Discrete Mathematics 98 (1991) 81–93
North-Holland


# Graph properties and hypergraph colourings

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Received 20 December 1988
Revised 13 June 1990

*Abstract*

Brown, J.I., D.G. Corneil, Graph properties and hypergraph colourings, Discrete Mathematics 98 (1991) 81–93.

Given a graph property $P$, graph $G$ and integer $k \geqslant 0$, a $P$ $k$-colouring of $G$ is a function $\pi : V(G) \to \{1, \ldots, k\}$ such that the subgraph induced by each colour class has property $P$. When $P$ is closed under induced subgraphs, we can construct a hypergraph $H_G^P$ such that $G$ is $P$ $k$-colourable iff $H_G^P$ is $k$-colourable. This correlation enables us to derive interesting new results in hypergraph chromatic theory from a 'graphic' approach. In particular, we build vertex critical hypergraphs that are not edge critical, construct uniquely colourable hypergraphs with few edges and find graph-to-hypergraph transformations that preserve chromatic numbers.

## 1. Introduction

A *property* $P$ is a collection of graphs (closed under isomorphism) containing $K_0$ and $K_1$ (all graphs considered here are finite); $P$ is *hereditary* iff $P$ is closed under induced subgraphs and *nontrivial* iff $P$ does not contain every graph. Any graph $G \in P$ is called a *$P$-graph*. Throughout, we shall only be interested in hereditary and nontrivial properties, and $P$ will always denote such a property. Given a graph $G$, integer $k \geqslant 0$ and property $P$, a *$P$ $k$-colouring of $G$* is a function $\pi : V(G) \to \{1, \ldots, k\}$ such that for all $i = 1, \ldots, k$, $\langle \pi^{-1}(i) \rangle_G \in P$ (for $X \subseteq V(G)$, $\langle X \rangle_G$ denotes the induced subgraph of $G$ on vertex set $X$). $G$ is *$P$ $k$-colourable* iff it has a $P$ $k$-colouring. $\chi(G:P)$ denotes the least $k$ for which $G$ is $P$ $k$-colourable, and $G$ is said to be $P$ $\chi(G:P)$-chromatic. The notion of a $P$ $k$-colouring is discussed by Hedetniemi [14], Cockayne [6] and Harary [13]. An in depth study of $P$ colourings can be found in [4–5].

Let $G$ be a graph and $P$ a property. We construct (as in [5]) a hypergraph $H_G^P$ as follows. Let the vertex set of $H_G^P$ be $V(G)$, and let $e \subseteq V(G)$ be an edge of $H_G^P$ iff $\langle e \rangle_G$ is vertex $P$ 2-critical (a subgraph $F$ of $G$ is *vertex $P$ 2-critical* iff $F \notin P$ but $F - v \in P$ for every vertex $v$ of $F$). The essential observation relating the hypergraph colourings of $H_G^P$ and the $P$ colourings of $G$ is the following.

**Theorem 1.1.** *$\pi : V(G) = V(H_G^P) \to \{1, \ldots, k\}$ is a $P$ $k$-colouring of $G$ iff it is a $k$-colouring of $H_G^P$.*

**Proof.** Note, first of all, that if a graph $F$ is not a $P$-graph, then $F$ contains a vertex $P$ 2-critical subgraph (we simply consider the subgraphs of $F$ that are not $P$-graphs and pick one of minimum order). Conversely, as $P$ is hereditary, if $F$ contains a vertex $P$ 2-critical subgraph, then $F$ cannot be a $P$-graph. Thus $\pi : V(G) \to \{1, \ldots, k\}$ is a $P$ $k$-colouring of $G$ iff no $\langle \pi^{-1}(i) \rangle_G$ contains a vertex $P$ 2-critical subgraph, and hence iff $\pi : V(G) \to \{1, \ldots, k\}$ is a $k$-colouring of $H_G^P$. □

In particular, $\chi(G : P) = \chi(H_G^P)$, and note that if $F$ is a graph of order $\geqslant 2$ and $P = -F = \{H : F \text{ is not an induced subgraph of } H\}$, then $H_G^P$ is $|V(F)|$-uniform. As an example, let $G$ be the graph shown in Fig. 1, formed by substituting a copy of $P_3$ for each vertex of degree 1 in a $P_3$ ($P_n$ denotes the path on $n$ vertices). The reader can verify that $G$ is $-P_3$ 3-chromatic, and hence $H_G^{-P_3}$ is a 3-uniform 3-chromatic hypergraph.

In the next two sections we illustrate how Theorem 1.1 can be utilized for constructing vertex $k$-critical and uniquely $k$-colourable hypergraphs. The last section uses $P$ colourings to define graph-to-hypergraph transformations that preserve chromatic numbers.

