

3.8. Oscillating functions

Consider the function

$$\frac{1}{s+\sigma},$$

where σ is an imaginary parameter. From transform T9, we know that its inverse is

$$B(\sigma, n+1) = \frac{n!}{\overline{\sigma^{n+1}}},$$

which is asymptotically a periodic function of $\log n$, since $B(\sigma, n+1) \sim \Gamma(\sigma) \mathrm{e}^{-\sigma \ln n}$.

In our applications, we will encounter functions of the form

$$F_n = \frac{1}{\ln p} \sum_{k \neq 0} B(\sigma_k, n+1) = \mathbf{B}_n \frac{1}{\ln p} \sum_{k \neq 0} \frac{1}{s+\sigma_k},$$

where $\sigma_k = \frac{2\pi i k}{\ln p}$, p is a real parameter $(0 < p < 1)$, and the summation ranges over all negative and positive integers. Besides this basic function, several others related to it will appear, and it will be convenient to define a family of functions

$$F_n^{[r]} = \frac{1}{\ln p} \sum_{k \neq 0} \sigma_k^{\overline{r}} B(\sigma_k, n+1)$$

such that

$$\hat{F}_s^{[r]} = \frac{1}{\ln p} \sum_{k \neq 0} \frac{\sigma_k^{\overline{r}}}{s + \sigma_k}.$$

This way of generalizing F_n is a natural one, if we observe that multiplying $n!/\sigma_k^{\overline{n+1}}$ by $\sigma_k^{\overline{r}}$ simply cancels or adds the appropriate factors, yielding

$$F_n^{[r]} = \frac{1}{\ln p} \sum_{k \neq 0} \frac{n!}{(\sigma_k + r)^{\overline{n-r+1}}}.$$

The three values of r that will appear often are 1, 0 and -1. The corresponding functions and their transforms are:

$$\begin{aligned} F_n^{[1]} &= \frac{1}{\ln p} \sum_{k \neq 0} \sigma_k B(\sigma_k, n+1), & \hat{F}_s^{[1]} &= \frac{1}{\ln p} \sum_{k \neq 0} \frac{\sigma_k}{s + \sigma_k}, \\ F_n^{[0]} &= \frac{1}{\ln p} \sum_{k \neq 0} B(\sigma_k, n+1), & \hat{F}_s^{[0]} &= \frac{1}{\ln p} \sum_{k \neq 0} \frac{1}{s + \sigma_k}, \\ F_n^{[-1]} &= \frac{1}{\ln p} \sum_{k \neq 0} \frac{B(\sigma_k, n+1)}{\sigma_k - 1}, & \hat{F}_s^{[-1]} &= \frac{1}{\ln p} \sum_{k \neq 0} \frac{1}{(\sigma_k - 1)(s + \sigma_k)}. \end{aligned}$$

Note that, by using a partial fraction expansion, we can rewrite $\hat{F}_s^{[-1]}$ as

$$\hat{F}_s^{[-1]} = \frac{1}{s+1} \frac{1}{\ln p} \sum_{k \neq 0} \left(\frac{1}{\sigma_k - 1} - \frac{1}{s + \sigma_k} \right).$$

But $\frac{1}{\ln p} \sum_{k \neq 0} \frac{1}{\sigma_k - 1} = \frac{1}{2} + \frac{p}{q} + \frac{1}{\ln p}$ (as can be seen by setting $s = -1$ in expansion E1 in Appendix A, and using the fact that $\sum_{k \neq 0} \frac{1}{\sigma_k + 1} = -\sum_{k \neq 0} \frac{1}{\sigma_k - 1}$), so we have that

$$\frac{\hat{F}_s^{[0]}}{s+1} = -\hat{F}_s^{[-1]} + \frac{1}{s+1} \left(\frac{1}{2} + \frac{p}{q} + \frac{1}{\ln p} \right). \tag{4}$$

Other summations involving the σ_k that will be useful are $\sum_{k \neq 0} \frac{1}{\sigma_k} = 0$, $\frac{1}{\ln^2 p} \sum_{k \neq 0} \frac{1}{\sigma_k^2} = -\frac{1}{12}$, and $\frac{1}{\ln^2 p} \sum_{k \neq 0} \frac{1}{(\sigma_k - 1)^2} = \frac{p}{q^2} - \frac{1}{\ln^2 p}$, which follows from formula 4.3.92 of [1].

Sometimes we will need to restrict these F functions to $[n > 0]$, or their transforms to $[s > 0]$. Observing that the function $\sigma^{\overline{r}} B(\sigma, n+1)[n > 0]$ can be rewritten as

$$\sigma^{\overline{r}} B(\sigma, n+1)[n > 0] = \sigma^{\overline{r}} \left(B(\sigma, n+1) - \frac{[n = 0]}{\sigma} \right)$$

and that its transform is

$$\sigma^{\overline{r}} \left(\frac{1}{s + \sigma} - \frac{1}{\sigma} \right) = -(\sigma + 1)^{\overline{r-1}} \frac{s}{s + \sigma},$$

we can then obtain the following general expression:

$$\mathbf{B}_s F_n^{[r]}[n > 0] = -\frac{1}{\ln p} \sum_{k \neq 0} (\sigma_k + 1)^{\overline{r-1}} \frac{s}{s + \sigma_k}.$$

Note that for the case $r = 1$, this implies that

$$\mathbf{B}_s F_n^{[1]}[n > 0] = -s\hat{F}_s^{[0]}. \tag{5}$$

On the other hand, if $r \leqslant 0$, we can write

$$\hat{F}_s^{[r]}[s > 0] = \hat{F}_s^{[r]} - \hat{F}_0^{[r]}[s = 0],$$

which implies that

$$\mathbf{B}_n \hat{F}_s^{[r]}[s > 0] = F_n^{[r]} - F_0^{[r]}.$$

The F functions are asymptotic to periodic functions of $\log n$. If p is not too small (say, $p > .3$), [3] these are functions of very small amplitude (less than 10^{-3}). Using the notation of [13], we have that

$$F_n^{[r]} \sim -f_{r,1/p}(n),$$

where

$$f_{r,1/p}(n) = -\frac{2}{\ln p}\sum_{k\geqslant 1} \Re\left\{\Gamma(r-\sigma_k)\mathrm{e}^{-\sigma_k \ln n}\right\}.$$

See [13,14] for further discussion of the f functions, and for a table of values for different values of p.

Consider now the effect of differentiating the equation

$$\mathbf{B}_n \frac{\sigma^{\overline{r}}}{s+\sigma} = \sigma^{\overline{r}} B(\sigma, n+1)$$

with respect to σ. Using the differential operator $\mathbf{D}_\sigma$ (derivative with respect to σ), we have that

$$\begin{aligned}\mathbf{D}_\sigma \mathbf{B}_n \frac{\sigma^{\overline{r}}}{s+\sigma} &= -\sigma^{\overline{r}} B(\sigma, n+1)(H_{\sigma+n} - H_{\sigma+r-1}) \\ &= -\sigma^{\overline{r}} B(\sigma, n+1)H_n + \sigma^{\overline{r}} B(\sigma, n+1)(H_n - H_{\sigma+n} + H_{\sigma+r-1}).\end{aligned}$$

Since $H_n - H_{\sigma+n} = \mathrm{O}(\frac{1}{n})$, the second term is another asymptotically periodic function of $\log n$. Based on it, we define a new family of functions

$$G_n^{[r]} = \frac{1}{\ln^2 p}\sum_{k\neq 0} \sigma_k^{\overline{r}} B(\sigma_k, n+1)(H_n - H_{\sigma_k+n} + H_{\sigma_k+r-1}).$$

These functions are also asymptotically periodic functions of $\log n$, and in the range of values of interest for p they are of very small amplitude.

Let us consider now some particular cases. For $r = 0$, we have that

$$\mathbf{D}_\sigma \frac{1}{s+\sigma} = -\frac{1}{(s+\sigma)^2} = \mathbf{D}_s \frac{1}{s+\sigma}$$

and therefore

$$\mathbf{B}_n \frac{\mathbf{D}_s \hat{F}_s^{[0]}}{\ln p} = -\frac{F_n^{[0]} H_n}{\ln p} + G_n^{[0]}.$$

Note that $\left.\frac{\mathbf{D}_s \hat{F}_s^{[0]}}{\ln p}\right|_{s=0} = \frac{1}{12}$, so

$$\mathbf{B}_n \frac{\mathbf{D}_s \hat{F}_s^{[0]}}{\ln p}[s > 0] = -\frac{F_n^{[0]} H_n}{\ln p} + G_n^{[0]} - \frac{1}{12}. \tag{6}$$

[3] The case of very small p is not very interesting for skip lists, as the data structure degenerates into a linear linked list.

