Operators Commuting with Translation by One

Part I. Representation Theorems

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SUMMARY

Part I. Representation Theorems

If $A$ is an operator on $L_2(-\infty, \infty)$ which commutes with the unit translation operator $S$ defined by $(Sf)(t) = f(t + 1)$ then the behavior of $A$ is determined by its behavior on the interval $(0, 1)$. To exploit this fact it is desirable to have a representation of $A$ in terms of its behavior on $(0,1)$ and such a representation is developed in Part I.

Let $T_j, j$ integer, be defined by $T_j f(t) = f(t + j), t \in [0, 1), T_j f(t) = 0, t \notin [0, 1)$. A representation theorem, developed in Section V, establishes a one-to-one correspondence between bounded operators $A$ on $L_2(-\infty, \infty)$ for which $AS = SA$ and $L_2(0, 1)$ operator valued functions $A'(\theta), 0 \leq \theta \leq 2\pi$ denoted by $A \sim A'(\theta)$, such that for $f$ in $L_2(-\infty, \infty)$

$$Af = \lim_{N_1 \to \infty} \sum_{j=-N_1}^{N_1} \sum_{k=-N_3}^{N_3} \frac{1}{2\pi} T_j \int_{\theta=0}^{2\pi} e^{i(\theta-k)} A'(\theta) T_k f d\theta.$$ 

It is shown that $|A| = \text{ess} \sup |A'(\theta)|$. For $f$ in $L_a(0, 1), A'(\theta) f$ is given by

$$A'(\theta) f = \sum_{k=-\infty}^{\infty} e^{-i\theta k} T_k Af$$

where the sense in which convergence is meant is made precise in Section V. If $A \sim A'(\theta), B \sim B'(\theta)$, then $AB \sim A'(\theta) B'(\theta)$. Necessary and sufficient conditions that $A$ be spectral in the sense of Dunford are given in terms of $A'(\theta)$. 

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Part II. Differential Operators with Periodic Coefficients

The application which prompted the investigation of the above representation is to the study of the spectral theory of differential operators with periodic coefficients. This application will be developed in Part II. In particular, by means of this representation sufficient conditions on the coefficients of a differential operator $\tau$ with coefficients of period 1 that the operator in $L_2(-\infty, \infty)$ defined by $\tau$ be spectral are obtained using perturbation methods of J. T. Schwartz and H. P. Kramer heretofore applicable only to operators with compact resolvents.

I. Introduction

The results presented here are the outgrowth of investigations in the spectral theory of differential operators with periodic coefficients having a common period which can be assumed to be one. Such operators on the infinite axis $(-\infty, \infty)$ and their inverses are distinguished by the fact that they commute with the operation $S$ of translation by one, $(Sf)(t) = f(t + 1)$.

The direct concern of this part is with those bounded operators in the Hilbert space $L_2(-\infty, \infty)$ which commute with $S$.

If $A$ is an operator operating on functions defined on $(-\infty, \infty)$ which commutes with $S$ (the terms "operator" and "commute" are used loosely here) then the behavior of $A$ is determined by its behavior on the interval $(0, 1)$. To exploit the fact that $AS = SA$ it is desirable to have methods of expressing the properties of $A$ on $(-\infty, \infty)$ in terms of its properties on $(0, 1)$. The classic example of such a tool is that due to Floquet who, for the case where $A$ is a formal differential operator with periodic coefficients, expressed any solution of $Af = 0$ on $(-\infty, \infty)$ in terms of the set of solutions of $Af = 0$ on $(0, 1)$.

If $A$ commutes with $S$ and is a bounded operator in $L_2(-\infty, \infty)$ it, and consequently its properties, are determined by the sequence of operators from $L_2(0, 1)$ to $L_2(j, j + 1), j = 0, \pm 1, \cdots$, defined by $\chi_{(j, j+1)}(t)(Af)(t), f \in L_2(0, 1)$ (where $\chi_E$ is the characteristic function of the set $E$ defined by $\chi_E(t) = 1, t \in E, \chi_E(t) = 0, t \notin E$), or equivalently by the sequence of operators from $L_2(0, 1)$ to $L_2(0, 1)$

$$A(j)f = S^j\chi_{(j, j+1)}Af, \quad f \in L_2(0, 1).$$

What is needed to give this statement more than superficial significance is an effective means of expressing the behavior of $A$ in terms of the behavior of

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1 To appear in a later issue of this journal.
the sequence $A(j)$. In Section V this is done by forming the $L_2(0, 1)$ operator valued function

$$A'(\theta) = \sum_{k=-\infty}^{\infty} e^{-i\theta k} A(k), \quad 0 \leq \theta \leq 2\pi,$$

where the sense in which convergence is to be taken is made precise in that section. The function $A'(\theta)$ is weakly (= strongly) measurable and $|A| = \text{ess sup} |A'(\theta)|$.

If $A'(\theta)$ is related to $A$ in the above fashion the notation $A \sim A'(\theta)$ is employed. There is also a converse result that if $A'(\theta)$ is a weakly measurable, $L_2(0, 1)$ operator valued function, $0 \leq \theta \leq 2\pi$, such that $\text{ess sup} |A'(\theta)| < \infty$ then there is a unique bounded operator $A$ such that $AS = SA$ and $A \sim A'(\theta)$.

An important property of the relation $A \sim A'(\theta)$ is that it is algebraic—if $B$ is another bounded operator such that $BS = SB$ and if $B'(\theta)$ is such that $B \sim B'(\theta)$ then $AB \sim A'(\theta) B'(\theta)$, the pointwise product of the functions $A'(\theta)$ and $B'(\theta)$. Since also $A^* \sim A'(\theta)^*$, it follows readily that $A$ is self-adjoint, normal, a projection, 0, or the identity if and only if the same is true of $A'(\theta)$ for almost all $\theta$, and that $A$ and $B$ commute if and only if $A'(\theta)$ and $B'(\theta)$ commute for almost all $\theta$.

A particular set of results of this nature relate the spectral reduction theory of $A$ to that of $A'(\theta)$, and a digression into spectral reduction theory is in order at this point.

For a survey of the theory of spectral operators the reader is referred to Dunford [1]. The basic definitions and theorems needed in this paper concerning spectral operators are stated below in Section II and only a brief introduction is presented here.

As is well known any self-adjoint or normal linear operator $T$ in a Hilbert space can be represented as $T = \int \lambda E(d\lambda)$ where $E(\delta)$ is a commuting self-adjoint projection valued measure defined on the field of Borel subsets of the complex plane $\mathbb{C}$ and such that $E(1) = 1$, the identity projection.

If $\sigma(T)$, the spectrum of $T$, is discrete this states that $T$ has an eigenvalue expansion $T = \sum_{\lambda \in \sigma(T)} \lambda E(\lambda)$ which is unconditionally convergent, and such that $\lambda_1 \neq \lambda_2$ implies $E(\lambda_1) E(\lambda_2) = 0$.

If $T$ is an operator in a finite dimensional space then this spectral representation is the statement in operator terminology of the fact that a suitable orthonormal base for the space can be found for which the matrix representation of $T$ is diagonal.

Suppose now that $T$ still operates in a finite dimensional space but is not necessarily normal (and hence not necessarily self-adjoint). Then it may no longer be possible to find a base in which the matrix representation of $T$ is diagonal, i.e., the Jordan Canonical Form for $T$ may involve off-diagonal
elements. Separating the Jordan Canonical Form into its diagonal and off-diagonal parts, it can readily be shown that, in operator terminology, $T$ has a representation as $T = S + N$ where $S$ has an eigenvalue expansion, $S = \sum_{\lambda \in \sigma(T)} \lambda E(\lambda)$, $N$ is a nilpotent of type less than or equal to the dimension of the space $n$, i.e., $N^n = 0$, and $S$ commutes with $T$. The basis for which the matrix representation of $T$ is in Jordan Canonical Form may not be orthogonal which means the projections $E(\lambda)$ may not be self-adjoint and hence may be of norm greater than 1, though still $\lambda_1 \neq \lambda_2$ implies $E(\lambda_1) \neq E(\lambda_2) = 0$.

Looking again at the infinite dimensional Hilbert space case, it is clear that if one wishes to generalize the theory of spectral representations to include nonnormal operators $T$ it will be necessary to accept projection valued measures $E(\delta)$ such that $|E(\delta)|$ may be greater than 1 and such that the difference $N = T - \int \lambda E(d\lambda)$ is not necessarily 0. Dunford in [2] has introduced the concept of spectral operators as a class for which a useful spectral reduction exists. A spectral operator, $T$, can be characterized as one for which there exists a projection valued measure $E(\delta)$ defined over the field of Borel subsets of the plane which is uniformly bounded in norm, i.e., $\sup_{\delta} |E(\delta)| < \infty$, such that $S = \int \lambda E(d\lambda)$ commutes with $T$, and such that $N = T - S$ is a generalized nilpotent, i.e., $\lim_{n \to \infty} |N^n|^{1/n} = 0$. The operator $N$ is called the radical part of $T$.

An extension of this theory to unbounded operators is to be found in Bade [3].

It will be shown in Section V below (Theorem 5.2) that if $A$ is a bounded operator in $L_2 (\mathbb{R}, \mathfrak{m})$ which commutes with $S$ and if $A'(\theta)$ is the $L_2 (0, 1)$ operator valued function such that $A \sim A'(\theta)$ then $A$ is a spectral operator with resolution of the identity $E(\delta)$, scalar part $B$, and radical part $N$ if and only if for almost all $\theta$, $A'(\theta)$ is a spectral operator with resolution of the identity $E'(\theta, \delta)$, scalar part $B'(\theta)$, and radical part $N'(\theta)$ which are all strongly measurable in $\theta$,

$$\sup_{\delta \in \mathfrak{B}^+} \sup_{\theta} |E'(\theta, \delta)| < \infty, \quad \text{and} \quad \lim_{n \to \infty} \sup_{\theta} |N''(\theta)|^{1/n} = 0.$$ 

Then $E(\delta) \sim E'(\theta, \delta)$, $B \sim B'(\theta)$, and $N \sim N'(\theta)$.

What makes these results useful in studying a differential operator $\tau$ of order $n$ with periodic coefficients is that if $A$ is the resolvent of such an operator (with suitable and natural domain of definition) then $A'(\theta)$ is the resolvent of the same operator on $[0, 1]$ subject to the boundary conditions

$$f^{(k)}(0) = e^{i\theta} f^{(k)}(1), \quad 0 \leq k \leq n - 1.$$ 

Consequently it is possible to study many of the properties of the singular operator $\tau$ in $L_2 (\mathbb{R}, \mathfrak{m})$, in particular its spectral theory, by studying the
properties of the regular differential operator $\tau$ in $L_2(0, 1)$ subject to a class of relatively simple boundary conditions. This application of the representation theory will be developed in Part II. By means of this representation sufficient conditions on the coefficients of $\tau$ that it define a spectral operator in $L_2(-\infty, \infty)$ are obtained using perturbation methods of J. T. Schwartz and H. P. Kramer heretofore applicable only to operators with compact resolvents. That the study of $\tau$ in $L_2(-\infty, \infty)$ is, after all, more difficult than that of $\tau$ in $L_2(0, 1)$ is manifested in the necessity of getting results "uniformly" in $\theta$.

The operator $S$ has its analogue in $L_2(-N, N)$, where $N$ is a positive integer, in the operation $S_N$ of translation by 1 modulo $2N$. In Section III a representation of bounded operators in $L_2(-N, N)$ that commute with $S_N$ is developed which is closely analogous to that developed in Section V for operators commuting with $S$. The development of Section III is much simpler that that of Section V due to the simpler nature of $S_N$. It is hoped that the more direct arguments of Section III will aid the reader, as they aided the author, in perceiving the pattern behind the technicalities of Section V.

