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Existence of perfect Mendelsohn designs with $k = 5$ and $\lambda > 1$

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Abstract

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Let v, k, and λ be positive integers. A (v, k, λ)-*Mendelsohn design* (briefly (v, k, λ)-MD) is a pair $(X, \mathcal{B})$ where X is a v-set (of *points*) and $\mathcal{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly λ blocks of $\mathcal{B}$. A set of k distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \cdots < a_k < a_1$ and the pair a_i, a_{i+t} is said to be t-apart in cyclic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k. It for all $t = 1, 2, \ldots, k-1$, every ordered pair of points of X is t-apart in exactly λ blocks of $\mathcal{B}$, then the (v, k, λ)-MD is called a *perfect* design and is denoted briefly by (v, k, λ)-PMD. In this paper, we shall be concerned mainly with the case where $k = 5$ and $\lambda > 1$. It will be shown that the necessary condition for the existence of a $(v, 5, \lambda)$-PMD, namely, $\lambda v(v-1) \equiv 0 \pmod 5$, is also sufficient for $\lambda > 1$ with the possible exception of pairs (v, λ) where $\lambda = 5$ and $v = 18$ and 28.

1. Introduction

The notion of a perfect cyclic design was introduced by N.S. Mendelsohn [12]. This concept was further developed and studied in subsequent papers by various

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authors (see, for example, [1–6, 10, 11, 18]). In what follows, we shall adapt the terminology and notation of [8, 10], where the designs have been called Mendelsohn designs.

A set of k distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \cdots < a_k < a_1$ and the pair a_i, a_{i+t} is said to be *t-apart* in a cyclic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k.

Let v, k and λ be positive integers. A *(v, k, λ)-Mendelsohn design* (briefly (v, k, λ)-MD) is a pair $(X, \mathcal{B})$ where X is a v-set (of *points*) and $\mathcal{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly λ blocks of $\mathcal{B}$. The (v, k, λ)-MD is called *r-fold perfect* if each ordered pair of points of X appears t-apart in exactly λ blocks for all $t = 1, 2, \ldots, r$. A $(k-1)$-fold perfect (v, k, λ)-MD is called *perfect* and is denoted briefly by (v, k, λ)-PMD. It is perhaps worth mentioning that a (v, k, λ)-MD is equivalent to the decomposition of the complete directed multigraph λK_v^* on v vertices into k-circuits and that a (v, k, λ)-PMD is equivalent to the decomposition of λK_v^* into k-circuits such that for any r, $1 \le r \le k-1$ and for any pair $(x, y) \in X \times X$ there are exactly λ circuits along which the distance from x to y is r.

It is easy to show that the number of blocks in a (v, k, λ)-PMD is $\lambda v(v-1)/k$, and hence an obvious necessary condition for its existence is $\lambda v(v-1) \equiv 0 \pmod{k}$. It is known [1, 13] that the necessary condition for the existence of a $(v, 3, \lambda)$-PMD is sufficient, except for $v = 6$ and $\lambda = 1$. It is also known [2, 6, 18] that the necessary condition for the existence of a $(v, 4, \lambda)$-PMD is sufficient except for $v = 4$ and λ odd, $v = 8$ and $\lambda = 1$, and possibly excepting $v = 12$ and $\lambda = 1$. For practical purposes, the necessary condition for the existence of a $(v, 5, \lambda)$-PMD can be reduced to the following.

Lemma 1.1. *A necessary condition for the existence of a* $(v, 5, \lambda)$-PMD *is*
(1) $v \equiv 0$ *or* $1 \pmod 5$ *for* $\lambda \not\equiv 0 \pmod 5$,
(2) $v \ge 5$ *for* $\lambda \equiv 0 \pmod 5$.

The problem of existence of a $(v, 5, 1)$-PMD was recently investigated in [5], where an almost complete solution was presented in the form of the following theorem.

Theorem 1.2. *A* $(v, 5, 1)$-PMD *exists for every positive integer* $v \equiv 0$ *or* $1 \pmod 5$ *with the exception of* $v = 6$ *and the possible exception of* $v = 10, 15, 20, 26, 30, 36, 46, 50, 56, 66, 86, 90, 110, 126, 130, 140, 146, 186, 206, 246$ *and* 286.

In this paper, we shall investigate the existence of a $(v, 5, \lambda)$-PMD where $\lambda > 1$, and show that the necessary condition in Lemma 1.1 is also sufficient except possibly when $\lambda = 5$ and $v = 18$ and 28.