Table 2
Special values for the F and G functions

	$n=0$	$n=1$
$F_n^{[-1]}$	$\frac{1}{2}+\frac{p}{q}+\frac{1}{\ln p}$	0
$F_n^{[0]}$	0	$\frac{1}{2}+\frac{p}{q}+\frac{1}{\ln p}$
$F_n^{[1]}$	∞	$-\frac{1}{2}-\frac{p}{q}-\frac{1}{\ln p}$
$G_n^{[-1]}$	$-\frac{1}{12}-\frac{p}{q^2}+\frac{1}{\ln^2 p}$	$-\frac{1}{12}-\frac{p}{q^2}+\frac{1}{\ln^2 p}$
$G_n^{[0]}$	$\frac{1}{12}$	$\frac{1}{12}+\frac{p}{q^2}+\frac{1}{2\ln p}+\frac{p}{q\ln p}$
$G_n^{[1]}$	0	$-\frac{p}{q^2}-\frac{1}{2\ln p}-\frac{p}{q\ln p}$

We can also find an identity related to the case $r=-1$. From the equations

$$\frac{1}{s+1}\frac{1}{(s+\sigma)^2}=\frac{1}{(\sigma-1)^2(s+1)}-\frac{1}{(\sigma-1)^2(s+\sigma)}-\frac{1}{(\sigma-1)(s+\sigma)^2}$$
$$=\frac{1}{(\sigma-1)^2(s+1)}+\mathbf{D}_\sigma\left(\frac{1}{(\sigma-1)(s+\sigma)}\right)$$

we have that

$$\mathbf{B}_n\frac{1}{s+1}\frac{\mathbf{D}_s\hat{F}_s^{[0]}}{\ln p}=\frac{F_n^{[-1]}H_n}{\ln p}-G_n^{[-1]}-\frac{1}{n+1}\left(\frac{p}{q^2}-\frac{1}{\ln^2 p}\right)$$

and also

$$\mathbf{B}_n\frac{1}{s+1}\frac{\mathbf{D}_s\hat{F}_s^{[0]}}{\ln p}[s>0]=\frac{F_n^{[-1]}H_n}{\ln p}-G_n^{[-1]}-\frac{1}{n+1}\left(\frac{p}{q^2}-\frac{1}{\ln^2 p}\right)-\frac{1}{12}.$$

To complete this discussion of oscillating functions we present in Table 2 some special values for these functions.

Note: As we have stated, this paper will focus on the application of these techniques to the analysis of algorithms, and in particular to the analysis of skip lists. The statistics associated with these algorithms usually involve terms consisting of linear combinations or squares of the oscillating functions studied in this section. For reasonable values of p, the contribution of these terms decreases dramatically as n increases, and asymptotically they are periodic functions of $\ln n$ of very small amplitude (but nonvanishing), with a period that depends on p. In our theorems, we give the exact form for all these terms, and additionally we provide asymptotic expressions. In these asymptotic expressions, we lump the contribution of all the oscillatory terms into a term we denote as $\chi(\cdot)$, in a manner analogous to the $O(\cdot)$ notation. Each occurrence of the χ symbol will denote a possibly different periodic function of very small amplitude.

4. Analysis of skip lists

4.1. Review of the data structure and known results

A (*probabilistic*) *skip list* [18] is a generalization of a linked list. Elements are stored in order of increasing key value, and the records are linked into several lists (see Fig. 1).

The bottom such list links all elements, and each successive list links a subset of the elements in the list below it. For each element in a given list, the decision to link it into the next level list is made probabilistically: a (possibly biased) coin is flipped, and if it lands "heads" the element is included in the next level list. In our analyses we will assume that the coin has probability p of landing "heads," and probability $q=1-p$ of landing "tails."

To search for a given element x, we start at the header of the top list, and compare x to the key of the next record. If x is less than or equal to that key, we move down to the pointer one level below. If not, we advance to the next pointer

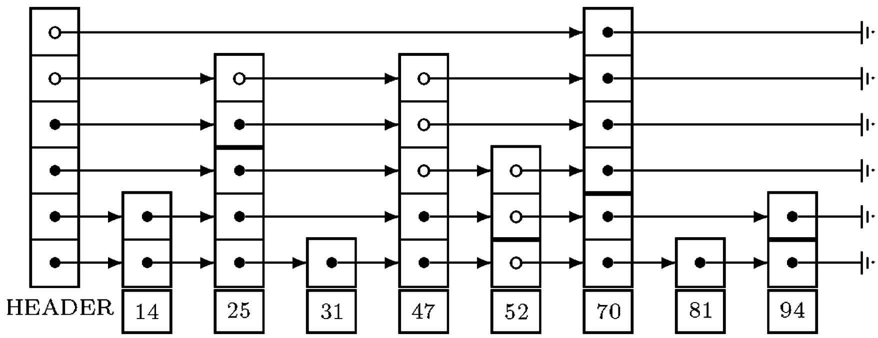

Fig. 1. A skip list showing a search for a key $x \in (52, 70]$.

to the right. We perform a final comparison for equality only when we have reached the bottom level. Fig. 1 shows (in white) the pointers inspected to search for any element in the range (52, 70]. We will use the number of pointers inspected as the measure of the cost of the search.

Note that, to simplify the description of the algorithm (and possibly its implementation), we assume the `nil` pointers point to a "trailer" record whose key is $+\infty$ (not shown in the figures).

We now define some random variables of interest for the analysis of skip lists. Let us assume that the skip list contains n keys

$$K_1 < K_2 < \cdots < K_n$$

(plus two fictitious keys $K_0 = -\infty$, $K_{n+1} = +\infty$). Then, we will write $C_n^{(m)}$ to denote the random variable "cost of searching for an element in the range $(K_m, K_{m+1}]$," S_n for the random variable "cost of searching successfully for a random element" (assuming all n keys are equally likely), and U_n for the random variable "cost of searching unsuccessfully for a random element" (assuming all $n+1$ intervals are equally likely). Note that $S_n = C_n^{(M_{n-1})}$ and $U_n = C_n^{(M_n)}$, where M_k is a uniform random variable taking integer values over the range $[0, k]$. Another random variable of interest is the total cost of searching for each of the n elements is a skip list, denoted I_n. (This last quantity is the equivalent for skip lists of the internal path length for binary search trees.) Note that the random variable S_n is *not* the same as the random variable I_n/n, although they have the same expected value.

Previous analyses have focused on obtaining asymptotic expressions for the expected values and some of the variances of these random variables. Asymptotic expressions for $\mathbf{E}C_n^{(0)}$, $\mathbf{E}C_n^{(m)}$, $\mathbf{E}S_n$ and $\mathbf{E}U_n$ can be found in [14]. Prodinger [17] studied the expected number of weak (or loose) left-to-right maxima of n i.i.d. geometrically distributed random variables. As shown in [14], the expected value of this random variable, plus the expected height is equal to $\mathbf{E}C_n^{(n)}$. The variance $\mathbf{V}C_n^{(n)}$ was studied by Papadakis, but appears only in [13]. Kirschenhofer and Prodinger [9] studied the random variable I_n, obtaining asymptotic expressions for $\mathbf{E}I_n$ and for $\mathbf{V}I_n$. Sen [20] and also Devroye [2,3] have obtained results about the distribution of $C_n^{(m)}$.

4.2. The height of a skip list

We begin our analyses by studying the height of a skip list, i.e. the number of pointers in the header of the data structure. The asymptotic behavior of this random variable is well known. It is basically the maximum of n independent geometric random variables, and it appears in the analysis of many algorithms. (See for example the analysis of tries in [11,21].) In our case, it will serve to illustrate the use of the binomial transform, and we will use it as a "warm up" before tackling more complicated problems, where we will use some of the results we will derive now.

Equivalently, the height of a skip list is the number of pointers of the tallest element in the skip list, and it is also equal to the cost of searching for any element less than or equal to the first element in the list. Fig. 2 shows the path followed by this kind of search. To simplify, only the "profile" of the skip list is shown (i.e. the stack of pointers for each element), and not the key values nor the arrows.

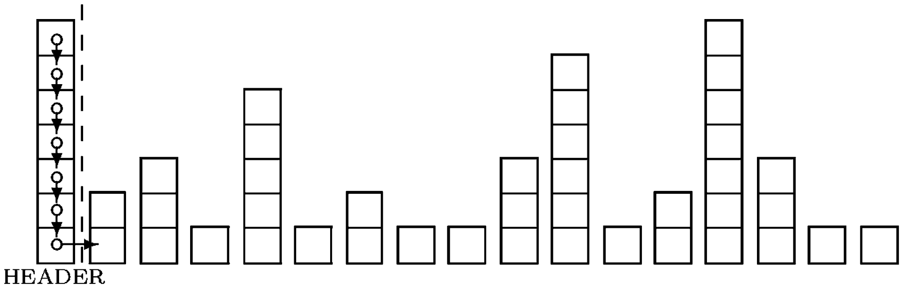

Fig. 2. The height of a skip list.

