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Wavelength routing of uniform instances in all-optical rings☆

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Abstract

We consider the problem of routing *uniform* communication instances in switched optical rings that use *wavelength-division multiplexing* technology. A communication instance is called uniform if it consists exactly of all pairs of nodes in the graph whose distance is equal to one from a specified set $S=\{d_1, d_2, \dots, d_k\}$. When $k=1$ or 2, we prove necessary and sufficient conditions on the values in S relative to n for the optimal wavelength index to be equal to the optimal load in the ring R_n. When $k=2$, we show that for any uniform instance specified by $\{d_1, d_2\}$, there is an optimal wavelength assignment on the ring R_n, if $n > (d_1/q-2)d_1+(d_1/q-1)d_2$, where $q=GCD(d_1, d_2)$. For general k and n, we show a $(\frac{3}{2})$-approximation for the optimal wavelength index; this is the best possible for arbitrary S. We also show that an optimal assignment can always be obtained provided n is large enough compared to the values in S.

Keywords: Optical ring; WDM; Wavelength index; Edge load; Uniform instance; Routing

1. Introduction

Communication networks based on optical fiber transmission systems provide capacities that are orders of magnitude higher than traditional networks. The high data transmission rate is achieved by transmitting information through optical signals, and maintaining the signal in optical form even for signal switching, thus avoiding the overhead of converting it into electronic form temporarily. *Wavelength-division multiplexing* (WDM) is one of the well-known and popular approaches to realize such high-capacity networks [3,4]. A switched optical network using the WDM approach consists of nodes connected by point-to-point fiber-optic links, each of which can support a fixed number of channels or wavelengths. Incoming data streams can be redirected at switches along different outgoing links based on wavelengths. Different messages can use the same link at the same time as long as they are assigned distinct wavelengths.

Optical networks are generally modeled by directed graphs, and more specifically by *symmetric digraphs*, that is, a directed graph G with vertex set $V(G)$ and edge set $E(G)$ such that if edge (x, y) is in $E(G)$, then edge (y, x) is also in $E(G)$. Thus, unless otherwise specified, whenever we talk about a graph, we always assume that we consider the associated *symmetric digraph* where any edge between x and y is replaced by two directed edges (x, y) and (y, x). The ring R_n is a network model with vertex set $\{v_0, v_1, \dots, v_{n-1}\}$ and edge set $\{(v_0, v_1), (v_1, v_2), \dots, (v_{n-2}, v_{n-1}), (v_{n-1}, v_0)\}$.

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In a network, a *request* is an ordered pair of nodes (x, y) which corresponds to a message to be sent from x to y. An *instance* I is a collection of requests. Given an instance I in the network, the *optical routing problem* is to determine a path through the network, and assign a wavelength to each request in I, so that no two requests whose paths share a link are assigned the same wavelength. Since the cost of an optical switch is proportional to the number of wavelengths it can handle, it is important to determine paths and wavelengths so that the total number of wavelengths required is minimized. Thus, an optical routing problem contains the related tasks of *route assignment* and *wavelength assignment*. A *routing* R for a given instance I is a set of paths $\{P(x, y) \mid (x, y) \in I\}$, where $P(x, y)$ is a path from x to y in the network. By representing a wavelength by a color, the wavelength assignment can be seen as a coloring problem where the same color is assigned to all the edges of a path given by the route assignment. We say that the coloring of a given set of paths is *conflict-free* if any two paths that share an edge are assigned different colors.

Given an instance I in a graph G, and a routing R for it, there are two parameters that are of interest. The *wavelength index* of the routing R, denoted $\vec{w}(G, I, R)$, is the *minimum* number of colors needed for a conflict-free assignment of colors to paths in the routing R of the instance I in G. The *edge-congestion* or *load* of the routing R for I, denoted by $\vec{\pi}(G, I, R)$, is the maximum number of paths that share the same edge. The parameters $\vec{w}(G, I)$, the *optimal wavelength index*, and $\vec{\pi}(G, I)$, the *optimal load* for the instance I in G are the minimum values over all possible routings for the given instance I in G. It is easy to see that $\vec{w}(G, I, R) \geqslant \vec{\pi}(G, I, R)$ for every routing R, thus $\vec{w}(G, I) \geqslant \vec{\pi}(G, I)$. It is known that the inequality can be strict [9]. A general upper bound for $\vec{w}(G, I)$ as a function of $\vec{\pi}(G, I)$ was given in [1]. Determining $\vec{\pi}(G, I)$ for arbitrary networks and instances is NP-hard, though for some specific networks such as trees and rings, or specific instances, such as the one-to-all instance, the problem can be solved efficiently. Finding $\vec{w}(G, I)$ is also NP-hard for arbitrary G and I. In fact, it is known to be NP-hard for specific graphs such as trees and rings [6]. Approximation algorithms for $\vec{w}(G, I)$ have been given for a variety of specific cases (see the survey paper [2] for references).

In this paper, we investigate the wavelength index of *uniform communication instances* on rings. We say that a set of requests I is a uniform communication instance in a network G if there exists a set of integers $S = \{d_1, d_2, \ldots, d_k\}$ such that I consists of all pairs of nodes in G whose distance is equal to d_i for some i where $1 \leqslant i \leqslant k$. We denote such an instance as I_S. It is easy to see that when $S = \{1, 2, \ldots, D_G\}$, where D_G is the diameter of G, then I_S is the all-to-all instance I_A [2]. Uniform communication instances can be used to implement some regular virtual networks. For example, a uniform communication instance $\{d_1, 1\}$ on a ring R_n corresponds to a virtual network of a chordal ring on n nodes with chord length d_1. Uniform communication instances and permutations may also be useful in doing certain systolic computations. Some specific uniform instances were studied in [12,13] and shown to be useful as sub-instances in solving other problems.

The problem of wavelength assignment in rings is equivalent to a coloring problem for circular-arc graphs. In [14], the problem of coloring circular-arc graphs [14] is studied, motivated by scheduling problems. The case of uniform instances restricts the types of circular-arc graphs that need to be considered. The results in [14] imply that $\vec{w}(G, I) \leqslant \frac{3}{2}\,\vec{\pi}(G, I)$ when the uniform instance contains exactly one path length, or when the sum of any three path lengths in the instance is always smaller than the size of the ring. Thus, the results in [14] apply only to upper bounds for some types of uniform instances, and optimal solutions are not studied.

It is easy to derive the optimal load of a uniform communication instance in a ring network; similar facts can be found in the literature. We state it briefly here for completeness.

Fact 1. *Let $S = \{d_1, d_2, \ldots, d_k\}$ be a set of integers such that $1 \leqslant d_i < n/2$ for $1 \leqslant i \leqslant k$. Then $\vec{\pi}(R_n, I_S) = \sum_{i=1}^{k} d_i$.*

This fact gives a lower bound on the wavelength index of $\vec{w}(R_n, I_S)$, and is used in the rest of the paper. As stated earlier, the task of optical routing involves both path assignment and color assignment to paths. In this paper, we always use shortest path routing, and unless otherwise stated, the path from u to v is the reverse of the path from v to u. The colors assigned to a path and its reverse are always the same, and hence we speak only about the color assigned to the path from u to v.

To derive bounds on the optimal wavelength index of a uniform instance, we first consider some special cases. In the next section, we consider I_S for S containing one or two elements. We show exact bounds on the wavelength index when $S = \{d_1\}$ is a singleton set. For $S = \{d_1, d_2\}$, we show necessary and sufficient conditions for the optimal wavelength index to be equal to the optimal load, and derive an upper bound on the wavelength index when the conditions are not met. Our main result is that for any choice of $S = \{d_1, d_2\}$, an optimal wavelength assignment is always possible, provided the ring size $n > (d_1/q - 2)d_1 + (d_1/q - 1)d_2$, where $q = GCD(d_1, d_2)$; this bound cannot be improved. To show the correctness of the assignment, we use a geometrical representation of the problem of routing I_S in R_n.

For the general case, in Section 3, we show that the optimal wavelength index is never more than $\frac{3}{2}$ times the optimal load; this is the best possible since there are instances where this many colors are required. We also show that the results for single and two path lengths can be used to obtain a wavelength assignment by decomposing the problem into subproblems. This gives the optimal wavelength index when n is large enough.