Section II lists known results needed concerning operator and spectral theory. The main purpose of Section II is to make it possible in later sections to refer unambiguously to a result or definition from the literature without stating the material needed in the midst of a proof.

Those results from the theory of vector valued functions that are needed in Section V are developed in Section IV.

II. LINEAR OPERATORS AND SPECTRAL OPERATORS

For ease of reference some results concerning linear operators that are needed subsequently are collected here. For results concerning linear operators in general and the theory of Banach spaces reference is made when possible to the first volume of Dunford and Schwartz [4]. One of the central purposes in this paper is to investigate linear operators commuting with translation by 1 in the context of the theory of spectral operators developed in Dunford [2] and Bade [3]. A statement of the results needed from these works is also presented below. The few results stated that do not appear in [2-4], or are not immediate corollaries of results therein, are presented with proofs.

At the time of this writing the second volume of Dunford and Schwartz [5] had not appeared in print and all references to that work are to the prepublication manuscript. Consequently reference numbers and phraseology may not coincide exactly with those in the published text.

If $T$ is a (bounded or unbounded) linear operator, $D(T)$ denotes the domain
of definition of $T$; $\rho(T)$ denotes the resolvent set of $T$, i.e., the set of all complex numbers $\lambda$ such that $(\lambda I - T)$ has a bounded everywhere defined inverse; this inverse, called the resolvent of $\lambda I - T$, is denoted by $R(\lambda, T)$ or $(\lambda I - T)^{-1}$; $\sigma(T)$ denotes the spectrum of $T$, i.e., the complement of $\rho(T)$.

2.1-definition. A $B$-space (Banach space) $X$ is called weakly complete if for every sequence $\{x_n\}$ in $X$ such that $\lim_n x^* x_n$ exists, $x^*$ in $X^*$, there exists an element $x$ in $X$ such that $\lim_n x^* x_n = x^* x$, $x^*$ in $X^*$.

Note that if $\lim_n x^* x_n$ exists, $x^* \in X^*$, then there is at most one $x$ such that $\lim_n x^* x_n = x^* x$, $x^* \in X^*$.

2.2 THEOREM. A reflexive space is weakly complete [4, Corollary II.3.29].

2.3 DEFINITION. Let $X$ and $Y$ be $B$-spaces. By $B(X, Y)$ will be meant the $B$-space of bounded linear operators $T$ with domain $X$ and range $Y$ and with norm $|T| = \sup_{x \in X} |T x|/|x|$. The symbol $R(X)$ will be used for $R(X, X)$.

The following is a special case of the uniform boundedness principle.

2.4 THEOREM. Let $X$ and $Y$ be $B$-spaces and $\{T_n\}$ a sequence of bounded linear operators on $X$ to $Y$. Then the limit $T x = \lim_n T_n x$ exists for every $x$ in $X$ if, and only if:

(i) the limit $T x$ exists for every $x$ in a fundamental set, and

(ii) for each $x$ in $X$ the supremum $\sup_n |T_n x| < \infty$.

When the limit $T x$ exists for each $x$ in $X$, the operator $T$ is bounded, and

$$|T| \leq \liminf_{n \to \infty} |T_n| \leq \sup_n |T_n| < \infty$$

[4, Theorem II.3.6].

A corollary to the uniform boundedness principle which will often be used follows.

2.5 COROLLARY. Let $X$ be a weakly complete $B$-space and let $\{T_n\}$ be a sequence of bounded operators on $X$ such that $\sup_n |T_n| < \infty$. If there exists a fundamental set $F$ of $X$ and a fundamental set $G$ of $X^*$ such that $\lim x^* T_n x$ exists for $x \in F$ and $x^* \in G$ then $\lim x^* T_n x$ exists for all $x \in X$ and $x^* \in X^*$ and there is a bounded linear operator $T$, $|T| \leq \liminf_{n \to \infty} |T_n|$, such that $\lim_n x^* T_n x = x^* T x$ for $x \in X$, $x^* \in X^*$.

PROOF. Let $x \in \text{span} F$ and consider the operators $S_n$ from $X^*$ to the field of scalars defined by $S_n x^* = x^* T_n x$. The hypotheses are such that the uniform boundedness principle may be applied to these operators to conclude that $\lim_n x^* T_n x$ exists for every $x^*$ in $X^*$. Let $x^* \in X^*$ and consider the
operators $R_n$ from $X$ to the field of scalars defined by $R_n x = x^* T_n x$. The operators $R_n$ are uniformly bounded in norm and, as was shown above, $\lim_n R_n x$ exists for $x$ in $\text{span} F$. Applying the uniform boundedness principle to $R_n$ yields the conclusion that $\lim_n x^* T_n x$ exists for every $x$ in $X$ and $x^*$ in $X^*$. Since $X$ is weakly complete this implies that for every $x$ in $X$ there exists an element $Tx$ in $X$ such that $x^* T_n x = x^* Tx, x^*$ in $X^*$. It follows readily that $T$ is a linear operator on $X$. Since $|x^* T x| = \lim_n |x^* T_n x| \leq \lim_n \inf |T_n| |x^*| |x|$, $T$ is bounded and $|T| \leq \lim_n \inf |T_n|$.

2.6 Theorem. If $T$ is a closed operator with nonempty resolvent set and $P(\lambda)$ a polynomial, then $P(\sigma(T)) = \sigma(P(T))$. [4, Theorem VII.9.10.]

2.7 Definition. A bounded operator $T$ on a $B$-space $X$ is quasi-nilpotent if $\lim_{n \to \infty} |T^n|^{1/n} = 0$.

The following theorem follows from Lemma VII.3.4 of [4].

2.8 Theorem. A bounded operator $T$ is quasi-nilpotent if and only if $\sigma(T) = \{0\}$.

Definitions and properties from the theory of operator valued measures that will be needed are developed below.

2.9 Definition. By a ring is meant a nonempty class of sets closed under the formation of unions and differences and by a $\sigma$-ring is meant a ring closed under the formation of countable unions. A field is a ring which contains the complement of every one of its members. If $R$ is a ring $S(R)$ denotes the smallest $\sigma$-ring containing $R$. A monotone class $M$ is a nonempty class of sets such that if $\{\delta_i\}_{i=1}^\infty$ is a sequence of sets in $M$ for which $\delta_i \subseteq \delta_{i+1}$ ($\delta_i \supseteq \delta_{i+1}$), $1 \leq i < \infty$, then

$$\bigcup_i \delta_i \in M \quad \left( \bigcap_i \delta_i \in M \right).$$

2.10 Theorem. Any monotone class containing a ring $R$ contains $S(R)$ [Halmos 6, p. 27].

2.11 Definition. Let $X$ be a Banach space and let $\Gamma$ be a total subspace of $X^*$. A mapping $P$ from a ring $R$ to $B(X)$ is a $\Gamma$-countably additive measure if for every $x \in X$ and $x^* \in \Gamma$, $x^* P(\cdot) x$ is a complex scalar valued countably additive measure on $R$ as defined in the usual way (see Halmos [6]). If, furthermore, for every disjoint sequence of sets $\{E_i\}_{i=1}^\infty$ in $R$ whose union is also in $R$,

$$\lim_{n \to \infty} P \left( \bigcup_{i=1}^n E_i \right) x = P \left( \bigcup_{i=1}^\infty E_i \right) x,$$

the limit on the left being in the norm topology, for every $x$ in $X$ then $P$ is strongly countably additive. The mapping $P$ is uniformly bounded with norm
M if $\sup_{\delta \in R} | P(\delta) | = M < \infty$. If $\Gamma = X^*$, $P(\cdot)$ is called \textit{weakly countably additive}. If $X \rightarrow Y^*$ for some $B$-space $Y$ and if $\Gamma$ is the natural imbedding of $Y$ into $Y^{**} = X^*$ then $P(\cdot)$ is called $Y$-\textit{countably additive}.

The following theorem of B. J. Pettis will be used frequently. In the reference it is assumed that $R$ is a $\sigma$-field. An examination of the proof reveals however that only the ring properties of $R$ are needed.

\textbf{2.12 Theorem.} A weakly countably additive measure from a ring $R$ to $B(X)$ is strongly countably additive [4, Theorem IV.10.1].

\textbf{2.13 Theorem.} If $R$ is a field then a weakly countably additive measure from a ring $R$ to $B(X)$ is uniformly bounded [4, Corollary IV.10.2].

\textbf{2.14 Theorem.} Let $R$ be a ring, $X$ a weakly complete $B$-space, $P_0$ a uniformly bounded weakly countably additive $B(X)$ valued measure on $R$ and $M$ the norm of $P_0$. There is a unique weakly countably additive measure $P(\cdot)$, $S(R)$ to $B(X)$ such that $P(\delta) = P_0(\delta)$, $\delta \in R$. Furthermore $P$ has the same norm $M$ as $P_0$, and $P$ is strongly countably additive.

\textbf{Proof.} The ordinary extension theory for scalar valued complex countably additive measures implies that for every $x$ in $X$ and $x^*$ in $X^*$ the measure $x^*P_0(\cdot)x$ has a unique extension to a complex measure on $S(R)$ which we denote by $u(x^*, x, \cdot)$. Furthermore the theory of scalar valued measures tells us that

$$\sup_{\delta \in R} | x^*P_0(\delta)x | = \sup_{\delta \in S(R)} | u(x^*, x, \delta) |.$$ 

From the uniqueness of this extension it follows that for $\delta \in S(R)$, $u(\cdot, \cdot, \delta)$ is a bilinear form. Furthermore since

$$| u(x^*, x, \delta) | \leq \sup_{\delta \in R} | x^*P_0(\delta)x | \leq \sup_{\delta \in R} | P_0(\delta) | \cdot | x^* | \cdot | x | ,$$

$u(\cdot, \cdot, \delta)$ is a bounded bilinear form and in particular for every $x^*$ in $X^*$, $u(x^*, \cdot, \delta)$ is a bounded linear functional on $X$ which we denote by $P'(\cdot)x^*$. There is no difficulty in establishing that the mapping $P'(\delta)$ of $X^*$ into $X^*$ is a bounded linear transformation with norm not exceeding $\sup_{\delta \in R} | P_0(\delta) | = M$. Since $u(x^*, x, \cdot) = xP'(\cdot)x^*$ is a scalar valued countably additive measure for all $x^*$ in $X^*$ and $x$ in $X$ we know that $P'(\cdot)$ is $X$-countably additive. Since $xP'(\cdot)x^*$ extends $xP_0(\cdot)x^*$, $P'(\cdot)$ extends $P_0(\cdot)*$. Let $M$ be the class of all $\delta \in S(R)$ for which $P(\delta)$ is the adjoint of an operator which we call $P(\delta)$. Let $\{\delta_i\}$ be a monotone set in $M$ and let $\delta = \lim_i \delta_i$. Then $xP(\delta)x^* = \lim_i xP(\delta_i)x^* = \lim_i x^*P(\delta_i)x$, all $x^* \in X^*$. Since $X$ is weakly complete, it follows that for each $x$ in $X$ there is an element which we denote by $Q(\delta)x$ in $X$ such that $x^*Q(\delta)x = \lim_i x^*P(\delta_i)x$, $x^* \in X^*$. The mapping $x \rightarrow Q(\delta)x$
is a bounded linear transformation which we denote by \( Q(\delta) \) and it is clear that \( Q(\delta)^* = P'(\delta) \), and hence using the notational convention adopted above, \( P(\delta) \) exists and \( P(\delta) = Q(\delta) \). Thus \( M \) is a monotone class and hence \( M = S(R) \). Since \( P'(\cdot)^* = P'(\cdot) \) is \( \mathcal{X} \)-countably additive, \( P(\cdot) \) is a weakly countably additive extension of \( P(\cdot) \). Hence by Theorem 12 \( P(\cdot) \) is a strongly countably additive extension of \( P(\cdot) \). Q.E.D.