Let $P_n(z)$ be the probability generating function for the random variable "height of the skip list." Since p is the probability that we keep adding pointers when inserting an element, then if we remove the bottom layer of pointers the probability that i elements will still have a nonempty stack of pointers, while j are left with no pointers is $(i, j)p^i q^j$, where $i + j = n$. Therefore,

$$P_n(z) = z \sum_{\substack{i,j \geqslant 0 \\ i+j=n}} (i, j)p^i q^j P_i(z) \quad \forall n > 0$$

and $P_0(z) = 1$.

Whenever we have an equation $x_n = \phi_n$ valid for $n > 0$, we can extend it to cover the case $n = 0$ by changing it into $x_n = [n > 0]\phi_n + [n = 0]x_0 = [n \geqslant 0]\phi_n + [n = 0](x_0 - \phi_0)$. Applying this rule to this equation, we get

$$P_n(z) = z \sum_{\substack{i,j \geqslant 0 \\ i+j=n}} (i, j)p^i q^j P_i(z) + (1 - z)[n = 0] \quad \forall n \geqslant 0.$$

Applying the binomial transform, we get

$$\hat{P}_s(z) = zp^s \hat{P}_s(z) + (1 - z)$$

or, equivalently

$$(1 - zp^s)\hat{P}_s(z) = (1 - z). \tag{7}$$

Note that we avoid taking the obvious step of solving this equation for $\hat{P}_s(z)$, because this would not be correct for the case $z = 1, s = 0$.

Now, to compute the average and the variance we have to differentiate Eq. (7) and evaluate at $z = 1$. Let us introduce the notation $a_n = P'_n(1)$, $b_n = P''_n(1)$. Note that the average is given by a_n, and the variance by $b_n + a_n - a_n^2$. Observing that $\hat{P}_s(1) = [s = 0]$ (because $P_n(1) = 1$), we obtain

$$\hat{a}_s = \frac{[s > 0]}{p^s - 1}$$

and

$$\hat{b}_s = \frac{2p^s \hat{a}_s}{1 - p^s} = -\frac{2p^s}{(p^s - 1)^2}\,[s > 0].$$

Alternatively, we could have solved Eq. (7) to find $\hat{P}_s(z) = \frac{1-z}{1-zp^s}$, but this would have been correct only for $s > 0$. From this we would have obtained expressions for $\hat{a}_s$ and $\hat{b}_s$, that would also be valid only for $s > 0$ (the difference from the correct ones being the lack of the factor "$[s > 0]$"). Noting that $\hat{a}_0 = a_0 = 0$ and $\hat{b}_0 = b_0 = 0$, we could then use the technique employed earlier to extend the solutions to expressions valid for all $s \geqslant 0$.

To invert these transforms the following expansions are useful (see Appendix A for their derivations):

E1. $\dfrac{1}{p^s-1} = -\dfrac{1}{2} + \dfrac{1}{s\ln p} + \dfrac{1}{\ln p}\sum_{k\neq 0}\dfrac{1}{s+\sigma_k} = -\dfrac{1}{2} + \dfrac{1}{\ln p} + \hat{F}_s^{[0]}.$

E2. $\dfrac{p^s}{(p^s-1)^2} = \dfrac{1}{s^2\ln^2 p} + \dfrac{1}{\ln^2 p}\sum_{k\neq 0}\dfrac{1}{(s+\sigma_k)^2} = \dfrac{1}{s^2\ln^2 p} - \dfrac{\mathbf{D}_s\hat{F}_s^{[0]}}{\ln p},$

where $\sigma_k = \frac{2\pi i k}{\ln p}$. Using E1, we have

$$\hat{a}_s = \left(-\frac{1}{2} + \frac{1}{s\ln p}\right)[s > 0] + \hat{F}_s^{[0]}$$

and applying the inverse transform we find the following (exact) expression for the expected height of a skip list for all $n \geqslant 0$:

$$a_n = -\frac{H_n}{\ln p} + \tfrac{1}{2}[n > 0] + F_n^{[0]}.$$

Let us look now at the variance. Using E2, we have that

$$\hat{b}_s = \left(-\frac{2}{s^2\ln^2 p} + \frac{2}{\ln p}\mathbf{D}_s\hat{F}_s^{[0]}\right)[s > 0] = -\frac{2}{s^2\ln^2 p}[s > 0] + \frac{2}{\ln p}\mathbf{D}_s\hat{F}_s^{[0]} - \frac{1}{6}[s = 0].$$

Applying the inverse transform, we have that, for $n \geqslant 0$,

$$b_n = \frac{H_n^2 + H_n^{(2)}}{\ln^2 p} - \frac{1}{6} - \frac{2}{\ln p}F_n^{[0]}H_n + 2G_n^{[0]}.$$

Using this, and after some simplification, we obtain the following result:

Theorem 1. *The expected value of the height of a skip list is*

$$\begin{aligned}\mathbf{E}C_n^{(0)} &= -\frac{H_n}{\ln p} + \frac{1}{2}[n > 0] + F_n^{[0]} \quad \text{for all } n \geqslant 0\\ &= \log_{1/p} n + \frac{1}{2} - \frac{\gamma}{\ln p} + \chi(\log n) + \mathrm{O}\left(\frac{1}{n}\right),\end{aligned}$$

and its variance is equal to

$$\begin{aligned}\mathbf{V}C_n^{(0)} &= \frac{H_n^{(2)}}{\ln^2 p} + \frac{1}{12} - \frac{1}{4}[n = 0] - (F_n^{[0]})^2 + 2G_n^{[0]} \quad \text{for all } n \geqslant 0\\ &= \frac{\pi^2}{6\ln^2 p} + \frac{1}{12} + \chi(\log n) + \chi(\log n)^2 + \mathrm{O}\left(\frac{1}{n}\right).\end{aligned}$$

Note that the $\chi(\log n)^2$ term has a nonzero mean value. We do not have a closed form for it, but for usual values of p its contribution is negligible (e.g., 10^{-11} for $p = \frac{1}{2}$).

4.3. Searching for $+\infty$

Let us now turn our attention to the opposite end of the data structure. A search for "$+\infty$" is (on the average) the most expensive one. We analyze now the cost of that search.

We will use the same notation as before, but the random variable we are studying is "number of pointers inspected when searching for $+\infty$." (We redefine P_n, a_n and b_n accordingly.) Fig. 3 shows the path followed in this case.

Consider the search path from right to left. At the rightmost end, there is some number, say k, of elements of height 1, all of whose bottom-level pointers had to be traversed during the search. In addition, the bottom-level pointer belonging to the element immediately preceding them was also traversed. Note that this last element must be of height > 1. If we

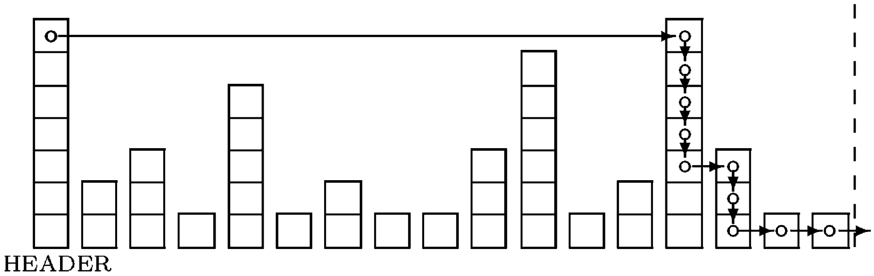

Fig. 3. Searching for $+\infty$ in a skip list.

now erase the bottom layer of pointers, all k rightmost elements will disappear. Of the other $n-k$ elements, some of them, say i, were of height >1, and the remaining ones, say j, were of height 1, so the latter also disappear. These i and j elements may be intermixed in any of $(i-1, j)$ possible orders, because one of the i "tall" elements is constrained to be in the rightmost position.

Therefore, the generating function for the search cost is

$$P_n(z) = \sum_{\substack{i,j,k \geqslant 0 \\ i+j+k=n}} (i-1, j) p^i q^{j+k} P_i(z) z^{k+1} \quad \forall n > 0. \tag{8}$$

Note that our previous argument assumes that there is at least one element of height >1 (i.e. $i>0$), but it is easy to verify that the equation we obtained holds also for the case $i=0$ (which implies $j=0, k=n$).

The boundary condition is $P_0(z)=1$ and, as before, we can fix the equation to hold for all $n \geqslant 0$ by adding to Eq. (8) the term $(1-z)[n=0]$.