2. Preliminaries

In order to establish our main result, we shall employ both direct and recursive constructions. Our recursive construction will involve product constructions and the motion of pairwise balanced designs (PBDs), which we briefly describe below. For more information on PBDs and auxiliary designs such as mutually orthogonal Latin squares (MOLS), the interested reader is referred to [7, 9, 16].

Let K be a set of positive integers. A *pairwise balanced design* (PBD) of index λ, $B(K, \lambda; v)$ is a pair $(X, \mathscr{B})$ where X is a v-set (of *points*) and $\mathscr{B}$ is a collection of subsets of X (called *blocks*) with sizes from K such that every pair of distinct points of X is contained in exactly λ blocks of $\mathscr{B}$. We shall denote by $B(K, \lambda)$ the set of all integers v for which there exists a PBD $B(K, \lambda; v)$. A $B(K, 1)$ will be denoted simply by $B(K)$. A PBD $B(\{k\}, \lambda; v)$ is essentially a *balanced incomplete block design* (BIBD) with parameters v, k and λ.

The following recursive construction is a general form of Theorem 2.9 in [12].

Lemma 2.1. *Let v, k, λ_1 and λ_2 be positive integers. Suppose there exists a* PBD $B(\{k_1, k_2, \ldots, k_r\}, \lambda_1; v)$ *and for each k_i there exists a (k_i, k, λ_2)*-PMD, *then there exists a $(v, k, \lambda_1\lambda_2)$*-PMD.

We shall also make use of the following obvious result.

Lemma 2.2. *If a (v, k, λ_1)*-PMD *and a (v, k, λ_2)*-PMD *exist, then there exists a $(v, k, m\lambda_1 + n\lambda_2)$*-PMD, *where m and n are nonnegative integers.*

For the most part, our direct method of construction will be a variation of the method using difference sets in the construction of BIBDs (see, for example, [7]). Instead of listing all of the blocks of a design, it suffices to give the group G acting on a set of base blocks. We shall adapt the following notation:

$$\text{dev}\, \mathscr{B} = \{B + g \colon B \in \mathscr{B} \text{ and } g \in G\},$$

where $\mathscr{B}$ is the collection of base blocks of the design.

Lemma 2.3. *If there exist $k-2$ idempotent* MOLS(n), *then there exists an (n, k, k)*-PMD. *In particular, if n is a prime power and $n \geqslant k$, then there exists an (n, k, k)*-PMD.

Proof. For $1 \leqslant i \leqslant k-2$, let $(Q, *_i)$ be the quasigroups corresponding to the $k-2$ idempotent MOLS(n) where $Q = \{1, 2, \ldots, n\}$. Let $\mathscr{B}$ be the following collection of blocks:

$$\mathscr{B} = \{(i, j, i *_1 j, i *_2 j, \ldots, i *_{k-2} j) \mid 1 \leqslant i, j \leqslant n, i \neq j\}.$$

Then it is readily checked that $(Q, \mathcal{B})$ is a (n, k, k)-PMD. Moreover, it is fairly well-known that if n is a prime power and $k \leqslant n$, then there exist $k-2$ idempotent MOLS(n), and so the conclusion follows. □

Lemma 2.4. *If $q = kn + 1$ is a prime power, then there exists a $(q+1, k, k)$-PMD.*

Proof. Suppose d is a primitive root of GF(q). Take $X = \mathrm{GF}(q) \cup \{\infty\}$. Let $\mathcal{B}$ consist of the following blocks:

$$(z, zd^n, zd^{2n}, \ldots, zd^{(k-1)n}), \quad z \in \mathrm{GF}(q)\backslash\{0, 1, d^n, \ldots, d^{(k-1)n}\},$$
$$(1, d^n, d^{2n}, \ldots, d^{(k-1)n}), \quad \text{twice},$$
$$(\infty, z, zd^n, \ldots, zd^{(k-2)n}), \quad z \in \{1, d^n, \ldots, d^{(k-1)n}\}.$$

It is readily checked that $(X, \mathrm{dev}\, \mathcal{B})$ is the required $(q+1, k, k)$-PMD. □

We shall make use of the following lemma (see [5]).

Lemma 2.5. *For any integer $n \geqslant 5$, $n \notin \{6, 10, 18, 22, 26, 28\}$, there exist* 3 *idempotent* MOLS(n).