2. Single and double path lengths

In this section, we consider the cases when S contains one or two elements. It is known that even when S is a singleton set, the wavelength index for I_S is not necessarily equal to the load. For example, for $S = \{2\}$, it can easily be seen that $\vec{\pi}(R_5, I_S) = 2$ but $\vec{w}(R_5, I_S) = 3$, since at most 4 paths can be assigned the same color, but there are 10 requests in the instance. The following theorem provides an exact condition for the wavelength index to equal the load. Although this theorem was not yet stated in the literature in the form given here, the statement of the theorem is implied in the techniques used to solve the all-to-all instance on rings in [2].We give the theorem mainly because we need the statement of the theorem in several places and its proof aids the understanding of proofs of subsequent theorems.

Theorem 2. *Let $S = \{d_1\}$, where $d_1 \leqslant n/2$. Then $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S) = d_1$ if and only if d_1 divides n. For $S = \{k\}$, $\vec{w}(R_{2k}, I_S) = \vec{\pi}(R_{2k}, I_S) = \lceil k/2 \rceil$.*

Proof. Let $d_1 < n/2$ be a factor of n, i.e., $n = qd_1$ for some integer $q > 2$. For any i, $0 \leqslant i \leqslant d_1 - 1$, one color can be used for paths between q pairs of vertices(v_{0+i}, v_{d_1+i}), $(v_{d_1+i}, v_{2d_1+i}), \ldots, (v_{(q-1)d_1+i}, v_{0+i})$. Thus, repeating this process for each i, $0 \leqslant i \leqslant d_1 - 1$, we can assign a color to any path of length d_1 in the cycle. Therefore, d_1 colors are sufficient to get a conflict free assignment of colors to all paths connecting vertices at distance d_1.

If d_1 is not a factor of n, i.e. $n = qd_1 + r$ with $0 < r < d_1$, then one color can be used for at most q paths of length d_1 and thus we can color with d_1 colors at most $qd_1 < n$ paths of length d_1, which implies that $\vec{w}(R_n, I_S) > d_1$.

If the ring is of length $2k$, and $S = \{k\}$, we cannot use a symmetric assignment of paths to obtain an optimal solution, since a single color would only be used on half the edges in the ring. Thus, we need to deal with this case differently. The total load generated by the instance I_S on R_{2k} is $2k^2$. Thus the load on any edge of the cycle is at least $\lceil (2k^2)/(4k) \rceil$. For a wavelength assignment that uses the same number of colors we do an asymmetric assignment of paths, that is, the path from v_i to v_{i+k} is *not* the reverse of the path from v_{i+k} to v_i. For vertex v_{2i}, we assign the path using clockwise edges from v_{2i} to v_{2i+k}, and continue clockwise from v_{2i+k} to v_{2i}, and assign the color i to both paths. Note that all directed edges in the clockwise direction have already been assigned the color i, but none of the edges in the anti-clockwise direction. Next, we assign the path using *anti-clockwise* edges from v_{2i+1} to v_{2i+1+k} and continue anti-clockwise from v_{2i+1+k} to v_{2i+1}. We use the same color i on this pair of paths. The total number of colors used is $\lceil k/2 \rceil = \vec{\pi}(R_{2k}, I_S)$. $\square$

We can derive the exact value of the wavelength index which is valid even in the case when d_1 does not divide n.

Theorem 3. *Let $S = \{d_1\}$, where $d_1 < n/2$ and $n = qd_1 + r$, where $0 \leqslant r < d_1$. Then $\vec{w}(R_n, I_S) = \lceil n/(\lfloor n/d_1 \rfloor) \rceil = d_1 + \lceil r/q \rceil \leqslant 3\,\vec{\pi}(R_n, I_S)/2$.*

Proof. We only need to consider the case when d_1 does not divide n, i.e., $n = qd_1 + r$ with $0 < r < d_1$. Then one color can be used for at most q paths of length d_1 and thus we need at least $\lceil n/q \rceil = \lceil n/(\lfloor n/d_1 \rfloor) \rceil$ colors.

Assume first that n and d_1 do not have a common factor. One color can be used for paths between the q pairs of vertices (v_0, v_{d_1}), $(v_{d_1}, v_{2d_1}), \ldots, (v_{(q-1)d_1}, v_{qd_1})$, the second color between the q pairs of vertices (v_{qd_1}, v_{d_1-r}),

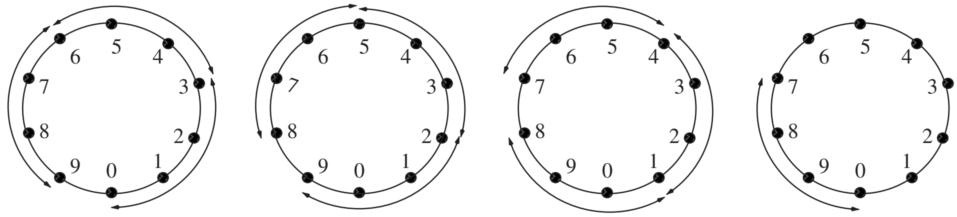

Fig. 1. Four colors suffice when $d_1 = 3$ and $n = 10$.

$(v_{d_1-r}, v_{2d_1-r}), \ldots, (v_{(q-1)d_1-r}, v_{qd_1-r})$, the third color between the q pairs of vertices (v_{qd_1-r}, v_{d_1-2r}), $(v_{d_1-2r}, v_{2d_1-2r}), \ldots, (v_{(q-1)d_1-2r}, v_{qd_1-2r})$, and so on. It is easy to check that the first n pairs of vertices we get in this process are distinct. Thus we need at most $\lceil n/q \rceil = \lceil n/(\lfloor n/d_1 \rfloor) \rceil$ colors.

If n and d_1 have a common factor then we start with the same process as above. However, due to n and d_1 having a common factor, we get the pair (v_0, v_{d_1}) in the process for the second time before all pairs are obtained. In that case we take the pair (v_1, v_{d_1+1}) instead and continue the process with pairs (v_{d_1+1}, v_{2d_1+1}), (v_{2d_1*1}, v_{3d_1+1}), etc. When we get the pair (v_1, v_{d_1+1}) in the process for the second time we again shift by 1 by taking the pair (v_2, v_{d_1+2}) instead and continue the process with pairs (v_{d_1+2}, v_{2d_1+2}), (v_{2d_1+2}, v_{3d_1+2}), etc. The first n pairs of vertices we get in this process are distinct and thus we need at most $\lceil n/q \rceil = \lceil n/(\lfloor n/d_1 \rfloor) \rceil$ colors. We have $\lceil n/(\lfloor n/d_1 \rfloor) \rceil = \lceil (qd_1 + r)/q \rceil = d_1 + \lceil r/q \rceil$.

Since $d_1 < n/2$ we obtain the largest difference between $\vec{w}(R_n, I_S)$ and $\vec{\pi}(R_n, I_S)$ when $n = 3d_1 - 1$. In this case $\vec{w}(R_n, I_S) = d_1 + \lceil (d_1 - 1)/2 \rceil = d_1 + \lceil (n-2)/6 \rceil = \vec{\pi}(R_n, I_S) + \lceil (n-2)/6 \rceil \leqslant 3\,\vec{\pi}(R_n, I_S)/2$. □

See Fig. 1 for an example of an assignment when $n = 10$ and $d_1 = 3$.

The previous theorem indicates that the difference between $\vec{w}(R_n, I_S)$ and $\vec{\pi}(R_n, I_S)$ will be large when d_1 and $(n \bmod d_1)$ are large. On the other hand the difference between $\vec{\pi}(R_n, I_S)$ and $\vec{\pi}(R_n, I_S)$ is small when d_1 is small as shown below.

