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The Discrete Dynamics of Nonlinear Infinite-Delay-Differential Equations

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Abstract—This paper deals with numerical stability of nonlinear infinite-delay systems of the form $y'(t) = f(t, y(t), y(pt))$ ($p \in (0,1)$, $t > 0$). Recently, linear stability properties of some numerical methods for infinite delay systems have been studied by several authors (cf. [1–9]). However, few results have been devoted to the nonlinear case. This paper considers global and asymptotic stability of one-leg θ-methods for the above nonlinear systems. Some stability criteria are obtained.

Keywords—Nonlinear stability, One-leg θ-methods, Infinite delay-differential.

1. INTRODUCTION

Discretization methods for infinite delay systems

$$\begin{aligned} y'(t) &= f(t, y(t), y(pt)), \qquad t > 0, \\ y(0) &= \eta, \end{aligned} \tag{1.1}$$

where $p \in (0,1)$ and $f : [0, +\infty) \times C^d \times C^d \to C^d$ are given, have been investigated in the last decades by several authors (cf. [1–9]). Those authors have noted that there exist very different numerical challenges between the infinite delay systems (1.1) and the finite delay systems

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau)), \qquad && t > 0, \\ y(t) &= \psi(t), && -\tau \le t \le 0. \end{aligned} \tag{1.2}$$

First, the solution of (1.1) is generally smooth and retains the degree of smoothness of f, but the solution of (1.2) is usually nonsmooth even when f is smooth. From this sense, (1.1) seems to be simpler to compute. However, (1.1) poses more computational complexities than (1.2) does. For

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instance, when ones solve (1.1) and (1.2) at $t = t^\star$, the past solutions in the intervals $[pt^\star, t^\star)$ and $[t^\star - \tau, t^\star)$ must be known, respectively. But, as $t^\star \to \infty$, $[pt^\star, t^\star)$ is unbounded while $[t^\star - \tau,\ t^\star)$ remains bounded. This creates a serious storage problem when the computation for (1.1) is run on any computer. However, this difficulty does not arise for (1.2). The computational difficulty for (1.1) was found first by Feldstein and Grafton [2]. To overcome the above storage obstacle, Liu [8] and Bellen, Guglielmi and Torelli [9] introduced some techniques, which will be used in the present paper. The nonlinear stability analysis of various numerical methods for (1.2) has been studied (cf. [10–15]), but all numerical stability analysis for (1.1) were devoted mainly to the linear case. Thus, this paper considers *nonlinear* stability of one-leg θ-methods for (1.1). Some stability results are obtained.

2. ONE-LEG θ-METHODS WITH VARIABLE STEPSIZE

The technique, introduced by Liu [8], yields the following one-leg θ-methods:

$$y_{n+1} = y_n + h_{n+1} f((1-\theta)t_n + \theta t_{n+1}, (1-\theta)y_n + \theta y_{n+1}, (1-\theta)y_{n-m} + \theta y_{n-m+1})), \qquad n \geq 0, \tag{2.1}$$

where $\theta \in [0,1]$, m is certain positive integer, y_n $(n \geq 0)$ are approximations to $y(t_n)$, stepsize $h_{n+1} = t_{n+1} - t_n$. The grid points t_n of methods (2.1) are selected as follows. First, divide $[0,\infty)$ into a set of infinite bounded intervals; that is,

$$[0,\infty) = \bigcup_{l=0}^{\infty} D_l,$$

where $D_0 = [0,\gamma]$ with a given positive number γ and $D_l = (T_{l-1}, T_l]$ $(l \geq 1)$ with $T_l = p^{-l}\gamma$. Then, partition every primary interval D_l $(l \geq 1)$ into m equal subintervals. Thus, the grid points on $[0,\infty)/D_0$ are determined by

$$t_n = T_{\lfloor (n-1)/m \rfloor} + \left(n - \left\lfloor \frac{n-1}{m} \right\rfloor m\right) h_{\lfloor (n-1)/m \rfloor + 1}, \qquad n \geq 0,$$

where $\lfloor \bullet \rfloor$ denotes the integer part. On D_0, choose $t_{-(m+1)} = 0$, $t_{-i} = pt_{m-i}$ $(i = m, m-1, \dots, 1)$ as grid points. The corresponding numerical solutions y_{-i} $(i = m+1, m, \dots, 1)$ are assumed to exist. So the function $\varphi(t) := pt$ has these properties:

