Cubic non-normal Cayley graphs of order $4p^2$

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Abstract

A Cayley graph $\text{Cay}(G, S)$ on a group $G$ is said to be normal if the right regular representation $R(G)$ of $G$ is normal in the full automorphism group of $\text{Cay}(G, S)$. In this paper all connected cubic non-normal Cayley graphs of order $4p^2$ are constructed explicitly for each odd prime $p$. It is shown that there are three infinite families of cubic non-normal Cayley graphs of order $4p^2$ with $p$ odd prime. Note that a complete classification of cubic non-Cayley vertex-transitive graphs of order $4p^2$ was given in [K. Kutnar, D. Marušič, C. Zhang, On cubic non-Cayley vertex-transitive graphs, J. Graph Theory 69 (2012) 77–95]. As a result, a classification of cubic vertex-transitive graphs of order $4p^2$ can be deduced.

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1. Introduction

For a finite, simple, undirected and connected graph $X$, we use $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For $u, v \in V(X)$, $u \sim v$ means that $u$ and $v$ are adjacent and denote by $[u, v]$ the edge incident to $u$ and $v$ in $X$. A graph $X$ is said to be vertex-transitive, edge-transitive and arc-transitive (or symmetric) if $\text{Aut}(X)$ acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively.

Let $G$ be a permutation group on a set $\Omega$ and $\alpha \in \Omega$. Denote by $G_\alpha$ the stabilizer of $\alpha$ in $G$, that is, the subgroup of $G$ fixing the point $\alpha$. We say that $G$ is semiregular on $\Omega$ if $G_\alpha = 1$ for every $\alpha \in \Omega$ and regular if $G$ is transitive and semiregular. Given a finite group $G$ and an inverse closed subset $S \subseteq G \setminus \{1\}$, the Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to $S$ is defined to have vertex set $G$ and edge set $\{g, sg \mid g \in G, s \in S\}$. A Cayley graph $\text{Cay}(G, S)$ is connected if and only if $S$ generates $G$. Given a $g \in G$, define the permutation $R(g)$ on $G$ by $x \mapsto gx$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the right regular representation of $G$, is a permutation group isomorphic to $G$. It is well-known that $R(G) \leq \text{Aut}(\text{Cay}(G, S))$. So, $\text{Cay}(G, S)$ is vertex-transitive. In general, a vertex-transitive graph $X$ is isomorphic to a Cayley graph on a group $G$ if and only if its automorphism group has a subgroup isomorphic to $G$, acting regularly on the vertex set of $X$ (see [2, Lemma 16.3]). A Cayley graph $\text{Cay}(G, S)$ is said to be normal if $R(G)$ is normal in $\text{Aut}(\text{Cay}(G, S))$.

For two inverse closed subsets $S$ and $T$ of a group $G$ not containing the identity $1$, if there is an $\alpha \in \text{Aut}(G)$ such that $S^\alpha = T$ then $S$ and $T$ are said to be equivalent, denoted by $S \equiv T$. One may easily show that if $S$ and $T$ are equivalent then $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ and then $\text{Cay}(G, S)$ is normal if and only if $\text{Cay}(G, T)$ is normal.

The concept of normal Cayley graph was first proposed by Xu [20], and following this article, the normality of Cayley graphs have been extensively studied from different perspectives by many authors. Note that Wang et al. [18] obtained all disconnected normal Cayley graphs. For this reason, it suffices to consider the connected ones when one investigates the normality of Cayley graphs. One of the standard problems in the studying of normality of Cayley graphs is to determine the normality of Cayley graphs with specific orders. It is well-known that every transitive permutation group of prime degree $p$ is either 2-transitive or solvable with a regular normal Sylow $p$-subgroup (see, for example, [4, Corollary 3.5B]). This implies
that a Cayley graph of prime order is normal if the graph is neither empty nor complete. The normality of Cayley graphs of order a product of two primes was determined by Dobson et al. [5,6,15].

There also has been a lot of interest in the study of the normality of small valent Cayley graphs. For example, Baik et al. [1] determined all non-normal Cayley graphs on abelian groups with valency at most 4, and Fang et al. [7] proved that the vast majority of connected cubic Cayley graphs on non-abelian simple groups are normal. Let Cay(G, S) be a connected cubic Cayley graph on a non-abelian simple group G. Praeger [16] proved that if N_{Aut(Cay(G, S))}(R(G)) is transitive on E(Cay(G, S)) then Cay(G, S) is normal. Let p and q be two primes. In [21–23], all connected cubic non-normal Cayley graphs of order 2pq are determined. Zhou and Mohsen [24] classified all connected cubic non-normal Cayley graphs of order 8p. Wang and Xu [19] determined all tetravalent non-normal 1-regular Cayley graphs on dihedral groups. Feng and Xu [11] proved that every connected tetravalent Cayley graph on a regular p-group is normal when p \neq 2, 5. Li et al. [8,14] investigated the normality of tetravalent edge-transitive Cayley graphs on G, where G is either a group of odd order or a finite non-abelian simple group. Recently, Kovács et al. [12] classified all connected tetravalent non-normal arc-transitive Cayley graphs on dihedral groups satisfying one additional restriction: the graphs are bipartite, with the two bipartition sets being the two orbits of the cyclic subgroup within the dihedral group. For more results on the normality of Cayley graphs, we refer the reader to [10,20].