Corollary 4. *Let $S = \{d_1\}$, where $1 \leqslant d_1 < n/2$, d_1 does not divide n, and $d_1 \leqslant \sqrt{in}$ for some positive integer i. Then $\vec{w}(R_n, I_S) \leqslant \vec{\pi}(R_n, I_S) + i$.*

Proof. Let $n = qd_1 + r$, where $0 \leqslant r < d_1$. By Theorem 3, $\vec{w}(R_n, I_S) = d_1 + \lceil \frac{r}{q} \rceil$. Since $d_1 \leqslant \sqrt{in}$, we have that $r < \sqrt{in}$ and $q \geqslant \sqrt{n/i}$ and, therefore, $\vec{w}(R_n, I_S) \leqslant d_1 + \lceil \sqrt{in}/\sqrt{n/i} \rceil \leqslant d_1 + i$. □

Next, we consider the case $S = \{d_1, d_2\}$. For convenience, we denote the optical routing problem of instance I_S on the ring R_n by $R(n, d_1, d_2)$ and assume that $d_1 > d_2$. Clearly if d_1 and d_2 both divide n, then we can solve the instances $I_{\{d_1\}}$ and $I_{\{d_2\}}$ separately using the results for the single path case, and then combine them to get a solution in which $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$. Therefore, we now consider the case when at least one of d_1 and d_2 does not divide n and find necessary and sufficient conditions for $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$.

If $d_1 = n/2$ and $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$ then every color is used on every edge of the ring. Let c be a color that is used for a path of length d_2. Then either this color is used only for paths of length d_2 and d_2 divides n, or c is used for one path of length d_1 and the remaining part of R_n is used for paths of length d_2 and thus d_2 divides $n/2$. In either case both d_1 and d_2 are factors of n, a case already dealt with. Therefore we assume in the rest of this section that $d_1 < n/2$.

First we derive a necessary condition for the optimal wavelength index to equal the load in the case when $GCD(d_1, d_2) = 1$.

Theorem 5. *Let $S = \{d_1, d_2\}$ such that $1 \leqslant d_2 < d_1 < n/2$, $GCD(d_1, d_2) = 1$, and at least one of d_1 and d_2 does not divide n. If $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$ then there exist integers $a > 0$ and $b \geqslant 0$ such that $n = ad_1 + bd_2$ and either $a = b$ or $a > b$, $a \geqslant d_2$ and $a - b \leqslant d_1 + d_2$.*

Proof. Since $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$, every color must be used on every edge of the ring. Assume that d_1 does not divide n. Let c_1 be a color that is assigned to the path of length d_1 from vertex v_0 to vertex v_{d_1}. Since d_1 does not divide n, color c must be assigned to at least one path of length d_2. Let a_1 be the number of paths of length d_1 that are assigned c_1 and b_1 be the number of paths of length d_2 thatare assigned the color c_1. Clearly, $a_1 > 0$, $b_1 > 0$, and $n = a_1d_1 + b_1d_2$. If $a_1 = b_1$ then we take $a = a_1$ and $b = b_1$ and the theorem is proved.

Assume that $a_1 > b_1$. Then the use of the pattern for color c assigns colors to more paths of length d_1 than to paths of length d_2. Thus, there must be a color, say c_2, which is used for every edge of the ring and which is assigned to a_2 paths of length d_1 and b_2 paths of length d_2 with $0 \leqslant a_2 < b_2$. Thus $n = a_2d_1 + b_2d_2 = a_1d_1 + b_1d_2$ and we have $(a_1 - a_2)d_1 = (b_2 - b_1)d_2$. Since $GCD(d_1, d_2) = 1$, integer d_2 is a factor of $(a_1 - a_2)$, i.e., $a_1 - a_2 = kd_2$ for some positive integer k and furthermore $b_2 - b_1 = kd_1$. In other words, $a_2 = a_1 - kd_2$ and $b_2 = b_1 + kd_1$. Since $a_1 > b_1$ and $(a_1 - kd_2) < b_1 + kd_1$, there exists an integer i, where $0 \leqslant i < k$ such that such that $(a_1 - id_2) > (b_1 + id_1)$ and $a_1 - (i + 1)d_2 < b_1 + (i + 1)d_1$. Take $a = a_1 - id_2$ and $b = b_1 + id_1$. We have $a > b$ and $a \geqslant d_2$ since $a_1 - (i + 1)d_2 \geqslant a_1 - kd_2 \geqslant 0$. Since $a_1 - (i+1)d_2 < b_1 + (i+1)d_1$, we have $a_1 - id_2 < b_1 + id_1 + d_1 + d_2$ and thus $a - b = a_1 - id_2 - (b_1 + id_1) < d_1 + d_2$.

If n is not a multiple of d_2, we get the result in a similar manner. □

The above theorem gives a necessary condition for $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$. We show below that these conditions are also sufficient. Therefore, we next consider the case when $n = ad_1 + bd_2$ with either $a = b$ or $a > b$, $a - b < d_1 + d_2$.

We first show that an optimal solution can be found when $a = b$, that is, $(d_1 + d_2)$ divides n. In fact, we prove a more general result below.

Theorem 6. *Let $S = \{d_1, d_2, \ldots, d_k\}$, where $1 \leqslant d_k < \cdots < d_2 < d_1 < n/2$ and $\sum_{i=1}^{k} d_i$ divides n. Then $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S) = \sum_{i=1}^{k} d_i$.*

Proof. Let $d_c = \sum_{i=1}^{k} d_i$ and $S' = \{d_c\}$. Since d_c divides n, if $d_c < n/2$, by Theorem 2, there exists an assignment using d_c colors for instance $I_{S'}$ on the ring. For vertex v_i of the ring and the path between v_i and v_{i+d_c} of length d_c, partition the path into k paths of length $d_1, d_2, \ldots, d_k$ in this order and assign them the same color that is assigned to the path between v_i and v_{i+d_c}. It is easy to see that this gives a conflict-free assignment of colors for the instance I_S.

Finally, if $d_c = n/2$, or $d_c = n$ it is easy to get an assignment using d_c colors, simply by using a pattern with two paths, or one path of each type, and shifting the origin of the pattern $n/2 - 1$, or $n - 1$ times, respectively. The paths are then partitioned into k paths of length $d_1, d_2, \ldots, d_k$ in this order. □

Now we consider the case when $d_1 + d_2$ does not divide n and $n = ad_1 + bd_2$, $a > b$, $a \geqslant d_2$, $a - b < d_1 + d_2$ and show that an optimal solution can be found for the routing problem $R(n, d_1, d_2)$.

Definition 7. For fixed d_1, d_2 and n, let a, b, a', b' be non-negative integers such that $n = ad_1 + bd_2 = a'd_1 + b'd_2$. Then we say (i, j) are valid multipliers for $(n, a, b, a'b')$ provided $ai + a'j = bi + b'j = n$.

Lemma 8. *Let $n = ad_1 + bd_2$, where $d_1 > d_2 > 1$, $a > b$, $a \geqslant d_2$, and $a - b < d_1 + d_2$, $a, b \geqslant 0$ and $GCD(d_1, d_2) = 1$. Then, there exist $a', b' \geqslant 0$, where $(a, b) \neq (a', b')$ such that $n = a'd_1 + b'd_2$, and valid multipliers $i, j > 0$ for (n, a, b, a', b').*

Proof. Let $a' = a - d_2$ and $b' = b + d_1$. Since $a \geqslant d_2$, $a' \geqslant 0$. Fix $j = (a - b) > 0$, and $i = b' - a' = d_1 + d_2 - (a - b) > 0$. It is straightforward to verify that i and j are valid multipliers. □

The next lemma shows that the second necessary case in Theorem 5 for the wavelength index being equal to the load is also sufficient.

Lemma 9. *Let $S = \{d_1, d_2\}$, where $1 < d_2 < d_1 < n/2$ and $GCD(d_1, d_2) = 1$. If $n = ad_1 + bd_2$ where $a > b \geqslant 0$, $a \geqslant d_2$ and $a - b < d_1 + d_2$ then $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$.*

Proof. By Lemma 8, there exist a' and b' where $(a, b) \neq (a', b')$, and $n = a'd_1 + b'd_2$, and valid multipliers $i, j > 0$ for (a, b, a', b'). We first give a brief description of the key idea, and then give the details later. Since $n = ad_1 + bd_2$, where

$a > b$, we can start at vertex v and alternate b paths of length d_2 and d_1 followed by $a - b$ paths of length d_1 to cover all edges of the ring and return to vertex v. All these paths may be assigned the same color without conflict. Using this *pattern*, we have assigned wavelengths to a paths of length $d_1$and b paths of length d_2. By shifting this pattern $i - 1$ times, that is, starting at i distinct appropriately chosen vertices, we can assign colors to ai paths of length d_1 and bi paths of length d_2.