[P1] $\varphi(t_n) = t_{n-m}$, $n \geq 0$,
[P2] $\varphi(D_{n+1}) = D_n$, $n \geq 1$,
[P3] $\varphi(h_n) = h_{n-m}$, $n \geq 1$,

where the stepsize sequence $\{h_n\}$ is determined by

$$h_n = \begin{cases} p\gamma, & n = -m, \\ \dfrac{(1-p)\gamma}{m}, & n = -m+1, -m+2, \dots, -1, 0, \\ \dfrac{(1-p)\gamma}{mp^{\lfloor (n-1)/m \rfloor + 1}}, & n = 1, 2, \dots. \end{cases} \tag{2.2}$$

Properties [P1]–[P3] imply that the choice of grid points has removed the computational storage problem for (1.1).

3. STABILITY OF THE METHODS

In order to study the stability of these methods, consider this related infinite-delay system on C^d

$$\begin{aligned} z'(t) &= f(t, z(t), z(pt)), \qquad t > 0, \qquad p \in (0,1), \\ z(0) &= \zeta. \end{aligned} \tag{3.1}$$

In both (1.1) and (3.1), assume that the function f satisfies

$$\Re\langle f(t,\mu_1,\nu)-f(t,\mu_2,\nu),\mu_1-\mu_2\rangle \le \alpha\|\mu_1-\mu_2\|^2, \qquad \forall\,\mu_1,\mu_2,\nu\in C^d, \quad t>0 \tag{3.2}$$

and

$$\|f(t,\mu,\nu_1)-f(t,\mu,\nu_2)\| \le \beta\|\nu_1-\nu_2\|, \qquad \forall\,\mu,\nu_1,\nu_2\in C^d, \quad t>0, \tag{3.3}$$

where $\langle\bullet,\bullet\rangle$ denotes an assigned inner product on C^d, $\|\bullet\|$ is the induced norm. In the following, all systems (1.1) with (3.2) and (3.3) will be called *class* $D_p(\alpha,\beta)$. For systems (1.1) and (3.1) of class $D_p(\alpha,\beta)$, it follows from the arguments given by Zennaro [16] that

$$\|y(t)-z(t)\| \le \|\eta-\zeta\|, \qquad \forall\,t>0$$

and

$$\lim_{t\to+\infty}\|y(t)-z(t)\|=0,$$

whenever $\beta\le-p\alpha$.

The following lemma will play an important role in the subsequent analysis.

LEMMA 3.1. *Assume* $\theta\ge 1/2$. *Then,*

$$\left(a_1^2-a_0^2\right) \le 2[\theta a_1+(1-\theta)a_0](a_1-a_0), \qquad \forall\,a_0,\ a_1\in R. \tag{3.4}$$

PROOF. By $\theta\ge 1/2$, it holds that

$$(1-2\theta)(a_0-a_1)^2\le 0, \qquad \forall\,a_0,a_1\in R.$$

Whereas the above inequality is equivalent to (3.4). Hence, this lemma is proved.

THEOREM 3.1. *Assume* $\theta\in[1/2,1]$ *and* $\beta\le-p\alpha$. *Then, the numerical solutions* y_n *and* z_n, *produced by the one-leg* θ*-method (2.1) applying to systems (1.1) and (3.1) of the class* $D_p(\alpha,\beta)$, *respectively, satisfy the global stability inequality*

$$\|y_n-z_n\| \le \left(1+\sqrt{\frac{\beta(1-p)\gamma}{p}}\right)\max_{-m\le l\le 0}\|y_l-z_l\|, \qquad n>0. \tag{3.5}$$

PROOF. Write

$$\omega_n=y_n-z_n, \quad \rho(\xi)=\xi-1, \quad \sigma(\xi)=\theta\xi+(1-\theta), \qquad \xi\in C.$$

