Automorphism groups of 2-groups

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Abstract

A well-known conjecture on \( p \)-groups states that every non-abelian \( p \)-group \( G \) has the property that \(|G|\) divides \(|\text{Aut}(G)|\). We exhibit periodic patterns in the automorphism group orders of the 2-groups of fixed coclass and we use this to show that for every positive integer \( r \) there are at most finitely many counterexamples to the conjecture among the 2-groups of coclass \( r \).

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1. Introduction

It is conjectured that \(|G|\) divides \(|\text{Aut}(G)|\) for every non-abelian \( p \)-group \( G \). This conjecture is still open despite various attempts to prove it or to find counterexamples for it. More generally, Mann [12] has asked how big is \( \text{Aut}(G) \) for a \( p \)-group \( G \)? In this paper we consider this question for 2-groups.

The divisibility conjecture has been proved for various special classes of \( p \)-groups. Gaschütz [8] proved that \( p \mid [\text{Aut}(G) : \text{Inn}(G)] \) for a non-abelian \( p \)-group \( G \). It follows that the conjecture holds for \( p \)-groups with centre of order \( p \). Otto [15] proved the divisibility conjecture for \( p \)-groups of maximal class, Faudree [6] for \( p \)-groups of class 2,
Davitt [2] for \( p \)-groups with centre of index at most \( p^4 \), Gavioli [9] for \( p \)-groups of order dividing \( p^7 \) and Flynn, MacHale and O’Brien [7] for 2-groups of order dividing \( 2^9 \).

For a group \( G \) of order \( p^n \) and class \( cl(G) \) the coclass is defined as \( cc(G) = n - cl(G) \). Leedham-Green and Newman [11] suggested to classify \( p \)-groups by coclass. This proposal initiated a major research project and has resulted in deep insights into \( p \)-groups, see the book of Leedham-Green and McKay [10] for details. The recent result [4] in this area yields that the (infinitely many) 2-groups of fixed coclass satisfy certain periodic patterns so that they can be described by a finite set of data.

We show here that the orders of the automorphism groups of the 2-groups of fixed coclass satisfy certain periodic patterns corresponding to those of the underlying groups. With every repetition of a pattern, the orders of the involved 2-groups grow by a constant factor. We prove that the orders of the corresponding automorphism groups also grow by a constant, but larger factor (see Section 3). This yields the following (see Theorem 2).

**Theorem.** For every \( s \in \mathbb{N} \) there exists \( o(r, s) \in \mathbb{N} \) such that \( 2^s|G| | |Aut(G)| \) for all 2-groups \( G \) of coclass \( r \) and order at least \( o(r, s) \).

**Corollary.** Almost all 2-groups of coclass \( r \) satisfy the divisibility conjecture.

Thus for every coclass there are at most finitely many counterexamples to the divisibility conjecture among the 2-groups of that coclass. We investigated the 2-groups of coclass at most 3 using computational methods and found no counterexamples (see Section 4).

### 2. The coclass graphs \( \mathcal{G}(p, r) \)

In this section we provide a brief introduction to coclass theory. For background and further information we refer to [10].

The finite \( p \)-groups of coclass \( r \) can be visualised by a graph \( \mathcal{G}(p, r) \): the vertices are identified with the isomorphism types of \( p \)-groups with coclass \( r \) and there is an edge between the vertices for \( G \) and \( H \) if there exists \( N \trianglelefteq G \) with \( |N| = p \) and \( G/N \cong H \). As \( cc(G) = cc(H) \) and \( |G| = |H|/p \), it follows that \( N \) is the last term of the lower central series of \( G \). Thus \( H \) is the unique ancestor for \( G \) in \( \mathcal{G}(p, r) \) and \( \mathcal{G}(p, r) \) is a forest.

The classification or investigation of \( p \)-groups now translates into an understanding of the coclass graphs \( \mathcal{G}(p, r) \). We outline some of the known features of these graphs in the remainder of this section.