After the substitution $h=i-1+j$, we can obtain the following somewhat simpler form:

$$P_n(z) = \sum_{0 \leqslant i \leqslant n} p^i q^{n-i} P_i(z) g_{i,n}(z) + (1-z)[n=0] \quad \forall n \geqslant 0, \tag{9}$$

where

$$g_{i,n}(z) = \sum_{i-1 \leqslant h \leqslant n-1} \binom{h}{i-1} z^{n-h}.$$

The summation defining the function $g_{i,n}(z)$ does not seem to have a simple closed form, but its derivatives at $z=1$ are not hard to find. It can be shown by induction that

$$g_{i,n}(1) = \binom{n}{i},$$
$$g_{i,n}^{(k)}(1) = k! \binom{n+1}{i+k} \quad \forall k \geqslant 1.$$

Differentiating Eq. (9) to find the average $a_n = P_n'(1)$, we get

$$\begin{aligned} a_n &= \sum_{0 \leqslant i \leqslant n} p^i q^{n-i} \left(\binom{n}{i} a_i + \binom{n+1}{i+1} \right) - [n=0] \\ &= \sum_{0 \leqslant i \leqslant n} \binom{n}{i} p^i q^{n-i} a_i + \frac{1-q^{n+1}}{p} - [n=0]. \end{aligned}$$

Note that this equation holds for all $n \geqslant 0$, but in itself does not determine a unique value for a_0. Therefore, we will have to handle the boundary condition $a_0 = 0$ separately. Applying the transform and using rule R8, we have

$$\hat{a}_s = p^s \hat{a}_s + \frac{[s=0]}{p} - qp^{s-1} - 1.$$

Again, this equation holds for $s = 0$, but it does not determine $\hat{a}_0$. To work around this problem, we will solve the equation for $s \neq 0$, getting

$$\hat{a}_s = \frac{q}{p} + \frac{1}{p(p^s - 1)}$$

and then fix it to incorporate the boundary condition. The resulting solution is

$$\begin{aligned}\hat{a}_s &= \left(\frac{q}{p} + \frac{1}{p(p^s - 1)}\right)[s > 0] \\ &= \frac{[s > 0]}{sp \ln p} + \left(\frac{1}{2p} - 1\right)[s > 0] + \frac{\hat{F}_s^{[0]}}{p}. \end{aligned} \tag{10}$$

Applying the inverse transform, we find that the expected cost of searching for $+\infty$ is

$$a_n = -\frac{H_n}{p \ln p} + \left(1 - \frac{1}{2p}\right)[n > 0] + \frac{F_n^{[0]}}{p}$$

for all $n \geqslant 0$.

To find the variance, we differentiate Eq. (9) twice and set $z = 1$, obtaining

$$\begin{aligned}b_n &= \sum_{0 \leqslant i \leqslant n} p^i q^{n-i} \left(\binom{n}{i} b_i + 2\binom{n+1}{i+1} a_i + 2\binom{n+1}{i+2}\right) \\ &= \sum_{0 \leqslant i \leqslant n} \binom{n}{i} p^i q^{n-i} b_i + 2 \sum_{0 \leqslant i \leqslant n} \binom{n+1}{i+1} p^i q^{n-i} a_i + \frac{2q}{p^2}(1 - q^n - npq^n)\end{aligned}$$

with $b_0 = 0$.

Applying the binomial transform and using rules R8 and R10, we get

$$\hat{b}_s = p^s \hat{b}_s + 2p^s \hat{a}_s - 2p^{s-1} q \sum_{0 \leqslant t < s} \hat{a}_t + \frac{2q}{p^2}[s = 0] - 2qp^{s-2} + 2sq^2 p^{s-2}.$$

Solving this first for $s > 0$, and then introducing the boundary condition $\hat{b}_0 = 0$, we obtain

$$\hat{b}_s = \left(-\frac{2p^s}{p^2(p^s - 1)^2} + \frac{2qp^s}{p^2(p^s - 1)} \sum_{1 \leqslant t \leqslant s} \frac{1}{p^t - 1}\right)[s > 0]. \tag{11}$$

The summation can be expanded as follows:

$$\begin{aligned}\sum_{1 \leqslant t \leqslant s} \frac{1}{p^t - 1} &= -\sum_{1 \leqslant t \leqslant s}\left(1 + \sum_{k \geqslant 1} p^{kt}\right) \\ &= -s - \sum_{k \geqslant 1} \frac{p^k(1 - p^{ks})}{1 - p^k} \\ &= -s - \sum_{k \geqslant 1} \frac{1 - p^{ks}}{p^{-k} - 1}.\end{aligned}$$

Replacing this in (11), we get

$$\begin{aligned}\hat{b}_s &= \left(-\frac{2p^s}{p^2(p^s - 1)^2} - \frac{2qsp^s}{p^2(p^s - 1)} - \frac{2q}{p^2} \sum_{k \geqslant 1} \frac{p^s(1 - p^{ks})}{(p^s - 1)(p^{-k} - 1)}\right)[s > 0] \\ &= \left(-\frac{2p^s}{p^2(p^s - 1)^2} - \frac{2qs}{p^2}\left(1 + \frac{1}{p^s - 1}\right) - \frac{2q}{p^2} \sum_{k \geqslant 1} \frac{p^s(1 - p^{ks})}{(p^s - 1)(p^{-k} - 1)}\right)[s > 0].\end{aligned}$$

Note that

$$\frac{p^s(1 - p^{ks})}{p^s - 1} = -(p^s + p^{2s} + \cdots + p^{ks}) = -k + \sum_{1 \leqslant j \leqslant k} (1 - p^{js}).$$

Therefore,

$$\hat{b}_s = \left(-\frac{2p^s}{p^2(p^s-1)^2} - \frac{2qs}{p^2}\left(1+\frac{1}{p^s-1}\right) + \frac{2q}{p^2}\sum_{k\geqslant 1}\frac{k}{p^{-k}-1} - \frac{2q}{p^2}\sum_{1\leqslant j\leqslant k}\frac{1-p^{js}}{p^{-k}-1}\right)[s>0].$$

We are almost ready to apply the inverse transform to this function. The first two terms have known expansions, and the summation in the third term is convergent. We will denote $\beta(p) = \sum_{k\geqslant 1}\frac{k}{p^{-k}-1}$ (for instance, $\beta(\frac{1}{2}) \approx 2.744\ldots$). We will consider the fourth term in more detail later. For the time being, let us denote $\hat{d}_s = -\sum_{1\leqslant j\leqslant k}(1-p^{js})/(p^{-k}-1)$ (note that this is a double summation).

Using our known expansions, we have that

$$\hat{b}_s = \left(-\frac{2}{p^2}\left(\frac{1}{s^2\ln^2 p} - \frac{\mathbf{D}_s\hat{F}_s^{[0]}}{\ln p}\right) - \frac{2q}{p^2}\left(\frac{s}{2}+\frac{1}{\ln p}+s\hat{F}_s^{[0]}\right) + \frac{2q}{p^2}\beta(p) + \frac{2q}{p^2}\hat{d}_s\right)[s>0],$$

whose inverse transform is

$$b_n = \left(\frac{H_n^2+H_n^{(2)}}{p^2\ln^2 p} - \frac{1}{6p^2} - \frac{2F_n^{[0]}H_n}{p^2\ln p} + \frac{2}{p^2}G_n^{[0]} \right.$$
$$\left. + \frac{q}{p^2}[n=1] + \frac{2q}{p^2\ln p} + \frac{2q}{p^2}F_n^{[1]} - \frac{2q}{p^2}\beta(p) + \frac{2q}{p^2}d_n\right)[n>0].$$

Before stating our results for this section, we need to study the behavior of the function d_n, whose transform is defined by the double summation

$$\hat{d}_s = -\sum_{1\leqslant j\leqslant k}\frac{1-p^{js}}{p^{-k}-1}.$$

Applying the inverse transform, we have that

$$d_n = -\sum_{1\leqslant j\leqslant k}\frac{[n=0]-(1-p^j)^n}{p^{-k}-1} = \sum_{1\leqslant j\leqslant k}\frac{(1-p^j)^n}{p^{-k}-1}[n>0].$$

Clearly, this function is always nonnegative for $p<1$. For an upper bound, observe that $p\leqslant 1$ implies that $p^{-j}-1\leqslant p(p^{-(j+1)}-1)$, and this in turn implies that $\frac{1}{p^{-(j+1)}-1}\leqslant\frac{p}{p^{-j}-1}$. Therefore, $\frac{1}{p^{-(j+k)}-1}\leqslant\frac{p^k}{p^{-j}-1}$ and

$$\sum_{k\geqslant j}\frac{1}{p^{-k}-1} = \sum_{k\geqslant 0}\frac{1}{p^{-(j+k)}-1} \leqslant \frac{1}{p^{-j}-1}\sum_{k\geqslant 0}p^k = \frac{p^j}{q(1-p^j)}$$

and

$$d_n \leqslant \frac{1}{q}\sum_{j\geqslant 1}p^j(1-p^j)^{n-1}[n>0] = \frac{1}{q}\sum_{j\geqslant 1}((1-p^j)^{n-1}-(1-p^j)^n).$$

To finish computing this bound, we now go back to the transformed domain. Since rule R7 allows us to find the transform of functions of the form $n(a_n-a_{n-1})$, it turns out to be simpler to bound nd_n rather than d_n. We have that

$$\mathbf{B}_s\sum_{j\geqslant 1}n\left((1-p^j)^{n-1}-(1-p^j)^n\right) = -s\sum_{j\geqslant 1}p^{js}[s>0] = \frac{sp^s}{p^s-1}[s>0].$$

Inverting this transform, we get

$$nd_n \leqslant \frac{1}{q}\left(-\frac{1}{2}[n=1] - \frac{1}{\ln p} - F_n^{[1]}\right)[n>0]$$

and therefore $d_n = \mathrm{O}(\frac{1}{n})$.