The content of the next lemma is that there is no difference between a commutative projection valued measure and a Boolean ring of measures. The lemma is purely algebraic.

2.15. LEMMA. Let \( Y \) be a linear space and \( Q \) a ring. Let \( E(\cdot) \) be a function from \( Q \) to the set of all projections on \( Y \) which are defined for all \( \gamma \) in \( Y \). Assume that if \( \delta_1 \) and \( \delta_2 \in Q \) then \( E(\delta_1) E(\delta_2) = E(\delta_2) E(\delta_1) \) and that if \( \delta_1 \cap \delta_2 = \Phi \) then \( E(\delta_1 \cup \delta_2) = E(\delta_1) + E(\delta_2) \). Then for \( \delta_1, \delta_2 \in Q \), \( E(\delta_1 \cap \delta_2) = E(\delta_1) E(\delta_2) \).

PROOF. Assume \( \delta_1, \delta_2 \in Q, \delta_1 \cap \delta_2 = \Phi \). Then \( E(\delta_1) + E(\delta_2) = E(\delta_1 \cup \delta_2) \) is a projection, i.e., \( E(\delta_1) + E(\delta_2) = (E(\delta_1) + E(\delta_2))^2 = E(\delta_1) + E(\delta_2) + 2E(\delta_1) E(\delta_2) \) and thus

\[
E(\delta_1) E(\delta_2) = 0, \delta_1, \delta_2 \in Q, \delta_1 \cap \delta_2 = \Phi. \quad (1)
\]

An arbitrary \( \delta_1 \) and \( \delta_2 \) in \( Q \) may be written as

\[
\delta_1 = (\delta_1 \cap \delta_2) \cup (\delta_1 - \delta_2)
\]
\[
\delta_2 = (\delta_1 \cap \delta_2) \cup (\delta_2 - \delta_1)
\]

where the decompositions are disjoint. Hence

\[
E(\delta_1) = E(\delta_1 \cap \delta_2) + E(\delta_1 - \delta_2)
\]
\[
E(\delta_2) = E(\delta_2 \cap \delta_2) + E(\delta_2 - \delta_1)
\]

and

\[
E(\delta_1) E(\delta_2) = E(\delta_1 \cap \delta_2)^2 + E(\delta_1 \cap \delta_2) [E(\delta_1 - \delta_2) + E(\delta_2 - \delta_1)]
\]

\[
+ E(\delta_1 - \delta_2) E(\delta_2 - \delta_1).
\]

Now

\[
(\delta_1 \cap \delta_2) \cap (\delta_1 - \delta_2) = (\delta_1 \cap \delta_2) (\delta_2 - \delta_1) = (\delta_1 - \delta_2) \cap (\delta_2 - \delta_1) = \Phi
\]

so, by using (1),

\[
E(\delta_1) E(\delta_2) = E(\delta_1 \cap \delta_2)^2 = E(\delta_1 \cap \delta_2). \quad \text{Q.E.D.}
\]
2.16 Theorem. Using the notation and hypotheses of Theorem 14, if \( P_0(\cdot) \) maps \( R \) into a commuting family of operators then \( P(\cdot) \) maps \( S(R) \) into a commuting family of operators. If \( P_0 \) maps \( R \) into a family of projections then \( P(\cdot) \) maps \( S(R) \) into a family of projections. If \( X \) is a Hilbert space and \( P_0(\delta) \) is self-adjoint, \( \delta \in R, \) then \( P(\delta) \) is self-adjoint, \( \delta \in S(R). \)

Proof. Suppose that \( P_0(\cdot) \) maps \( R \) into a commuting family of operators. Let \( M \) be the class of all sets \( \delta \) in \( S(R) \) such that \( P(\delta)P(\delta) = P(\delta)P(\delta). \) Let \( \{\delta_n\} \) be a monotone sequence of sets in \( M \) and let \( \delta = \lim_n \delta_n. \) Then for any \( \delta_0 \) in \( R \) and \( x \) in \( X, \) \( P(\delta_0)P(\delta_0)x = P(\delta_0)P(\delta_0)x \) and thus \( P(\delta)P(\delta_0)x = \lim_n P(\delta_n)P(\delta_0)x = \lim_n P(\delta_0)P(\delta_n)x \) so \( P(\delta)P(\delta_0) = P(\delta_0)P(\delta) \) and \( M \) is a monotone class. Since \( M \) contains \( R \) by hypothesis, \( M \) contains \( S(R). \) Thus \( P(\delta)P(\delta_0) = P(\delta_0)P(\delta), \delta \in S(R), \delta_0 \in R. \)

Assume now that \( P_0(\cdot) \) maps \( R \) into a family of projections and let \( M \) be the class of all sets \( \delta \in S(R) \) such that \( P(\delta) \) is a projection. Let \( \{\delta_n\} \) be a monotone sequence of elements in \( M \) and let \( \delta = \lim_n \delta_n. \) Then for \( x \) in \( X, \)

\[
(P(\delta) - P(\delta_n))x = (P(\delta) - P(\delta_n))P(\delta)x + P(\delta_n)(P(\delta) - P(\delta_n))x.
\]

The first term on the right converges to 0 since \( P(\cdot) \) is strongly countably additive. In the second term, \( (P(\delta) - P(\delta_n))x \to 0 \) for the same reason and, since \( |P(\delta_n)| \) is uniformly bounded, the whole second term also converges to 0. Thus \( P(\delta)x = \lim_n P(\delta_n)x = \lim_n P(\delta_0)x = P(\delta)x, \) i.e., \( P(\delta) \) is a projection and hence \( M \) is a monotone class containing \( R \) by hypothesis and thus containing \( S(R). \)

Suppose that \( X \) is a Hilbert space, let \( M \) be the class of all \( \delta \) in \( S(R) \) such that \( P(\delta) \) is self-adjoint, and assume \( R \subset M. \) Let \( \{\delta_i\} \) be a monotone sequence in \( M \) with limit \( \delta. \) Then for \( x \) in \( X, \) \( P(\delta)x = \lim_i P(\delta_i)x = \lim_i P^*(\delta_i)x = P^*(\delta)x, \) i.e., \( P(\delta) = P(\delta)^* \). Thus \( M = S(R). \) Q.E.D.

2.17 Corollary. Using the notation and hypotheses of Theorem 14, if, for every \( \delta_1 \) and \( \delta_2 \) in \( R, \) \( P_0(\delta_1)P_0(\delta_2) = P_0(\delta_1 \cap \delta_2), \) then the same is true of \( P(\cdot), \) i.e., \( P(\delta_1)P(\delta_2) = P(\delta_1 \cap \delta_2), \) \( \delta_1, \delta_2 \in S(R). \)

Proof. It is trivial to verify that the hypothesis implies that \( P_0(\cdot) \) maps \( R \) into a commuting family of projections and thus by Theorem 16 \( P(\cdot) \) maps \( S(R) \) into a commuting family of projections. That \( P(\delta_1)P(\delta_2) = P(\delta_1 \cap \delta_2), \) \( \delta_1, \delta_2 \in S(R) \) now follows from Lemma 15. Q.E.D.

The following definitions and results concerning bounded spectral operators are due to Dunford [2]. In [2] fields other than the Borel field and convergence other than strong convergence are introduced but only these two will be considered here.
2.18 Definition. If \( S \) is a Borel subset of the complex plane \( Z \) then \( \beta(S) \) will be used to denote the \( \sigma \)-field of Borel subsets of \( S \). If \( S = Z \) the symbol \( \beta \) will be used for \( \beta(Z) \).

2.19 Definition. By a spectral measure is meant a strongly countably additive, commutative, projection valued measure defined on the \( \sigma \)-field \( \beta \) of Borel sets of the complex plane. If furthermore \( E(Z) = I \), \( E \) will be called a resolution of the identity.

If \( E \) is a spectral measure then by Theorem 13 \( \sup_{\delta \in \beta} | E(\delta) | < \infty \).

2.20 Definition. Let \( T \) be a bounded or unbounded linear operator on a B-space \( X \) and let \( X' \) be a subspace of \( X \) such that \( Tx \in X' \) for every \( x \in X' \cap D(T) \). Then by \( T \mid X' \) will be meant the restriction of \( T \) to \( X' \cap D(T) \) considered as an operator in \( X' \).

2.21 Definition. A bounded linear operator \( T \) on a B-space \( X \) is said to be a spectral operator if there exists a resolution of the identity \( E \) such that \( E(\delta)T = TE(\delta) \), \( \delta \in \beta \), and \( \sigma(T \mid E(\delta)X) \subseteq \delta \), \( \delta \in \beta \). The resolution of the identity \( E \) will be called the resolution of the identity for \( T \).

2.22 Theorem. The resolution of the identity for a spectral operator \( T \) is unique.

2.23 Theorem. Let \( T \) be a bounded spectral operator and \( A \) a bounded linear transformation commuting with \( T \). Then \( A \) commutes with the resolution of the identity for \( T \).

If \( E \) is a self-adjoint spectral measure in Hilbert space the following theorem is well known. For the general case see [2, Theorem 7].

2.24 Theorem. Let \( E \) be a spectral measure and let \( K = \sup_{\delta \in \beta} | E(\delta) | \). Let \( \delta \) be a compact subset of the plane and let \( f \) be a scalar valued function continuous on \( \delta \). Then the Riemann integral \( \int_\delta f(\lambda)E(d\lambda) \) exists in the uniform operator topology and

\[
\left| \int_\delta f(\lambda)E(d\lambda) \right| \leq 4K \sup_{\lambda \in \delta} | f(\lambda) | .
\]

If \( E \) is a self-adjoint spectral measure in a Hilbert space the constant 4 in the above inequality may be replaced by 1.

2.25 Definition. A bounded spectral operator \( S \) is of scalar type if \( S = \int \lambda E(d\lambda) \) where \( E \) is the resolution of the identity for \( S \).
2.26 Theorem. A bounded operator $T$ is spectral if and only if $T = S + N$ where $S$ is a spectral operator of scalar type and $N$ is a quasi-nilpotent operator commuting with $S$. This decomposition is unique, $\sigma(T) = \sigma(S)$, and $T$ and $S$ have the same resolution of the identity.

2.27 Definition. If $T$ is a bounded spectral operator the unique decomposition $T = S + N$ of the above theorem is called the canonical decomposition of $T$. The operators $S$ and $N$ are called the scalar part of $T$ and the radical or quasi-nilpotent part of $T$ respectively.

Bade in [3] has extended the concept of spectral operator to include unbounded closed operators.

2.28 Definition. A closed (not necessarily bounded) operator $T$ will be called a spectral operator if there is a resolution of the identity $E$ such that

1. The domain $D(T)$ of $T$ contains the dense subspace $X_0 = \{x \mid x = E(\sigma)x, \sigma \in \beta, \sigma \text{ bounded}\};$
2. If $\delta \in \beta$, $E(\delta)D(T) \subseteq D(T)$ and $E(\delta)Tx = TE(\delta)x, x \in D(T);$ and
3. $\sigma(T) \mid E(\delta)X \subseteq \delta, \delta \in \beta.$

2.29 Definition. A spectral operator $S$ with resolution of the identity $E$ will be called of scalar type if

$$D(S) = \left\{ x \mid \lim_n \int_{|\lambda| \leq n} \lambda E(d\lambda) \text{ exists} \right\}$$

and

$$Sx = \lim_n \int_{|\lambda| \leq n} \lambda E(d\lambda)x, \quad x \in D(S).$$

The following theorem is due to Bade [3] although he erroneously omits the condition $F(\{0\}) = 0$. A corrected proof is to be found in the second volume of Dunford and Schwartz [5].