In a similar way, a second pattern is achieved, by using the relation $n = a'd_1 + b'd_2$: we can start at a particular vertex v, and alternate a' paths of length d_1 and d_2 followed by $b' - a'$ paths of length d_2 to return to the same vertex v. All these paths may be assigned the same color without conflict. By shifting this second pattern $j - 1$ times, we can assign colors to $a'j$ paths of length d_1 and $b'j$ paths of length d_2. Since i and j are valid multipliers for (n, a, b, a', b'), if all the paths assigned above are distinct, we have finished the routing and wavelength assignment for the instance I_S. Since we need a new color each time we shift a pattern, we need a total of $i + j = d_1 + d_2$ colors. By Fact 1, we conclude that $\overrightarrow{w}(R_n, I_S) = \overrightarrow{\pi}(R_n, I_S)$.

It remains to show how to shift the first pattern and second pattern in such a manner that we obtain a path of length d_1 and d_2 from each vertex. We claim that starting the second pattern at vertex at 0, and shifting by d_1 an additional $j - 1$ times, and starting the first pattern at $(d_1 + d_2)(a' - b) \bmod n$ and shifting it by d_2 a total of $i - 1$ times will result in a valid assignment. To show this, we use a planar geometrical representation of the routing problem $R(n, d_1, d_2)$. The idea stems from a geometrical representation for distributed loop graphs that was already used in [7,8,15] and was subsequently used to derive results on compact routing and wavelength routing in chordal rings [10,12]. (See Fig. 4 for an example of the geometrical representation we are using for the case $n = 68$, $d_1 = 7$, $d_2 = 5$, $a = 9$, $b = 1$, $a' = 4$, $b' = 8$, $i = 4$ and $j = 8$.)

First we define a representation in the Euclidean plane of the vertices in R_n. Consider the Euclidean plane divided into square units. Label each square with the index of a vertex of R_n as follows. The square in coordinates $[0, 0]$ is labeled 0. For any square in coordinates $[i, j]$ with label u, the square to the right in coordinates $[i+1, j]$ is labeled $(u+d_2) \bmod n$, the square to the left in coordinates $[i - 1, j]$ is labeled $(u - d_2) \bmod n$, the square above it in coordinates $[i, j + 1]$ contains the label $(u+d_1) \bmod n$ and the square below it in coordinates $[i, j-1]$ contains the label $(u-d_1) \bmod n$. Since for any u, $(((u+d_1) \bmod n)+d_2) \bmod n = (((u+d_2) \bmod n)+d_1) \bmod n$ and $(((u-d_1) \bmod n)-d_2) \bmod n = (((u-d_2) \bmod n)-d_1) \bmod n$, the labeling of the plane is well-defined. Thus, paths of type (v_i, v_{i+d_2}), (v_i, v_{i-d_2}) in the ring are represented by horizontal lines of length 1, and paths of type (v_i, v_{i+d_1}), (v_i, v_{i-d_1}) are represented by vertical lines of length 1.

Consider a square in the plane in coordinates $[x, y]$. It follows from the geometrical representation that the label of the square is equal to $u = xd_2 + yd_1$. Furthermore, there is a path from v_0 to v_u in the ring consisting of $|x|$ sub-paths of length d_2 each and $|y|$ sub-paths of length d_1 each.

We now focus on one part of the plane, a portion which is shown to contain every vertex exactly once. Following the established terminology, we call this portion *a tile* [7,8,15]. Let $n = ad_1 + bd_2$ and $n = a'd_1 + bd_2$ satisfying the conditions in the statement of the lemma such that (i, j) are valid multipliers for (a, b, a', b'). Let A_1 consist of all squares whose coordinates $[x, y]$ satisfy $0 \leqslant x < a'$ and $x \leqslant y < x + j$. Let A_2 consist of all squares whose coordinates $[x, y]$ satisfy $a' \leqslant x < b'$, $a' \leqslant y < a' + j$, and finally A_3 consist of all squares whose coordinates $[x, y]$ satisfy $a' - b \leqslant y < a'$ and $y < x \leqslant y + i$. We define $Tile(n, d_1, d_2)$ to be $A_1 \cup A_2 \cup A_3$. Fig. 2 demonstrates the area covered by $Tile(n, d_1, d_2)$ in cases $a' \geqslant b$ and $a' < b$.

The following two technical claims are crucial to proving that using the two patterns as described results in a correct assignment.

Claim 10. *Every label in* $\{0, 1, 2, \ldots, n - 1\}$ *appears exactly once in* $Tile(n, d_1, d_2)$.

Proof. It is easy to observe that if coordinates $[x, y]$ and $[x', y']$ contain the same label then for any integers k_1 and k_2 the coordinates $[x + k_1, y + k_2]$ and $[x' + k_1, y' + k_2]$ also contain the same label. This implies that if any label appears twice in the tile, the label 0 would also appear twice in the tile. Thus it suffices to show that the label 0 appears only in coordinates $[0, 0]$ and no other coordinates in $Tile(n, d_1, d_2)$. Clearly, for any $[x, y]$ such that $0 \leqslant x \leqslant b$ and $0 \leqslant y < a$, we have $yd_1 + xd_2 < ad_1 + bd_2 = n$, thus 0 cannot appear at any such coordinate unless $[x, y] = [0, 0]$. Similarly, it cannot appear at any coordinate for any $[x, y] \neq [0, 0]$ such that $0 \leqslant x < b'$ and $0 \leqslant y \leqslant a'$, since we have $yd_1 + xd_2 < a'd_1 + b'd_2 = n$. From the area covered by $Tile(n, d_1, d_2)$, this leaves only the areas B_1 when $a' \geqslant b$ and B_2

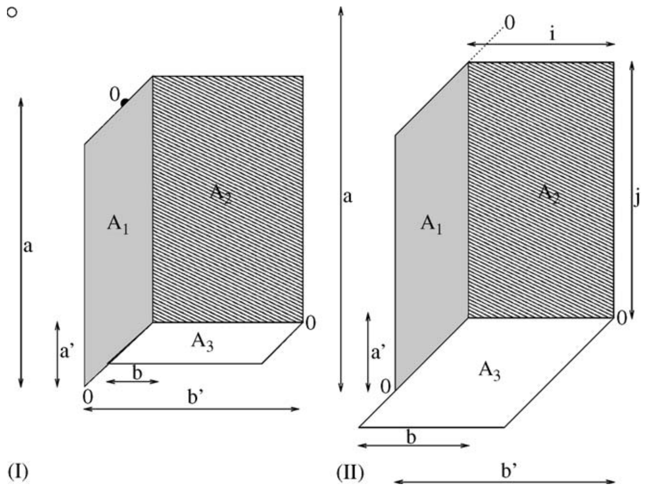

Fig. 2. $Tile(n, d_1, d_2)$ for (I) the case $a' \geqslant b$ and (II) the case $a' < b$.

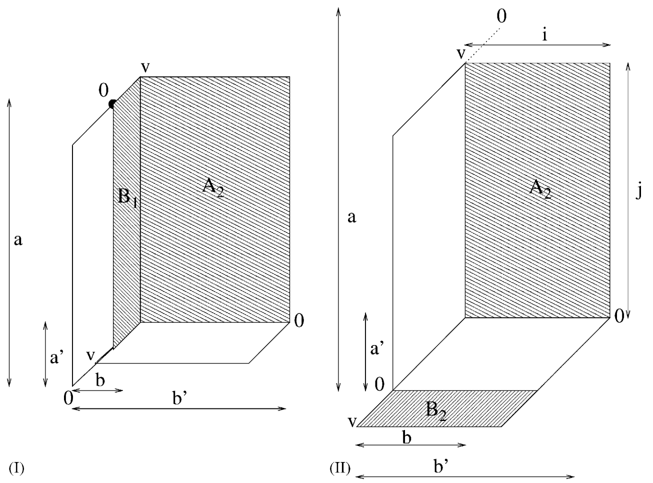

Fig. 3. The areas B_1, B_2 and A_2.

when $a' < b$, as well as A_2 in both cases (see Fig. 3). Here, B_1 is that part of A_1 given by $b < x < a'$ and $x \leqslant y \leqslant j + x$, and B_2 is that part of A_3 given by $a - b' \leqslant y < 0$ and $y \leqslant x \leqslant y + i$. Assume, for the purposes of contradiction that 0 does appear at coordinates $[x, y]$ in one of these areas. Then we know that $ad_1 + bd_2 = a'd_1 + b'd_2 = yd_1 + xd_2 = n$. Clearly, $(a - a')/(b' - b) = (a - y)/(x - b) = (y - a')/(b' - x) = d_2/d_1$ and $a - a' = d_2$, $b' - b = d_1$.