3. The construction of $(v, 5, \lambda)$-PMD, $\lambda \not\equiv 0 \pmod 5$

In this section we shall show that the necessary condition for the existence of a $(v, 5, \lambda)$-PMD for $\lambda > 1$ and $\lambda \not\equiv 0 \pmod 5$, namely, $v \equiv 0$ or $1 \pmod 5$, is also sufficient. In view of Lemma 2.2, we need only establish the result for the cases $\lambda = 2$ and $\lambda = 3$. The following known result (see, for example, [7, IX(7.6.d); 16, Proposition 5.2]) will be very useful here.

Lemma 3.1. *For any positive integer $v \equiv 0$ or $1 \pmod 5$, $v \in B(\{5, 6, 10, 11, 15, 16, 20, 35, 40\})$.*

Combining Lemma 2.1 and Lemma 3.1 the existence problem now can be reduced to a finite number of cases, some of which are obvious from Theorem 1.2.

Lemma 3.2. *If $\lambda = 2$ or 3, then there exists a $(v, 5, \lambda)$-PMD where $v = 5$, 11, 16, 35, 40.*

Proof. Apply Lemma 2.2 with the result in Theorem 1.2. □

We are now in a position to prove the following.

Theorem 3.3. *A $(v, 5, 2)$-PMD exists for any positive integer $v \equiv 0$ or $1 \pmod 5$.*

Proof. From Lemma 3.1 and Lemma 3.2, we need only prove the result when $v = 6, 10, 15, 20$.

For $v = 6$, let $G = \mathbb{Z}_5$ and $X = \mathbb{Z}_5 \cup \{\infty\}$. Let $\mathscr{B}_1$ be the following blocks:

$$\mathscr{B}_1 = \{(0, 1, 2, 3, 4), (0, 4, 3, 2, 1)\}.$$

Let $\mathscr{B}_2$ be the following base blocks:

$$\mathscr{B}_2 = \{(\infty, 0, 2, 3, 1\}, (\infty, 0, 2, 1, 4)\}.$$

It is readily checked the $(X, \mathscr{B}_1 \cup \text{dev}\, \mathscr{B}_2)$ is a $(6, 5, 2)$-PMD.

For the remaining three cases, we let G be the cyclic group $\mathbb{Z}_{v-1} = \{0, 1, \ldots, v-2\}$ and $X = \mathbb{Z}_{v-1} \cup \{\infty\}$. We then present a collection of base blocks $\mathscr{B}$, and it is readily checked that $(X, \text{dev}\, \mathscr{B})$ is the required $(v, 5, 2)$-PMD. The last construction is due to Wu [17].

(1) $v = 10$, $G = \mathbb{Z}_9$,

$$\mathscr{B} = \{(0, 1, 4, 2, 5), (0, 5, 2, 4, 1), (\infty, 0, 1, 3, 7), (\infty, 0, 5, 3, 2)\}.$$

(2) $v = 15$, $G = \mathbb{Z}_{14}$,

$$\mathscr{B} = \{(0, 1, 2, 11, 9), (0, 12, 1, 4, 10), (0, 13, 12, 7, 9), (0, 11, 13, 9, 3), (\infty, 0, 6, 13, 9), (\infty, 0, 4, 11, 5)\}.$$

(3) $v = 20$, $G = \mathbb{Z}_{19}$,

$$\mathscr{B} = \{(0, 1, 16, 7, 15), (0, 12, 18, 2, 14), (0, 11, 1, 5, 4), (0, 7, 4, 17, 5), (0, 3, 4, 9, 11), (0, 17, 12, 11, 8), (\infty, 0, 2, 11, 5), (\infty, 0, 6, 16, 14)\}. \quad \square$$

Theorem 3.4. *A* $(v, 5, 3)$-PMD *exists for any positive ingeger* $v \equiv 0$ *or* $1 \pmod 5$.

Proof. From Lemma 3.1 and Lemma 3.2, we need only consider the existence of a $(v, 5, 3)$-PMD for $v = 6, 10, 15, 20$.

For $v = 6$, let $G = \mathbb{Z}_5$ and $X = \mathbb{Z}_5 \cup \{\infty\}$. Let

$$\mathscr{B}_1 = \{(0, 1, 2, 3, 4), 3 \text{ times}\} \quad \text{and}$$

$$\mathscr{B}_2 = \{(\infty, 0, 2, 4, 1), (\infty, 0, 3, 1, 4), (\infty, 0, 4, 3, 2)\}.$$

It is readily checked that $(X, \mathscr{B}_1 \cup \text{dev}\, \mathscr{B}_2)$ is a $(6, 5, 3)$-PMD.