Motivated by the facts listed above, we aim to consider the normality of cubic Cayley graphs with specific order. In view of the fact that a classification of cubic vertex-transitive non-Cayley graphs of order 4p^2 (p a prime) was given in [13], one may expect to give a classification of cubic Cayley graphs of order 4p^2. To do this, the crucial step is to classify all cubic non-normal Cayley graphs of order 4p^2. Note that connected cubic non-normal Cayley graphs of order 16 are classified in [24]. In this article, we classify all cubic non-normal Cayley graphs of order 4p^2 with p an odd prime. Before stating our main result we introduce three families of Cayley graphs.

Construction 1. Let G_0^{4p^2} = \langle a, b \mid a^{2p^2} = b^2 = 1, b^{-1}ab = a^{-1} \rangle. Set \Omega = \{b, ba, ba^2 \}. Define C_0^{4p^2} = Cay(G_0^{4p^2}, \Omega).

Construction 2. Let G_1^{4p^2} = \langle a, b, c \mid a^{2p} = b^p = c^2 = 1, ab = ba, ac = ca, c^{-1}bc = b^{-1} \rangle. Set \Theta = \{ab, a^{-1}b^{-1}, c \}, and define C_1^{4p^2} = Cay(G_1^{4p^2}, \Theta).

Construction 3. Let G_2^{4p^2} = \langle a, b, c, d \mid a^{2p} = b^p = c^2 = d^2 = 1, ab = ba, bc = cb, ad = da, cd = dc, c^{-1}ac = a^{-1}, d^{-1}bd = b^{-1} \rangle. Take \lambda \in \mathbb{Z}_p^* such that 2\lambda \equiv 1(\text{mod } p). Set \Lambda = \{cd, cdab, ca^\lambda \}, and define C_2^{4p^2} = Cay(G_2^{4p^2}, \Lambda).

The following theorem is the main result of this paper.

Theorem 1.1. Let p be an odd prime. A connected cubic Cayley graph Cay(G, S) on a group G of order 4p^2 is non-normal if and only if one of the following happens:

1. G = C_0^{4p^2} and S \equiv \Omega;
2. G = C_1^{4p^2} and S \equiv \Theta;
3. G = C_2^{4p^2} and S \equiv \Lambda.

Furthermore, C_1^{4p^2} \cong C_2^{4p^2}, \text{Aut}(C_0^{4p^2}) \cong \mathbb{Z}_p^* \times D_{2p^2} and \text{Aut}(C_1^{4p^2}) \cong \mathbb{Z}_p^* \times \langle D_6 \times \mathbb{Z}_2 \rangle.

The paper is organized as follows. After this introductory section, in Section 2 we will give some preliminary results. In Section 3, the non-normality of the graphs C_1^{4p^2} is proved. In Section 4, we consider the normality of cubic Cayley graphs on the semidirect product \mathbb{Z}_p^* \times \mathbb{Z}_2^2 and in Section 5, Theorem 1.1 is proved.

2. Preliminaries

We start by some notational conventions used throughout this paper. For a regular graph X, use d(X) to represent its valency, and for any subset B of V(X), the subgraph of X induced by B will be denoted by X[B]. For a connected vertex-transitive graph X, let G \leq \text{Aut}(X) be vertex-transitive on X, and let N be a normal subgroup of G. The quotient graph X_N of X relative to N is defined as the graph with vertices the orbits of N on V(X) and with two orbits adjacent if there is an edge in X between those two orbits.

Let n be a positive integer. Denote by \mathbb{Z}_n the cyclic group of order n as well as the ring of integers modulo n, by \mathbb{Z}_n^* the multiplicative group of \mathbb{Z}_n consisting of numbers coprime to n, by D_{2n} the dihedral group of order 2n, and by C_n and K_n the cyclic and the complete graph of order n, respectively. We call C_n an n-cycle. For two groups M and N, N \leq M means that N is a subgroup of M, N < M means that N is a proper subgroup of M, and N \times M denotes a semidirect product of N by M. For a subgroup H of a group G, denote by C_G(H) the centralizer of H in G and by N_G(H) the normalizer of H in G. Then C_G(H) is normal in N_G(H).

For a finite group G, denote by G' and Z(G) the derived subgroup and the center of G, respectively. In view of [17, 10.1.7], we have the following result.
Proposition 2.1. Let the finite group $G$ have an abelian Sylow $p$-subgroup $P$. Then $p \mid |G \cap Z(G)|$.

Let Cay$(G, S)$ be a Cayley graph on a group $G$ with respect to a subset $S$ of $G$. Set $A = \text{Aut(Cay}(G, S))$ and $\text{Aut}(G, S) = \{ \alpha \in \text{Aut}(G) \mid S^\alpha = S \}$.

Proposition 2.2 ([20, Proposition 1.5]). The Cayley graph Cay$(G, S)$ is normal if and only if $A_1 = \text{Aut}(G, S)$, where $A_1$ is the stabilizer of the identity $1$ of $G$ in $A$.

To end this section, we give an easy observation about the regular representation of a finite group.

Proposition 2.3. Let $R(G)$ be the right regular representation of the finite group $G$. Then $R(g)^\alpha = R(g^\alpha)$ for any $\alpha \in \text{Aut}(G)$ and $g \in G$.