The following theorem summarizes the results obtained in this section:

Theorem 2. *The expected value of the cost of searching for* $+\infty$ *in a skip list is*

$$\mathbf{E}C_n^{(n)} = -\frac{H_n}{p\ln p} + \left(1 - \frac{1}{2p}\right)[n > 0] + \frac{F_n^{[0]}}{p} \quad \textit{for all } n \geqslant 0$$
$$= \frac{1}{p}\log_{1/p} n + 1 - \frac{1}{2p} - \frac{\gamma}{p\ln p} + \chi(\log n) + \mathrm{O}\left(\frac{1}{n}\right)$$

and its variance is

$$\mathbf{V}C_n^{(n)} = -\frac{q}{p^2\ln p} H_n + \frac{q}{p^2}[n = 1] + \left(\frac{H_n^{(2)}}{p^2\ln^2 p} + \frac{1}{2p} - \frac{5}{12p^2} + \frac{2q}{p^2\ln p} - \frac{2q}{p^2}\beta(p)\right.$$
$$\left. + \frac{q}{p^2}F_n^{[0]} - \frac{(F_n^{[0]})^2}{p^2} + \frac{2q}{p^2}F_n^{[1]} + \frac{2}{p^2}G_n^{[0]} + \frac{2q}{p^2}d_n\right)[n > 0] \quad \textit{for all } n \geqslant 0$$
$$= \frac{q}{p^2}\log_{1/p} n$$
$$+ \frac{\pi^2}{6p^2\ln^2 p} + \frac{1}{2p} - \frac{5}{12p^2} + \frac{q(2-\gamma)}{p^2\ln p} - \frac{2q}{p^2}\beta(p)$$
$$+ \chi(\log n) + \chi(\log n)^2 + \mathrm{O}\left(\frac{1}{n}\right) \quad \textit{as } n \to \infty,$$

where $\beta(p) = \sum_{k\geqslant 1}\frac{k}{p^{-k}-1}$.

4.4. Searching for an arbitrary element

We consider now a more general case. Suppose the element we are looking for is not necessarily the first or the last one in the list, but an arbitrary one (see Fig. 4). If we assume that the rank of the element is known to us, we can view the list as partitioned in two sections: the left part, of size n_1 contains all elements smaller than the given one; the right part, of size n_2, contains all elements greater than or equal to the given one. (Of course, $n_1 + n_2 = n$.)

Generalizing our notation, we write $P_{n_1,n_2}(z)$ for the probability generating function of the search cost. In a similar way, we write a_{n_1,n_2} and b_{n_1,n_2}.

Note that we have already studied the cases $n_1 = 0$ (height of the skip list, or cost of searching for the first element) and $n_2 = 0$ (cost of searching for $+\infty$). Also note that this means that we already know the results for the case $s_1 = 0$ and for the case $s_2 = 0$ in the transformed domain.

The recurrence equation for $P_{n_1,n_2}(z)$ is

$$P_{n_1,n_2}(z) = \sum_{\substack{i,j,k,\ell,m\geqslant 0\\ i+j+k=n_1\\ \ell+m=n_2}} (i-1, j)p^i q^{j+k} z^{k+1}(\ell, m)p^\ell q^m P_{i,\ell}(z) + (1-z)[n_1 = 0][n_2 = 0]$$

with $P_{0,0}(z) = 1$.

This can be rewritten as

$$P_{n_1,n_2}(z) = \sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant \ell\leqslant n_2}} p^i q^{n_1-i} g_{i,n_1}(z)\binom{n_2}{\ell}p^\ell q^{n_2-\ell}P_{i,\ell}(z) + (1-z)[n_1 = 0][n_2 = 0].$$

Applying a binomial transform to n_2, we get

$$\grave{P}_{n_1,s_2}(z) = p^{s_2}\sum_{0\leqslant i\leqslant n_1} p^i q^{n_1-i} g_{i,n_1}(z)\grave{P}_{i,s_2}(z) + (1-z)[n_1 = 0][s_2 \geqslant 0].$$

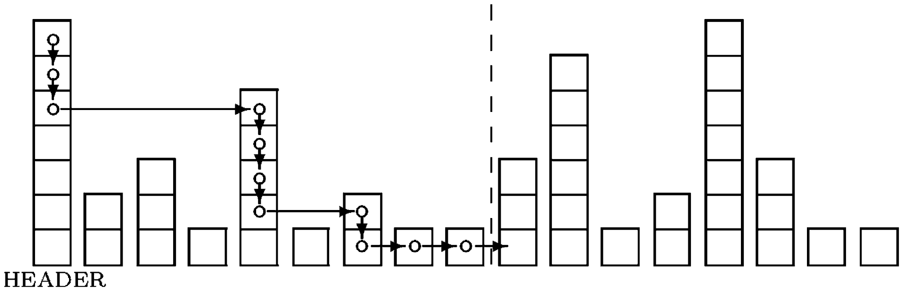

Fig. 4. Searching for an arbitrary element.

It does not seem possible to simplify this further, so we differentiate this equation to obtain equations for the moments:

$$\grave{a}_{n_1,s_2} = p^{s_2}\left(\sum_{0\leqslant i\leqslant n_1}\binom{n_1}{i}p^i q^{n_1-i}\grave{a}_{i,s_2} + \frac{1-q^{n_1+1}}{p}[s_2=0]\right) - [n_1=0][s_2\geqslant 0].$$

We now apply the transform to n_1, using rule R8 to obtain

$$\hat{a}_{s_1,s_2} = p^{s_1+s_2}\hat{a}_{s_1,s_2} + \frac{1}{p}[s_1=0][s_2=0] - \frac{q}{p}\,p^{s_1}[s_2=0] - [s_1\geqslant 0][s_2\geqslant 0].$$

We solve this equation first assuming $s_1+s_2>0$, and then introducing the boundary condition $\hat{a}_{0,0}=0$. The result is

$$\hat{a}_{s_1,s_2} = \frac{q}{p}\left(1+\frac{1}{p^{s_1}-1}\right)[s_1>0][s_2=0] + \frac{1}{p^{s_1+s_2}-1}[s_1+s_2>0].$$

This expression resembles the one we found for $\hat{a}_s$ when studying the problem of searching for $+\infty$ (Eq. (10)). In our current notation, Eq. (10) can be rewritten as

$$\acute{a}_{s,0} = \left(\frac{q}{p}+\frac{1}{p(p^s-1)}\right)[s>0].$$

Using this connection with our earlier results, we can rewrite $\hat{a}_{s_1,s_2}$ as follows:

$$\hat{a}_{s_1,s_2} = q\acute{a}_{s_1,0}[s_2=0] + p\acute{a}_{s_1+s_2,0} - q[s_1\geqslant 0][s_2>0]$$

and computing its inverse (using property P1) we find that the average search cost is

$$\begin{aligned} a_{n_1,n_2} &= qa_{n_1,0} + pa_{n_1+n_2,0} + q[n_1=0][n_2>0] \\ &= \left(-\frac{H_{n_1+n_2}}{\ln p}+\frac{1}{2}+F^{[0]}_{n_1+n_2}\right)[n_1+n_2>0] + \frac{q}{p}\left(-\frac{H_{n_1}}{\ln p}-\frac{1}{2}+F^{[0]}_{n_1}\right)[n_1>0] \end{aligned}$$

for $n_1, n_2 \geqslant 0$.