For the other three cases, we let $G = \mathbb{Z}_{v-1}$ and $X = \mathbb{Z}_{v-1} \cup \{\infty\}$. It is readily checked that $(X, \text{dev } \mathcal{B})$ is the required $(v, 5, 3)$-PMD where $\mathcal{B}$ is listed below.

(1) $v = 10$, $G = \mathbb{Z}_9$,

$$\mathcal{B} = \{(0, 1, 7, 3, 5), (0, 4, 1, 3, 2),$$
$$(0, 7, 4, 3, 8), (\infty, 0, 2, 5, 6),$$
$$(\infty, 0, 5, 8, 6), (\infty, 0, 8, 2, 6)\}.$$

(2) $v = 15$, $G = \mathbb{Z}_{14}$,

$$\mathcal{B} = \{(0, 5, 3, 11, 4), (0, 1, 2, 5, 6),$$
$$(0, 7, 13, 1, 10), (0, 13, 10, 9, 8),$$
$$(0, 3, 6, 8, 10), (0, 12, 10, 7, 4),$$
$$(\infty, 0, 5, 13, 3), (\infty, 0, 5, 12, 7),$$
$$(\infty, 0, 10, 2, 11)\}.$$

(3) $v = 20$, $G = \mathbb{Z}_{19}$,

$$\mathcal{B} = \{(0, 1, 3, 9, 10), (0, 2, 17, 13, 5),$$
$$(0, 7, 4, 17, 5), (0, 10, 18, 6, 16),$$
$$(0, 11, 15, 12, 2), (0, 12, 5, 10, 16),$$
$$(0, 15, 16, 1, 11), (0, 17, 16, 11, 3),$$
$$(0, 18, 5, 17, 6), (\infty, 0, 2, 11, 5),$$
$$(\infty, 0, 3, 8, 7), (\infty, 0, 4, 9, 7)\}. \quad \square$$

Applying Lemma 2.2 with the results in Theorem 3.3 and Theorem 3.4, we obtain the main theorem of this section.

Theorem 3.5. *For any given integer $\lambda > 1$ and $\lambda \not\equiv 0 \pmod 5$, there exists a $(v, 5, \lambda)$-PMD for any positive integer $v \equiv 0$ or $1 \pmod 5$.*

For the sake of completeness, we wish to remark that there are some useful generalizations of results contained in [5] which can be applied in the study of PMDs with $\lambda > 1$. While utilizing Lemmas 2.5, 3.1 and 3.2, the following constructions provide a proof of Theorem 3.5 with very few (5) exceptions. For the definitions of MOLS with holes (HMOLS) and a PMD with holes (HPMD), the reader is referred to [5]. We state the following obvious generalization of [5, Theorem 3.3].

Theorem 3.6. *Let k be an odd prime and λ be a positive integer. If there exist $k-2$ HMOLS(n) of type $(n_1, n_2, \ldots, n_h)$, then there exists a (kn, k, λ)-HPMD of type $(kn_1, kn_2, \ldots, kn_h)$.*

For the case $k = 5$, we have the corresponding generalizations of [5, Corollaries 3.5 and 3.6].

Corollary 3.7 (The $5n$-construction). *If there exist* 3 HMOLS(n) *of type* $(n_1, n_2, \ldots, n_h)$ *and a* $(5n_i, 5, \lambda)$-PMD *for* $1 \leq i \leq h$, *then there exists a* $(5n, 5, \lambda)$-PMD.

Corollary 3.8 (The $5n+1$-construction). *If there exist* 3 HMOLS(n) *of type* $(n_1, n_2, \ldots, n_h)$ *and a* $(5n_i + 1, 5, \lambda)$-PMD *for* $1 \leq i \leq h$, *then there exists a* $(5n+1, 5, \lambda)$-PMD.

4. The construction of $(v, 5, 5)$-PMD

In this section we establish the existence of a $(v, 5, 5)$-PMD for any integer $v \geq 5$, with the possible exception of 18 and 28.

Lemma 4.1. *There exists a* (22, 5, 5)-PMD.

Proof. Let $X = \mathbb{Z}_{21} \cup \{\infty\}$ and let $\mathcal{B}$ be the development of the following base blocks modulo 21.