3. Non-normality of the graphs $C^4_{4p}$

In this section, we shall show that all Cayley graphs given in Constructions 1–3 are non-normal. From [24, Theorem 2.2] we obtain that $C^1_{4p}$ is non-normal. To show that $C^4_{4p}$ and $C^2_{4p}$ are non-normal, we first introduce a family of cubic graphs which first appeared in [13]. For an odd prime $p$, the graph $\Gamma'(4p^2)$ is defined to have vertex set $V = \{(i_k, j_k) \mid i, j \in \mathbb{Z}_4, k \in \mathbb{Z}_p\}$, and edge set. See Fig. 1 for the small one in this family of graphs.

$$E = \bigcup \left\{ \{i_k, (i + 1)_k\} \mid i \in \mathbb{Z}_4 \setminus \{3\}, j, k \in \mathbb{Z}_p \bigcup \left\{ \{0_k, 1_k\} \mid j, k \in \mathbb{Z}_p \bigcup \left\{ \{2_k, 3_k^{-1}\} \mid j, k \in \mathbb{Z}_p \right\} \right\} \right\}.$$  

Theorem 3.1. The Cayley graphs $C^1_{4p}$ and $C^2_{4p}$ are isomorphic to $\Gamma'(4p^2)$, and they are also non-normal.

Proof. Define a map from $V(\Gamma'(4p^2))$ to $V(C^1_{4p^2})$ as following:

$$\beta : \quad 0_k \mapsto (a^2 b^{-2})^k (a^2 b^{-2})^k, \quad 1_k \mapsto (a b) (a^2 b^{-2}) (a^2 b^{-2})^k, \quad 2_k \mapsto (a^2 b^{-2})^k (a^2 b^{-2})^k, \quad 3_k \mapsto (a^2 b^{-2})^k (a^2 b^{-2})^k, \quad \forall j, k \in \mathbb{Z}_p.$$

Note that $P = \langle a^2 b^{-2} \rangle \times \langle a^2 b^{-2} \rangle$ is the normal Sylow $p$-subgroup of $C^1_{4p^2}$, and $\{1, ab^{-1} c, a^2 c\}$ is a left transversal to $P$ in $C^1_{4p^2}$. From this one may see that $\beta$ is a bijection. Furthermore, one may easily check that $\beta$ maps each edge of $\Gamma'(4p^2)$ to an edge of $C^1_{4p^2}$. Since both $\Gamma'(4p^2)$ and $C^1_{4p^2}$ are cubic graphs, $\beta$ is an isomorphism from $\Gamma'(4p^2)$ to $C^1_{4p^2}$. So, $\Gamma'(4p^2) \cong C^1_{4p^2}$. To show $\Gamma'(4p^2) \cong C^2_{4p^2}$, we define a map from $V(\Gamma'(4p^2))$ to $V(C^2_{4p^2})$ as following:

$$\gamma : \quad 0_k \mapsto (a^{-1} b)(a^{-1} b^{-1})^k, \quad 1_k \mapsto (cd)(a^{-1} b)(a^{-1} b^{-1})^k, \quad 2_k \mapsto (da^{-1})(a^{-1} b)(a^{-1} b^{-1})^k, \quad 3_k \mapsto (ca^{-1})(a^{-1} b)(a^{-1} b^{-1})^k, \quad \forall j, k \in \mathbb{Z}_p.$$

With a similar argument as above, one may see that $\gamma$ is an isomorphism from $\Gamma'(4p^2)$ to $C^2_{4p^2}$. So, $\Gamma'(4p^2) \cong C^2_{4p^2}$.

Let $\alpha$ be a map on $V$ defined as following:

$$\alpha : 0_k \mapsto 0_k, \quad 1_k \mapsto 1_k, \quad 2_k \mapsto 2_k, \quad 3_k \mapsto 3_k, \quad \forall j, k \in \mathbb{Z}_p.$$

Clearly, $\alpha$ is also a permutation. By easy checking, we see that $\alpha$ is also a non-identity automorphism of $\Gamma'(4p^2)$ which fixes $0_k, 1_k, 2_k, 3_k$ for all $k \in \mathbb{Z}_p$. Since $\Gamma'(4p^2) \cong C^1_{4p^2}$, $C^1_{4p^2}$ also has a non-identity automorphism fixing $\{1\} \cup \emptyset$ pointwise. By Proposition 2.2, $C^1_{4p^2}$ is non-normal. Similarly, $C^2_{4p^2}$ is non-normal. \(\square\)
4. Cubic Cayley graphs on $\mathbb{Z}_p^2 \times \mathbb{Z}_2^2$

Let $G$ be a non-abelian group of order $4p^2$, where $p$ is an odd prime. If $G$ is a semidirect product of $\mathbb{Z}_p \times \mathbb{Z}_p$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$, then by elementary group theory, either $G \cong G_{4p^2}^1$, $G_{4p^2}^2$ (see Constructions 1, 2) or $G$ is isomorphic to the following group

$$G_{4p^2}^3 = \langle a, b, c \mid a^{2p} = b^p = c^2 = 1, ab = ba, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1} \rangle.$$  

(1)

**Lemma 4.1.** Let $X$ be a connected cubic Cayley graph on $G_{4p^2}^3$. Then $|\text{Aut}(X)| \leq 8p^2$, and hence $X$ is normal.