#### 2.1. Coclass trees

Let \( L \) be an infinite pro-\( p \)-group of coclass \( r \) and denote its lower central series with \( L = \gamma_1(L) > \gamma_2(L) > \cdots \). Let \( t \in \mathbb{N} \) be minimal such that \( L/\gamma_t(L) \) has coclass \( r \) and \( L/\gamma_t(L) \) does not arise as quotient of an infinite pro-\( p \)-group of coclass \( r \) not isomorphic to \( L \). Then the full subtree \( T = T(L) \) of \( \mathcal{G}(p, r) \) consisting of all descendants of \( L/\gamma_t(L) \) is called a **maximal coclass tree** in \( \mathcal{G}(p, r) \).

By construction, the sequence of groups \( L/\gamma_t(L), L/\gamma_{t+1}(L), \ldots \) contains every infinite path of \( T(L) \). It is also called its *main line*. The groups on the main line are also denoted with \( G_0, G_1, \ldots \) so that \( G_i = L/\gamma_{t+i}(L) \).
The group $L$ can be reconstructed from the maximal coclass tree $T(L)$ as the inverse limit of the groups on the main line of $T(L)$. Hence the maximal coclass trees in $\mathcal{G}(p, r)$ correspond 1–1 to the isomorphism types of infinite pro-$p$-groups of coclass $r$.

### 2.2. The coclass theorems

Leedham-Green and Newman [11] proposed five conjectures on the structure of $\mathcal{G}(p, r)$. These coclass-conjectures have been proved in several steps and hence they are now theorems; see [10] for an overview and references.

The coclass theorems yield that there are only finitely many isomorphism types of infinite pro-$p$-groups of coclass $r$ and hence there are only finitely many maximal coclass trees in $\mathcal{G}(p, r)$. It is not difficult to show that all but finitely many $p$-groups of coclass $r$ are contained in a maximal coclass tree of $\mathcal{G}(p, r)$.

Thus the proof of the main theorem of this paper reduces to proving that a maximal coclass tree $T$ in $\mathcal{G}(p, r)$ can contain only finitely many counterexamples to the divisibility conjecture.

### 2.3. Periodicity of $\mathcal{G}(2, r)$

Let $T$ be a maximal coclass tree in $\mathcal{G}(2, r)$ and let $G_0, G_1, \ldots$ denote its main line. For $i \in \mathbb{N}_0$ let the branch $T_i$ be the subgraph of $T$ containing all descendants of $G_i$ which are not descendants of $G_{i+1}$. As $T$ contains only one main line, it follows that $T_i$ is a finite subtree of $T$.

Recall that the infinite pro-$p$-group $L$ corresponding to the tree $T$ is an extension of an infinite group $T \cong \mathbb{Z}_2^d$ (called translation subgroup) by a finite 2-group $P$ (called point group). The dimension $d$ of $T$ is also called the rank of $T$.

The main result of [4] can now be summarised as follows (see [4, Theorem 7]).

**Theorem 1.** Let $T$ be a maximal coclass tree in $\mathcal{G}(2, r)$ of rank $d$. Then $T$ is virtually periodic with periodicity $d$; that is, there exists $f \in \mathbb{N}$ such that for every $i \geq f$ there is an isomorphism

$$\pi_i : T_i \mapsto T_{i+d} : G \mapsto G^{\pi_i}$$

and $|G^{\pi_i}| = |G|2^d$ for all $G \in T_i$.

The main line group $G_f$ is called a periodicity root of $T$ and the full subtree of $T$ with root $G_f$ is the corresponding periodic part of $T$. Whenever there is no ambiguity we write $\pi$ for $\pi_i$ and we assume that $i \geq f$ if $\pi$ is used.