$(\infty, 0, 4, 11, 3)$, $(\infty, 4, 11, 3, 9)$, $(\infty, 11, 3, 9, 0)$,
$(\infty, 3, 9, 0, 4)$, $(\infty, 9, 0, 4, 11)$, $(0, 1, 6, 8, 18)$ (5 times),
$(0, 15, 2, 16, 12)$ (5 times), $(0, 19, 14, 13, 10)$ (5 times),
$(0, 4, 11, 3, 9)$ (2 times).

It is readily checked that $(X, \mathcal{B})$ is a (22, 5, 5)-PMD. □

Theorem 4.2. *A* $(v, 5, 5)$-PMD *exists for any integer* $v \geq 5$, *with the possible exception of* $v = 18$ *and* 28.

Proof. If $v \in \{6, 10, 26\}$, then we combine the $(v, 5, 2)$-PMD and the $(v, 5, 3)$-PMD constructed in Theorem 3.5 to produce a $(v, 5, 5)$-PMD by Lemma 2.2. If $v = 22$, the result is obtained in Lemma 4.1. If $v \geq 5$ and $v \notin \{6, 10, 18, 22, 26, 28\}$, then we apply Lemmas 2.3 and 2.5 to obtain a $(v, 5, 5)$-PMD. This completes the proof of the theorem. □

5. Main result

To obtain our main results of this paper, we need only consider the existence of a $(v, 5, \lambda)$-PMD when $\lambda \geq 10$ and $\lambda \equiv 0 \pmod 5$. The problem can be reduced to the cases $\lambda = 10$ and 15 by using Lemma 2.2. Moreover, by Lemma 2.2 and the result in Theorem 4.2, the problem can further be reduced to the existence of a $(v, 5, 10)$-PMD and a $(v, 5, 15)$-PMD for $v = 18$ and 28. We shall make use of the following lemma, which is taken from [9, Lemma 5.33].

Lemma 5.1. *A* $(v, 6, 5)$-PBD *exists if* $v \equiv 0$ *or* $1 \pmod 3$ *and* $v \geq 6$.

We can now prove the following.

Lemma 5.2. *There exist a* $(v, 5, 10)$-PMD *and a* $(v, 5, 15)$-PMD *for* $v = 18$ *and* 28.

Proof. For $v = 18, 28$, there exists a $(v, 6, 5)$-PBD from Lemma 5.1. For $\lambda = 2$ or 3, a $(6, 5, \lambda)$-PMD exists from Theorem 3.3 and Theorem 3.4. We then apply Lemma 2.1 to get a $(v, 5, 10)$-PMD and a $(v, 5, 15)$-PMD. □

As an immediate consequence, we have the following.

Lemma 5.3. *There exist a* $(v, 5, 10)$-PMD *and a* $(v, 5, 15)$-PMD *for any integer* $v \geq 5$.

Proof. Combine the results in Theorem 4.2 and Lemma 5.2. □

Using Lemma 2.2, we have

Theorem 5.4. *There exists a* $(v, 5, \lambda)$-PMD *for any integer* $v \geq 5$ *when* $\lambda \equiv 0 \pmod 5$ *and* $\lambda \geq 10$.

Now, we come to the main result of this paper.

Theorem 5.5. *Let* v *and* λ *be positive integers satisfying* $\lambda v(v-1) \equiv 0 \pmod 5$ *and* $\lambda > 1$. *Then there exists a* $(v, 5, \lambda)$-PMD *except possibly when* $\lambda = 5$ *and* $v = 18$ *and* 28.

Proof. Combine Theorem 3.5, Theorem 4.2 and Theorem 5.4. □

6. Concluding remark

From Theorems 1.2 and 5.5, we conclude that the necessary condition for the existence of a $(v, 5, \lambda)$-PMD, namely, $\lambda v(v-1) \equiv 0 \pmod 5$, it also sufficient, except for $\lambda = 1$ and $v = 6$, and possibly excepting $\lambda = 1$ and 21 values of v shown in Theorem 1.2, and $\lambda = 5$ and $v \in \{18, 28\}$.

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Note added in proof. Recently, R.J.R. Abel [J. Combin. Theory Ser. A 58 (1991) 306–309] has established the existence of 4 MOLS(28). Consequently, the number 28 can now be removed from the list of possible exceptions in Lemma 2.5, Theorems 4.2 and 5.5.

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