**Proof.** Let $X = \text{Cay}(G_{4p^2}^3, S)$, where $S = \{c^ih_1, c^jh_2, c^kh_3\}$ with $i, j, k \in \mathbb{Z}_2$ and $h_1, h_2, h_3 \in \langle a, b \rangle$. By the connectivity of $X$, we can let $k = 1$. Then $G = \langle a, b \rangle \ltimes \langle \text{ch}_3 \rangle$ and $(\text{ch}_3)^{-1}g(\text{ch}_3) = g^{-1}$ for any $g \in \langle a, b \rangle$. Thus, there is an automorphism of $G$ which maps $\text{ch}_3$ to $c$ and hence we may let $S = \{c^ih_1, c^jh_2\}$. Then $G = \langle S \rangle = \langle h_1, h_2 \rangle \ltimes \langle c \rangle$. This implies that $h_1, h_2 \notin \langle a^p \rangle$ and $(h_1, h_2) \neq (h_2, h_1)$. As a result, $S = \{c^ih_1, c^jh_2\}$.

If $p = 3$ then with the help of MAGMA [3], we can obtain that $|\text{Aut}(X)| \leq 8 \cdot 3^2$, and hence $X$ is normal. Let $p > 3$. We depict the subgraph of $X$ induced by the vertices at distance less than 4 from the identity element 1 (see Fig. 2). Let $A = \text{Aut}(X)$. Let $A_1$ be the subgroup of $A$ fixing $S$ pointwise. From Fig. 2 one may see that $X$ has girth 6 and for any $s_1, s_2 \in S$, passing through $1, s_1, s_2$ there is a unique 6-cycle. As $A_1$ fixes $S$ pointwise, $A_1$ fixes each vertex in each of these 6-cycles. In particular, $A_1$ fixes each vertex at distance 2 from 1. By the connectivity and vertex-transitivity of $X$, we find that $A_1$ fixes every vertex of $X$ which implies $A_1 = 1$. Since $p > 2$, by [9, Theorem 6.2], $X$ is non-symmetric. This implies that $A_1$ is a 2-group, and so $A_1 = A_1/A_1^2 \leq \mathbb{Z}_2$. Therefore, $|A| \leq 8p^2$ and hence $R(G) \not\subseteq A$. \hfill $\Box$

**Lemma 4.2.** Let $X = \text{Cay}(G_{4p^2}^1, S)$ be a connected cubic Cayley graph on $G_{4p^2}^1$. Then $X$ is non-normal if and only if $S = \{ab, a^{-1}b^{-1}, c\}$.

**Proof.** The sufficiency follows Theorem 3.1. We only need to show the necessity. Let $X$ be non-normal and set $S = \{x, y, z\}$. Note that $G_{4p^2}^1 = \langle a \rangle \times \langle b, c \rangle \cong \mathbb{Z}_{2p} \times \mathbb{D}_{2p}$. All involutions of $G_{4p^2}^1$ are contained in $\langle a^p, b, c \rangle$. Since $X$ is connected, we may assume $x = y^{-1}$ order greater than 2, and $z$ is an involution. Furthermore, $z = cb\ell a^{\ell}$ for some $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_2$. The map $a \mapsto a, b \mapsto b, z \mapsto c$ can induce an automorphism of $G_{4p^2}^1$. So, we may let $z = c$. Clearly, $x \in \{a^ib^jc^k \mid i \in \mathbb{Z}_{2p}, j \in \mathbb{Z}_p, k \in \mathbb{Z}_2\}$. Note that for each $i \in \mathbb{Z}_p, j \in \mathbb{Z}_p$, the map $a^i \mapsto a^ib^j, b^j \mapsto b^j, c \mapsto c$ can also induce an automorphism of $G_{4p^2}^1$. Consequently, we may let $x = ab$ or $abc$, and hence $S = \{abc, a^{-1}bc, c\}$ or $\{ab, a^{-1}b^{-1}, c\}$.

Suppose $S = \{abc, a^{-1}bc, c\}$. The map $\alpha : a \mapsto a^{-1}, b \mapsto b, c \mapsto c$ can induce an automorphism of $G_{4p^2}^1$ of order 2. Clearly, $S^\alpha = S$. So, $\alpha \in \text{Aut}(G_{4p^2}^1, S)$. By Proposition 2.3, we have $R(a)^\alpha = R(c)$ and so $R(c)\alpha$ has order 2. Furthermore, $R(a)R(\alpha)\alpha = R(a)^{-1}$ and $R(b)R(\alpha)\alpha = R(b)^{-1}$. So, $H = \langle R(a), R(b), R(c)\alpha \rangle \cong G_{4p^2}^3$. Note that $\langle R(a), R(b) \rangle$ has two orbits on $V(X) = G_{4p^2}^1$, that is, $(a, b)$ and $(a, b)c$. Since $1R(c)\alpha = c, H$ is transitive on $V(X)$, and hence it is regular on $V(X)$. So, $X$ is isomorphic to a Cayley graph on $G_{4p^2}^1$. By Lemma 4.1, $X$ is normal, a contradiction. \hfill $\Box$

**Lemma 4.3.** Let $X = \text{Cay}(G_{4p^2}^2, S)$ be a connected cubic Cayley graph on $G_{4p^2}^2$. Then $S$ is equivalent to one of the following sets:

- $W_1 = \{ad, a^{-1}d, cdb\}$
- $W_2 = \{cd, ca, db\}$
- $W_3 = \{cd, cda, b^i d^j \mid i \in \mathbb{Z}_p^*\}$
- $W_4 = \{cd, cdba, c^i d^j \mid i \in \mathbb{Z}_p^*\}$

Furthermore, if $S \equiv W_1$ or $W_3$, then $X$ is normal.