2.30 Theorem. Let $T$ be a closed operator with nonempty resolvent set. Then $T$ is a spectral operator if and only if $R(\lambda, T)$ is a spectral operator for some (and hence all) $\lambda$ in $\rho(T)$ with resolution of the identity $F$ such that $F(\{0\}) = 0$. Let $g(z) = (\lambda - z)^{-1}$. Then the resolution of the identity $E$ of $T$ is given by $E(\delta) = F(g(\delta)), \delta \in \beta$. The operator $T$ is of scalar type if and only if $R(\lambda, T)$ is of scalar type.

If $T$ is a compact operator (and hence bounded) or if $T$ is the inverse of a compact operator (and hence unbounded), then the simple nature of the
spectrum of $T$ simplifies the spectral theory of $T$. In particular, if $T$ is spectral then its resolution of the identity will be the contour integral of the resolvent, a statement which will be made precise in Theorem 35 below. First the type of subsets of $\sigma(T)$ around which it is possible to integrate the resolvent must be specified.

2.31 DEFINITION. An *admissible domain* in the complex plane is an open set bounded by a finite number of rectifiable Jordan curves. The boundary of an admissible domain is called an *admissible contour*.

2.32 DEFINITION. For a (not necessarily bounded) operator $T$ a bounded subset of $\sigma(T)$ which is both open and closed in the relative topology of $\sigma(T)$ is called a *spectral set*.

2.33 DEFINITION. Let $T$ be an operator on a $B$-space $X$. For $S$ a subset of the complex plane such that $S \cap \sigma(T)$ is a spectral set let

$$E(S, T) = \frac{1}{2\pi i} \int_{S - \delta} R(u, T)du$$

where $\delta$ is any admissible domain such that

$$\delta \cap \sigma(T) = \delta \cap \sigma(T) \quad \text{and} \quad (\delta - \delta) \cap \sigma(T) = \emptyset$$

and the integral is taken in the positive sense of complex variable theory.

The following theorem is an immediate corollary of Theorem VII.9.5 in [4].

2.34 THEOREM. Let $T$ be a bounded or unbounded linear operator on a $B$-space $X$ and let $\delta_1$ and $\delta_2$ be such that $E(\delta_1, T)$ and $E(\delta_2, T)$ are defined. Then $E(\delta_1 \cap \delta_2, T)$, $E(\delta_1 \cup \delta_2, T)$ and $E(\delta_1 - \delta_2, T)$ are all defined and

$$E(\delta_1 \cap \delta_2, T) = E(\delta_1, T) E(\delta_2, T),$$

$$E(\delta_1 \cup \delta_2, T) = E(\delta_1, T) + E(\delta_2, T) - E(\delta_1, T) E(\delta_2, T),$$

$$E(\delta_1 - \delta_2, T) = E(\delta_1) (I - E(\delta_2)).$$

If $T$ is a compact operator, its spectrum consists of a sequence of points whose only possible limit point is 0. Hence $\beta(\sigma(T))$ is the smallest $\sigma$-field containing the field of spectral sets.

The following theorem is due to Dunford [2].
2.35 Theorem. If $T$ is a compact operator in a reflexive space $X$, then $T$ is a spectral operator if and only if

$$\sup_{\delta} |E(\delta, T)| < \infty$$

where $\delta$ varies over all spectral sets. If this is the case and $\delta$ is a spectral set, $E(\delta, T)$ is the value of the resolution of the identity of $T$ on $\delta$.

The next theorem is a corollary of Theorems 30 and 35.

2.36 Theorem. Let $T$ be a closed operator in a reflexive space $X$ with non-empty resolvent set and assume that for some $\lambda$ in $\rho(T)$, $R(\lambda, T)$ is compact. Then $T$ is a spectral operator if and only if

$$\sup_{\delta} |E(\delta, T)| < \infty$$

where $\delta$ varies over all spectral sets of $T$ and

$$\lim_{n \to \infty} E(\delta_n, T)x = x, \quad x \in X,$$

where $\{\delta_n\}$ is an increasing sequence of spectral sets such that $\bigcup_n \delta_n = \sigma(T)$. If $T$ is a spectral operator and $\delta$ a spectral set, then $E(\delta, T)$ is the value of the resolution of the identity of $T$ on $\delta$.

III. OPERATORS COMMUTING WITH TRANSLATION BY ONE (mod 2N) IN $L_2(-N, N)$

Although some of the results of this section are used later the primary purpose is to present the ideas which motivate Section V in a form unobscured by technical difficulties.

Let $N$ be a positive integer. The unitary operator $S_N$ of translation by one (mod 2N) operating on $L_2(-N, N)$ and the operators that commute with $S_N$ are discussed below. A representation of an arbitrary operator that commutes with $S_N$ is given in terms of 2N operators on $L_2(0, 1)$. The results are analogous to those of Section V where operators commuting with $S$ are studied.

The spectrum of $S_N$ is the finite number of 2N-th roots of unity and the decomposition

$$L_2(-N, N) = \bigoplus_{j=-N}^{N-1} (L_2(j, j + 1))$$
involves only a finite summation whereas the spectrum of $S$ is the whole unit circle $\{z \mid |z| = 1\}$ and the decomposition

$$L_\theta(-\infty, \infty) = \sum_{j=-\infty}^{\infty} L_\theta(j, j + 1)$$

is an infinite summation. Because of these differences the integrals of vector valued functions and infinite summations of Section V have their analogues here in finite sums. Consequently the properties of operators commuting with $S_N$ can be developed in a more direct and intuitive fashion than can the analogous properties of operators commuting with $S$.$^2$

The positive integer $N$ will be fixed throughout.

In this and the following sections if $a$, $b$, $c$, and $d$ satisfy $-\infty \leq a \leq b \leq c \leq d \leq \infty$ the space $L_\theta(b, c)$ will often be thought of as the subspace of $L_\theta(a, d)$ of functions vanishing almost everywhere outside the interval $(b, c)$.

3.1 Definition. For real number $t$ let $\langle t \rangle$ be the unique real number satisfying $-N \leq \langle t \rangle < N$, $t \equiv \langle t \rangle \pmod{2N}$.

Observe that, for any integer $j$, $e^{\pi ij/N} = e^{\pi i\langle t \rangle/N}$.

3.2 Definition. Let $S_N$ be the operator in $L_\theta(-N, N)$ defined by $(S_N f)(t) = f(t + 1)$, $f \in \mathcal{L}_\theta(-N, N)$.

The following lemma is easily verified.

3.3 Lemma. The operator $S_N$ is unitary and $S_N^{2N} = I$. A bounded operator $A$ on $L_\theta(-N, N)$ commutes with $S_N$ if and only if $A^*$ commutes with $S_N$.

It follows from the spectral mapping theorem, Theorem 2.6, that $\sigma(S_N) \subseteq \{e^{\pi ij/N}, j = 0, 1, \ldots, 2N - 1\}$. That equality holds, i.e., that $\sigma(S_N) = \{e^{\pi ij/N}, j = 0, 1, \ldots, 2N - 1\}$, follows from the observation that if $f \in L_\theta(-N, N)$ has the form $f(t) = p(t)e^{\pi ij/N}$, $t \in (-N, N)$, where $p(t)$ has period $1$, then $(S_N f)(t) = p(t)e^{\pi ij\langle t + 1 \rangle/N} = p(t)e^{\pi ij/N}e^{\pi ij/N} = e^{\pi ij/N}f(t)$. In fact one has the following lemma.

3.4 Lemma. The spectrum of $S_N$ is the $2N$th roots of unity. A necessary and sufficient condition that $f \in \mathcal{L}_\theta(-N, N)$ be an eigenvector of the eigenvalue $e^{\pi ij/N}$ of $S_N$ is that $f(t) = p(t)e^{\pi ij/N}$, $-N \leq t < N$, where $p(t)$ has period $1$.

Proof. Sufficiency has already been established. To prove the necessity let $f$ be such that $S_N f = e^{\pi ij/N}f$. Let $p(t) = e^{-\pi ij/N}f(t)$. Then $(S_N p)(t) = e^{-\pi ij\langle t + 1 \rangle/N}(S_N f)(t) = e^{-\pi ij/N}e^{-\pi ij/N}e^{\pi ij/N}f(t) = p(t)$ and hence $p(t) = p(t + 1)$, $-N \leq t < N$. Q.E.D.

Since $S_N$ is normal it has a self-adjoint resolution of the identity with carrier $\sigma(S_N) = \{e^{\pi ij/N}, j = 0, 1, \ldots, 2N - 1\}$.

$^2$ In applications $S_N$ may be physically more meaningful than $S$. 
It is convenient to consider the points $\pi j/N, j = 0, \ldots, 2N - 1$, rather than the points $e^{i\pi j/N}, j = 0, \ldots, 2N - 1$, and thus the following definition is made.

3.5 Definition. Let $P_N$ be the unique self-adjoint resolution of the identity with carrier $\{\pi j/N, j = 0, \ldots, 2N - 1\}$ such that

\[ S_N = \sum_{j=0}^{2N-1} e^{i\pi j/N} P_N(\pi j/N). \]

The resolution of the identity $P_N$ is, except for a change of measure, the resolution of the identity of $S_n$. Thus a bounded operator $A$ commutes with $S_N$ if and only if it commutes with $P_N$ (Theorem 2.23), and thus if and only if it sends each of the orthogonal subspaces $P_N(\pi j/N)L_2 (-N, N)$ into itself, $j = 0, \ldots, 2N - 1$. Thus a natural decomposition of an operator $A$ that commutes with $S_N$ is into $A = \sum_{j=0}^{2N-1} A P_N(\pi j/N)$. Since also for $j \neq k$, $A P_N(\pi j/N) [P_N(\pi k/N)L_2 (-N, N)] = 0$, it follows from the following lemma that $|A| = \max_{0 \leq i \leq 2N-1} |AP_N(\pi j/N)|$.

3.7 Lemma. Let $A_1, \ldots, A_n$ be $n$ operators in a Hilbert space $H$ such that $H$ may be written as $H = \sum_{i=1}^{n} H_i$, $\oplus$ denoting orthogonal direct sum of the subspaces $H_i$, where $A_i H_i \subseteq H_i$, $1 \leq i \leq n$, and $A_i H_i = \{0\}$, $i \neq j$. Let $A = \sum_{i=1}^{n} A_i$. Then $|A| = \max_{1 \leq i \leq n} |A_i|$.

Proof. Without loss of generality, assume $|A_1| = \max_{1 \leq i \leq n} |A_i|$. Let $x = \sum_{i=1}^{n} x_i$ where $x_i$ belongs to $H_i$, $1 \leq i \leq n$. Then $|Ax|^2 = \sum_{i=1}^{n} |A_i x_i|^2 \geq |A_1 x_1|^2$ so $|A| \geq |A_1|$. Also

\[ \frac{|Ax|^2}{|x|^2} = \frac{\sum_{i=1}^{n} |A_i x_i|^2}{\sum_{i=1}^{n} |x_i|^2} \leq \frac{\sum_{i=1}^{n} |A_i|^2 |x_i|^2}{\sum_{i=1}^{n} |x_i|^2} = |A_1|^2 \]

and hence $|A| \leq |A_1|$, Q.E.D.

So far we have effected a reduction of an operator $A$ on $L_2 (-N, N)$ which commutes with $S_N$ into a set of $2N$ operators, $AP_N(\pi j/N), j = 0, \ldots, 2N - 1$, all on $L_2 (-N, N)$, whereas what has been promised is a reduction of $A$ into operators on $L_2 (0, 1)$. To proceed further some notation for mapping into and out of $L_2 (0, 1)$ is necessary. The following definition will also be used extensively in the $L_2 (-\infty, \infty)$ case in Section V.