Let us look now at the variance. We have

$$\begin{aligned} \grave{b}_{n_1,s_2} &= p^{s_2}\sum_{0\leqslant i\leqslant n_1} p^i q^{n_1-i}\left(\binom{n_1}{i}\grave{b}_{i,s_2} + 2\binom{n_1+1}{i+1}\grave{a}_{i,s_2} + 2\binom{n_1+1}{i+2}[s_2=0]\right) \\ &= p^{s_2}\sum_{0\leqslant i\leqslant n_1}\binom{n_1}{i}p^i q^{n_1-i}\grave{b}_{i,s_2} + 2p^{s_2}\sum_{0\leqslant i\leqslant n_1}\binom{n_1+1}{i+1}p^i q^{n_1-i}\grave{a}_{i,s_2} \\ &\quad + \frac{2q}{p^2}\left(1-q^{n_1}-n_1 p q^{n_1}\right)[s_2=0] \end{aligned}$$

with $\grave{b}_{0,0} = 0$. Applying the transform to n_1, we get

$$\hat{b}_{s_1,s_2} = p^{s_1+s_2}\hat{b}_{s_1,s_2} + 2p^{s_1+s_2}\hat{a}_{s_1,s_2} - 2qp^{s_1+s_2-1}\sum_{0\leqslant t<s_1}\hat{a}_{t,s_2} + \left(\frac{2q}{p^2}[s_1=0] - 2qp^{s_1-2} + 2s_1q^2p^{s_1-2}\right)[s_2=0].$$

Solving this equation, we obtain

$$\hat{b}_{s_1,s_2} = -\frac{2q}{p}\frac{p^{s_1}}{(p^{s_1}-1)^2}[s_1>0][s_2=0] + \frac{2q^2}{p^2}\frac{p^{s_1}}{p^{s_1}-1}\sum_{1\leqslant t<s_1}\frac{1}{p^t-1}[s_2=0][s_1>0] - 2\frac{p^{s_1+s_2}}{(p^{s_1+s_2}-1)^2}[s_1+s_2>0] + \frac{2q}{p}\frac{p^{s_1+s_2}}{p^{s_1+s_2}-1}\sum_{s_2\leqslant t<s_1+s_2}\frac{1}{p^t-1}[t>0][s_1+s_2>0].$$

As we did for $\hat{a}_{s_1,s_2}$, this can be expressed in terms of $\acute{b}_{s,0}$ from (11). In terms of this function, $\hat{b}_{s_1,s_2}$ can be expressed as follows:

$$\hat{b}_{s_1,s_2} = q\acute{b}_{s_1,0}[s_2=0] + p\acute{b}_{s_1+s_2,0} - \frac{2q}{p}\frac{p^{s_1+s_2}}{p^{s_1+s_2}-1}\sum_{1\leqslant t<s_2}\frac{1}{p^t-1}[s_1+s_2>0].$$

From this expression, we can obtain an explicit formula for b_{n_1,n_2} (and therefore for the variance) using the definition of the inverse transform, but we have been unable to find a closed form. In spite of this, the expression we have found here for $\hat{b}_{s_1,s_2}$ will enable us in the next section to find the variance of the cost of searching for a random element.

Therefore, we state only our results for the expected cost:

Theorem 3. *The expected cost of searching for an element in the range* $(K_m, K_{m+1}]$ *in a skip list is*

$$\mathbf{E}C_n^{(m)} = \left(-\frac{H_n}{\ln p} + \frac{1}{2} + F_n^{[0]}\right)[n>0] + \frac{q}{p}\left(-\frac{H_m}{\ln p} - \frac{1}{2} + F_m^{[0]}\right)[m>0] \quad \textit{for all } 0\leqslant m\leqslant n$$
$$= \log_{1/p} n + \frac{q}{p}\log_{1/p} m + 1 - \frac{1}{2p} - \frac{\gamma}{p\ln p} + \chi(\log n) + \chi(\log m) + \mathrm{O}\left(\frac{1}{m}\right) + \mathrm{O}\left(\frac{1}{n}\right)$$

as $m, n \to \infty$.

4.5. Searching for a random element

We consider the case of an unsuccessful search for a random element. When the search stops, the list has been split in (n_1, n_2) elements as in the previous section, and the $n+1$ possible splits are equally likely.

If we write $\bar{P}_n(z)$ to denote the probability generating function for the search cost, we have that

$$\bar{P}_n(z) = \frac{1}{n+1}\sum_{0\leqslant k\leqslant n} P_{k,n-k}(z)$$

where the P in the right-hand side is from the previous section. Using property P2 of two-dimensional binomial transforms, we have that a similar relationship holds between their transforms:

$$\hat{\bar{P}}_s(z) = \frac{1}{s+1}\sum_{0\leqslant t\leqslant s}\hat{P}_{t,s-t}(z).$$

Similar equations hold for $\bar{a}_n$ and $\bar{b}_n$.

Using this and the results from the previous section we can analyze the cost of this search. For the average, using (4) we get

$$\hat{\bar{a}}_s = \left(p + \frac{q}{s+1}\right)\acute{a}_{s,0} - \frac{qs}{s+1}$$
$$= \left(\frac{1}{sp\ln p} - \frac{1}{2} + \hat{F}_s^{[0]} - \frac{q}{p}\hat{F}_s^{[-1]} + \frac{1}{p(s+1)}\right)[s>0],$$

and therefore the average cost for the unsuccessful search for a random element is

$$\bar{a}_n = -\frac{H_n}{p\ln p} + 1 - \frac{1}{2p} + \frac{q}{p\ln p} - \frac{1}{2}[n=0] + F_n^{[0]} - \frac{q}{p}F_n^{[-1]} + \frac{1}{p(n+1)}.$$

Consider now the variance. We have

$$\begin{aligned}\hat{\hat{b}}_s &= \frac{1}{s+1}\sum_{0\leqslant t\leqslant s}\hat{b}_{t,s-t}\\ &= \frac{1}{s+1}\sum_{0\leqslant t\leqslant s}\left(q\acute{b}_{t,0}[s-t=0] + p\acute{b}_{s,0} - \frac{2q}{p}\frac{p^s}{p^s-1}\sum_{1\leqslant r<s-t}\frac{1}{p^r-1}[s>0]\right)\\ &= \left(p+\frac{q}{s+1}\right)\acute{b}_{s,0} - \frac{2q}{p}\frac{p^s}{p^s-1}\frac{1}{s+1}\sum_{1\leqslant r\leqslant s}\frac{s-r}{p^r-1}[s>0],\end{aligned}$$

where

$$\acute{b}_{s,0} = \left(-\frac{2}{p^2}\frac{p^s}{(p^s-1)^2} + \frac{2q}{p^2}\frac{p^s}{p^s-1}\sum_{1\leqslant r\leqslant s}\frac{1}{p^r-1}\right)[s>0].$$

Replacing this in the previous expression and simplifying, we get

$$\hat{\hat{b}}_s = \left(-\frac{2}{p^2}\left(p+\frac{q}{s+1}\right)\frac{p^s}{(p^s-1)^2} + \frac{2q}{p^2}\frac{1}{s+1}\frac{p^s}{p^s-1}\sum_{1\leqslant r\leqslant s}\frac{1+pr}{p^r-1}\right)[s>0].$$

We use now the expansion $\frac{1}{p^r-1} = -1 - \sum_{k\geqslant 1}p^{kr}$, obtaining

$$\begin{aligned}\hat{\hat{b}}_s = &\left(-\frac{2}{p^2}\left(p+\frac{q}{s+1}\right)\frac{p^s}{(p^s-1)^2} + \left(-\frac{q}{p}s + \frac{2q}{p^2}(p\alpha(p)-1)\left(1-\frac{1}{s+1}\right)\right)\frac{p^s}{p^s-1}\right.\\ &\left.-\frac{2q}{p^2}\frac{p^s}{p^s-1}\left(p\sum_{k\geqslant 1}\frac{1-p^{ks}}{p^{-k}-1} + \frac{1}{s+1}\sum_{k\geqslant 1}\frac{1-p^{ks}}{p^{-k}-1} + \frac{p}{s+1}\sum_{k\geqslant 1}\frac{1-p^{ks}}{(p^{-k}-1)^2}\right)\right)[s>0],\end{aligned}$$

where $\alpha(p) = \sum_{k\geqslant 1}\frac{1}{p^{-k}-1}$. Now we substitute $\frac{p^s}{p^s-1}(1-p^{ks}) = -k + \sum_{1\leqslant j\leqslant k}(1-p^{js})$, to obtain

$$\begin{aligned}\hat{\hat{b}}_s = &\left(-\frac{2}{p^2}\left(p+\frac{q}{s+1}\right)\frac{p^s}{(p^s-1)^2} + \left(-\frac{q}{p}s + \frac{2q}{p^2}(p\alpha(p)-1)\left(1-\frac{1}{s+1}\right)\right)\frac{p^s}{p^s-1}\right.\\ &\left.+\frac{2q}{p^2}\left(p\beta(p) + \frac{\beta(p)+p\gamma(p)}{s+1}\right) + \frac{2q}{p^2}(p\hat{d}_s + \hat{e}_s + p\hat{f}_s)\right)[s>0],\end{aligned}$$

where $\beta(p) = \sum_{k\geqslant 1}\frac{k}{p^{-k}-1}$, $\gamma(p) = \sum_{k\geqslant 1}\frac{k}{(p^{-k}-1)^2}$, $\hat{d}_s = -\sum_{1\leqslant j\leqslant k}\frac{1-p^{js}}{p^{-k}-1}$, $\hat{e}_s = -\frac{1}{s+1}\sum_{1\leqslant j\leqslant k}\frac{1-p^{js}}{p^{-k}-1}$, and $\hat{f}_s = -\frac{1}{s+1}\sum_{1\leqslant j\leqslant k}\frac{1-p^{js}}{(p^{-k}-1)^2}$.