Then (2.1) implies that

$$\rho(E)\omega_n=h_{n+1}[f(\sigma(E)t_n,\sigma(E)y_n,\sigma(E)y_{n-m})-f(\sigma(E)t_n,\sigma(E)z_n,\sigma(E)z_{n-m})], \quad n\ge 0, \tag{3.6}$$

where E denotes the shift operator. Let $\{e_v\}_{v=1}^d$ be a set of orthonormal basis on C^d such that

$$\omega_n=\sum_{v=1}^d x_n^v e_v.$$

Write

$$a_n^v=\Re(x_n^v), \qquad b_n^v=\mathrm{Im}\,(x_n^v).$$

It follows from Lemma 3.1 that

$$\begin{aligned}\|\omega_{n+1}\|^2 - \|\omega_n\|^2 &= \sum_{v=1}^{d}\left(\left|x_{n+1}^v\right|^2 - |x_n^v|^2\right)\\ &= \sum_{v=1}^{d}\left\{\left[\left(a_{n+1}^v\right)^2 - (a_n^v)^2\right] + \left[\left(b_{n+1}^v\right)^2 - (b_n^v)^2\right]\right\}\\ &\le 2\sum_{v=1}^{d}\left[\sigma(E)a_n^v\rho(E)a_n^v + \sigma(E)b_n^v\rho(E)b_n^v\right]\\ &= 2\Re\left\langle\sigma(E)\omega_n, \rho(E)\omega_n\right\rangle.\end{aligned} \tag{3.7}$$

Whereas conditions (3.2), (3.3), and (3.6) imply that

$$\begin{aligned}2\Re\left\langle\sigma(E)\omega_n, \rho(E)\omega_n\right\rangle &\le 2h_{n+1}\alpha\|\sigma(E)\omega_n\|^2 + 2h_{n+1}\beta\|\sigma(E)\omega_n\|\|\sigma(E)\omega_{n-m}\|\\ &\le h_{n+1}(2\alpha+\beta)\|\sigma(E)\omega_n\|^2 + h_{n+1}\beta\|\sigma(E)\omega_{n-m}\|^2.\end{aligned} \tag{3.8}$$

Inserting (3.8) into (3.7) yields

$$\|\omega_{n+1}\|^2 \le \|\omega_n\|^2 + h_{n+1}(2\alpha+\beta)\|\sigma(E)\omega_n\|^2 + h_{n+1}\beta\|\sigma(E)\omega_{n-m}\|^2, \qquad n \ge 0.$$

Further, an induction yields

$$\|\omega_{n+1}\|^2 \le \|\omega_0\|^2 + (2\alpha+\beta)\sum_{i=0}^{n} h_{i+1}\|\sigma(E)\omega_i\|^2 + \beta\sum_{i=0}^{n} h_{n+1}\|\sigma(E)\omega_{i-m}\|^2, \qquad n \ge 0. \tag{3.9}$$

Moreover, (2.2) leads to

$$\begin{aligned}\sum_{i=0}^{n} h_{i+1}\|\sigma(E)\omega_{i-m}\|^2 &= \sum_{i=-m}^{n-m} h_{m+i+1}\|\sigma(E)\omega_i\|^2\\ &= \frac{1}{p}\sum_{i=-m}^{n-m} h_{i+1}\|\sigma(E)\omega_i\|^2\\ &\le \frac{1}{p}\left(\sum_{i=0}^{n} h_{i+1}\|\sigma(E)\omega_i\|^2 + \sum_{i=-m}^{-1} h_{i+1}\|\sigma(E)\omega_i\|^2\right)\\ &\le \frac{1}{p}\left(\sum_{i=0}^{n} h_{i+1}\|\sigma(E)\omega_i\|^2 + (1-p)\gamma\max_{-m\le i\le -1}\|\sigma(E)\omega_i\|^2\right)\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9) and using conditions $p \in (0,1)$ and $\beta \le -p\alpha$, yield that

$$\begin{aligned}\|\omega_{n+1}\|^2 &\le \|\omega_0\|^2 + \frac{2}{p}(p\alpha+\beta)\sum_{i=0}^{n} h_{i+1}\|\sigma(E)\omega_i\|^2 + \frac{\beta(1-p)\gamma}{p}\max_{-m\le i\le -1}\|\sigma(E)\omega_i\|^2\\ &\le \|\omega_0\|^2 + \frac{\beta(1-p)\gamma}{p}\max_{-m\le i\le -1}\|\sigma(E)\omega_i\|^2, \qquad n \ge 0,\end{aligned} \tag{3.11}$$

which implies that

$$\|\omega_{n+1}\| \le \|\omega_0\| + \sqrt{\frac{\beta(1-p)\gamma}{p}}\max_{-m\le i\le -1}\|\sigma(E)\omega_i\|, \qquad n \ge 0. \tag{3.12}$$