**Proof.** Set $S = \{x, y, z\}$. Note that $G_{4p^2}^2 = \langle a, c \rangle \times \langle b \rangle \cong \mathbb{D}_{2p} \times \mathbb{D}_{2p}$. It is easy to check that each of the following maps:

- $\gamma_1 : a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto d$ (i $\in \mathbb{Z}_p^*$)
- $\delta_1 : a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto d$ (i $\in \mathbb{Z}_p^*$)
- $\phi_1 : a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto d$ (j $\in \mathbb{Z}_p^*$)
- $\psi_1 : a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto d$ (j $\in \mathbb{Z}_p$)
- $\theta : a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto c$

can induce an automorphism of $G_{4p^2}^2$.

Let $S$ contain an element, say $x$, of order greater than 2. Let $y = x^{-1}$, then $z$ is an involution. Observe that every subgroup of $G_{4p^2}^2$ of order $p$ is normal. We get that $x$ has order $2p$. Then, $x = a^ib^j$ or $b^k c^l$ for some $i, j \in \mathbb{Z}_p^*$ and $k, l \in \mathbb{Z}_p$. Since $(a^i b^j)^{x^{i^2}} = ad$ and $(b^k c^l)^{x^{i^2}} = cb$, one may let $x = ad$ or $cb$. Also, since $(ad)^x = bc$, one may let $x = ad$. By the connectivity of $X, z = cda^ib^j$ for some $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_p^*$. If $i \equiv 0 \pmod{p}$ then $z^j = cdb$, and if $i \not\equiv 0 \pmod{p}$ then $z^{x^j} = cdb$. So, we may let $z = cdb$ or $cda$. Since $(ad, a^{-1}d, cda)^{x^j} = (ad, a^{-1}d, cda)$, one may let $S = \{ad, a^{-1}d, cda\} = W_1$. 


Lemma 5.1.

5. Proof of Theorem 1.1

In view of the fact that (R(a), R(b)) has four orbits on V(X) = G(a,b), that is, \{(a, b), (a, b)c, (a, b)d\} and \{(a, b)cd\}.

Assume that any two elements of S are in distinct two orbits of (R(a), R(b)). Without loss of generality, assume that x ∈ cd(a, b), y ∈ c(a, b) and z ∈ d(a, b). Let x = cda, y = ca and z = db with i, j, k, ℓ ∈ Zp. Since x^(\ell) = cd, we may let x = cd. By the connectivity of X, we have k, ℓ ∈ Zp*. Then S^{\ell} = {cd, ca, db} \{cd, ca, db\} = W_2.

Now assume that S has two elements which are in the same orbit of (R(a), R(b)). If x, y ∈ c(a, b), then x, y ∈ c(a, c). In view of the fact that (a, c) ≤ G_4^2, we get that (x, y, z) ≤ (a, c) \times (z) < G_4^2, a contradiction. Similarly, x, y \notin d(a, b).

Therefore, x, y ∈ cd(a, b). Clearly, x = cda for some i, j ∈ Zp. Since x^j = (ca)^j = (db)^j = cd, one may let x = cd. Then y = cda, where i, j ∈ Zp and at least one of i and j is in Zp*. If both i and j are in Zp*, then (x, y) \notin \{cd, cda\}. If p | i, then i ∈ Z_p and then (x, y)^i = (cd, cda). If p \mid j, then j ∈ Z_p and then (x, y)^j = (cd, cdb). Since \{cd, cda\} \{cd, cdb\}, one has \{x, y\} = \{cd, cda\} or \{cd, cdb\}.

Let (x, y) = \{cd, cda\}. If z ∈ cd(a, b), then (x, y, z) ≤ (a, b) \times (cd) < G_4^2, contrary to the connectivity of X. If z ∈ c(a, b), then (x, y, z) ≤ (a) \times (c, d) < G_4^2. This is also impossible. Thus, z ∈ d(a, b), and hence z = b'd for some i ∈ Zp*. So, S = \{cd, cda, b'd\} = W_3.

Let (x, y) = \{cd, cdb\}. By the connectivity of X, z ∈ g(a, b) with g = c or d. It follows that z = ca or db for some i ∈ Z_p.

Since \theta maps ca to db and fixes cd and cdb, one may let z = ca. So, S = \{cd, cda, b'd\} = W_4.

Suppose that S = W_1 or W_3. To show that X is normal, by Lemma 4.1 it suffices to show that Aut(X) has a regular subgroup isomorphic to G_4^2.

First, let S = W_1 = \{ad, a^{-1}d, cd\}. Then g^y = \{ad, a^{-1}d, cd\}. Then g^y = \{ R(a)R(c)g^y, y = R(b) and z = R(d)g^y \}. By Proposition 2.3, we obtain that the orders of x, y and z are 2p, 2p and 2, respectively. Furthermore, xyz = yxz, x^y = x^{-1} and y^x = y^{-1}. So, H = \langle x, y, z \rangle \cong G_4^2. Recall that (x^y, y^x = (R(a), R(b))) has four orbits on V(X) = G_4^2, namely, \langle a, b \rangle g with g = 1, c, d, cd. An easy computation gives that 1_R(c)g^{-1} = c, 1_R(d)g^{-1} = d and 1_R(c)g^{-1}R(d)g^{-1} = cd. This implies that H is regular on V(X), as required.