Our graph and hypergraph notation will, in general, follow that of [2]. All graphs and hypergraphs are finite, undirected and without loops or multiple edges. The *order* and *size* of a hypergraph $H$ are $|V(H)|$ and $|E(H)|$ respectively,

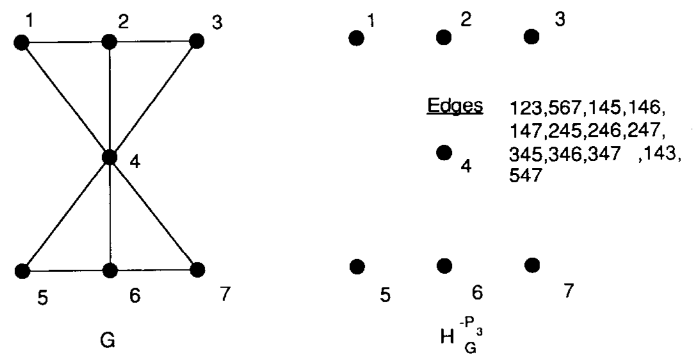


Fig. 1.

and $H$ is *r-uniform* if every edge of $H$ contains exactly $r$ elements. The term *subgraph* will always stand for 'induced subgraph' and we write $F \trianglelefteq G$ to denote $F$ is a subgraph of $G$; for $U \subseteq V(G)$, $\langle U \rangle_G$ (or $\langle U \rangle$ if no ambiguity arises) denotes the subgraph of $G$ induced by $U$. If $H$ and $K$ are hypergraphs, then $K$ is a *partial subhypergraph* of $H$ if $V(K) \subseteq V(H)$ and $E(K) \subseteq E(H)$, and $K$ is a *subhypergraph* of $H$ if $V(K) \subseteq V(H)$ and $E(K)$ consists of all edges $e$ of $H$ with $e \subseteq V(K)$. If $e$ is an edge of $H$ and $v$ is a vertex of $H$, then $H - e$ denotes the partial subhypergraph with vertex set $V(H)$ and edge set $E(H) - \{e\}$, and $H - v$ denotes the subhypergraph with vertex set $V(H) - \{v\}$. Let $\omega$ denote the *clique number* of a graph. A $k$-colouring of a hypergraph $H$ is a function $\pi : V(H) \to \{1, \ldots, k\}$ such that no edge of $H$ is monochromatic under $\pi$. The chromatic number of $H$, $\chi(H)$, is the least nonnegative integer $k$ for which $H$ is $k$-colourable. The notation of a hypergraph colouring was initially introduced as a generalization of graph colourings [7, 15], and it appears that it is an indispensable tool in constructing graphs with certain colouring properties [15, 18]. The chromatic theory of hypergraphs has been developed in the work of Toft and others [2, 20–21].

Unless otherwise stated, different graphs or hypergraphs are assumed to be disjoint. For hypergraphs $H_1$ and $H_2$, $H_1 \cup H_2$ denotes their disjoint union. Given graphs $G_1$ and $G_2$, $G_1 + G_2$ is formed from $G_1 \cup G_2$ by adding all the edges $\{g_1, g_2\}$, where $g_1 \in V(G_1)$ and $g_2 \in V(G_2)$. If $F$ and $G$ are graphs and $v$ is a vertex of $G$, then the graph $K$ arising by substituting $F$ for $v$ in $G$ is formed from $(G - v) \cup F$ by adding in all edges $\{g, f\}$, where $\{g, v\} \in E(G)$ and $f \in V(F)$.

We remark that Abbott [1] used Ramsey's Theorem on edge colourings of complete hypergraphs to prove the existence of $k$-chromatic $r$-uniform hypergraphs with no 2-cycles for all $k \geq 1$ and $r \geq 3$. Also, results from the chromatic theory of hypergraphs have been applied to prove results in $P$ chromatic theory as well [17, 4–5].

## 2. Vertex critical hypergraphs

In hypergraph chromatic theory, $k$-critical hypergraphs have attracted much attention. A hypergraph $H$ is *(edge) k-critical* iff it is $k$-chromatic but any proper partial subhypergraph is $(k-1)$-colourable. Examples of constructions of such hypergraphs may be found in Toft's work [20–21], where it is also shown how to associate, for a given $k$-critical hypergraph, a set of $k$-critical graphs. While the structure of 3-critical graphs is well known (to be odd cycles), the class of 3-critical $r$-uniform hypergraphs does not appear to have a 'nice' description for any $r \geq 3$.

The weaker notion of vertex criticality for hypergraphs, however, has not been examined to any extent (a hypergrpah $H$ is *vertex k-critical* iff $H$ is $k$-chromatic but $H - v$ is $(k-1)$-colourable for all vertices $v$ of $H$). In this section, we will be concerned with the construction of vertex critical uniform hypergraphs via $P$ colourings and graphs. Before stating the key result, we will need another definition from [5]. A graph $G$ is *vertex $P$ $k$-critical* iff $G$ is $P$ $k$-chromatic but $G-v$ is $P$ $(k-1)$-colourable for all vertices $v$ of $G$. The theory of vertex $P$ $k$-critical graphs and examples are discussed in [5].

**Theorem 2.1.** *Let $G$ be a vertex $P$ $k$-critical graph. Then $H=H_G^P$ is vertex $k$-critical.*

In [5] examples of vertex $P$ $k$-critical graphs for properties $P$ closed under substitution are described, and these yield vertex $k$-critical hypergraphs via Theorem 2.1. Also, we leave it to the reader to verify that the graph $G$ of Fig. 1 is indeed vertex $-P_3$ 3-critical, and hence its associated hypergraph is a vertex 3-critical 3-uniform hypergraph.