$(x, y) \in A_2$: Since $[x, y] \in A_2$, we know that $y \geqslant a'$. By assumption, $(a - y)/(x - b) = d_2/d_1$. Since $a - y \leqslant a - a'$, and since $x \neq b'$, it must be that $a - y' = d_2/r$ and $x - b = d_1/r$ for some $r > 1$, that is, r is a common factor of d_1 and d_2. But d_1 and d_2 are co-prime, a contradiction.

$[x, y] \in B_1, a' \geqslant b$: Since $[x, y] \in B_1$, we know that $x > b$. By assumption, $(y - a')/(b' - x) = d_2/d_1$. Since $b' - x < b' - b$, it must be that $a - y' = d_2/r$ and $x - b = d_1/r$ for some $r > 1$, that is, r is a common factor of d_1 and d_2. But d_1 and d_2 are co-prime, a contradiction.

$[x, y] \in B_2, a' < b$: Since $[x, y] \in B_2$, we know that $y < 0$ and $x < y + i < i = b' - a' \leqslant b'$. By assumption, $(a' - y)/(x - b') = d_2/d_1$, but since $a' - y > 0$ and $x - b' < 0$, we have a contradiction.

We see that in all the cases, 0 cannot appear at the coordinates $[x, y]$ in the specified areas. Thus, 0 cannot appear at any coordinates other than $[0, 0]$ within $Tile(n, d_1, d_2)$ as claimed. $\square$

Claim 11. *Repeating the two patterns as described results in every vertex of the tile being assigned a path of type d_1 and d_2.*

Proof. The second pattern, defined by the equation $n = a'd_1 + b'd_2 = (a - d_2)d_1 + (b + d_1)d_2$, is done $a - b$ times, starting from the squares at the left border of the area A_1 of the tile. Informally, using the geometrical representation, we can say that the second pattern initially forms a' "stairs" in the area A_1, each stair beginning with the vertical part, followed by a horizontal line of length $b' - a'$ through area A_2. More formally, when the second pattern starts in the square with label 0, it assigns a path of type d_1 to the squares with labels $0, d_1 + d_2, 2d_1 + 2d_2, \ldots, (a' - 1)(d_1) + (a' - 1)d_2$ in the area A_1 of the tile, a path of type d_2 to the squares with labels $d_1, 2d_1 + d_2, 3d_1 + 2d_2, \ldots, a'd_1 + (a' - 1)d_2$ in the area A_1 of the tile and to vertices $a'd_1 + a'd_2, a'd_1 + (a' + 1)d_2, \ldots, a'd_1 + (b' - 1)d_2$ in the area A_2 of the tile. When the second pattern starts in the square with label d_1, it assigns a path of type d_1 to the squares with labels $d_1, 2d_1 + d_2, 3d_1 + 2d_2, \ldots, a'd_1 + (a' - 1)d_2$ in the area A_1 of the tile, a path of type d_2 to the squares with labels $2d_1, 3d_1 + d_2, 4d_1 + 2d_2, \ldots, (a' + 1)d_1 + (a' - 1)d_2$ in the area A_1 of the tile and to squares with labels $(a' + 1)d_1 + a'd_2, (a' + 1)d_1 + (a' + 1)d_2, \ldots, (a' + 1)d_1 + (b' - 1)d_2$ in the area A_2 of the tile.

Notice that those squares in area A_1 that are assigned a path of type d_2 by the second pattern that starts in square with 0 are assigned a path of type d_1 by the second pattern that starts in square with d_1. Clearly, this is also true for patterns that start in squares kd_1 and in $(k + 1)d_1$ for any k, $0 \leqslant k \leqslant a - b - 1$. From this we can conclude that all squares in area A_1 are assigned a path of type d_1, and all squares in area A_1, except for those at the bottom border, are assigned a path of type d_2. Furthermore, when the second pattern starts in the square with label $d_1(a - b - 1)$ then this pattern assigns paths of type d_2 to squares that do not necessarily belong to A_1. In case $a' < b$, it assigns a path of type d_2 to a' squares that are at the right border of area A_3 starting from its bottom corner. In case $a' \geqslant b$ it assigns a path of type d_2 to $a' - b$ squares at the bottom of area A_1 from the left corner to the beginning of the area A_3 and to all b squares that are at the right border of area A_3. Clearly, all squares in area A_2 are assigned path of type d_2.

The first pattern, defined by the equation $n = ad_1 + bd_2$, is done $b' - a' = b + d_1 - a + d_2$ times, starting from the squares at the horizontal bottom border of the tile, i.e., in squares in coordinates from $[a' - b, a' - b]$ to $[a' - b, a' - b + (b' - a') - 1] = [a' - b, b' - b - 1]$. Informally, using the geometrical representation, we can say that the first pattern initially forms b "stairs" in the area A_3, followed by a vertical line of length $a - b$ through area A_2. More formally, when the first pattern starts in the square $[a' - b, a' - b]$ labeled with $d = (d_1 + d_2)(a' - b)$, it assigns a path of type d_2 to the squares labeled $d, d + d_1 + d_2, d + 2(d_1 + d_2), \ldots, d + (b - 1)(d_1 + d_2)$, a path of type d_1 to the squares labeled $d + d_2, d + d_1 + 2d_2, \ldots, d + (b - 1)d_1 + bd_2$ in the area A_3 of the tile and to the squares labeled $d + b(d_1 + d_2), d + b(d_1 + d_2) + d_1, \ldots, d + b(d_1 + d_2) + (a - b - 1)d_1$ in the area A_2 of the tile. If $a' \geqslant b$ then the squares labeled $d, d + d_1 + d_2, d + 2(d_1 + d_2), \ldots, d + (b - 1)(d_1 + d_2)$ are the remaining squares on the bottom border of area A_1 that were not yet assigned a path of type d_2 by the second pattern. If $a' < b$ then these squares are the remaining squares on the right border of A_3 not assigned a path of type d_2 by the second pattern, and all squares on the bottom border of area A_1.

When the first pattern starts in the square with label $d + d_2$, it assigns a path of type d_2 to the squares with labels $d + d_2, d + 2d_2 + d_1, d + 3d_2 + 2d_1, \ldots, bd_2 + (b - 1)d_1$ in the area A_3 of the tile, a path of type d_2 to the squares with labels $d + 2d_2, d + 3d_2 + d_1, d + 4d_2 + 2d_1, \ldots, d + (b + 1)d_2 + (b - 1)d_1$ in the area A_3 of the tile and to squares with labels $d + (b + 1)d_2 + bd_1, (b + 1)d_2 + (b + 1)d_1, \ldots, d + (b + 1)d_2 + (a - b - 1)d_1$ in the area A_2 of the tile.

Notice that those squares in area A_3 that are assigned a path of type d_1 by the first pattern that starts in a square labeled with d are assigned a path of type d_2 by the first pattern that starts in $d + d_2$. Clearly, this is also true for patterns that start in $d + kd_2$ and in $d + (k + 1)d_1$ for any k, $0 \leqslant k \leqslant b' - a' - 1$. From this we can conclude that all squares in area A_3 are assigned a path of type d_1 and all squares in area A_3 are assigned a path of type d_2. Furthermore, all squares in area A_2 are assigned path of type d_1.