Secondly, let S = W_3 = \{cd, cda, b'd\} (i ∈ Z_p^*). Let \beta = g^{-1}y. Then, \beta^a = a^{-1}, \beta^b = b, \beta^c = ca and \beta^d = d. Also, \beta ∈ Aut(G_4^2, S). Let k ∈ Z_p such that 2k ≡ 1 (mod p). Then (ca)^kb = ca^{-k} = ca^k. Let h = R(a)R(ca^k), y = R(b) and v = R(d)\beta. With a similar argument as the above paragraph, we get that the subgroup generated by h, y, v is isomorphic to G_4^2 and acts regularly on V(X), as required. □

5. Proof of Theorem 1.1

First, we prove the first part of Theorem 1.1. The sufficiency can be obtained from [24, Theorem 2.2] and Theorem 3.1. We only need to prove the necessity. Let X = Cay(G, S) be a connected cubic non-normal Cayley graph of order 4p. Set A = Aut(X). Then R(G) is non-normal in A. Since p is odd, by [9, Theorem 6.2], X is non-symmetric. Then the vertex-stabilizer A_v of v ∈ V(X)^3 in A is a 2-group. It follows that |A| = |V(X)| |A_v| = 2^{x+y}p for some positive integer s ≥ 2. By Burnside p^e q^f theorem (see [17, 8.5.3]), A is solvable. Let P be a Sylow p-subgroup of A. We shall finish the proof by the following two lemmas.

Lemma 5.1. If F ⊆ Z_p^* and P ⊆ A, then either G = G_4^2 and S = \emptyset, or G = G_4^2 and S = A. Furthermore, let Q be the Sylow 2-subgroup of A such that A_1 ≤ Q. Then Q = A_1 × (Q \cap R(G)) ≤ A_1 × Z_2^2.

Proof. Consider the quotient graph X_P of X relative to P, and let K be the kernel of A acting on V(X_P). Then |X_P| = 4. Set V(X_P) = \{A_0, A_1, A_2, A_3\}, and let 1 ∈ A_0. Since P ⊆ A, one has P ≤ R(G). Since R(G) acts on V(X) by right multiplication, one has P = R(A_0), and A_1 (i = 1, 2, 3) are right cosets of A_0 in G. Clearly, X_P isomorphic to K_4 or C_4.

Suppose X_P isomorphic to K_4. Then the stabilizer K_v of v ∈ V(X) in K fixes the neighborhood of v in X pointwise because K fixes each orbit of P setwise. By the connectivity and vertex-transitivity of X, K_v fixes each vertex in V(X), forcing K_v = 1. Hence,
$K = P$. Since $A/P$ is a 2-group, one has $|A/P| \leq 8$ because $A/P \leq \text{Aut}(X_P) \cong S_4$. It follows that $|A| \leq 8p^2$, and hence $R(G) \trianglelefteq A$, a contradiction.

Let $X_P \cong C_2$. Let $\Delta_i \sim \Delta_i+1$ with $i \in Z_4$. Since $d(X) = 3$ and $X$ is connected, one has $d(X[\Delta_0]) = 0$ because $|\Delta_0| = p^2$ is odd. Without loss of generality, we may assume $X[\Delta_0 \cup \Delta_3] \cong X[\Delta_1 \cup \Delta_2] \cong p^2K_2$. Then $X[\Delta_0 \cup \Delta_1] \cong X[\Delta_2 \cup \Delta_3] \cong p^2C_2$, because $P \cong Z_p$. This implies that $A/K$ is not edge-transitive on $X_P$. So, $A/K \cong Z_{p^2}$, and hence $R(G)K/K = A/K$. Noting that $K = P \times K_1$, one has $K_1 = A_i = Q \cap K \leq Q$. Then, $R(G)/P \cong R(G)K/K \cong Q/K_1 \cong Z_{2^2}$. Since $|QR(G)| = |A|$, one has $|Q \cap R(G)| = 4$, and so $Q = K_1 \times (Q \cap R(G))$. By [1, Theorem 1.2], $G$ is non-abelian. Since $P \cong Z_{p^2}$, by elementary group theory, $G$ is isomorphic to one of $G_{4p^2}$ with $i = 1, 2, 3$. By Lemma 4.1, $G \cong G_3^{4p^2}$ and by Lemma 4.2, if $G \cong G_4^{4p^2}$ then $S \equiv \{a, a^{-1}b^{-1}, c\} = \Theta$.

Let $G \cong G_{4p^2}$. By Lemma 4.3, $S \equiv \{cd, ca\}$ or $\{cd, cdab, ca\}$ ($i \in Z_{p^2}$). Since any two of $cd, ca$ and $ab$ are in distinct orbits of $P$, then $S \equiv \{cd, cdab, ca\}$ ($i \in Z_{p^2}$). Since $A/P$ is a 2-group, $R(G)/P < A/P$ implies that $R(G) < N_4(R(G))$. From Proposition 2.2 we get that $\text{Aut}(G, S) = \{o\} \cong Z_2$. Observe that $ca$ commutes with $b$, and both $cd$ and $cdab$ map each element of order $p$ to its inverse under conjugacy. This implies that $a$ fixes $cd$ and interchanges $cd$ and $cdab$. It follows that $(ab)^a = (ab)^{-1}$. Then $ca = [(ca(ab)^{-1})^a] = (ca(ab)) = ca$, and so $d^a = [c(d(c))a = (ca)^2(cdab) = a^{-1}b^{-1}$. Since $d$ has order 2, one has $2i - 1 \equiv 0(\text{mod } p)$. Consequently, $S \equiv \Theta$. □

Lemma 5.2. If either $P \cong Z_{p^2}$ or $P \not\trianglelefteq A$, then $G \cong G_{4p^2}^0$ and $S \equiv \Omega$.