### 3. Periodicity of automorphism groups

Let $T$ be a maximal coclass tree in $\mathcal{G}(2, r)$ with rank $d$ and periodicity root $G_f$. In this section we consider the automorphism groups of the groups in the periodic part of $T$ and we prove the following theorem.
Theorem 2. Let $T$ be a maximal coclass tree in $G(2, r)$ with rank $d$ and with periodicity root $G_f$. Let $L$ be the infinite pro-$2$-group corresponding to $T$ and let $P$ be a point group for $L$ with translation subgroup $T$. Denote $i = \dim \text{End}_P(T)$. Then for every group $G \in T_i$ with $i \geq f$ it follows that

$$|\text{Aut}(G^\pi)| = |\text{Aut}(G)|2^{d+l}.$$  

Theorems 1 and 2 imply that

$$\frac{|\text{Aut}(G^\pi)|}{|G^\pi|} = \frac{|\text{Aut}(G)|2^{d+l}}{|G|2^d} = \frac{|\text{Aut}(G)|2^l}{|G|}.$$  

Note that $l > 0$. Hence the automorphism group orders grow with each application of the periodicity map $\pi$ and they grow faster than the orders of the underlying groups. By the coclass theorems, the graph $G(2, r)$ consists of finitely many maximal coclass trees and only finitely many groups are not contained in a maximal coclass tree. Thus the main theorem of this paper and its corollary follow from Theorem 2.

3.1. The groups in $G(2, r)$ as extensions

As a first step in the proof of Theorem 2, we recall the relation of $G$ and $G^\pi$ from [4] in more detail. For this purpose let $L$ be the infinite pro-$2$-group of coclass $r$ associated with the maximal coclass tree $T$. Note that we can choose the translations $T$ of $L$ as a subgroup in the lower central series $T = \gamma_k(L)$ for all large enough $k \in \mathbb{N}$. We define $T_0 := T$ and $T_{i+1} := [T_i, L]$ for $i \in \mathbb{N}_0$.

Theorem 3. (See [4, Theorem 5].) Let $T = \gamma_k(L)$ and $P = L/T$. If $k$ is chosen large enough, then every group $G$ in the periodic part of $T$ can be written as an extension of $T/T_j$ by $P$ where $j$ is determined by $|G| = |P|2^j$.

Every extension of $T/T_j$ by $P$ is defined by an element of $H^2(P, T/T_j)$. The structure of this cohomology group is investigated in [4] and we recall the main results here briefly. Let $\mu : H^3(P, T_j) \to H^3(P, T_{j+d})$ be the isomorphism induced from $T_j \to T_{j+d} : t \mapsto 2t$.

Theorem 4. (See [4, Theorem 6].) If $j$ is chosen large enough, then

(a) there exists a canonical isomorphism $H^2(P, T/T_j) \cong H^2(P, T) \oplus H^3(P, T_j)$, and
(b) the map $(\text{id} \oplus \mu) : H^2(P, T/T_j) \to H^2(P, T/T_{j+d})$ mapping the element $(\alpha, \beta) \in H^2(P, T) \oplus H^3(P, T_j)$ to $(\alpha, \mu(\beta)) \in H^2(P, T) \oplus H^3(P, T_{j+d})$ is an isomorphism.

Suppose that $G$ is a group in the periodic part of $T$. By Theorem 3, the group $G$ is an extension of $T/T_j$ by $P$ for suitable $j$. Let $\gamma_G \in H^2(P, T/T_j)$ be a cocycle defining $G$. The following result of [4] gives an explicit construction for the periodicity map.
Theorem 5. (See [4, Theorem 7].) Let $G$ be a group in the periodic part of $T$. Then $G^\pi$ is an extension of $T/T_{j+d}$ by $P$ defined by the cocycle $(id \oplus \mu)(\gamma_G)$.

3.2. The automorphism group of an extension

We recall the well-known relation between automorphism groups and group extensions in the following. For this purpose let $M$ be a $P$-module and let $P \to Aut(M) : g \mapsto \bar{g}$ denote the corresponding action of $P$ on $M$. Then the group of compatible pairs of $M$ and $P$ is defined as

$$Comp(P, M) = \{ (\beta, \delta) \in Aut(P) \times Aut(M) \mid g^\beta = g^\delta \text{ for all } g \in P \}.$$  

This group acts on $Z^2(P, M)$ via $\gamma^{(\beta, \delta)}(g, h) = \gamma(g^{\beta^{-1}}, h^{\beta^{-1}})\delta$ for $\gamma \in Z^2(P, M)$. The coboundaries $B^2(P, M)$ are setwise invariant under this action and thus we obtain an induced action of $Comp(P, M)$ on $H^2(P, M)$. This setup is used in the following theorem to determine the automorphism group of an extension. We refer to [16] for a proof.