For $k=1, 2$ and 3, the vertex $k$-critical graphs and $k$-critical graphs coincide, but as shown in Fig. 2, there is a vertex 4-critical graph that is not 4-critical (the removal of the solid edge does not make this vertex 4-critical graph 3-colourable). Our application of $P$ colourings in this section is to provide explicit constructions of vertex $k$-critical $r$-uniform hypergraphs that are not $k$-critical hypergraphs for each $k$, $r \geq 3$ (for $k=1$ and 2 they clearly coincide).

**Theorem 2.2.** *For all $k$, $r \geq 3$ there is a vertex $k$-critical $r$-uniform hypergraph of order $\frac{1}{2}k(k-1)(r-2)+k$ that is not $k$-critical.*

**Proof.** Let $G_k = X_0 + X_1 + \cdots + X_{k-1}$, where $X_i \cong \overline{K_{i(r-2)+1}}$ $(k \geq 2, r \geq 3)$. We first show by induction on $k \geq 2$ that:

(i) $G_k$ is vertex $-K_{1,r-1}$ $k$-critical.

(ii) there is a $-K_{1,r-1}$ $k$-colouring of $G_k$ such that (the subgraph induced by) each colour class has independence number less than $r-1$.

For $k=2$, $G_2 = K_{1,r-1}$ and both (i) and (ii) are clear. Assume $k \geq 2$ and consider $G_{k+1} = G_k + X_k$. If $\pi : V(G_{k+1}) \to \{1, \ldots, k\}$ is any map, then as $X_k$ has $k(r-2)+1$ points, there is an $S \subseteq V(X_k)$ of order $r-1$ that is monochromatic, say $\pi(S) = \{k\}$. If $k \in \pi(V(G_k))$ then there is a $K_{1,r-1}$ monochromatically

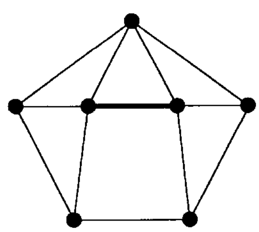

Fig. 2.

coloured $k$; otherwise, $\pi(V(G_k)) \subseteq \{1, \ldots, k-1\}$, so there is a monochromatic $K_{1,r-1}$ in $G_k$, and hence in $G_{k+1}$. In any event, we see that $G_{k+1}$ is not $-K_{1,r-1}$ $k$-colourable.

Now let $v$ be a vertex of $G_{k+1}$; we now show that $G_{k+1} - v$ is $-K_{1,r-1}$ $k$-colourable, and from this and the previous statement it will follow that $G_{k+1}$ is vertex $-K_{1,r-1}$ $(k+1)$-critical. If $v \in G_k$, then by (i) there is a $-K_{1,r-1}$ $(k-1)$-colouring of $G_k - v$, and by colouring $X_k$ with $k$ we get a $-K_{1,r-1}$ $k$-colouring of $G_{k+1} - v$. If $v \in X_k$, we partition $X_k$ into sets $S_1, \ldots, S_k$, each of order $r-2$. By (ii) there is a $-K_{1,r-1}$ $k$-colouring $\rho$ of $G_k$ such that each colour class has independent number less than $r-1$; extending $\rho$ to all of $G_{k+1} - v$ by setting $\rho(S_i) = \{i\}$, we get a $-K_{1,r-1}$ $k$-colouring of $G_{k+1} - v$ since each colour class, being the join of graphs of independence number less than $r-1$, also has independence number less than $r-1$. In either case, $G_{k+1} - v$ is $-K_{1,r-1}$ $k$-colourable, and as noted above, we conclude that (i) holds for $k+1$, that is $G_{k+1}$ is vertex $-K_{1,r-1}$ $(k+1)$-critical. Further, if we extend $\rho$ to all of $G_{k+1}$ by setting $\rho(v) = k+1$, then (ii) is satisfied for $k+1$ by $\rho$.