We can now use our known expansions to get $\hat{\hat{b}}_s$ in a form suitable for inverting the transform, and from it we can obtain $\bar{b}_n$. The resulting expressions are quite lengthy, so we will omit them and state our final results in the following theorem:

Theorem 4. *The expected value of the cost of an unsuccessful search in a skip list is*

$$\begin{aligned}\mathbf{E}U_n &= -\frac{H_n}{p\ln p} + 1 - \frac{1}{2p} + \frac{q}{p\ln p} - \frac{1}{2}[n=0] + F_n^{[0]} - \frac{q}{p}F_n^{[-1]} + \frac{1}{p(n+1)} \quad \textit{for all } n\geqslant 0\\ &= \frac{1}{p}\log_{1/p}n + 1 - \frac{1}{2p} + \frac{q-\gamma}{p\ln p} + \chi(\log n) + \mathrm{O}\left(\frac{1}{n}\right)\end{aligned}$$

and the variance is

$$\begin{aligned}
\mathbf{V}U_n = &-\frac{q}{p^2\ln p}\,H_n \\
&+\left(\frac{H_n^{(2)}}{p^2\ln^2 p}+\frac{1}{2p}-\frac{5}{12p^2}+\frac{3q}{p^2\ln p}+\frac{q(1+p)}{p^2\ln^2 p}-\frac{2q\alpha(p)}{p\ln p}-\frac{2q(1+p)\beta(p)}{p^2}-\frac{2q\gamma(p)}{p}\right.\\
&+\left(\frac{q(p-2)}{p^2}-\frac{2q}{p\ln p}+\frac{2q\alpha(p)}{p}\right)F_n^{[0]}+\left(\frac{q(p-3)}{p^2}+\frac{2q^2}{p^2\ln p}+\frac{2q\alpha(p)}{p}\right)F_n^{[-1]}\\
&-\left(F_n^{[0]}-\frac{q}{p}F_n^{[-1]}\right)^2+\frac{q}{p}F_n^{[1]}+\frac{2}{p}G_n^{[0]}-\frac{2q}{p^2}G_n^{[-1]}+\frac{q}{2p}[n=1]\\
&+\left(\frac{p^2-6p+3}{qp^2}-\frac{2q}{p^2\ln p}-\frac{2\alpha(p)}{p}+\frac{2q\beta(p)}{p^2}+\frac{2q\gamma(p)}{p}-\frac{2}{p}F_n^{[0]}+\frac{2q}{p^2}F_n^{[-1]}\right)\frac{1}{n+1}\\
&\left.+\frac{2}{p^2\ln p}\frac{H_n}{n+1}+\frac{2q}{p}d_n+\frac{2q}{p^2}e_n+\frac{2q}{p}f_n-\frac{1}{p^2(n+1)^2}\right)[n>0]\quad \textit{for all } n\geqslant 0\\
= &\frac{q}{p^2}\log_{1/p} n\\
&+\frac{\pi^2}{6p^2\ln^2 p}+\frac{1}{2p}-\frac{5}{12p^2}+\frac{q(3-\gamma)}{p^2\ln p}+\frac{q(1+p)}{p^2\ln^2 p}-\frac{2q\alpha(p)}{p\ln p}-\frac{2q(1+p)\beta(p)}{p^2}-\frac{2q\gamma(p)}{p}\\
&+\chi(\log n)+\chi(\log n)^2+\mathrm{O}\left(\frac{\log n}{n}\right)
\end{aligned}$$

where $\alpha(p)=\sum_{k\geqslant 1}\frac{1}{p^{-k}-1}$, $\beta(p)=\sum_{k\geqslant 1}\frac{k}{p^{-k}-1}$, $\gamma(p)=\sum_{k\geqslant 1}\frac{k}{(p^{-k}-1)^2}$.

Note that the first line in the huge expression for the variance is the leading term, that coincides with that of the variance of the cost of searching for $+\infty$. The second line shows the constant term, while lines 3–4 contain the oscillating part. We already know that $d_n=\mathrm{O}(\frac{1}{n})$. A similar analysis shows that $e_n=\mathrm{O}(\frac{\log n}{n})$ and $f_n=\mathrm{O}(\frac{1}{n})$. Therefore, the three final lines contain terms that are either transient, or can be bounded by $\mathrm{O}(\frac{\log n}{n})$.

5. Conclusions and further work

We believe the main contribution of this paper is the development of a theory of the binomial transform, that brings it to a level comparable to that of other, better known, tools for solving recurrence equations. We have shown its usefulness by deriving a number of results for the behavior of (probabilistic) skip lists. These results confirm previously known asymptotic results. In one case (variance of the search cost for a random element) we have solved a problem that was still open. While we do not claim that this method is always better than alternative ones, there are certainly cases where the binomial transform should be the method of choice. Since no method is uniformly best, the availability of a new tool should enhance our chances of solving a given problem.

There are several ways in which this work can be continued. One obvious line of development is the application of this transform to the analysis of other problems. Some preliminary results indicate that some hashing algorithms can be analyzed using the binomial transform. Another interesting line of research is to try to find ways of writing down the equations in the domain of the binomial transform (the "s domain") directly from the algorithm being analyzed, without going through the intermediate step of writing down recurrence equations. This is the approach that Flajolet and his group have been investigating for several kinds of generating functions, with great success (for instance, see [5]).

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Appendix A

A.1. Rules for the manipulation of binomial transforms

R1. $\mathbf{B}_s(\alpha a_n + \beta b_n) = \alpha \mathbf{B}_s a_n + \beta \mathbf{B}_s b_n$.

Proof. Direct from the definition. □

R2. $\mathbf{B}_s \frac{a_{n+1}}{n+1} = -\frac{\hat{a}_{s+1}}{s+1}$.

Proof. $\mathbf{B}_s \frac{a_{n+1}}{n+1} = \sum_n (-1)^n \binom{s}{n} \frac{a_{n+1}}{n+1} = \sum_n (-1)^n \frac{1}{s+1} \binom{s+1}{n+1} a_{n+1} = -\frac{\hat{a}_{s+1}}{s+1}$. □

R3. $\mathbf{B}_s \sum_{0 \leqslant k < n} a_k = -\hat{a}_{s-1}$.

Proof. Define the operator $\mathbf{S}a_n = \sum_{0 \leqslant k < n} a_k$. This operator satisfies $\mathbf{\Delta S} \equiv \mathbf{I}$. In terms of these operators, we have $\mathbf{B}_s \mathbf{S} \equiv -\mathbf{Z}(-\mathbf{\Delta})^{s-1} \equiv -\mathbf{B}_{s-1}$.

$$\mathbf{B}_s \sum_{0 \leqslant k \leqslant n} a_k = \hat{a}_s - \hat{a}_{s-1}.$$

Proof. $\mathbf{B}_s(\mathbf{I} + \mathbf{S}) \equiv \mathbf{B}_s - \mathbf{B}_{s-1}$.

$$\mathbf{B}_s a_{n-1} = -\sum_{0 \leqslant t < s} \hat{a}_t \quad \mathbf{B}_s(a_n - a_{n-1}) = \sum_{0 \leqslant t \leqslant s} \hat{a}_t.$$

Proof. These are mirror images of the two previous formulas. □

R4. $\mathbf{B}_s \frac{1}{n+1} \sum_{0 \leqslant k \leqslant n} a_k = \frac{\hat{a}_s}{s+1}$.

Proof. Direct from rules R3 and R2 above.

$$\mathbf{B}_s \frac{a_n}{n+1} = \frac{1}{s+1} \sum_{0 \leqslant t \leqslant s} \hat{a}_t.$$

Proof. This is the mirror image of the previous formula. □

R5. $\mathbf{B}_s(a_{n+1} - a_n) = -\hat{a}_{s+1}$.

Proof. $\mathbf{B}_s \mathbf{\Delta} \equiv -\mathbf{Z}(-\mathbf{\Delta})^{s+1} \equiv -\mathbf{B}_{s+1}$.

$$\mathbf{B}_s a_{n+1} = \hat{a}_s - \hat{a}_{s+1}.$$

Proof. This is the mirror image of the previous formula. □

R6. $\mathbf{B}_s n^{\underline{k}} a_{n-k} = (-1)^k s^{\underline{k}} \hat{a}_{s-k}$.

Proof. $\mathbf{B}_s n^{\underline{k}} a_{n-k} = \sum_{n\geqslant k} (-1)^n \binom{s}{n} n^{\underline{k}} a_{n-k} = (-1)^k s^{\underline{k}} \sum_{n\geqslant k} (-1)^{n-k} \binom{s-k}{n-k} a_{n-k}$. □

R7. $\mathbf{B}_s n a_n = s(\hat{a}_s - \hat{a}_{s-1})$.

Proof. $\mathbf{B}_s n a_n = \mathbf{B}_s \mathbf{E} n a_{n-1} - \mathbf{B}_s a_n = (-s\hat{a}_{s-1} + (s+1)\hat{a}_s) - \hat{a}_s$.