Theorem 6. Let $E$ be an extension of $M$ by $P$ where $M$ embeds as a characteristic subgroup in the extension $E$. Let $\nu: Aut(E) \to Aut(P) \times Aut(M) : \alpha \mapsto (\alpha_{E/M}, \alpha_M)$ be the natural homomorphism.

(a) $\ker(\nu) \cong Z^1(P, M)$.
(b) $\text{Im}(\nu) = \text{Stab}_{Comp(P, M)}(\gamma)$, where $\gamma \in H^2(P, M)$ is a cocycle defining $E$.

Thus $|Aut(E)| = |Z^1(P, M)||\text{Stab}_{Comp(P, M)}(\gamma)|$ follows.

Theorem 6 yields the following for groups $G$ and $H = G^\pi$ in the periodic part of $T$, where $G$ is written as an extension of $T/T_j$ by $P$ defined by $\gamma_G$ and the cocycle $\gamma_H = (id \oplus \mu)(\gamma_G)$ as in Theorem 5:

- $|Aut(G)| = |Z^1(P, T/T_j)||\text{Stab}_{Comp(P, T/T_j)}(\gamma_G)|$, and
- $|Aut(H)| = |Z^1(P, T/T_{j+d})||\text{Stab}_{Comp(P, T/T_{j+d})}(\gamma_H)|$.

In order to prove Theorem 2, it remains to relate the orders of the 1-cocycles and the stabilisers in the compatible pairs. This is pursued in the following subsections.

3.3. The group of one-cocycles

The following theorem determines the relation between

$$|Z^1(P, T/T_j)| \quad \text{and} \quad |Z^1(P, T/T_{j+d})|$$

for $d = \text{rank}(T)$.

Theorem 7. If $j$ is large enough, then $|Z^1(P, T/T_{j+d})| = 2^d|Z^1(P, T/T_j)|$.

Proof. As a first step, we show that $|B^1(P, T/T_j)| = 2^{j-1}$ for $j \in \mathbb{N}$. By definition, the group $B^1(P, T/T_j)$ is the image of the homomorphism $\varphi: T/T_j \to C^1(P, T/T_j)$:
\[ m \mapsto \delta_m \text{ with } \delta_m(g) = m^g - m. \] The kernel of \( \varphi \) are the fixed points in \( T/T_j \) under the action of \( P \). As \( P \) acts uniserially on \( T/T_j \), we obtain that \( |\text{Fix}_P(T/T_j)| = 2 \) and thus \( |B^1(P, T/T_j)| = |T/T_j|/2 = 2^{j-1} \).

As a next step, we show that \( |H^1(P, T/T_j)| = |H^1(P, T/T_{j+d})| \). For this purpose we consider the exact sequence of cohomology groups as in [1, Proposition 6.1]:

\[
H^1(P, T) \xrightarrow{\alpha} H^1(P, T/T_j) \xrightarrow{\beta_j} H^2(P, T_j).
\]

Let \( \exp(H^1(P, T)) = 2^e \). Then \( Z^1(P, T_{de}) = Z^1(P, 2^e T) \leq 2^e Z^1(P, T) \leq B^1(P, T) \).

Thus \( \alpha \) is injective if \( j \geq ed \). Next, we consider \( \beta_j \).

As a next step, we consider \( \beta_j \) for all integers \( j \geq 1 \). Let \( k = j \mod d \). Then \( T_j \cong P T_k \) and \( H^2(P, T_j) \cong H^2(P, T_k) \). Thus \( \text{Im}(\beta_j) \) embeds as a subgroup \( I_j \) into \( H^2(P, T_k) \).