By Theorem 2.1, $H_{r,k} = H_{G_k}^{-K_{1,r-1}}$ is a vertex $k$-critical $r$-uniform hypergraph, and a simple calculation shows that $H_{r,k}$ has order $\frac{1}{2}k(k-1)(r-2)+k$. It remains to show that it is not $k$-critical for $k \geqslant 3$. Note that $H_{r,k}$ is $k$-critical if and only if for every copy $F$ of $K_{1,r-1}$ in $G_k$, there is a map $\pi: V(G_k) \to \{1, \ldots, k-1\}$ such that the only monochromatic $K_{1,r-1}$ is $F$. Consider a fixed subset $S_{k-1}$ of $V(X_{k-1})$ of order $r-1$; we will show that there is no map $\pi: V(G_k) \to \{1, \ldots, k-1\}$ such that the only monochromatic $K_{1,r-1}$ is $X_0 + S_{k-1}$. For suppose there is such a map, with $\pi(X_0 + S_{k-1}) = \{k-1\}$. Clearly $k-1$ is not used on $X_1 \cup \cdots \cup X_{k-2}$, so since $\pi(X_{k-2}) \subseteq \{1, \ldots, k-2\}$ and $X_{k-2}$ has order $(k-2)(r-2)+1$, there is a subset $S_{k-2} \subseteq V(X_{k-2})$ of order $r-1$ that is monochromatic, coloured say $k-2$. Again, $k-2$ appears nowhere else, so recursively we find for $i = 1, \ldots, k-1$ subsets $S_i \subseteq V(X_i)$ of order $r-1$ such that each $S_i$ is monochromatic with a different colour, say $\pi(S_i) = \{i\}$. Now pick $x \in V(X_{k-1}) - S_{k-1}$ (this can be done as $k \geqslant 3$). If $\pi(x) = k-1$ then for any $w \in S_{k-1}$, $\langle X_0 \cup (S_{k-1} - w) \cup \{x\} \rangle$ is another monochromatic $K_{1,r-1}$. If $\pi(x) = i$ for some $1 \leqslant i \leqslant k-2$, then $\langle \{x\} \cup S_i \rangle$ is another monochromatic $K_{1,r-1}$. In any case, we have a contradiction to the fact that $X_0 + S_{k-1}$ is the only $K_{1,r-1}$ in $G_k$ monochromatic under $\pi$. It follows that $H_{r,k}$ is not $k$-critical. □

(We digress to mention that a single example of a vertex 3-crtical 3-uniform hypergraph $H$ that is not 3-critical can be constructed via Ramsey theory by taking $V(H)$ to be the edge set of $K_6$ and $E(H)$ to be those sets of edges that form a triangle. That $H$ is vertex 3-critical follows from $r(3,3) = 6$ and the fact that the size Ramsey number for $K_3$ is 20 [8]; that $H$ is not 3-critical follows from Goodman's result that in any 2-colouring of the edges of $K_6$ there must be at least two monochromatic triangles [11] (see also [12]).)

We concentrate now on the smallest case, $r = 3$, and construct (via $-P_3$

colourings) *infinitely* many examples of vertex $k$-critical 3-uniform hypergraphs that are not 3-critical. We begin with $k = 3$.

**Theorem 2.3.** *For all $n \geq 3$, there is a vertex 3-critical 3-uniform hypergraph of order $2n + 1$ that is not 3-critical.*

**Proof.** Let $n \geq 3$, and let $G$ be the circulant graph of order $2n + 1$ with distances 1 and 3, i.e., $V(G) = \mathbb{Z}_{2n+1}$ and for each $i \in \mathbb{Z}_{2n+1}$, $G$ has edges $\{i, i+1\}$ and $\{i, i+3\}$ (see Fig. 3). We will show that $G$ is vertex $-P_3$ 3-critical.

Suppose $G$ had a $-P_3$ 2-colouring $\pi$. As $2n + 1$ is odd, some edge $\{i, i+1\}$ is monochromatic, say $\pi(0) = \pi(1) = 1$. Since

$$\langle\{0, 1, 2\}\rangle \cong \langle\{0, 1, 3\}\rangle \cong \langle\{2n, 0, 1\}\rangle \cong \langle\{2n-1, 0, 1\}\rangle \cong P_3,$$

we must have $\pi(2) = \pi(3) = \pi(2n-1) = \pi(2n) = 2$. However, $\langle\{2n, 2, 3\}\rangle \cong P_3$ if $n \geq 4$, and if $n = 3$, then $\langle\{2n-1, 2, 3\}\rangle = \langle\{5, 2, 3\}\rangle \cong P_3$. In any event, a copy of $P_3$ is monochromatic under $\pi$, a contradiction. Thus $G$ has no such $-P_3$ 2-colouring, so $\chi(G : -P_3) \geq 3$.

Let $v \in V(G)$; we must show $G - v$ is $-P_3$ 2-colourable. By symmetry, we may assume $v = 0$. We construct a function $\pi : V(G - v) \to \{1, 2\}$ by setting $\pi(i) = 1$ if $i = 1, 3, \ldots, 2n-1$ and $\pi(i) = 2$ if $i = 2, 4, \ldots, 2n$. Note that neither colour class contains an edge $\{i, i+1\}$. Moreover, if $1 \leq i \leq 2n-3$, then $i$ and $i+3$ have opposite parity, so the only monochromatic edges are $\{2n-1, 1\}$ (coloured 1) and $\{2n, 2\}$ (coloured 2). As each colour class has only one edge, the colour classes are $P_3$-free, so $\pi$ is a $-P_3$ 2-colouring of $G - v$. Note that $\pi$ can clearly be extended to a $-P_3$ 3-colouring of $G$, so $G$ is $-P_3$ 3-chromatic.

It follows that $G$ is vertex $-P_3$ 3-critical, so by Theorem 2.1, $H = H_G^{-P_3}$ is a vertex 3-critical 3-uniform hypergraph of order $2n + 1$. It remains to be shown that $H$ is not 3-critical.