Proof. We first claim that $A$ has a normal 2-subgroup, that is, $O_2(A) > 1$. Suppose on the contrary that $O_2(A) = 1$. Since $A$ is solvable, $A$ has a minimal normal p-subgroup, say $M$. In view of our assumption, we get that $M \cong Z_p$. Let $C = C_M(A)$. Clearly, $P \leq C$. If $C = P$, then $A/P \leq \text{Aut}(X_P) \cong Z_{2^2}$ (see [17, 16, 13]). It follows that $R(G)/P \leq A/P$ and hence $R(G) \leq A$, a contradiction. Therefore, $C > P$ and so $2 \not\mid |C|$. If the derived subgroup $C'$ of $C$ is identity, then $C$ is abelian, and hence the Sylow 2-subgroup of $C$ is normal in $A$, a contradiction. Let $C' > 1$. By the assumption, $C'$ is a $(2, p)$-group with $O_2(C') = 1$, and moreover, the center $Z(C)$ of $C$ must be a $p$-group. So, $O_2(C') > 1$. Since $p \not\mid |C' \cap Z(C)|$ by Proposition 2.1, one has $Z(C) = M$ and $O_2(C') \cong Z_{2^2}$. This forces that $P = M \times O_2(C') \cong Z_{2^p}$ is normal in $A$, a contradiction.

Now we observe that $O_2(A) > 1$. Still use $M$ to denote $O_2(A)$. Consider the quotient graph $X_M$ of $X$ relative to $M$, and let $K$ be the kernel of $A$ acting on $V(X_M)$. Then $A/K \leq \text{Aut}(X_M)$.

Suppose $M$ has an orbit of length 4. Then $|X_M| = p^2$. Since $p^2$ is odd, one has $d(X_M) = 2$. Let $X_M = \{\Delta_0, \Delta_1, \ldots, \Delta_{p^2-1}\}$ with $\Delta_0 \sim \Delta_{p^2+1}$ ($i \in Z_{p^2}$). Again since $p^2$ is odd, one has $X[\Delta_0] \equiv 2K_2$. Then the stabilizer $K_0$ of $\Delta_0 \in V(X)$ in $K$ fixes the neighborhood of $v$ in $X$ pointwise because $K$ fixes each orbit of $M$ setwise. By the connectivity and vertex-transitivity of $X$, $K_0$ fixes each vertex in $V(X)$, forcing $K_0 = 1$. Then $|K| = 4$, and $|A| \leq 8p^2$ because $A/K \not\leq \text{Aut}(X_M) \cong Z_{2^p}$. Consequently, $R(G) \not\leq A$, a contradiction.

Assume that every orbit of $M$ has length 2. Suppose $d(X_M) = 3$. It is easy to show that $K$ is semifield, and so $K = M \cong Z_{2^2}$. Let $H/M$ be a minimal normal subgroup of $A/M$. Then $H/M \cong Z_2 \times Z_{p^2}$ or $Z_{p^2}$, clearly, $M$ is in the center of $A$. It follows that $H = M \times R$ where $R$ is a Sylow $p$-subgroup of $H$. Since $H \leq A$, one has $R \leq A$. By our assumption, we have $R \cong Z_2 \times Z_2$, and hence $H \cong Z_{2^4}$. Let $C = C_M(H)$. Then $A/C \leq \text{Aut}(H) \cong Z_{2^2}$ (see [17, 16, 13]). This implies that $p^2 \mid |C|$ and hence $P \leq C$. Let $|C| = 2p^2$. Since $(A/P)P/C \leq Z_{2^2}$ and $C/P$ is in the center of $A/P$, we get that $A/P$ is abelian. It follows that $R(G)/P \leq A/P$ and hence $R(G) \leq A$, a contradiction. Thus, $|C| > 2p^2$. If $C' = 1$ then $C$ is abelian, and so every Sylow subgroup of $C$ must be normal in $H$. By the maximality of $M$, we have $|C' \leq 2p^2$; a contradiction. Thus, $C' > 1$ and $O_2(C') \leq M$. So, $O_2(C') > 1$. Clearly, $R \cong Z(C)$. By Proposition 2.1, $p \mid |C' \cap Z(C)|$. It follows that $P = O_2(C') \cong R \cong Z_{2^p}$ is normal in $A$, a contradiction.