By repeating the second pattern $j = a - b$ times and the first pattern $i = b' - a'$ times, we have altogether assigned $ia + ja' = n$ paths of type d_1 and $ib + jb' = n$ paths of type d_2. See Fig. 4 for an example of paths covered by both types of patterns for the case $n = 68$, $d_1 = 7$, $d_2 = 5$. It is easy to verify that $a = 9$, $b = 1$, $a' = 4$, $b' = 8$, $i = 4$ and $j = 8$ in this case. □

The claims above show that paths of length d_1 and d_2 are assigned to every vertex in R_n, and can be colored with $d_1 + d_2$ colors, completing the proof of the lemma. □

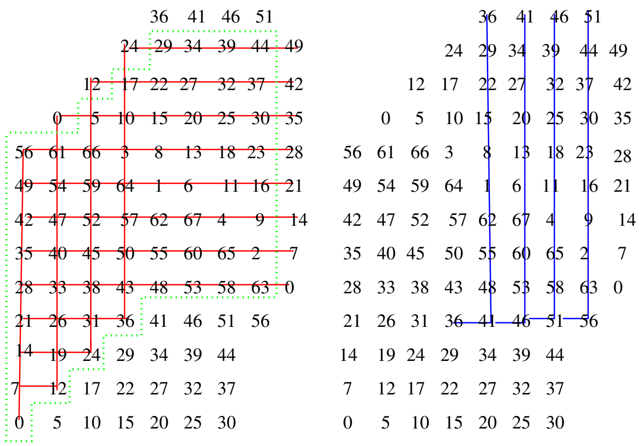

Fig. 4. Patterns cover all paths.

Given d_1, d_2 and n, we can easily verify whether there are constants a and b satisfying Lemma 9. First verify that at least one of d_1, d_2 does not divide n and $GCD(d_1, d_2) = 1$. Then let $q_1 = \lfloor n/d_1 \rfloor$ and $r = n \bmod d_1$. If $r = 0$ then the conditions of the lemma are satisfied if and only if $q_1 > d_2$. In particular, take $a = q_1 - d_2 j$ and $b = jd_1$, where $j = \lfloor q_1/(d_1 + d_2) \rfloor$ i.e., j is the largest positive integer such that $q_1 - d_2 j > jd_1$. If instead, $r > 0$ then determine the smallest i, $0 \leqslant i < q$, if any, such that $(id_1 + r) \bmod d_2 = 0$. If no such i exists then no constants satisfying the lemma exist. When such an i exists, $n = (q_1 - i)d_1 + q_2 d_2$, where $q_2 = (id_1 + r)/d_2$. The constants satisfying the conditions of the lemma exist if and only if $q_1 - i > q_2$ and $q_1 - i \geqslant d_2$. In particular, $a = q_1 - i - d_2 j$ and $b = q_2 + jd_1$, where $j = \lfloor (q_1 - i - q_2)/(d_1 + d_2) \rfloor$ i.e., j is the largest positive integer such that $q_1 - i - d_2 j > q_2 + jd_1$. Clearly the above procedure needs only need a constant number of operations in any of the cases.

In the next lemma we show that given fixed values of d_1 and d_3, the conditions on constants in Lemma 9 are always satisfied when n is large enough.

Lemma 12. *Let n be a positive integer and $S = \{d_1, d_2\}$, where $1 < d_2 < d_1 < n/2$, $GCD(d_1, d_2) = 1$, and $n > (d_1 - 2)d_1 + (d_1 - 1)d_2$. Then $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$.*

Proof. Since $GCD(d_1, d_2) = 1$ and $n \geqslant 2d_1 d_2$, there exist non-negative integers a and b such that $n = ad_1 + bd_2$. If $a = b$ or if $a > b$, $a \geqslant d_2$ and $a - b < d_1 + d_2$, by Theorem 6 and Lemma 8, respectively, an optimal solution for the routing problem $R(n, d_1, d_2)$ exists. If not, then we show the existence of integers $p, q \geqslant 0$ that satisfy these conditions. The following cases are exhaustive:

$b < a < d_2$: in this case, $n = ad_1 + bd_2 \leqslant (d_2 - 1)d_1 + (d_2 - 2)d_2 \leqslant (d_1 - 2)d_1 + (d_1 - 1)d_2$, a contradiction.
$a < b$: let i be the minimum integer such that $b - a \leqslant i(d_1 + d_2)$, and let $p = a + id_2$ and $q = b - id_1$. First, we show that $q \geqslant 0$. Suppose instead that $b < id_1$. Since $b - a > (i-1)(d_1 + d_2)$, we have that $a < b - (i-1)(d_1 + d_2) \leqslant id_1 - 1 - (i - 1))(d_1 + d_2) = d_1 - (i-1)d_2 - 1$. Then, $n = ad_1 + bd_2 \leqslant (d_1 - (i-1)d_2 - 2)d_1 + (id_1 - 1)d_2 = (d_1 - 2)d_1 + d_1 - 1)d_2$, a contradiction. Thus $q \geqslant 0$ as required.

If $b - a = i(d_1 + d_2)$ then $p = q$ and since $n = pd_1 + qd_2$, by Lemma 6, there is an optimal solution for $R(n, d_1, d_2)$. If instead $b - a < i(d_1 + d_2)$, then note that since $i > 0$, $p > d_2$. Also it is easy to see that $p > q$. Finally, $p - q = a + id_2 - b + id_1 = (a - b) + i(d_1 + d_2) < d_1 + d_2$ since $(b - a) > (i - 1)(d_1 + d_2)$ by choice of i as being minimum. Thus, p and q satisfy the conditions for Lemma 8, and by Lemma 9, an optimal solution for $R(n, d_1, d_2)$exists.
$a > b$ and $a \geqslant d_2$, but $(a - b) \geqslant d_1 + d_2$: let i be the maximum integer such that $a - b \geqslant i(d_1 + d_2)$, and let $p = a - id_2$ and $q = b + id_1$. Again, if $a - b = i(d_1 + d_2)$ then $p = q$ and by Lemma 6, there is an optimal solution for $R(n, d_1, d_2)$. Otherwise $a - b > i(d_1 + d_2)$, and thus $p > q$. Further, since $a, b \geqslant 0$, we know that $a > i(d_1 + d_2)$, thus $p = a - id_2 > id_1 \geqslant d_2$, since $i > 0$. Finally, $p - q < d_1 + d_2$ since $a - b < (i + 1)(d_1 + d_2)$, since i is chosen to be the maximum

possible. Thus, p and q satisfy the conditions for Lemma 8, and by Lemma 9, an optimal solution for $R(n, d_1, d_2)$ exists. □

Note that when $n = (d_1 - 2)d_1 + (d_1 - 1)d_2$ then n cannot be expressed as $n = ad_1 + bd_2$ with a, b satisfying the condition of Lemma 9 and thus the bound from Lemma 12 cannot be improved.

Now we consider the case when $d_1 + d_2$ does not divide n, $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$, and $GCD(d_2, d_2) > 1$. Let $GCD(d_2, d_2) = q$. As we argued before, there must exist integer constants a and b such that $n = ad_1 + bd_2$. However, if $n = ad_1 + bd_2$ then q must be also a factor of n. We show below that in this case can use a solution of the wavelength index for the problem "scaled down" by a factor of q to derive a solution for $R(n, d_1, d_2)$, as shown in the next lemma.

Lemma 13. *Let $S = \{d_1, d_2\}$, where $1 < d_2 < d_1 < n/2$, $d_1 + d_2$ does not divide n and $gcd(n, d_1, d_2) = q$. Then $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$ if and only if $\vec{w}(R_{n/q}, I_{S'}) = \vec{\pi}(R_{n/q}, I'_S)$, where $S' = \{d_1/q, d_2/q\}$.*

Proof. Suppose $\vec{w}(R_{n/q}, I_{S'}) = \vec{\pi}(R_{n/q}, I_{S'}) = d_1/q + d_2/q$, and fix an optimal color assignment to paths that achieves this bound. We show how to build an optimal solution for the problem $R(n, d_1, d_2)$. Let i be the color used for the path from v_i to $v_{i+d_j/q}$, where $j \in \{1, 2\}$. Then we can use the same color in R_n for a path from v_{fi} to v_{fi+d_j}. This will assign a coloring to all paths of length d_1 or d_2 for vertices v_i where $i \bmod q = 0$. The coloring for all paths of length d_1 or d_2 for vertices v_i where $i \bmod q = 1$ can be obtained by "shifting the pattern", using the same type of coloring as that done for vertices v_i, where $i \bmod d = 0$, but using new colors. Since we need to shift the pattern $q - 1$ times, we obtain a solution for $R(n, d_1, d_2)$ that uses $q * (d_1/q + d_2/q) = d_1 + d_2$ colors, which is optimal.