As in the proof of Theorem 2.2, we show that $H$ is not 3-critical by proving that there is no map $\pi : V(G) \to \{1, 2\}$ such that $\langle\{0, 1, 2\}\rangle$ is the only monochromatic $P_3$; if such a $\pi$ exists, without loss, $\pi(0) = \pi(1) = \pi(2) = 1$. It follows that $\pi(3) = \pi(4) = 2$. If $n \geq 4$, then $\langle\{1, 2, 5\}\rangle \cong P_3$, so $\pi(5) = 2$; if $n = 3$, then

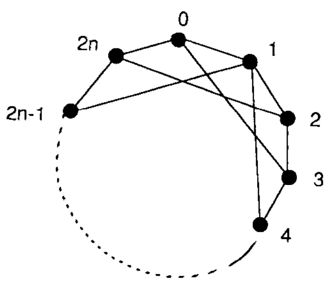


Fig. 3.

$\langle\{0,1,5\}\rangle \cong P_3$, so $\pi(5)=2$. In any case, $\langle\{3,4,5\}\rangle$ is another monochromatic copy of $P_3$, a contradiction. Thus $H$ is not 3-critical. □

To extend Theorem 2.3 to higher values of $k$, we will need a hypergraph operation that was suggested by a construction of vertex $P$ $k$-critical graphs described in [5]. If $H_1, \ldots, H_{r-1}$ are disjoint hypergraphs and $v$ is a new vertex, then let $H(v, H_1, \ldots, H_{r-1})$ denote the hypergraph on $\{v\} \cup H_1 \cup \cdots \cup H_{r-1}$ formed by adding in all edges of the form $\{v, v_1, \ldots, v_{r-1}\}$, where $v_i \in V(H_i)$. We leave the proof of the following proposition to the reader (the argument can be found in [4]).

**Proposition 2.4.** *Let $H_1, \ldots, H_{r-1}$ be vertex $k$-critical hypergraphs. Then $H(v, H_1, \ldots, H_{r-1})$ is vertex $(k+1)$-critical and, $H(v, H_1, \ldots, H_{r-1})$ is $(k+1)$-critical iff $H_1, \ldots, H_{r-1}$ are $k$-critical.*

We interrelate the hypergraph operation of Proposition 2.4 and the construction of vertex 3-critical 3-uniform hypergraphs (via vertex $-P_3$ 3-critical graphs) in the next two results.

**Proposition 2.5.** *For all $n \geq 13$ there is a vertex 4-critical 3-uniform hypergraph of order $n$ that is not 4-critical.*

**Proof.** From Theorem 2.3 we have vertex 3-critical 3-uniform hypergraphs that are not 3-critical of all odd orders greater than 6. Theorem 2.2 yields an example of a vertex 3-critical 3-uniform hypergraph $H'$ of order 6.

Let $n \geq 13$. We take two cases.

*Case* (i) *$n$ is odd.*

Then we can write $n = 1 + l + m$, where either (a) $n = 13$ and $l = m = 6$, or (b) $n \geq 15$, $l = 7$ and $m \geq 7$ is odd. In either case, we conclude from Theorem 2.3 and the remark above that there are vertex 3-critical 3-uniform hypergraphs $H_1$ and $H_2$ of order $l$ and $m$ respectively that are not 3-critical. By Proposition 2.4, $H(v, H_1, H_2)$ is a vertex 4-critical 3-uniform hypergraph (of order $n$) that is not 4-critical.

*Case* (ii) *$n$ is even.*

Then $n = 1 + 6 + m$, where $m \geq 7$ is odd. Let $H_2$ be a vertex 3-critical 3-uniform hypergraph of order $m$ that is not 3-critical, as guaranteed by Proposition 2.3. The previous proposition shows that $H(v, H', H_2)$ is a vertex 4-critical 3-uniform hypergraph (of order $n$) that is not 4-critical. □

**Theorem 2.6.** *For all $k \geq 4$ and all $n \geq 7 \cdot 2^{k-3} - 1$, there is a vertex $k$-critical 3-uniform hypergraph of order $n$ that is not $k$-critical.*

**Proof.** For $k = 4$ the result is Proposition 2.5. We assume $k \geq 5$ and proceed by induction on $k$. If $n \geq 7 \cdot 2^{k-3} - 1$, then $n = 1 + (7 \cdot 2^{k-4} - 1) + m$, where $m \geq$

$7 \cdot 2^{k-4} - 1$. There are, by induction, vertex $(k-1)$-critical 3-uniform hypergraphs $H_1$ and $H_2$, of orders $7 \cdot 2^{k-4} - 1$ and $m$ respectively, that are not $(k-1)$-critical. Proposition 2.4 shows that $H(v, H_1, H_2)$ is a vertex $k$-critical 3-uniform hypergraph of order $n$ that is not $k$-critical. □