It is straightforward to observe that \( I_k \leq I_{k+d} \leq \cdots \). As \( H^2(P, T_k) \) is a finite group, there exists \( w \in \mathbb{N} \) such that \( I_{w+d} = \cdots \). Choosing \( j \geq \max\{w, ed\} \) we obtain that \( |H^1(P, T/T_j)| = |H^1(P, T)||I_w| = |H^1(P, T/T_{j+d})| \).

The desired result now follows from the first two steps as

\[
|Z^1(P, T/T_{j+d})| = |H^1(P, T/T_{j+d})||B^1(P, T/T_{j+d})| = |H^1(P, T/T_j)|2^{j+d-1} = 2^d|H^1(P, T/T_j)||B^1(P, T/T_j)| = 2^d|Z^1(P, T/T_j)|.
\]

This completes the proof. \( \square \)

### 3.4. The group of compatible pairs

The aim in this section is to prove the following theorem which exhibits the relation between \( |\text{Comp}(P, T/T_j)| \) and \( |\text{Comp}(P, T/T_{j+d})| \).

**Theorem 8.** If \( j \) is large enough, then \( |\text{Comp}(P, T/T_{j+d})| = 2^l|\text{Comp}(P, T/T_j)| \) where \( l = \dim \text{End}_P(T) \).

As a first step, we introduce some maps between groups of compatible pairs. If \( k \leq j \), then \( T/T_k \cong_P (T/T_j)/(T_k/T_j) \) and thus there is a natural projection \( T/T_j \to T/T_k \) which is compatible with the action of \( P \). Similarly, there is a natural projection \( T \to T/T_k \).

These projections induce the following maps:

\[
\pi_{j,k} : \text{Comp}(P, T/T_j) \to \text{Comp}(P, T/T_k) : (\beta, \delta) \mapsto (\beta, \delta T/T_k),
\]

\[
\pi_k : \text{Comp}(P, T) \to \text{Comp}(P, T/T_k) : (\beta, \delta) \mapsto (\beta, \delta T/T_k).
\]

**Lemma 9.** The maps \( \pi_{j,k} \) and \( \pi_k \) are well defined for \( k \in \mathbb{N} \) and \( j \geq k \).

**Proof.** Let \( (\beta, \delta) \in \text{Comp}(P, T/T_j) \). We show that \( \delta \) leaves \( T_k/T_j \) invariant and thus induces an automorphism \( \delta T/T_k \in \text{Aut}(T/T_k) \) such that \( (\beta, \delta T/T_k) \in \text{Comp}(P, T/T_k) \). For this purpose note that the definition of compatible pairs implies that \( \delta \in \mathcal{N}_{\text{Aut}(T/T_j)}(\overline{P}) \).

Thus \( \delta \) permutes the \( P \)-invariant subgroups of \( T/T_j \). As \( P \) acts uniserially on \( T/T_j \),
with unique maximal $P$-invariant series $T/T_j > T_{j+1}/T_j > \cdots > T_j/T_j$, it follows that the subgroups $T_k/T_j$ are invariant under $\delta$. Thus $\delta_{T/T_k} \in \text{Aut}(T/T_k)$ and $(\beta, \delta_{T/T_k}) \in \text{Comp}(P, T/T_k)$ follows directly. Hence $\pi_{j,k}$ is well defined. The proof that $\pi_k$ is well defined follows similarly. □

Next we observe that the induced projections are compatible with each other. The proof of the following is straightforward.

**Lemma 10.** For $j \geq k \geq l$ it follows that $\pi_{j,k} \circ \pi_{k,l} = \pi_{j,l}$ and $\pi_k \circ \pi_{k,l} = \pi_l$.