$\mathbf{B}_s n(a_n - a_{n-1}) = s\hat{a}_s$.

Proof. This is the mirror image of the previous formula. □

R8. $\mathbf{B}_s \sum_k \binom{n}{k} p^k q^{n-k} a_k = p^s \hat{a}_s$.

Proof. $\sum_{n,k} (-1)^n \binom{s}{n}\binom{n}{k} p^k q^{n-k} a_k = \sum_k (-1)^k \binom{s}{k} a_k p^k \sum_n (-1)^{n-k} \binom{s-k}{n-k} q^{n-k} = p^s \hat{a}_s$. □

R9. $\mathbf{B}_s \sum_k \binom{n}{k} p^k q^{n-k} a_k b_{n-k} = \sum_k \binom{s}{k} p^k q^{s-k} \hat{a}_k \hat{b}_{s-k}$.

Proof. $\sum_{n,k} (-1)^n \binom{s}{n}\binom{n}{k} p^k q^{n-k} a_k b_{n-k} = \sum_k \binom{s}{k} p^k a_k (-1)^k \sum_n \binom{s-k}{n-k} (-1)^{n-k} q^{n-k} b_{n-k}$. But the inner summation is just $\mathbf{B}_{s-k} q^n b_n$, and we can write it as a convolution using the mirror image of rule R8 so the whole summation can be rewritten as $\sum_{k,j} (-1)^k \binom{s}{k}\binom{s-k}{j} p^{s-j} q^j a_k \hat{b}_j = \sum_j \binom{s}{j} p^{s-j} q^j \hat{b}_j \sum_k \binom{s-j}{k} (-1)^k a_k$, and now the inner summation is equal to $\hat{a}_{s-j}$. The result follows by making the change of variable $k = s - j$. □

R10. $\mathbf{B}_s \sum_k \binom{n+1}{k+1} p^k q^{n-k} a_k = p^s \hat{a}_s - p^{s-1} q \sum_{0\leqslant t<s} \hat{a}_t$.

Proof. $\mathbf{B}_s \sum_k \binom{n+1}{k+1} p^k q^{n-k} a_k = \mathbf{B}_s \sum_k \left(\binom{n}{k} + \binom{n}{k+1}\right) p^k q^{n-k} a_k = p^s \mathbf{B}_s (a_n + \frac{q}{p} a_{n-1})$. □

R11. $\mathbf{B}_s (a_n - a_0) = \mathbf{B}_s (a_n - a_0)[n > 0] = \hat{a}_s [s > 0]$.

Proof. $\hat{a}_s[s > 0] = \hat{a}_s - \hat{a}_0[s = 0] = \hat{a}_s - a_0[s = 0] = \mathbf{B}_s(a_n - a_0) = \mathbf{B}_s(a_n - a_0)[n > 0]$. □

A.2. Table of transforms

T1. $1 \rightleftharpoons [n = 0]$.

Proof. $\mathbf{B}_s[n \geqslant 0] = \sum_{n\geqslant 0} (-1)^n \binom{s}{n} = [s = 0]$. □

T2. $[n > 0] \rightleftharpoons -[n > 0]$.

Proof. $\mathbf{B}_s[n > 0] = \mathbf{B}_s([n\geqslant 0] - [n = 0]) = [s = 0] - [s\geqslant 0] = -[s > 0]$. □

T3. $p^n \rightleftharpoons q^n$.

Proof. $\mathbf{B}_s p^n = \sum_n (-1)^n \binom{s}{n} p^n = (1 - p)^s$. □

T4. $\binom{n}{k} \rightleftharpoons (-1)^k [n = k]$.

Proof. $\mathbf{B}_s \binom{n}{k} = \mathbf{B}_s \binom{n}{k} [n\geqslant 0] = (-1)^k \binom{s}{k} [s - k = 0]$. □

T5. $\binom{n}{k} p^{n-k} \rightleftharpoons (-1)^k \binom{n}{k} q^{n-k}$.

Proof. $\mathbf{B}_s \binom{n}{k} p^{n-k} = (-1)^k \binom{s}{k} (1 - p)^{s-k}$. □

T6. $\frac{1}{n+1} \rightleftharpoons \frac{1}{n+1}$.

Proof. From rule R2 with $a_n = [n > 0]$. □

T7. $H_n \rightleftharpoons -\frac{[n>0]}{n}$.

Proof. $\mathbf{B}_s H_n = \mathbf{B}_s \mathbf{S} \frac{[n \geqslant 0]}{n+1} = -\frac{[n-1 \geqslant 0]}{n}$. □

T8. $\frac{1}{2}(H_n^2 + H_n^{(2)}) \rightleftharpoons -\frac{[n>0]}{n^2}$.

Proof. From formula (6.21) of [8], $\mathbf{B}_s \frac{1}{2}(H_n^2 + H_n^{(2)}) = \mathbf{B}_s \mathbf{S} \frac{H_{n+1}}{n+1} = -\frac{[s>0]}{s^2}$. □

T9. $\frac{1}{n+x} \rightleftharpoons \frac{n!}{x^{\overline{n+1}}} = B(x, n+1)$.

Proof. From exercise 1.2.6–48 of [10] or formula (5.41) of [8], we have that $\sum_n (-1)^n \binom{s}{n} \frac{1}{n+x} = \frac{s!}{x^{\overline{s+1}}}$. □

T10. $\frac{1}{(n+x)^2} \rightleftharpoons B(x, n+1)(\psi(x+n+1) - \psi(x))$.

Proof. Differentiate the previous formula with respect to x. □

T11. $n(H_n - 1) \rightleftharpoons -\frac{[n>1]}{n-1}$.

Proof. From rule R7 and from transform T7. □

T12. $Q_r(m, n) \rightleftharpoons (-1)^n (n, r) \frac{n!}{m^n}$.

Proof. $\mathbf{B}_s Q_r(m, n) = \sum_n (-1)^n \binom{s}{n} \sum_{k \geqslant 0} (k, r) \frac{n^{\underline{k}}}{m^k} = \sum_{k \geqslant 0} \frac{(r+1)^{\overline{k}}}{m^k} \sum_n (-1)^n \binom{s}{n} \binom{n}{k}$. But the inner summation is equal to $[s = k]$. The result follows. □

A.3. Properties of two-dimensional binomial transforms

P1. $a_{n_1,n_2} = b_{n_1+n_2}\ \forall n_1, n_2 \geqslant 0 \iff \hat{a}_{s_1,s_2} = \hat{b}_{s_1+s_2}\ \forall s_1, s_2 \geqslant 0$.

Proof. $\hat{a}_{s_1,s_2} = \sum_{n_1,n_2} (-1)^{n1+n2} \binom{s_1}{n_1} \binom{s_2}{n_2} a_{n_1,n_2} = \sum_n (-1)^n b_n \sum_{n_1+n_2=n} \binom{s_1}{n_1} \binom{s_2}{n_2}$. Now use Vandermonde's convolution formula to replace the inner summation by $\binom{s_1+s_2}{n}$. The result follows. □

P2. $b_n = \frac{1}{n+1} \sum_{0 \leqslant k \leqslant n} a_{k,n-k}\ \forall n \geqslant 0 \iff \hat{b}_s = \frac{1}{s+1} \sum_{0 \leqslant t \leqslant s} \hat{a}_{t,s-t}\ \forall s \geqslant 0$.

Proof. $\hat{b}_s = \sum_n \frac{(-1)^n}{n+1} \binom{s}{n} \sum_{i+j=n} a_{i,j} = \sum_{i,j} \frac{(-1)^{i+j}}{i+j+1} \binom{s}{i+j} a_{i,j} = \frac{1}{s+1} \sum_{i,j} (-1)^{i+j} \binom{s+1}{i+j+1} a_{i,j}$. Now, using formula 1.2.6–25 from [10], we replace $\binom{s+1}{i+j+1}$ by $\sum_{0 \leqslant k \leqslant s} \binom{s-k}{i} \binom{k}{j}$. The results follows after interchanging the order of the two summations. □

A.4. Useful expansions

E1. $\frac{1}{p^s - 1} = -\frac{1}{2} + \frac{1}{s \ln p} + \frac{1}{\ln p} \sum_{k \neq 0} \frac{1}{s + \sigma_k}$.

Proof. We use the identities $\frac{1}{p^s - 1} = \frac{1}{2}(\coth(\frac{s \ln p}{2}) - 1)$, that follows from the definition $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, and $-\frac{1}{2y} + \frac{\pi}{2} \coth \pi y = \sum_{n \geqslant 1} \frac{y}{n^2 + y^2}$ (formula 6.3.13 in [1]), and using partial fraction decomposition. □

E2. $\frac{p^s}{(p^s-1)^2} = \frac{1}{s^2 \ln^2 p} + \frac{1}{\ln^2 p} \sum_{k \neq 0} \frac{1}{(s+\sigma_k)^2}$.

Proof. Differentiate the previous formula with respect to s. □

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