Finally, we consider an old conjecture, due to Dirac, which asks whether every vertex $k$-critical graph ($k \geq 2$) has a critical edge (i.e., an edge whose removal decreases the chromatic number). We can now show that the corresponding question for hypergraphs has a negative answer. Let $G$ be the circulant graph of order 7 described in Theorem 2.3, and let $H = H_G^{-P_3}$. By symmetry, if $e$ is an edge of $H$, then we may assume $e = \{0, 1, 2\}$ or $\{0, 1, 3\}$. In the proof of Theorem 2.3 we have seen $\chi(H - \{0, 1, 2\}) = 3$. If $H - \{0, 1, 3\}$ has a 2-colouring $\pi$, then $\langle\{0, 1, 3\}\rangle$ is the only monochromatic copy of $P_3$ in $G$. Without loss, $\pi(0) = \pi(1) = \pi(3) = 1$. It follows that $\pi(4) = \pi(5) = \pi(6) = 2$, a contradiction as $\langle\{4, 5, 6\}\rangle \cong P_3$. Thus $\chi(H - \{0, 1, 3\}) = 3$ as well, so $H$ is a vertex 3-critical 3-uniform hypergraph with no critical edge. In fact, in much the same way it can be verified that all the hypergraphs in the proof of Theorem 2.3 have no critical edge, so we have the following result.

**Theorem 2.7.** *For all $n \geq 3$ there is a vertex 3-critical 3-uniform hypergraph of order $2n + 1$ with no critical edge.*

## 3. Uniquely colourable hypergraphs

A hypergraph $H$ is *uniquely $k$-colourable* iff $H$ is $k$-chromatic and, up to a permutation of colours, $H$ has only one $k$-colouring. While the notion of unique colourability for graphs has been well studied (see [3] for an interesting extension related to perfect graphs), the corresponding hypergraph analogue has only received a few passing references. Our primary aim here is not to develop the theory of uniquely $k$-colourable hypergraphs, but to briefly illustrate how $P$ chromatic theory can be applied to construct such hypergraphs. Again, the result below follows from Theorem 1.1.

**Theorem 3.1.** *$G$ is uniquely $P$ $k$-colourable iff $H_G^P$ is uniquely $k$-colourable.*

Thus any construction of uniquely $P$ $k$-colourable graphs yields a construction for uniquely $k$-colourable hypergraphs. Examples of the former are presented in [5] (and in a more generalized form, in [4]). Here we shall use $P$ chromatic theory to build uniquely $k$-colourable uniform hypergraphs with few edges. It is not hard to see that any uniquely $k$-colourable $r$-uniform hypergraph $H$ ($r \geq 3$) of order $n$ must have at least $(k-1)n$ edges. For if $V_1, \ldots, V_k$ are the unique colour classes of some $k$-colouring of $H$, then for each $i \in \{1, \ldots, k\}$ and every

$v \in V(H) - V_i$, there is an edge $e_i^v \subseteq \{v\} \cup V_i$ (as otherwise $H$ is not uniquely $k$-colourable). Now $e_i^v = e_j^w$ iff $i = j$ and $v = w$ (since $|e_l^z \cap V_m| = r - 1 \geqslant 2$ for all $z$, $l$ and $m \neq l$), so $H$ has size at least $\sum_{i=1}^{k} \sum_{j \neq i} |V_j| = (k-1)n$. It is not hard to verify that if $H$ is the $r$-uniform hypergraph on vertex set $V_1 \cup \cdots \cup V_k$ ($|V_i| = m \geqslant r$) whose edges are all the subsets of cardinality $r$ not contained in any $V_i$, then $H$ is uniquely $k$-colourable. However, $H$ has order $n = km$ and size $O(n^r)$. Thus the question remains as to whether there are uniquely $k$-colourable $r$-uniform hypergraphs of order $n$ and size $O(n)$, for all sufficiently large $n$. We use $P$ colourings to provide a partial answer.

**Theorem 3.2.** *Let $r \geqslant 6$ and $k \geqslant 2$. For all sufficiently large $n$ there is an $r$-uniform unqiuely $k$-colourable hypergraph of order $n$ and size $Cn$, where $C$ is a constant depending only on $r$ and $k$.*

**Proof.** Let $P = -(K_1 + C_{r-1})$, where $C_{r-1}$ is the cycle of length $r-1$. By a theorem of Folkman [9] (see also [5]), there is a graph $F$ such that:

(i) $\omega(F) = \omega(C_{r-1}) = 2$, and

(ii) $\chi(F : -C_{r-1}) > k$.

Let $p = |V(F)|$ and $q = |E(H_F^{-C_{r-1}})|$ = number of $C_{r-1}$'s in $F$; both $p$ and $q$ are constants (depending on $k$ and $r$, of course).

Let $n \geqslant kp$ and write $n = kN + l$, where $0 \leqslant l < k$ and $N \geqslant p$. Form graphs $G_1, \ldots, G_k$ as follows. For $i = 1, \ldots, l$, $G_i$ is a copy of $F \cup \overline{K_{N-p+1}}$, and for $i = l+1, \ldots, k$, $G_i$ is a copy of $F \cup \overline{K_{N-p}}$. Note that the number of $C_{r-1}$'s in $G_i$ is the same as the number of $C_{r-1}$'s in $F$, namely $q$.