Our ultimate aim is to investigate kernel and image of the map $\pi_{j+d,j}$ to obtain a proof for Theorem 8. The kernels of the projections $\pi_{j,k}$ are considered in the following theorem. If $(\beta, \delta) \in \text{Ker}(\pi_{j,k})$, then $\beta = \text{id}$ and $\delta : T/T_j \mapsto T/T_j : t \mapsto t + e_t$ for certain $e_t \in T_k/T_j$. We denote $e_\delta : T/T_j \mapsto T_k/T_j : t \mapsto e_t$.

**Theorem 11.** Let $k \in \mathbb{N}$ and $j \geq k$.

(a) $\sigma_{j,k} : \text{Ker}(\pi_{j,k}) \to \text{Hom}_P(T/T_j, T_k/T_j) : (1, \delta) \mapsto e_\delta$ is a bijection.

(b) $\sigma_k : \text{Ker}(\pi_k) \to \text{Hom}_P(T, T_k) : (1, \delta) \mapsto e_\delta$ is a bijection.

(c) $\sigma_{j,k}$ is a homomorphism of abelian groups if $2k \geq j$.

**Proof.**

(a) First we observe that $\sigma_{j,k}$ is well defined. As $\delta \in \text{Aut}(T/T_j)$, it follows directly that $e_\delta \in \text{Hom}(T/T_j, T_k/T_j)$. As $(1, \delta) \in \text{Comp}(P, T/T_j)$, we find that $e_{t^g} = (e_t)^g$ for every $g \in P$ and $t \in T/T_j$. Thus $e_\delta$ is compatible with the action of $P$. This yields that $\sigma_{j,k}$ is well defined. Next, it is obvious that $\sigma_{j,k}$ is injective. The surjectivity of $\sigma_{j,k}$ follows by similar arguments.

(b) This follows by similar arguments as (a).

(c) Let $(1, \delta_i) \in \text{Ker}(\pi_{j,k})$ for $i = 1, 2$ and denote $e_i = e_{\delta_i}$. Then $t^{\delta_1 \delta_2} = t + e_{1,t} + e_{2,t} + e_{e_1 e_2}$. Thus $\sigma_{j,k}$ is a homomorphism if and only if $T_k/T_j \subseteq \text{Ker}(\delta)$ for all $\delta \in \text{Ker}(\pi_{j,k})$. As $\delta : T_i/T_j \mapsto T_{k+i}/T_j$ for all $i$, this yields the desired result. □

From now on we assume that $j \geq d$. We define the group of central automorphisms by

$$\text{Cent}(P, T/T_j) = \{ \delta \in \text{Aut}(T/T_j) \mid (1, \delta) \in \text{Ker}(\pi_{j,d}) \}.$$  

$$\text{Cent}(P, T) = \{ \delta \in \text{Aut}(T) \mid (1, \delta) \in \text{Ker}(\pi_d) \}.$$  

This definition is used in the following theorem.

**Theorem 12.** If $j$ is large enough, then $|\text{Cent}(P, T/T_{j+d})| = 2^l |\text{Cent}(P, T/T_j)|$. 
Proof. The compatibility of the maps $\pi_{j,k}$ and $\pi_k$ induces a commutative diagram

$$
\cdots \rightarrow \text{id} \rightarrow \text{Cent}(P,T) \rightarrow \text{id} \rightarrow \text{Cent}(P,T) \rightarrow \text{id} \rightarrow \cdots
$$

$$
\cdots \rightarrow \pi_{j,k} \rightarrow \pi_k \rightarrow \pi_{j,k} \rightarrow \cdots
$$

Theorem 11 implies that the groups of central automorphisms are in bijection to their corresponding groups of homomorphisms. This induces an equivalent commutative diagram

$$
\cdots \rightarrow \text{id} \rightarrow \text{Hom}_P(T,T) \rightarrow \text{id} \rightarrow \text{Hom}_P(T,T) \rightarrow \text{id} \rightarrow \cdots
$$

$$
\cdots \rightarrow \tilde{\pi}_{j,k} \rightarrow \tilde{\pi}_k \rightarrow \tilde{\pi}_{j,k} \rightarrow \cdots
$$