Conversely, suppose that $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$, and fix a color assignment to paths that achieves this bound. Any color e_1 that is used in the assignment must necessarily be used on every edge of the ring. Thus, the paths that are assigned color e_1 break the ring into segments of length d_1 and d_2. Notice that if one of these segments starts in a vertex v_j where $j \bmod q = k$ then, since d_1 and d_2 have a common factor q, any other segment that is assigned color e_1 starts in a vertex v_l such that $l \bmod q = k$.

Let $e_1, e_1, \ldots, e_t$ be the colors that are assigned to segments that start in vertices whose indices are $0 \bmod q$. Then, $t = \vec{w}(R_n, I_S)/q = \vec{\pi}(R_n, I_S)/q$. The assignment of colors $e_1, e_1, \ldots, e_t$ to segments that start in vertices whose indices are $0 \bmod q$ can be used to obtain a route and wavelength assignment to $I_{S'}$ on $R_{n/q}$ by scaling down by a factor q as follows. If a segment of length d_i, $i \in \{1, 2\}$ that starts in vertex v_{lq} of R_n is assigned color e_j then assign this color to the segment of length d_i/q that starts in vertex v_l of $R_{n/q}$. By doing this for each color $e_1, e_1, \ldots, e_t$, we obtain a solution for $R(n/q, d_1/q, d_2/q)$ that uses $t = \vec{\pi}(R_n, I_S)/q = (d_1 + d_2)/q = \vec{\pi}(R_{n/q}, I_{S'})$ colors. □

This leads us to the main theorem of this section in which we summarize all cases in which the load and the wavelength index are the same.

Theorem 14. *Let n be a positive integer and $S = \{d_1, d_2\}$, where $1 \leqslant d_2 < d_1 \leqslant n/2$, and let $q = GCD(d_1, d_2)$. Then $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S)$ if and only if one of the following holds*:

(1) *d_1 divides n and d_2 divides n.*
(2) *$(d_1 + d_2)$ divides n.*
(3) *$n > (d_1/q - 2)d_1 + (d_1/q - 1)d_2$.*
(4) *There are integers a and b such that $n/q = ad_1/q + bd_2/q$, $a > b$, $a \geqslant d_2/q$ and $a - b < (d_1 + d_2)/q$.*

Proof. Follows from Theorem 6, Lemmas 2, 12, and 13. □

Note that neither the first nor the second condition is strictly stronger than the other. For example, when $d_1 = 7$, $d_2 = 5$, $n = 35$, the first condition is satisfied, but the second one is not. On the other hand, when $d_1 = 7$, $d_2 = 2$, $n = 27$, the second condition is satisfied, but the first is not.

We now consider the case when none of the conditions in Theorem 14 above is satisfied. This is the case, for example, when $n = d_1 d_2 + 1$, $d_2 > 1$, and $GCD(d_1, d_2) = 1$. The next theorem shows that in some cases we can obtain a solution for R_n and I_S from a solution of $R_{n'}$ and I_S for n' that is "slightly smaller" than n, by extending the existing color assignment for the smaller problem, by adding a color and $n - n'$ vertices.

Theorem 15. *Let n' be an integer and $S = \{d_1, d_2\}$ be a set of distances such that $\vec{w}(R_{n'}, I_S) = \vec{\pi}(R_{n'}, I_S)$, and $1 \leqslant d_2 < d_1 < n/2$. Then $\vec{w}(R_n, I_S) \leqslant \vec{\pi}(R_n, I_S) + 1$ for any n such that $n' < n \leqslant n' + \lfloor n'/(d_1 + d_2) \rfloor$.*

Proof. Assume that we have an optimal solution to the instance I_S in the ring $R_{n'}$.Let $n = n' + 1$, and consider the ring R_n obtained from $R_{n'}$ by inserting one vertex v_n between vertices v_{n-1} and v_0. We construct an assignment of colors to paths of lengths d_1 and d_2 from every vertex in R_n as follows.

For any path of length d_1 that originates in vertices $v_0, v_1, \ldots, v_{n-1}$ assign it the same color as in the optimal solution of I_S for $R_{n'}$. For any path of length d_2 that originates in vertices $v_0, v_1, \ldots, v_{n-d_2}$ assign it the same color as in a solution of I_S for $R_{n'}$. For any path of length d_2 that originates in vertices $v_{n-d_2+1}, v_{n-d_2+2}, v_n$ assign it the color given to the path of length d_2 originating in vertex $v_{n-d_2}, v_{n-d_2+1}, v_{n-1}$, respectively. It is obvious that this assignment of colors is conflict-free and it remains only to assign colors to the path of length d_1 that originates in vertex v_n and to the path of length d_2 that originates in vertex v_{n-d_2}. Since these two paths do not share any edge, they can both be assigned the same additional color, say c. Thus, $\vec{w}(R_n, I_S) \leqslant \vec{\pi}(R_n, I_S) + 1$.

Since the color c is used only on a contiguous segment of the ring R_n of length $d_1 + d_2$, the process of inserting an additional vertex and coloring the additional paths of length d_1 and d_2 using the color c can be repeated at least $\lfloor n'/(d_1 + d_2) \rfloor$ times by inserting a new vertex in a section of $R_{n'}$ where c has not been used yet. □

Since the number of colors used for a solution of the problem I_S on R_n is only one more than the value of $\vec{w}(R_{n'}, I_S)$, in cases where none of the conditions of Theorem 14 is not satisfied, the value of $\vec{\pi}(R_{n'}, I_S) + 1$ is actually the best possible value of $\vec{w}(R_n, I_S)$. Thus, the above theorem actually gives us an optimal coloring in those cases.

Corollary 16. *Let $S = \{d_1, d_2\}$, where $1 \leqslant d_2 < d_1 < n/2$. Then $\vec{w}(R_n, I_S) \leqslant \vec{\pi}(R_n, I_S) + 1$ whenever $n \geqslant 2d_1 d_2/q$, where $q = GCD(d_1, d_2)$.*

Proof. It is easy to see that if $n' = 2d_2 d_1 + kd_2$ where k is a positive integer, then we can express $n' = ad_1 + bd_2$ with a, b satisfying Lemma 9. Thus $\vec{w}(R_{n'}, I_S) = \vec{\pi}(R_{n'}, I_S)$ for any $n' = 2d_1 d_2 + kd_2$, where k is any positive integer. For any other value of n, we have $n = n' + i$ where $i \leqslant d_2$ and the wavelength index equals the load for I_S on the ring $I_{n'}$. Furthermore, for $n \geqslant 2d_1 d_2$, we have $\lfloor n/(d_1 + d_2) \rfloor \geqslant n/2d_1 \geqslant d_2$. It follows as a consequence of Theorem 15 that for all values of $n \geqslant 2d_1 d_2$, we have $\vec{w}(R_n, I_S) \leqslant \vec{\pi}(R_n, I_S) + 1$. □

3. The general case

In this section, we consider the case $S = \{d_1, d_2, \ldots, d_k\}$ for $k > 2$. Although the previous two sections dealt mostly with the special cases $S = \{d_1\}$ and $S = \{d_1, d_2\}$, these results can be used to obtain the exact value of $\vec{w}(R_n, I_S)$ in many instances of the general case and an approximation can be obtained in any case. The proof of the following theorem is based on Theorems 1 and 3. It is possible to construct instances, where $\vec{w}(R_n, I_S) \geqslant 3\,\vec{\pi}(R_n, I_S)/2 - \Theta(1)$, where the size of S is a constant. For example, this is true when $n = 3d_k - 1$, $d_{k-1} = d_k + 1$, $d_{k-2} = d_{k-1} + 1$ and so on. Thus the approximation is the best possible for arbitrary S of constant size.