Set $F_n = G_1 + \cdots + G_k$; $F_n$ is clearly of order $n$. Moreover,

$$\omega(G_i) = \omega(F) < \omega(K_1 + C_{r-1}) \quad \text{and} \quad \chi(G_i : -C_{r-1}) = \chi(F : -C_{r-1}) > k.$$

By an argument similar to that given in the proof of Lemma 3.4 of [5], $F_n$ is uniquely $-(K_1 + C_{r-1})$ $k$-colourable; for the sake of completeness, we sketch a proof here. First, if $\pi : V(F_n) \to \{1, \ldots, k-1\}$ is any map, then in each $G_i$, there is a monochromatic copy of $C_{r-1}$, coloured say $\pi_i$, since $\chi(G_i : -C_{r-1}) > k$. By the pigeonhole principle, for some $i \neq j$, $\pi_i = \pi_j$, and so there is a monochromatic $C_{r-1} + C_{r-1}$ (and hence a monochromatic $K_1 + C_{r-1}$) under $\pi$. Thus $F_n$ is not $-(K_1 + C_{r-1})$ $(k-1)$-colourable. $F_n$ is $-(K_1 + C_{r-1})$ $k$-chromatic as the map $\pi : V(F_n) \to \{1, \ldots, k\}$ such that $\pi(G_i) = \{i\}$ $(i = 1, \ldots, k)$ is a $-(K_1 + C_{r-1})$ $k$-colouring of $F_n$ (since $\omega(G_i) = 2 < \omega(K_1 + C_{r-1})$ implies each $G_i$ is $(K_1 + C_{r-1})$-free). Moreover, if $\pi : V(F_n) \to \{1, \ldots, k\}$ is a $-(K_1 + C_{r-1})$ $k$-colouring of $F_n$, then again in each $G_i$ there is a monochromatic copy of $C_{r-1}$ coloured say $\pi_i$. Now $\pi_i \neq \pi_j$ for all $i \neq j$, since otherwise there is a monochromatic $C_{r-1} + C_{r-1}$, and hence a monochroamtic $K_1 + C_{r-1}$, under $\pi$. Thus $\{\pi_i : i = 1, \ldots, k\} = \{1, \ldots, k\}$, and $\pi(G_i) = \pi_i$, for if $u \in V(G_i)$ is coloured $c \neq \pi_i$, then colour class $c$ contains a copy of $K_1 + C_{r-1}$, a contradiction. Thus every $-(K_1 + C_{r-1})$

$k$-colouring of $F_n$ has colour classes $V(G_1), \ldots, V(G_k)$, and $F_n$ is therefore uniquely $-(K_1 + C_{r-1})$ $k$-colourable.

Theorem 3.1 now shows that $H_n = H^P_{F_n}$ is uniquely $k$-colourable. $H_n$ is $r$-uniform and of order $n$. The size of $H_n$ is the number of $K_1 + C_{r-1}$'s in $F_n$. Each $K_1 + C_{r-1}$ in $F_n$ consists of a $C_{r-1}$ in some $G_i$ and a vertex in some other $G_j$. This follows as no $K_1 + C_{r-1}$ has more than one vertex in each of two $G_i$'s, or has vertices in at least three $G_i$'s, since $C_{r-1}$ is not of the form $F_1 + F_2$ (where each $F_i$ is non-empty). Therefore,

$$\begin{aligned}|E(H_n)| &= \text{number of } K_1 + C_{r-1}\text{'s in } F_n \\ &= \sum_{i \neq j} (\text{number of } C_{r-1}\text{'s in } G_i)\, |V(G_j)| \\ &= q(k-1)n = Cn,\end{aligned}$$

where $C = q(k-1)$ is a constant. □

We remark that in [4] it is shown that for all $r \geq 3$, $k \geq 2$, $g \geq 2$, any $\varepsilon \in (0, 1/4g)$ and all sufficiently large $n$, there is a uniquely $k$-colourable $r$-uniform hypergraph of girth at least $g$ with order $n$ and size $O(n^{1+\varepsilon})$; in fact, this was utilized to build uniquely $-G$ $k$-colourable graphs for those $G$ that are 2-connected.

## 4. Graph-to-hypergraph transformations

We end our applications of $P$ colourings to hypergraphs by considering the relationship between graph and hypergraph colourings. Lovász [16] described a graph-to-hypergraph construction such that the original graph is $k$-colourable iff the resulting hypergraph is 2-colourable. Phelps and Rődl [19] presented for each $k \geq 2$ and $r \geq 3$ a transformation that produces for each graph $G$ an $r$-uniform hypergraph $H(G)$ such that $G$ is $k$-colourable iff $H(G)$ is $k$-colourable. All of these reductions depend on the given value of $k$, and the question remains as to whether there is a transformation from graphs to $r$-uniform hypergraphs that preserves chromatic numbers. No such transformation has appeared in the literature, and we now use $P$ colourings to describe such a construction (this again illustrates our basic philosophy that $P$ chromatic theory is useful even if one is only interested in graph and hypergraph colourings).

**Construction 4.1.** Let $r \geq 3$ be given and let $F = K_1 + F'$ be a fixed graph of order $r$. If $G$ is a graph of order $n$, they by Folkman's result [9] (see Section 3) there is a graph $G_n$ with clique number $\omega(F')$ that is not $-F'$ $n$-colourable. Form $G'$ by (successively) substituting a copy of $G_n$ for each vertex of $G$, and set $H = H^{-F}_{G'}$.