As $\text{Hom}_P(T/T_{j+k}, T_{d+j}/T_{d+k}) \cong \text{Hom}_P(T/T_j, T/T_j) = \text{End}_P(T/T_j)$ for all $j$, it follows that the above commutative diagram is equivalent to

$$
\cdots \rightarrow \text{id} \rightarrow \text{End}_P(T) \rightarrow \text{id} \rightarrow \text{End}_P(T) \rightarrow \text{id} \rightarrow \cdots
$$

$$
\cdots \rightarrow \tilde{\pi}_{j,k} \rightarrow \tilde{\pi}_k \rightarrow \tilde{\pi}_{j,k} \rightarrow \cdots
$$

where all maps in the diagram are natural projections. As $T \cong \mathbb{Z}_2^d$, it follows that $\text{End}(T) \cong T \otimes T \cong \mathbb{Z}_2^{d^2}$. Let $P \rightarrow \text{GL}(d^2, \mathbb{Z}_2) : g \mapsto M_g$ denote the action of $P$ on $\text{End}(T)$. Then $\text{End}_P(T) = \text{Fix}_{\text{End}(T)}(P)$ can be determined as the kernel of the matrix

$$
M := \left( \begin{array}{cccc}
M_{g_1} - I \\
& \ddots \\
& & M_{g_m} - I
\end{array} \right) \in \mathbb{Z}^{md^2 \times d^2},
$$

where $P = \{g_1, \ldots, g_m\}$. Similarly, the group $\text{End}_P(T/T_j)$ corresponds to the kernel of $M$ in $T/T_j$. Let $M = PDQ$ for a diagonal matrix $D$ and invertible matrices $P$ and $Q$. Then for $v \in \text{End}(T)$ we find that

$$
v + T_j \otimes T_j \in \text{End}_P(T/T_j) \iff Mv \equiv 0 \mod T_j \otimes T_j
$$

$$
\iff PDQv = w \in T_j \otimes T_j
$$

$$
\iff D(Qv) = P^{-1}w \in P^{-1}(T_j \otimes T_j)
$$

$$
\iff Dv' \equiv 0 \mod P^{-1}(T_j \otimes T_j)
$$
and

\[ v \in \text{End}_P(T) \iff Mv = 0 \iff Dv' = 0. \]

Let \( d_1, \ldots, d_s, e_1, \ldots, e_l \) be the elementary divisors of \( D \) sorted such that \( d_i > 0 \) and \( e_j = 0 \) for all \( i, j \). Then \( \dim \text{End}_P(T) \) corresponds to the number of elementary divisors \( 0 \) in \( D \). Suppose that \( j \) is large enough, so that \( \exp(T/T_j) \geq d_i \) for all \( i \). Then

\[ \text{End}_P(T/T_j) \cong C_{d_1} \times \cdots \times C_{d_s} \times \text{Im}(\tilde{\pi}_{j+d}). \]

As \( T_{j+d} = 2T_j \), it follows that \( P^{-1}(T_j \otimes T_j) = 2P^{-1}(T_{j-d} \otimes T_{j-d}) \) and hence

\[ |\text{Im}(\tilde{\pi}_{j+d})| = 2^l|\text{Im}(\tilde{\pi}_j)|. \]

Thus \( |\text{End}_P(T/T_j)| = 2^l|\text{End}_P(T/T_{j-d})| \) and the result follows.

Now we can summarise all the results of this section to obtain a proof for Theorem 8.

**Proof.** (For Theorem 8.) Let \( I_j = \text{Im}(\pi_{j,d}) \leq \text{Comp}(P, T/T_d) \). Then by Lemma 10 it follows that \( \cdots \geq I_{j-d} \geq I_j \geq I_{j+d} \geq \cdots \). As \( \text{Comp}(P, T/T_d) \) is a finite group independent of \( j \), we deduce that there exists \( k \) such that \( I_j = I_{j+d} \) for all \( j \geq k \).