Now let $d(X_M) = 2$. Let $V(X_M) = \{B_i \mid i \in Z_{2^2}\}$ with $B_i \sim B_{i+1}$. Since $d(X) = 3$ and $X$ is connected, $d(X[B_i]) = 0$ or 1. Assume $d(X[B_i]) = 1$. Then the stabilizer $K_0$ of $u \in V(X)$ in $K$ fixes the neighborhood of $u$ in $X$ pointwise because $K$ fixes each orbit of $M$. By the connectivity of $X$, one has $K_0 = 1$ and hence $K = M \cong Z_{2^2}$. Since $X_M \cong C_{p^2}$, one has $A/K \not\leq \text{Aut}(C_{p^2}) \cong D_{p^2}$, and hence $|A| \leq 8p^2$. It follows that $R(G) \leq A$, a contradiction. Assume $d(X[B_i]) = 0$. Since $d(X) = 3$, one may let $X[B_i \cup B_{i+1}] \equiv 2K_2$ and $X[B_i \cup B_{2p-1}] \equiv C_4$. Let $B_i = \{x_i, y_i\}$ for each $i \in Z_{2^p}$. By the transitivity of $A$ on $X$, we may assume that $x_i \sim x_{i+1}, y_i \sim y_{i+1}, x_{2i} \sim y_{2i+1}$ and $y_{2i} \sim x_{2i+1}$ for each $i \in Z_{2^2}$. From [24, Theorem 2.2], we obtain that $G \cong G_{4p^2}^0$ and $S \equiv \Omega$. □

So far, we have proved the first part of Theorem 1.1. For the second part, by [24, Theorem 2.2] we know that $\text{Aut}(G_{4p^2}) \cong Z_{2^2} \times D_{2p^2}$. From Theorem 3.1 we know that $G_{4p^2} \cong C_{2p^2} \cong \Gamma(4p^2)$. To complete the proof, it suffices to prove the following lemma.

Lemma 5.3. $\text{Aut}(\Gamma(4p^2)) \cong Z_{2^2} \times (D_8 \times Z_2)$.

Proof. Let $A = \text{Aut}(\Gamma(4p^2))$. Define two permutations on $V(\Gamma(4p^2))$ as follows:

$$x = \prod_{i=0}^{3} \prod_{k=0}^{p-1} (i_k \ i_k \ i_k^{p-1}) \quad y = \prod_{i=0}^{3} \prod_{j=0}^{p-1} (i_k \ i_j \ i_{k,j}^{p-1}).$$
It is easy to check that $x, y \in A$ such that $P = \langle x, y \rangle \cong \mathbb{Z}_p^2$ is a Sylow $p$-subgroup of $A$. Since $\Gamma(4p^2)$ is a non-normal Cayley graph, by Lemma 5.2, one has $P \not\leq A$ because any Sylow $p$-subgroup of $\text{Aut}(\mathbb{C}_p^0)$ is cyclic. Clearly, $\Delta_i = \{i^j_k | j \in \mathbb{Z}_p\} (i \in \mathbb{Z}_4)$ are four orbits of $P$ on $V(\Gamma(4p^2))$. Observe that for each $j, k \in \mathbb{Z}_p$,
\begin{align*}
C^j_{01} &= \{0^0_1, 1^0_1, 0^1_1, \ldots, 0^j_{p-1}, 1^j_{p-1}\}, \\
C^k_{23} &= \{(2^0_k, 3^0_k, 2^1_k, \ldots, 2^{k-1}_k, 3^{k-1}_k)\}
\end{align*}
are two 2p-cycles. Furthermore, $\Gamma(4p^2)[\Delta_0 \cup \Delta_1] = \bigcup_{i=0}^{p-1} C^i_{01}$ and $\Gamma(4p^2)[\Delta_2 \cup \Delta_3] = \bigcup_{k=0}^{p-1} C^k_{23}$. For each $j, k \in \mathbb{Z}_p$, between $C^j_{01}$ and $C^k_{23}$ there are exactly two edges that are $\{1^j_k, 2^k_j\}$ and $\{3^{j-k}_{k+1}, 0^{j-k}_{k+1}\}$.

Let $T = A^0_{01}$ and let $T^* \leq T$ such that $T/T^* \leq \mathbb{Z}_2$. Clearly, $T^*$ acts on $\{3^0_0, 3^{-1}_0\}$. Take $g \in T^*$ such that $g$ fixes each neighbor of $1^0_0$. Then $T/T^* \leq \mathbb{Z}_2$. Let $T^*$ acts on $\{3^0_0, 3^{-1}_0\}$.

Let $Q$ be a Sylow 2-subgroup of $A$ containing $T$. By Lemma 5.1, $Q = T \times N$ where $N \cong \mathbb{Z}_2^2$. Note that the map $\alpha : a \mapsto a\cdot b$, $b \mapsto b\cdot c$, $c \mapsto c$ can induce an automorphism of $G_{p^2}$ of order 2. Since $\text{Aut}(\mathbb{C}_p^0)$, one has $\alpha \in \text{Aut}(\mathbb{C}_p^0)$. This implies that $T$ cannot be cyclic. So, $T \cong \mathbb{Z}_2^2$. Again since $\Gamma(4p^2)$ is a non-normal Cayley graph, $Q$ is non-abelian. Then $T \cap \mathbb{Z}(Q) \cong \mathbb{Z}_2$. Set $T = (a, b)$ with $a \in T \cap \mathbb{Z}(Q)$. Then $Q = \langle a \rangle \times \langle b, N \rangle$. Since $Q$ is non-abelian, one has $\langle b, N \rangle \cong \mathbb{Z}_2$ and so $Q \cong \mathbb{Z}_2 	imes \mathbb{Z}_2$. Therefore, $A = P \times Q \cong \mathbb{Z}_p^2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

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