3.9 Definition. For any integer $j, -\infty < j < \infty$, and $f$ in $L_2 (-\infty, \infty)$ let the operator $T_j$ be defined by $(T_j f)(s) = f(s + j), s \in [0,1)$, $(T_j f)(s) = 0, s \notin [0,1)$. The operator $T_j$ cuts $f$ down to the interval $[j,j+1)$ and shifts this piece to $[0,1)$. The adjoint $T_j^*$ cuts $f$ down to $[0,1)$ and shifts this piece to $[j,j+1)$. One has $T_j T_k = 0$ if $j \neq k$, $(T_j T_j^*) (t) = \chi_{[0,1)}(t) f(t)$, and $(T_j^* T_j f)(t) = \chi_{[k,k+1)}(t) f(t + j - k)$.
In this section attention is restricted to $T_j$ for $-N \leq j \leq N - 1$ and it is more convenient to think of $T_j$ and $T_j^*$ for such $j$ as being operators in $L_2(-N, N)$. One has, for $-N \leq j \leq N - 1$, $T_j S_N = T_{(j+1)}$, and

$$I = \sum_{l=-N}^{N-1} S_{N-1}^l T_l = \sum_{k=-N}^{N-1} T^*_k S_N^k.$$

The following elementary fact will be used below.

3.10. **Lemma.** If $k$ is an integer,

$$\sum_{l=-N}^{N-1} e^{inkl/N} = \sum_{l=0}^{2N-1} e^{inkl/N} = 0, \quad k \neq 0$$

$$= 2N, \quad k = 0.$$

Observe that $f \in L_2(-N, N)$ has the form $f(t) = p(t)e^{\pi j t / N}$ where $p(t)$ has period 1 and hence satisfies $P_N(\pi j / N) f = f$ if and only if

$$f = \sum_{k=-N}^{N-1} e^{ni j / NT_j^*} f_0$$

where $f_0$ is an element in $L_2(0, 1)$. In this case $|f|^2 = 2N \ |f_0|^2$.

3.11 **Lemma.** For $j = 0, \ldots, 2N - 1$,

$$P_N(\pi j / N) = (2N)^{-1} \sum_{k=-N}^{N-1} \sum_{l=-N}^{N-1} T^*_k e^{\pi j (k-l)/N} T_l.$$

**Proof.** Since $\sum_{j=0}^{2N-1} P_N(\pi j / N) = I$, it follows that every $f$ in $L_2(-N, N)$ has a unique decomposition $f = \sum_{m=0}^{2N-1} f_m$ where $f_m = \sum_{k=-N}^{N-1} e^{\pi km / NT_j^*} f_{m0}$ and $f_{m0} \in L_2(0, 1)$. Since $P_N(\pi j / N) P_N(\pi j / N) = 0, \ m \neq j, \ P_n(\pi j / N) f_m = 0, \ m \neq j, \ P_n(\pi j / N) f_j = f_j$. One has, for any $j$ and $m$,

$$\frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{l=-N}^{N-1} T^*_k e^{\pi j (k-l)/N} T_l f_m$$

$$= \frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{l=-N}^{N-1} \sum_{n=-N}^{N-1} T^*_k e^{\pi j (k-l)/N} T_l e^{\pi lm / N} T^*_n f_{m0}$$

$$= \frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{l=-N}^{N-1} T^*_k e^{\pi j k / N} e^{\pi lm / N} T^*_n f_{m0}$$

since $T_l T^*_n = 0, \ l \neq n$, $T_l T^*_l g = g, \ g \in L_2(0, 1)$. 

Using Lemma 10 in the summation over \( l \),
\[
\frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{l=-N}^{N-1} T_j^* e^{\pi ij(k-l)/N} T_l f_m = 0, \quad m \neq j
\]
\[
= \sum_{k=-N}^{N-1} T_k^* e^{\pi ik} f_0
\]
\[
= f_j, \quad m = j,
\]
\[
= P_N(\pi j/N) f_m, \quad 0 < m < 2N - 1.
\]
Q.E.D.

3.12 DEFINITION. For a bounded operator \( A \) in \( L_2(-N, N) \) or in \( L_2(-\infty, \infty) \) and integer \( j \) let \( A(j) \) be that linear operator from \( L_2(0, 1) \) to \( L_2(0, 1) \) defined by
\[
D(A(j)) = L_2(0, 1)
\]
\[
A(j)f = T_j Af, \quad f \in L_2(0, 1).
\]

The operator \( A(j) \) takes that piece of \( Af \) that lies in the interval \( (j, j+1) \) and shifts it to the interval \( (0, 1) \). The reader should test his conception of this definition and that of \( T_j, -\infty < j < \infty \), by verifying that an operator \( A \) in \( L_2(-N, N) \) commutes with \( S_N \) if and only if
\[
A = \sum_{j=-N}^{N-1} \sum_{k=-N}^{N-1} T_j^* A(\langle j-k \rangle) T_k.
\]

Suppose now that \( A \) does commute with \( S_N \) and consider the product \( A P_N(\pi j/N) \). Using the form for \( P_N(\pi j/N) \) given in Lemma 11,
\[
A P_N \left( \frac{\pi j}{N} \right) = \frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{l=-N}^{N-1} \sum_{m=-N}^{N-1} \sum_{n=-N}^{N-1} T_k^* A(\langle k-l \rangle) T_l T_j^* e^{\pi ij(m-n)/N} T_n.
\]

Since \( T_l T_k^* = 0, \) if \( l \neq m \), and maps elements in \( L_2(0, 1) \) into themselves if \( l = m \), this becomes upon rearranging summation,
\[
A P_N \left( \frac{\pi j}{N} \right) = \frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{m=-N}^{N-1} \sum_{n=-N}^{N-1} T_k^* A(\langle k-l \rangle) e^{\pi ij(m-n)/N} T_n.
\]

Now
\[
\sum_{l=-N}^{N-1} A(\langle k-l \rangle) e^{\pi ij(l-n)/N} = e^{\pi ij(k-n)/N} \sum_{l=-N}^{N-1} A(l) e^{-\pi ij l/N}
\]

\[
= P_N(\pi j/N) f_m, \quad 0 < m < 2N - 1.
\]
Q.E.D.
and hence

\[ A_P N \left( \frac{\pi j}{N} \right) = \frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{n=-N}^{N-1} T^*_k e^{\pi ij(k-n)/N} \left[ \sum_{v=-N}^{N-1} A(v) e^{-\pi ivj/N} \right] T_n. \]  

(2)

The term in brackets on the right in (2) is of sufficient importance to give it a special name. The somewhat artificial terminology to follow is used to emphasize the analogy with results in Section V.

3.13 DEFINITION. For an operator \( A \) such that \( AS_N = S_N A \), and for integer \( j \), let

\[ A_N' \left( \frac{\pi j}{N} \right) = \sum_{v=-N}^{N-1} A(v) e^{-\pi ivj/N}. \]

Considering \( A_N' \) as a function on the set \( I_N = \{ \pi j/N, j = 0, \ldots, 2N - 1 \} \), the symbolism \( A \sim A_N' \) will be used to denote that \( A_N' \) is obtained from \( A \) by the above formula.

Since \( A \sum_{j=0}^{2N-1} P_N(\pi j/N) = A I = A \), (2) above yields

\[ A = \sum_{k=-N}^{N-1} \sum_{n=-N}^{N-1} T^*_k \left[ \frac{1}{2N} \sum_{j=0}^{2N-1} e^{\pi ij(k-n)/N} A_N' \left( \frac{\pi j}{N} \right) \right] T_n. \]  

(3)

Since the bracketed term on the right in (3) is a function of \( \langle k - n \rangle \) it is easy to verify that if \( A_N' \) is any \( L_2(0, 1) \) operator valued function on \( I_N \) then (3) defines an operator \( A \) which commutes with \( S_N \). Furthermore, if \( A_N'(\pi j/N) \neq 0 \) for some \( j \) then from (2), \( A_P N(\pi j/N) \neq 0 \) and so \( A \neq 0 \). It follows easily that the relationship \( A \sim A_N' \) is an isomorphism between the space of all bounded \( L_2(-N, N) \) operators commuting with \( S_N \) onto the space of all \( L_2(0, 1) \) operator valued functions on \( I_N \).

3.14 THEOREM. The isomorphism \( A \sim A_N'(\pi j/N) \) defined above between bounded linear operators \( A \) on \( L_2(-N, N) \) which commute with \( S_N \) and \( L_2(0, 1) \) operator valued functions \( A_N' \) defined on \( I_N \) has the properties:

(a) If also \( B \sim B_N'(\pi j/N) \) then \( AB \sim A_N'(\pi j/N) B_N'(\pi j/N). \)

(b) \( A^* \sim A_N'(\pi j/N)^*. \)

(c) \( |A| = \max_{0 \leq j \leq 2N-1} |A_N'(\pi j/N)|. \)

PROOF. To prove (a) observe that

\[ AB_P N(\pi j/N) = (A_P N(\pi j/N) (BP_N(\pi j/N)). \]
so that, using Eq. (2) above,

\[ AB \mathbf{P}_N \left( \frac{\pi j}{N} \right) \]

\[ = \frac{1}{4N^2} \sum_{k=-N}^{N-1} \sum_{l=-N}^{N-1} T_k^* e^{\pi i (k-l)j/N} A'_{N} \left( \frac{\pi j}{N} \right) T_l \times \]

\[ \times \sum_{m=-N}^{N-1} \sum_{n=-N}^{N-1} T_m^* e^{\pi i (m-n)j/N} B'_N \left( \frac{\pi j}{N} \right) T_n \]

\[ = \frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{m=-N}^{N-1} T_k^* e^{\pi i (k-m)j/N} A'_{N} \left( \frac{\pi j}{N} \right) B'_N \left( \frac{\pi j}{N} \right) T_n \]

and thus \( AB \sim A'_{N}(\pi j/N)B'_N(\pi j/N) \).

Part (b) may be verified by taking adjoints in equality (3).

Since \( |A| = \max_{0 \leq j \leq 2N-1} |A\mathbf{P}_N(\pi j/N)| \), to prove (c) it suffices to prove that \( |A\mathbf{P}_N(\pi j/N)| = |A'_{N}(\pi j/N)| \).

Since \( A \) commutes with \( \mathbf{P}_N(\pi j/N) \) and \( P_N(\pi j/N) \) is an orthogonal projection,

\[ |A\mathbf{P}_N(\pi j/N)| = \sup_{f \in \mathbf{P}_N(\pi j/N)L_2(-N,N)} |A\mathbf{P}_N(\pi j/N)f|/\|f\|. \]

As was observed above, any \( f \) in \( \mathbf{P}_N(\pi j/N)L_2(-N,N) \) has the form

\[ f = \sum_{m=-N}^{N-1} e^{\pi i jm/N}T_m^*f_0 \]

where \( f_0 \) is an arbitrary function in \( L_2(0,1) \) and \( |f| = (2N)^{1/2} |f_0| \). For such an \( f \), Eq. (2) gives

\[ A\mathbf{P}_N \left( \frac{\pi j}{N} \right) f = \frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{m=-N}^{N-1} T_k^* e^{\pi i (k-m)j/N} A'_{N} \left( \frac{\pi j}{N} \right) T_m \sum_{m=-N}^{N-1} e^{\pi i jm/N}T_m^*f_0 \]

\[ = \frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{m=-N}^{N-1} T_k^* e^{\pi i (k-m)j/N} A'_{N} \left( \frac{\pi j}{N} \right) \]

and thus \( |A\mathbf{P}_N(\pi j/N)f| = (2N)^{1/2} |A'_{N}(\pi j/N)f_0| \).