In addition, note that

$$\|\sigma(E)\omega_i\| = \|\theta\omega_{i+1} + (1-\theta)\omega_i\| \le \max\{\|\omega_{i+1}\|, \|\omega_i\|\}. \tag{3.13}$$

Combining (3.12) with (3.13) yields (3.5). This concludes the proof.

The following studies further the asymptotic stability of method (2.1). For simplicity, continue the notations introduced in the proof of Theorem 3.1.

THEOREM 3.2. *Suppose $\theta \in (1/2, 1]$ and $\beta < -p\alpha$. Then the one-leg θ-method (2.1) is asymptotically stable for the class $D_p(\alpha, \beta)$, i.e., $\lim_{n\to\infty} \|y_n - z_n\| = 0$.*

PROOF. It follows from the first inequality of (3.11) that

$$-\frac{2}{p}(p\alpha + \beta)\sum_{i=0}^{n} h_{i+1}\|\sigma(E)\omega_i\|^2 \le \|\omega_0\|^2 + \frac{\beta(1-p)\gamma}{p}\max_{-m\le i\le -1}\|\sigma(E)\omega_i\|^2. \tag{3.14}$$

Further, (3.14) and $\beta < -p\alpha$ imply that

$$\lim_{i\to\infty}\sqrt{h_{i+1}}\|\sigma(E)\omega_i\| = 0.$$

Whereas it follows from (2.2) that $\lim_{i\to\infty} 1/h_{i+1} = 0$. Hence,

$$\lim_{i\to\infty}\|\sigma(E)\omega_i\| = \lim_{i\to\infty}\frac{1}{\sqrt{h_{i+1}}}\lim_{i\to\infty}\sqrt{h_{i+1}}\|\sigma(E)\omega_i\| = 0,$$

which implies for all $\epsilon > 0$ that there is an $l > 0$ such that

$$\|\sigma(E)\omega_i\| < \epsilon, \qquad i \ge l. \tag{3.15}$$

On the other hand, by

$$\sigma(E)\omega_n = \theta\omega_{n+1} + (1-\theta)\omega_n,$$

it follows that

$$\omega_{n+1} = -\frac{1-\theta}{\theta}\omega_n + \frac{1}{\theta}\sigma(E)\omega_n. \tag{3.16}$$

An induction argument applied to (3.16) yields

$$\omega_n = \left(-\frac{1-\theta}{\theta}\right)^{n-l}\omega_l + \sum_{i=l}^{n-1}\left(-\frac{1-\theta}{\theta}\right)^{n-i-1}\sigma(E)\omega_i, \qquad l > 0. \tag{3.17}$$

Since $\theta \in (1/2, 1]$, then $|(1-\theta)/\theta| < 1$. Thus, there exists an N with $N > l$ such that

$$\left|\frac{1-\theta}{\theta}\right|^{n-l} < \epsilon, \qquad n > N. \tag{3.18}$$

A combination of (3.15), (3.17), and (3.18) yields

$$\|\omega_n\| \le \left[\|\omega_l\| + \frac{1}{1-|(1-\theta)/\theta|}\right]\epsilon, \qquad n > N.$$

This implies that

$$\lim_{n\to\infty}\|y_n - z_n\| = 0.$$

Hence, the proof is completed.

4. CONCLUSION

This paper deals with the discrete dynamics of the one-leg θ methods for nonlinear infinite-delay-differential equations of the class $D_p(\alpha, \beta)$ and gives some new nonlinear stability criteria. In the future, we intend to work on effective implementation and error analysis of the presented algorithms. Moreover, we are also interested in the following open problems (proposed by Professor Feldstein):

(1) what happens when there are multiple delay;
(2) what happens for systems where each component of the solution could have a different delay;
(3) what can be said when the delay pt is replaced by a variable delay function $\alpha(t)$;
(4) what can be said when the delay function is state dependent, as in $\alpha(t, y(t))$.

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