By construction, we have that \( \text{Cent}(P, T/T_j) \cong \text{Ker}(\pi_{j,d}) \). By Theorem 12, we know that \( |\text{Cent}(P, T/T_{j+d})| = 2^l|\text{Cent}(P, T/T_j)| \) provided that \( j \) is large enough. Thus in this case we find that

\[
|\text{Comp}(P, T/T_{j+d})| = |I_{j+d}| |\text{Cent}(P, T/T_{j+d})| = 2^l |I_j| |\text{Cent}(P, T/T_j)|
= 2^l |\text{Comp}(P, T/T_j)|
\]

and this completes the proof.

3.5. The stabilizer of a cocycle

Finally, we show that the relation between the orders of the compatible pairs yields the same relation between the stabilisers of cocycles as used in Theorem 6.

**Theorem 13.** Let \( G \in T_i \) with \( H = G^\sigma \). Let \( j \in \mathbb{N} \) such that \( G \) is an extension of \( T/T_j \) by \( P \). If \( j \) is large enough, then

\[ |\text{Stab}_{\text{Comp}}(P, T/T_{j+d})(\gamma_H)| = 2^l |\text{Stab}_{\text{Comp}}(P, T/T_j)(\gamma_G)|. \]

**Proof.** By Theorem 8, it is known that \( |\text{Comp}(P, T/T_{j+d})| = 2^l |\text{Comp}(P, T/T_j)| \). Thus it remains to show that the orbits of \( \gamma_G \) and \( \gamma_H \) under their respective acting groups have equal lengths. This was proved in [4, Theorems 17 and 24].
3.6. The proof of Theorem 2

The proof of the main theorem can now be read off. Let \( G \) be a group in the periodic part of a maximal coclass tree \( T \). Let \( \gamma_G \) be a cocycle defining \( G \) as extension of \( T/T_j \) by \( P \). Let \( H = G^\pi \) and let \( \gamma_H = (id \oplus \mu)(\gamma_G) \) as in Theorem 5.

Then Theorems 7 and 13 yield for large enough \( j \) that

\[
|Aut(H)| = |Z^1(P, T/T_{j+d})| |Stab_{Comp}(P, T/T_{j+d})(\gamma_H)|
\]

\[
= 2^d |Z^1(P, T/T_j)| 2^l |Stab_{Comp}(P, T/T_j)(\gamma_G)|
\]

\[
= 2^{d+l} |Aut(G)|.
\]

4. Groups of coclass at most 3

In this section we consider \( G(2, r) \) for \( r \leq 3 \). We use computational tools to investigate these graphs and observe experimentally that the divisibility conjecture holds for the groups in these graphs.

The groups of 2-coclass at most 3 have been investigated in detail in \cite{13}. As part of this investigation the infinite pro-2-groups of coclass at most 3 have been determined. Thus it is known that:

- there is 1 infinite pro-2-group of coclass 1;
- there are 5 infinite pro-2-groups of coclass 2;
- there are 54 infinite pro-2-groups of coclass 3.

We constructed the 60 infinite pro-2-groups of coclass at most 3 using the information in \cite{13} and the algorithm in \cite{3}. The \( p \)-group generation algorithm \cite{14} was employed to construct all groups in a branch \( T_i \) of a maximal coclass tree \( T \) described by its corresponding infinite pro-2-group. The method of \cite{5} was used to determine automorphism groups of \( p \)-groups.

Using these tools, we determined experimentally a periodicity root \( G_f \) for a maximal coclass tree corresponding to a infinite pro-2-group of coclass at most 3. We found periodicity roots \( G_f \) of order at most \( 2^{13} \). Then for the 60 maximal coclass trees \( T \) we checked the divisibility conjecture for all the groups in the branches \( T_{f+1}, \ldots, T_{f+d} \) of \( T \). Finally, we constructed the 8173 non-abelian groups of \( G(2, r) \) which are not contained in the periodic part of a maximal coclass tree and we checked that they satisfy the divisibility conjecture.

This computation was performed using GAP \cite{17}. It took about 2 days to complete.

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