It follows that \( |A\mathbf{P}_N(\pi j/N)| = |A'_{N}(\pi j/N)| \) which proves (c). O.E.D.
3.15 Corollary. Let $A$ and $B$ be bounded operators in $L_2(-N,N)$ which commute with $S_N$, $A \sim A'_N$, $B \sim B'_N$. Then

(a) $A$ is a projection if and only if $A_N(\theta)$ is a projection, $\theta \in I_N$,
(b) $A$ is normal if and only if $A'_N(\theta)$ is normal, $\theta \in I_N$,
(c) $A$ is self-adjoint if and only if $A'_N(\theta)$ is self-adjoint, $\theta \in I_N$,
(d) $AB = BA$ if and only if $A'_N(\theta)B'_N(\theta) = B'_N(\theta)A'_N(\theta)$, $\theta \in I_N$,
(e) the scalar $\lambda \in \rho(A)$ if and only if $\lambda \in \rho(A'_N(\theta))$, $\theta \in I_N$ in which case $R(\lambda, A) \sim R(\lambda, A'_N(\theta))$, and

(f) $A$ is quasi-nilpotent if and only if $A'_N(\theta)$ is quasi-nilpotent, $\theta \in I_N$, or, equivalently, if and only if

$$\lim_{n \to \infty} (\max_{\theta \in I_N} |A'_N(\theta)^n|)^{1/n} = 0.$$ 

3.16 Theorem. Let $A$ be a bounded operator in $L_2(-N,N)$ such that $AS_N = S_NA$ and let $A \sim A'_N(\theta)$. Then $A$ is a spectral operator if and only if $A'_N(\theta)$ is a spectral operator for every $\theta \in I_N$. If this is the case and if the decompositions into scalar and quasi-nilpotent parts of $A$ and $A'_N(\theta)$ are $A = T + M$, $A'_N(\theta) = T'_N(\theta) + M'_N(\theta)$ then $TS_N = S_NT$, $MS_N = S_NM$ and $T \sim T'_N(\theta)$, $M \sim M'_N(\theta)$. If $E(\delta)$ is the resolution of the identity of $A$ and $E'_N(\theta, \delta)$ the resolution of the identity of $A'_N(\theta)$ then $E(\delta)S_N = S_NE(\delta)$ and $E(\delta) \sim E'_N(\theta, \delta)$, $\delta \in \beta$.

Proof. Let $A = T + M$ be a spectral operator with scalar part $T$, quasi-nilpotent part $M$, and resolution of the identity $E(\delta)$ and let $AS_N = S_NA$. It follows from Theorem 2.23 that $S_N$ commutes with $T$, $M$, and $E(\delta)$. Let $T'_N(\theta)$, $M'_N(\theta)$, and $E'_N(\theta, \delta)$ be such that $T \sim T'_N(\theta)$, $M \sim M'_N(\theta)$, and $E(\delta) \sim E'_N(\theta, \delta)$. Since $T$, $M$, and $E(\delta)$ all commute it follows from Corollary 16 (d) that $T'_N(\theta)$, $M'_N(\theta)$, and $E'_N(\theta, \delta)$ all commute, $\theta \in I_N$.

Corollary 15 (c) implies that $M'_N(\theta)$ is quasi-nilpotent, $\theta \in I_N$.

Since $E(\delta)$ is a commutative, projection valued measure, Corollary 15 (a), (d) implies that $E'_N(\theta, \delta)$ is a commutative, projection valued measure, $\theta \in I_N$.

By expressing $E'_N(\theta, \delta)$ in terms of $E(\delta)$ by means of Definition 13 it is seen readily that the strong countable additivity of $E(\delta)$ implies that $E'_N(\theta, \delta)$ is strongly countably additive, $\theta \in I_N$. Since $I$, the identity in $L_2(-N,N)$, satisfies $I \sim I'_N(\theta)$ where $I'_N(\theta)$ is the identity in $L_2(0,1)$, $\theta \in I_N$, $E(Z) = I$ implies that $E'_N(\theta, Z)$ is the identity in $L_2(0,1)$, $\theta \in I_N$. Thus $E'_N(\theta, \delta)$ is a resolution of the identity, $\theta \in I_N$.

Using Definition 13,

$$T_N(\theta)f = \sum_{k=-N}^{N-1} e^{-i\theta k/N} T_k T f, \quad f \in L_2(0,1), \quad \theta \in I_N.$$
Since $T = \int \lambda E(d\lambda)$,

$$T_N'(\theta) f = \sum_{k=-N}^{N-1} e^{-i\theta k/N} T_k \int \lambda E(d\lambda) f$$

$$= \int \lambda \sum_{k=-N}^{N-1} e^{-i\theta k/N} T_k E(d\lambda) f$$

$$= \int \lambda E_N'(\theta, d\lambda) f, \quad f \in L_2(0,1), \quad \theta \in \mathbb{N}$$

and hence $T_N'(\theta)$ is a spectral operator of scalar type with resolution of the identity $E_N'(\theta, \delta), \theta \in \mathbb{N}$. Thus for $\theta \in \mathbb{N}$, $A_N'(\theta) = T_N'(\theta) + M_N'(\theta)$ is the sum of a scalar type spectral operator $T_N'(\theta)$ and a quasi-nilpotent operator $M_N'(\theta)$ which commutes with $T_N'(\theta)$. Hence (Theorem 2.26) $A_N'(\theta)$ is a spectral operator, $\theta \in \mathbb{N}$.

The proof that if $A_N'(\theta)$ is a spectral operator, $\theta \in \mathbb{N}$, then $A$ is a spectral operator is left to the reader. Q.E.D.

IV. VECTOR VALUED FUNCTIONS

In Section V there is need for some concepts and results concerning vector valued functions. While many of these are contained in [4] they are phrased there in a more general setting. Consequently it is easier to develop here most of the results needed.

It is assumed below that $H$ is a separable Hilbert space and measurability is with respect to either Borel or Lebesque measure on $(0,2\pi)$.

4.1 Definition. By a measurable $H$ valued function is meant a function $x'(\theta)$ defined for almost all $\theta$ in $(0,2\pi)$, with values in $H$, and such that for every $y$ in $H(x(\theta), y)$ is a measurable scalar valued function. The linear space of all measurable $H$ valued functions is called $M(H)$. By a strongly measurable $B(H)$ valued function is meant a function $B'(\theta)$ defined for almost all $\theta$, with values in $B(H)$, and such that $B'(\theta)x$ is a measurable $H$ valued function for every $x$ in $H$. The linear space of all strongly measurable $B(H)$ valued functions is called $M(B(H))$.

4.2 Lemma. If $x'(\theta)$ and $y'(\theta)$ are in $M(H)$ then $(x'(\theta), y'(\theta))$ is a measurable scalar valued function.

Proof. Let $\{z_i\}_{i=1}^\infty$ be an orthonormal basis for $H$ and write $x'(\theta) = \sum_{i=1}^\infty a_i(\theta) z_i$, $y'(\theta) = \sum_{i=1}^\infty b_i(\theta) z_i$. Since $a_i(\theta) = (x'(\theta), z_i)$ and $b_i(\theta) = (y'(\theta), z_i)$,
$a_i(\theta)$ and $b_i(\theta)$ are measurable scalar valued functions of $\theta$, $1 \leq i < \infty$, and thus $(x'(\theta), y'(\theta)) = \lim_{n \to \infty} \sum_i a_i(\theta) b_i(\theta)$ is a measurable scalar valued function of $\theta$. Q.E.D.

4.3 Corollary. If $x'(\theta) \in M(H)$ then $|x'(\theta)|$ is measurable.

4.4 Lemma. If $B'(\theta) \in M(B(H))$ then $B'^*(\theta) \in M(B(H))$.

Proof. The $B(H)$ valued function $B'(\theta) \in M(B(H))$ if and only if $(B'(\theta)x, y)$ is measurable, all $x, y \in H$, i.e., if and only if $(x, B'^*(\theta)y)$ is measurable, all $x, y \in H$, i.e., if and only if $B'^*(\theta) \in M(B(H))$. Q.E.D.

4.5 Lemma. If $B'(\theta)$ and $C'(\theta)$ belong to $M(B(H))$ then $C'(\theta)B'(\theta)$ belongs to $M(B(H))$.

Proof. Since $(C'(\theta)B'(\theta)x, y) = (B'(\theta)x, C'^*(\theta)y)$ the assertion follows from Lemmas 2 and 4. Q.E.D.

4.6 Corollary. If $B'(\theta) \in M(B(H))$ and $P(u, v)$ is a polynomial in the variables $u$ and $v$ then $P(B'(\theta), B'^*(\theta)) \in M(B(H))$.

4.7 Lemma. Let $\{x'_i(\theta)\}$ be a sequence of elements in $M(H)$ such that $x'(\theta) = \lim_i x'_i(\theta)$ exists almost everywhere in the weak topology [i.e., $(x'(\theta), y) = \lim_i (x'_i(\theta), y)$ all $y \in H$, a.a. $\theta$]. Then $x'(\theta) \in M(H)$.

Proof. Since pointwise limits of sequences of scalar valued functions are measurable, $(x'(\theta), y)$ is measurable, $y \in H$. Q.E.D.

4.8 Lemma. If $B'(\theta) \in M(B(H))$ then $|B'(\theta)|$ is measurable.

Proof. Let $\{x_i\}$ be a countable dense subset of $\{x \mid |x| = 1\}$. Then $|B'(\theta)| = \sup_i |B'(\theta)x_i|$. Since the supremum of a countable number of real valued measurable functions is measurable the lemma follows from Corollary 3. Q.E.D.

4.9. Definition. For $x'(\theta) \in M(H)$ let $x'$ denote the equivalence class of elements in $M(H)$ that are equal to $x'(\theta)$ except possibly on a set of measure 0. The notation $x'(\theta)$ is used to denote an arbitrary element in $x'$ and is called a representation of $x'$. In a similar fashion $B'$ will denote an equivalence class of functions in $M(B(H))$ and $B'(\theta)$ will denote a particular representation for $B'$.

Lower case primed letters will be used to denote vector $(H)$-valued functions and upper case primed letters will be used to denote operator $(B(H))$-valued functions.
4.10. DEFINITION. For $1 \leq p \leq \infty$ let $L_p((0, 2\pi); H)$ be the linear space of equivalence classes of functions $x'$ in $M(H)$ such that $|x'(\theta)| \in L_p(0, 2\pi)$. Let the norm of $x'$ in $L_p((0, 2\pi); H)$ be given by $|x'|_p = ||x'(\theta)||_p$, the norm of the function $|x'(\theta)|$ in $L_p(0, 2\pi)$. Let $L_p((0, 2\pi); B(H))$ be the linear space of equivalence classes $B'$ of functions in $M(B(H))$ such that $|B'(\theta)| \in L_p(0, 2\pi)$. Let the norm of $B'$ in $L_p((0, 2\pi); B(H))$ be given by $|B'|_p = ||B'(\theta)||_p$, the norm of the function $|B'(\theta)|$ in $L_p(0, 2\pi)$.

4.11 THEOREM. For $1 \leq p \leq \infty$, $L_p((0, 2\pi); H)$ is complete.

PROOF. The proof for $1 < p < \infty$ follows from Theorems III.6.11 and III.6.4 in [4].

For $p = \infty$ let $\{x'_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $L_\infty((0, 2\pi); H)$ and let $x'_i(\theta)$ be a representation of $x'_i$, $1 \leq i < \infty$. The convergence of $\{x'_i\}$ in $L_\infty((0, 2\pi), H)$ implies that for almost all $\theta \{x'_i(\theta)\}$ is a Cauchy sequence in the norm topology of $H$ and hence $x'_i(\theta)$ converges to an element $x'(\theta)$ for almost all $\theta$. But convergence in the norm of $H$ implies weak convergence and thus by Lemma 7 $x'(\theta) \in M(H)$. It is easy to verify that $x'_i \to x'$ in $L_\infty((0, 2\pi); H)$. Q.E.D.