**Proposition 4.2.** *The hypergraph $H$ described in Construction 4.1 is $r$-uniform and $\chi(H) = \chi(G)$.*

**Proof.** The fact that $H$ is $r$-uniform is clear from the definition of $H_{G'}^{-F}$. We now show that $\chi(G':-F)=\chi(G)$. Let $\pi:V(G)\to\{1,\ldots,k\}$ be a $k$-colouring of $G$. We define $\pi_0:V(G')\to\{1,\ldots,k\}$ by colouring the copy $G_n^v$ of $G_n$ substituted for $v$, by $\pi(v)$. Then $\pi_0$ is a $-F$ $k$-colouring of $G'$, since each colour class, being the disjoint union of copies of $G_n$, has clique number $\omega(F')$, and hence is $F$-free. Thus $\chi(G':-F)\leqslant\chi(G)$. Let $\pi:V(G')\to\{1,\ldots,m\}$ be a $-F$ $m$-colouring of $G'$, where $m=\chi(G':-F)\leqslant\chi(G)\leqslant n$. Since $\chi(G_n:-F')>n\geqslant m$, in each copy $G_n^v$ there is a monochromatic copy of $F'$, coloured say $\pi^v$. Now if $\{u,v\}\in E(G)$, then $\pi^u\neq\pi^v$, for otherwise there is a monochromatic $F'+F'$ (and hence a monochromatic $F$) under $\pi$. Thus $\pi:V(G)\to\{1,\ldots,m\}:u\mapsto\pi^u$ is an $m$-colouring of $G$, so $\chi(G)\leqslant m=\chi(G':-F)$ and thus $\chi(G)=\chi(G':-F)$.

Now utilizing Theorem 1.1, $\chi(H)=\chi(G':-F)=\chi(G)$. □

Thus Construction 4.1 provides many graph-to-hypergraph transformations that preserve chromaticity. However, the word "construction" is perhaps a misnomer here, as there is no known way of producing "small" graphs $G_n$ as described. Folkman's original proof of his result produced extremely large graphs, and a subsequent proof by Nešetřil and Rődl [17] does not solve this problem. What we are after is a polynomial transformation of graphs to $r$-uniform hypergraphs. We can salvage such a transformation from Construction 4.1 by a clever choice for the graph $F'$. Let $F'=\overline{K_{r-1}}$. Then $\overline{K_{n(r-2)+1}}$ is a graph with clique number $1<\omega(K_1+\overline{K_{r-1}})$ that is not $-\overline{K_{r-1}}$ $n$-colourable. For a graph $G$ of order $n$, we set $G_n=\overline{K_{n(r-2)+1}}$. The order of the graph $G'$ produced by Construction 4.1 (and hence the order of $H$) is $n(n(r-2)+1)=\mathrm{O}(n^2)$. Thus we have shown the following.

**Theorem 4.3.** *If $F=K_1+\overline{K_{r-1}}$ and $G=\overline{K_{n(r-2)+1}}$ in Construction* 4.1, *then the transformation from graphs to r-uniform hypergraphs preserves chromaticity and is polynomial.*

As a corollary, we can also describe a hypergraph-to-hypergraph transformation that preserves two different kinds of chromatic numbers. The *strong chromatic number* of a hypergraph $H$, $\chi_s(H)$, is the fewest number of colours needed to colour the vertices of $H$ such that if $x$ and $y$ are in some edge of $H$, then $x$ and $y$ receive different colours. Strong chromatic numbers are discussed in [2]. Theorem 4.3 can be used to reduce strong chromatic numbers to chromatic numbers.

**Proposition 4.4.** *Let $r,s\geqslant 2$. Then there is a polynomial transformation $\rho$ from r-uniform hypergraphs to s-uniform hypergraphs such that $\chi(\rho(H))=\chi_s(H)$ for all r-uniform hypergraphs $H$.*

**Proof.** Suppose $H$ is an $r$-uniform hypergraph. It is well known (and obvious) that the graph $G$ on vertex set $V(H)$, formed by joining $x$ to $y$ iff they belong to some edge of $H$, is $\chi_s(H)$-chromatic. From Theorem 4.3 we can efficiently transform $G$ into an $s$-uniform hypergraph $H'$ such that $\chi(H') = \chi(G) = \chi_s(H)$ (if $s = 2$, take $H = G$). Now the order of $H'$ is $O(n^2)$, where $n = |V(G)| = |V(H)|$, so the transformation is polynomial. □

As a final note, we remark that it is unlikely there is a hypergraph-to-graph polynomial transformation that preserves chromaticity since hypergraph 2-colourability is NP-complete, while graph 2-colourability is polynomial (see [10]).

## Note added in proof

The first author has recently solved Dirac's conjecture for graphs; the result will appear as a note in this journal.

## Acknowledgements

Most of the material presented in this paper has also appeared in the first author's doctoral dissertation [4] written at the University of Toronto under the supervision of the second author. The authors wish to thank the Natural Sciences and Engineering Research council of Canada for financial assistance. The first author would also like to acknowledge the support of the Faculty of Arts at York University.

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