Theorem 17. *Let $S = \{d_1, d_2, \ldots, d_k\}$, where $1 \leqslant d_k < \cdots < d_2 < d_1 \leqslant n/2$. Then $\vec{w}(R_n, I_S) \leqslant 3\,\vec{\pi}(R_n, I_S)/2$.*

The results in [14] imply that $\vec{w}(R_n, I_S) \leqslant \frac{3}{2}\,\vec{\pi}(R_n, I_S)$ when $k = 1$ or when for any distinct x, y, z between 1 and k, we have $d_x + d_y + d_z < n$. Our result applies to any uniform instance and hence, is more general in this case.

In many cases, we can obtain a value for $\vec{w}(R_n, I_S)$ that equals the load. For example, Theorem 6 states that if $\sum_{i=1}^{k} d_i$ divides n then $\vec{w}(R_n, I_S) = \sum_{i=1}^{k} d_i$. Otherwise, if n is large enough, we can partition S into pairs and apply the results of Section 3. We have to deal slightly differently with the cases when k is even and when k is odd. We immediately get the following corollaries:

Corollary 18. *Let n be a positive integer and k be an even integer, $S = \{d_1, d_2, \dots, d_k\}$, $1 \leqslant d_k < \cdots < d_2 < d_1 \leqslant n/2$. If there is a partition of S into a set of pairs $S_1, S_2, \dots, S_{k/2}$ such that every pair S_i, $2 \leqslant i \leqslant (k-1)/2$ satisfies one of* 1 *to* 4 *in Theorem* 14 *then* $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S) = \sum_{i=1}^{k} d_i$.

When k is odd, we can once again decompose the problem into sub-problems. If at least one of $d_1, d_2, \dots, d_k$, say d_j divides n, then we can consider partitions of $S' = \{d_1, d_2, \dots, d_k\} - \{d_j\}$ into pairs. This case arises, for example, when $d_k = 1$. We can proceed similarly when the sum of any odd number of elements in S divides n. We therefore assume below that none of the elements in S divides n. We can convert this case to k being even by adding two of elements of S and consider this modified situation.

Corollary 19. *Let n be a positive integer and k be an odd integer, $S = \{d_1, d_2, \dots, d_k\}$, $1 \leqslant d_k < \cdots < d_2 < d_1 \leqslant n/2$. If there is a partition of S into a set of $(k-3)/2$ pairs $S_1, S_2, \dots, S_{(k-3)/2}$ and one triple $S_{(k-1)/2} = \{d_{i1}, d_{i2}, d_{i3}\}$ for some integers $i1, i2, i3$ such that every pair S_i, $2 \leqslant i \leqslant (k-3)/2$ and the pair $\{d_{i1} + d_{i2}, d_{i3}\}$ satisfy one of* 1 *to* 4 *in Theorem* 14 *then* $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S) = \sum_{i=1}^{k} d_i$.

When n is large enough then the wavelength index is always equal to the load as shown in the next theorems.

Theorem 20. *Let k be an even integer, $S = \{d_1, d_2, \dots, d_k\}$, $1 \leqslant d_k < \cdots < d_2 < d_1 < n/2$ and $n > \max\{(d_i - 2)d_i + (d_i - 1)d_{k-i+1} : 1 \leqslant i \leqslant k/2\}$. Then $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S) = \sum_{i=1}^{k} d_i$.*

Proof. Since k is even, we decompose the problem into several subproblems, each subproblem being a pair of the distances from S, i.e, let $S_1 = \{d_1, d_k\}$, $S_2 = \{d_2, d_{k-1}\}, \dots, S_{k/2} = \{d_{k/2}, d_{k/2+1}\}$. Since $n > \max\{(d_i - 2)d_i + (d_i - 1)d_{k-i+1} : 1 \leqslant i \leqslant k/2\}$, it follows that for every i such that $1 \leqslant i \leqslant k/2$, $n > (d_i/q_i - 2)d_i + (d_i/q_i - 1)d_{k-i+1}$, where $q_i = GCD(d_i, d_{k-i+1})$. Therefore, S_i satisfies the condition of Theorem 14 for $1 \leqslant i \leqslant k/2$ and $\vec{w}(R_n, I_{S_i}) = d_i + d_{k-i+1}$. Thus, $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S) = \sum_{i=1}^{k} d_i$. $\square$

When k is odd we proceed similarly as in Corollary 19 by adding two of the lengths together, namely d_1 and d_k and convert the case to the one in which we have even number of different lengths in the instance.

Theorem 21. *Let k be an odd integer, $S = \{d_1, d_2, \dots, d_k\}$, where $1 \leqslant d_k < \cdots < d_2 < d_1 < n/2$. If $n > \max\{(d_1 + d_k - 2)(d_1 + d_k) + (d_1 + d_k - 1)d_{k-1}, (d_2 - 2)d_2 + (d_2 - 1)d_{k-2}, \dots, (d_{(k-1)/2} - 2)d_{(k-1)/2} + (d_{(k-1)/2} - 1)d_{(k+1)/2}\}$ then $\vec{w}(R_n, I_S) = \vec{\pi}(R_n, I_S) = \sum_{i=1}^{k} d_i$.*

Proof. It follows immediately from partitioning the instance S into $\{d_1 + d_k, d_{k-1}\}, \{d_2, d_{k-2}\}, \dots, \{d_{(k-1)/2}, d_{(k+1)/2}\}$ and applying Theorem 14. $\square$

Theorem 20 applies when $d_1 < \sqrt{n/2}$, and Theorem 21 applies when $d_1 + d_k$ is smaller than $\sqrt{n/2}$. If a solution equal to the lower bound cannot be found approximations can be used. Similarly as we generalized above Theorem 14, we can generalize Corollary 16 for $k > 2$.

Theorem 22. *Let k be an even integer, $S = \{d_1, d_2, \dots, d_k\}$, where $1 \leqslant d_k < \cdots < d_2 < d_1 < n/2$ and $n > \max\{2d_i d_{k-i+1} : 1 \leqslant i \leqslant k/2\}$. Then $\vec{w}(R_n, I_S) \leqslant \vec{\pi}(R_n, I_S) + k/2 = \sum_{i=1}^{k} d_i + k/2$.*

Theorem 23. *Let k be an odd integer, $S = \{d_1, d_2, \dots, d_k\}$, where $1 \leqslant d_k < \cdots < d_2 < d_1 < n/2$ and $n > \max\{2(d_1 + d_k)d_{k-1}, 2d_2 d_{k-2}, \dots, 2d_{(k-1)/2} d_{(k+1)/2}\}$. Then $\vec{w}(R_n, I_S) \leqslant \vec{\pi}(R_n, I_S) + (k-1)/2 = \sum_{i=1}^{k} d_i + (k-1)/2$.*

In case when none of the above theorems can be used we can partition S into subsets $S_1, S_2, \dots, S_l$ so that s_i, the sum of integers in S_i is a factor of n or $n \bmod s_i$ is small. We can then apply either Theorem 2 or Theorem 3 to get the value or an approximation of $\vec{w}(R_n, I_S)$.

4. Discussion

We studied the problem of routing uniform communication instances in optical rings using WDM technology. When the number of path lengths in the instance is 1 or 2, we characterized exactly the instances for which the optimal wavelength index equals the optimal load. For the general case, we gave a $\frac{3}{2}$-approximation, and described techniques to obtain better approximations in many cases.

Finding a precise characterization of instances for which the optimal wavelength index equals the optimal load is an interesting avenue of further research. Having optimal solutions for instances with three path lengths would give another technique to improve the solution for the general case, when the number of path lengths is odd. However, it would appear that the case of three path lengths is harder than the case of two path lengths. Indeed the tile representation that we used in Section 2 is not as well-understood for the three-dimensional case.

The techniques studied in this paper should also be applicable to the study of uniform instances in other classes of graphs. Some results for tori have been obtained in [5]. Related to the uniform communication instances studied in this paper is a *uniform permutation*, which contains for every $u \in V(G)$ and every i, exactly one pair (u, v) at distance d_i. These could be studied on both the unidirectional and the bidirectional ring. Some results for such permutations can be found in [11], but the problem deserves further study.

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