4.12 THEOREM. The space $L_\infty((0, 2\pi); B(H))$ is complete.\(^3\)

PROOF. Let $\{B'_i\}$ be a Cauchy sequence in $L_\infty((0, 2\pi); B(H))$. Let $B'_i(\theta)$ be a representation of $B'_i$. Then for almost all $\theta \{B'_i(\theta)\}$ is a Cauchy sequence in the uniform operator topology. Let $B'(\theta) = \lim_i B'_i(\theta)$. Then $B'(\theta)$ is in $M(B(H))$ as may be readily verified using Lemma 7 and $\chi_{\theta}(\theta)$ where $x(B)$ is an element

\[^3\]So are the spaces $L_p((0, 2\pi); B(H))$, $1 \leq p < \infty$ but this is harder to prove and not needed. Similar remarks apply to Theorems 14 (for $1 \leq p < \infty$) and 15 (1 $\leq p \leq \infty$) below.
in $H(B(H))$, and $\Theta$ is a Borel measurable set. An interval simple function in $M(H)$ is a linear combination of functions of the form $\chi_{a}(\theta)\chi_{B(H)}(\theta)\chi_{B}$ where $x(B)$ is in $H(B(H))$ and $\Theta$ is an interval of the form $\Theta = [a, b)$, $0 \leq a \leq b \leq 2\pi$.

4.14 Theorem. Interval simple functions are dense in $L_{2}(0, 2\pi); H)$.

Proof. Let $\{\alpha_{i}\}$ be an orthonormal base for $H$ and let $x'(\theta) = \sum_{i=1}^{\infty} \alpha_{i}(\theta)z_{i}$ be an arbitrary element in $L_{2}(0, 2\pi); H)$.

Then

$$|x'|_{2} = \int_{0}^{2\pi} |x'(\theta)|^{2}d\theta$$

$$= \sum_{i=1}^{\infty} \int_{0}^{2\pi} |\alpha_{i}(\theta)|^{2}d\theta$$

$$= \sum_{i=1}^{\infty} |\alpha_{i}|_{2}^{2}.$$ 

Hence for $\epsilon > 0$ there is an $N$ such that

$$|x' - \sum_{i=1}^{N} \alpha_{i}(\theta)z_{i}|_{2} \leq \frac{\epsilon}{2}.$$ 

Scalar valued interval simple functions are dense in $L_{2}(0, 2\pi)$ and hence for each $i$, $1 \leq i \leq N$, there exists a scalar valued interval simple function $\alpha_{i}(\theta)$ such that

$$|\alpha_{i}(\theta) - \alpha_{i}(\theta)|_{2} \leq \epsilon/2N^{1/2}.$$ 

The function $y'(\theta) = \sum_{i=1}^{N} \alpha_{e}(\theta)z_{i}$ is an $H$-valued interval simple function and

$$|x'(\theta) - y'(\theta)|_{2} \leq |x'(\theta) - \sum_{i=1}^{N} \alpha_{i}(\theta)z_{i}|_{2}$$

$$+ \sum_{i=1}^{N} \alpha_{i}(\theta)z_{i} - \sum_{i=1}^{N} \alpha_{i}(\theta)z_{i}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon.$$ 

Q.E.D.

4.15 Theorem. If $H$ is finite dimensional then Borel simple functions are dense in $L_{2}(0, 2\pi); B(H))$. 
OPERATORS COMMUTING WITH TRANSLATION BY ONE

PROOF. If $\dim H = n < \infty$ then $B(H)$ is equivalent to the class of $n \times n$ matrices and the result follows from the fact that each of the $n^2$ entries in an essentially bounded measurable matrix valued function of $\theta$ can be approximated in $L_\infty(0, 2\pi)$ by a Borel simple function. Q.E.D.

4.16 LEMMA. Let $A'(x)$ be a linear mapping from $H$ into the space of equivalence classes of functions in $M(H)$. Then for every $x$ in $H$ there exists a representation $A'(\theta, x)$ of $A'(x)$ in $M(H)$ such that for almost all $\theta$, $A'(\theta, x)$ is linear in $x$.

PROOF. Actually, a stronger result will be proved. Let $\{z_a\}$ be a Hamel basis for $H$. For each $z_a$ pick a representation $A'(\theta, z_a)$ of $A'(z_a)$ which is defined everywhere. Let $x = \sum b_a z_a$ be an arbitrary element in $H$ where the summation is finite and let $A'(\theta, x) = \sum b_a A'(\theta, z_a)$. Then it is trivial to verify that for every $\theta$, $A'(\theta, x)$ is linear in $x$. Q.E.D.

4.17 LEMMA. Suppose in addition to the hypotheses of the preceding lemma that $A'(x) \in L_\infty((0, 2\pi); H)$ for every $x$ in $H$ and $\sup_{|x| = 1} |A'(x)|_\infty = K < \infty$. For $x \in H$ let $A'(\theta)x$ be a representation of $A'(x)$ satisfying the conditions of the preceding lemma and defined only for those $\theta$ for which it is linear in $x$. Then $A'(\theta) \in M(B(H))$, the equivalence class $A'$ of $A'(\theta)$ is in $L_\infty((0, 2\pi); B(H))$ and $|A'|_\infty = K$.

PROOF. If the linear operator $A'(\theta)$ is unbounded let $|A'(\theta)| = \infty$. Let $S_0$ be a measurable null set such that $A'(\theta)$ is defined, $\theta \notin S_0$ and such that $|A'|_\infty = \text{ess sup} |A'(\theta)| = \sup_{\theta \notin S_0} |A'(\theta)|$. Let $\{x_i\}_{i=1}^\infty$ be a countable dense subset of $\{x \mid |x| = 1\}$ such that $\sup_{|x| = 1} |A'(x)|_\infty = \sup_i |A'(x_i)|_\infty$. For each $i$, $1 \leq i < \infty$, let $S_i$ be a measurable null set such that $|A'(x_i)|_\infty = \text{ess sup} |A'(\theta)x_i| = \sup_{\theta \notin S_i} |A'(\theta)x_i|$. Let $S = \cup_{i=1}^\infty S_i$. Then $S$ is a measurable null set, $|A|_x = \sup_{\theta \notin S} |A'(\theta)|$, and $|A'(x_i)|_\infty = \sup_{\theta \notin S} |A'(\theta)x_i|$. Thus

\[
|A'|_\infty = \sup_{\theta \notin S} |A'(\theta)| = \sup_{\theta \notin S} \sup_i |A'(\theta)x_i| = \sup_i \sup_{\theta \notin S} |A'(\theta)x_i| = \sup_i |A'(x_i)|_\infty = K. \quad \text{Q.E.D.}
\]
4.18 Lemma. Let $A'(\theta) \in M(B(H))$. Then

$$\text{ess, sup} |A'(\theta)| = \sup_{|x|=1} \text{ess, sup} |A'(\theta)x| = \sup_{|y|=1} \text{ess, sup} |A'(\theta)x, y|.$$ 

Proof. The first equality is a corollary of the preceding lemma. The second can be proved in the same manner as the preceding lemma by taking two sequences $\{x_i\}, \{y_j\}$ dense in $\{x \mid |x| = 1\}$ and such that

$$\sup_{|x|=1} \text{ess, sup} |(A'(\theta)x, y)| = \sup_{i,j} \text{ess, sup} |(A'(\theta)x_i, y_j)|. \quad \text{Q.E.D.}$$

V. Operators Commuting with Translation

by One in $L_2(-\infty, \infty)$

We are now ready to develop the central results of this part. Our concern is with bounded operators in $L_2(-\infty, \infty)$ that commute with translation by one and it is well to recall the definition of the unit translation operator.

5.1 Definition. Let $S$ be the operator in $L_2(-\infty, \infty)$ defined by $(Sf)(t) = f(t+1)$.

5.2 Lemma. The operator $S$ is unitary, i.e. $S^* = S^{-1}$. A bounded operator $A$ commutes with $S$ if and only if $A^*$ commutes with $S$.

Technical difficulties force the reasoning in this section to be circuitous. Consequently, though the results established are analogous to those of Section III our proofs often must take a different tack.

The reader should verify that

$$Af = \text{lim} \sum_{N_2} \sum_{N_1} T^*_j A(j-k)T_k f, \quad f \in L_2(-\infty, \infty),$$

if and only if $AS = SA$, where $\text{lim}$ denotes limit in mean.

5.3 Lemma. Let $\{x_i\}, -\infty < i < \infty, i \text{ integer},$ be a sequence of elements in a linear space $X$. Then for integer $N > 0$,

$$\frac{1}{2N} \sum_{k=0}^{2N-1} \sum_{j=-k}^{k} x_j = \frac{1}{2N} \sum_{l=-N}^{N-1} \sum_{k=-N}^{N-1} x_{l-k}.$$
5.4 Lemma. If $A$ is a bounded operator in $L_2(-\infty, \infty)$ such that $AS = SA$ then there is a unique $A'$ in $L_\infty(0, 2\pi); B(L_2(0, 1))$ such that for $f$ in $L_2(-\infty, \infty)$, 

$$Af = \lim_{N \to \infty} \frac{1}{2\pi} \sum_{j=-N}^{N} \sum_{k=-N}^{N} T_j^* \int_0^{2\pi} e^{i\theta(\theta - k)} A'(... \theta) f \, d\theta$$

the limit being in the $L_2(-\infty, \infty)$ norm. Furthermore $\|A'\|_\infty \leq |A|$. For every $f$ in $L_2(0, 1)$, $A'$ is the limit in $L_2((0, 2\pi); L_2(0, 1))$ of $\sum_{k=-n}^{n} e^{-ik\theta} A(k) f$ and $|A'f|_2 = (2\pi)^{1/2} |Af|$.

Proof. Let $f$ be in $L_2(0, 1)$ and consider the partial sums $A'_n f = A'_n(\theta) f \equiv \sum_{-n}^{n} e^{-ik\theta} A(k) f$. It follows from the orthogonality over $(0, 2\pi)$ of $e^{ik\theta}, e^{ik\theta}$, $j \neq k$ that for $m > n$, $||A_m f - A_n f||_2 \leq 2\pi \sum_{m \geq |k| \geq n+1} |A(k)f|^2$. Now $|A(k)f|^2 = |\chi_{(k,k+1)}A f|^2$ so $|A(k)f|^2 = \sum_{k=-\infty}^{\infty} |\chi_{(k,k+1)}A f|^2 = |A f|^2 < \infty$ and hence $\{A'_n f\}$ is Cauchy in $L_2((0, 2\pi); L_2(0, 1))$. Since this latter space is complete (Theorem 4.11) there is a unique element $A'(f)$ in $L_2((0, 2\pi); L_2(0, 1))$ such that $\lim_{n \to \infty} |A'(f) - A_n f|_2 = 0$ and $|A'(f)|_2 = 2\pi \sum_{k=-\infty}^{\infty} |A(k)f|^2 = 2\pi |A f|^2$. Since $A'_n \in L_2((0, 2\pi); B(L_2(0, 1)))$, it follows that $A'(f)$ is linear in $f$. It follows from Lemma 4.16 that for every $f \in L_2(0, 1)$ there is a representation $A'(\theta) f$ of $A'(f)$ such that $A'(\theta) f$ is linear in $f$, a.a. $\theta$.

For arbitrary $f$ and $g$ in $L_2(0, 1)$ of norm 1 and integer $N > 0$ let

$$f_N(\theta) = (2N)^{-1/2} \sum_{k=-N}^{N-1} e^{ik\theta} T_k^* f,$$

$$g_N(\theta) = (2N)^{-1/2} \sum_{k=-N}^{N-1} e^{ik\theta} T_k^* g.$$

Then $|f_N(\theta)| = |g_N(\theta)| = 1, 0 \leq \theta \leq 2\pi$, and $|(A f_N(\theta), g_N(\theta))| \leq |A|$. Now

$$Af_N(\theta) = (2N)^{-1/2} \sum_{k=-N}^{N-1} e^{ik\theta} S^{-k} Af$$

$$= (2N)^{-1/2} \sum_{l=-\infty}^{\infty} \sum_{k=-N}^{N-1} e^{ik\theta} S^{-k} S^{-l} A(l) f$$

$$= (2N)^{-1/2} \sum_{l=-\infty}^{\infty} \sum_{k=-N}^{N-1} e^{ik\theta} S^{-l} A(l \quad k) f$$

4 The use of functions of this form is suggested by the general form given following Lemma 3.10 for any eigenvector of the eigenvalue $e^{i\pi j/N}$ of $S_N$, $\sum_{k=-N}^{N-1} e^{ik\theta/N} T_k f_0$. 
and since $S^{-1}A(l - k)f$ has carrier contained in $(l, l + 1)$,

$$(Af_N(\theta), g_N(\theta)) = (2N)^{-1} \sum_{l=-N}^{N-1} \sum_{k=-N}^{N-1} e^{-i\theta(l-k)}(A(l - k)f, g)$$

and hence by Lemma 3,

$$(Af_N(\theta), g_N(\theta)) = \frac{1}{2N} \sum_{k=-k}^{2N-1} (e^{-i\theta}A(j)f, g)$$

which is just the average of the partial sums $(A_N^k(\theta)f, g)$, $0 \leq k \leq 2N - 1$. Thus by a result in the theory of Fourier series (see Zygmund [7, p. 49])

$$\lim_{N \to \infty} (Af_N(\theta), g_N(\theta)) = (A'\theta)f, g)$$

for almost all $\theta$. Therefore $|A'(\theta)f, g)| \leq |A|$, a.a. $\theta$, and hence by Lemma 4.18 $A'$ is in $L_{\infty}((0, 2\pi); B(L_2(0, 1)))$ and $|A'| \leq |A| /

Since $A'f$ is the limit in $L_2((0, 2\pi); L_2(0, 1))$ of $A'_N f$ it follows that for integer $j$,

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} A'_N(\theta)f d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} A'(\theta)f d\theta.$$

For $n \geq |j|$, 

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} A'_N(\theta)f d\theta = \frac{1}{2\pi} \sum_{k=-n}^{n} \int_0^{2\pi} e^{i\theta(j-k)} A(k)f d\theta$$

$$- A(j)f$$

and thus

$$A(j)f = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} A'(\theta)f d\theta, \quad f \in L_2(0, 1).$$

Since

$$Af \to \lim_{N \to \infty} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T^*_j A(j - k)T_k f, \quad f \in L_2(-\infty, \infty),$$

it follows that

$$Af = \lim_{N \to \infty} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T^*_j \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta(j-k)} A'(\theta)T_k f d\theta.$$
Suppose there is another function $A_1'$ in $L_2((0, 2\pi); B(L_2(0, 1)))$ such that for $f$ in $L_2(-\infty, \infty)$

$$Af = \lim_{N_4 \to \infty} \sum_{j=-N_4}^{N_4} \sum_{k=-N_3}^{N_3} \frac{1}{2\pi} T_j \int_{0}^{2\pi} e^{i\theta(j-k)} A_1'(\theta) T_k f d\theta.$$ 

Letting $f$ be an arbitrary element in $L_2(0, 1)$ this implies that

$$\int_{0}^{2\pi} e^{ij\theta} A_1'(\theta) f d\theta = \int_{0}^{2\pi} e^{ij\theta} A_1'(\theta) f d\theta,$$

and hence for $f, g$ in $L_2(0, 1)$,

$$\int_{0}^{2\pi} e^{ij\theta} A_1'(\theta) f d\theta = \int_{0}^{2\pi} (e^{ij\theta} A_1'(\theta) f, g) d\theta$$

and hence by the uniqueness of the Fourier coefficients of functions in $L_2(0, 2\pi)$ that $(A_1'(\theta)f, g) = (A_1'(\theta)f, g)$ a.a. $\theta$. It follows that $A'(\theta) = A_1'(\theta)$, a.a. $\theta$. Q.E.D.

The converse to Lemma 4, with the inequality $|A| \leq |A'|_\infty$, will be proved for progressively larger classes of functions $A'$ in Lemmas 9, 11, and 12.

5.5 Lemma. Let $f'$ and $g'$ be in $L_2((0, 2\pi); L_2(0, 1))$. Extend $f'$ and $g'$ to be zero outside $(0, 2\pi)$. Then, for $u$ real,

$$\lim_{u \to 0} \int_{0}^{2\pi} (f'(u + v), g'(v)) dv = \int_{0}^{2\pi} (f'(v), g'(v)) dv.$$ 

Proof. Let $f'$ be fixed and arbitrary. For $g'$ in $L_2((0, 2\pi); L_2(0, 1))$ let $T_u g' = \int_{0}^{2\pi} (f'(u + v), g'(v)) dv$. Since

$$|T_u g'| \leq \int_{0}^{2\pi} |f'(u + v)| |g'(v)| dv \leq |f'|_z |g'|_z,$$

$\{T_u\}$ is a uniformly bounded set of linear functionals on $L_2((0, 2\pi); L_2(0, 1))$. Thus by the uniform boundedness principle, Theorem 2.4, to show $\lim_{u \to 0} T_u g' = T_0 g'$ which is the assertion of the lemma it is sufficient to show $\lim_{u \to 0} T_u g' = T_0 g'$ for a fundamental set of $g'$ in $L_2((0, 2\pi); L_2(0, 1))$, and hence (Theorem 4.14) for $g' = x \chi_{(a,b)}$ where $a$ and $b$ are arbitrary numbers in $(0, 2\pi)$ and $x$ is an arbitrary element in $L_2(0, 1)$. For such a $g'$,

$$|T_u g' - T_0 g'| = \left| \int_{a}^{b} (f'(u + v), x) dv - \int_{a}^{b} (f'(v), x) dv \right|$$

$$\leq |x| \left[ \int_{a}^{a+u} |f'(v)| dv + \int_{b}^{b+u} |f'(v)| dv \right].$$
Since \( |f'(\varphi)| \in L_1((0, 2\pi)) \subseteq L_1((0, 2\pi)) \) it follows from the absolute continuity of the integral of an integrable function that the two terms on the right approach 0 as \( u \) approaches 0. Q.E.D.

5.6 **Lemma.** Let \( f' \) and \( g' \) be in \( L_1((0, 2\pi); L_2(0, 1)) \). Let

\[
B_n = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \sum_{j=-\infty}^{\infty} e^{i\theta_1+\theta_2} (f'(\theta_1), g'(\theta_2))d\theta_1d\theta_2.
\]

Then \( B_n \) is \( (C, 1) \) summable to \( \int_0^{2\pi} (f'(\theta), g'(\theta))d\theta \), i.e.,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} B_n = \int_0^{2\pi} (f'(\theta), g'(\theta))d\theta.
\]

**Proof.** Extend \( f' \) and \( g' \) by periodicity. By making the change of variables \( u = \theta_1 - \theta_2, \quad v = \theta_2 \) write

\[
B_n = \frac{1}{2\pi} \int_{\theta_2 = -\pi}^{\pi} \sum_{j=-\infty}^{\infty} e^{iu} \int_{\varphi = 0}^{2\pi} (f'(u + \varphi), g'(\varphi))d\varphi
\]

\[
= \frac{1}{2\pi} \int_{\theta_2 = -\pi}^{\pi} \sum_{j=-\infty}^{\infty} e^{iu} H(u)du
\]

where \( H(u) = \int_{\varphi = 0}^{2\pi} (f'(u + \varphi), g'(\varphi))d\varphi, \quad -\pi \leq u \leq \pi \). How \( H(u) \) is continuous at \( u = 0 \) by the preceding lemma and thus it follows from a theorem of Fejér (Zygmund, [7, p. 45]) that \( B_n \) is \( (C, 1) \) summable to \( H(0) = \int_{\varphi = 0}^{2\pi} (f'(\varphi), g'(\varphi))d\varphi \). Q.E.D.

5.7 **Corollary.** Under the assumptions of the preceding lemma if \( \lim_{n \to \infty} B_n \) exists then \( \lim_{n \to \infty} B_n = \int_0^{2\pi} (f'(\theta), g'(\theta))d\theta \).

**Proof.** This is a consequence of the fact that the \( (C, 1) \) limit of a sequence always agrees with the \( (C, 0) \) limit whenever the latter exists. Q.E.D.

5.8 **Definition.** Let \( H' \) be the dense subspace of \( L_2(-\infty, \infty) \) of functions having compact carrier.

5.9 **Lemma.** For every Borel set \( \Theta \) of \([0, 2\pi)\) and \( f \) in \( L_2(-\infty, \infty) \) the limit in mean

\[
P(\Theta) f = \frac{1}{2\pi} \lim_{N_1, N_2, \infty} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T_j * T_k f d\theta.
\]
exists and defines a self-adjoint projection $P(\Theta)$. The projections $P(\Theta)$, $\Theta$ in $\beta([0, 2\pi])$, form a resolution of the identity and $P(\Theta) = 0$ if and only if $\Theta$ is a null set.

**Proof.** Let $R$ be the ring generated by half-closed intervals of the form $[a, b)$ where $0 \leq a \leq b \leq 2\pi$. Then every element in $R$ can be written as the union of a finite number of disjoint half-closed intervals and $S(R) = \beta([0, 2\pi])$. For $\Theta$ in $R$ consider the bilinear form $P(\Theta, f, g)$ defined for $f$ and $g$ in $H'$ by

$$P(\Theta, f, g) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left( \int_{\theta} e^{i\theta(j-k)} T_k f d\theta, T_k g \right)$$

$$= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left( \int_{0}^{2\pi} e^{i\theta(j-k)} \chi_{\Theta}(\theta) T_k f d\theta, T_k g \right)$$

where it should be observed that for fixed $f$ and $g$ in $H'$ this is actually a finite sum,

$$P(\Theta, f, g) = \frac{1}{2\pi} \sum_{j=-M}^{M-1} \sum_{k=-M}^{M-1} \left( \int_{0}^{2\pi} e^{i\theta(j-k)} \chi_{\Theta}(\theta) T_k f d\theta, T_k g \right)$$

where $M$ is any integer so large that $f$ and $g$ are both in $L_2(-M, M)$. Since $\Theta$ is in $R$, $\chi_{\Theta}(\theta)$ is continuous except at a finite set of points and hence the integrands above are actually Riemann integrable. In particular,

$$P(\Theta, f, g) = \lim_{N \to \infty} \sum_{j=-M}^{M-1} \sum_{k=-M}^{M-1} \frac{1}{2N} \sum_{l=0}^{2N-1} \left( e^{i\pi l(j-k)/N} \chi_{\Theta} \left( \frac{\pi l}{N} \right) \right) T_k f, T_k g$$

Now for $N > M$ this is just $P(\Theta, f, g) = \lim_{N \to \infty} (P_N(\Theta)f, g)$ where $P_N(\Theta)$ is the operator in $L_2(-N, N)$ given by

$$P_N(\Theta) = \sum_{j=-N}^{N-1} \sum_{k=-N}^{N-1} T_j^* \frac{1}{2N} \sum_{l=0}^{2N-1} \left( e^{i\pi l(j-k)/N} \chi_{\Theta} \left( \frac{\pi l}{N} \right) \right) T_k.$$ 

It follows from Lemma 3.11 that $P_N$ is (except for a change of measure) the resolution of the identity for $S_N$, and hence

$$|P_N(\Theta)| = \max_{0 \leq l \leq 2N-1} |\chi_{\Theta} \left( \frac{\pi l}{N} \right)| \leq 1$$

and hence $|P(\Theta, f, g)| \leq |f| \cdot |g|$. Thus there is a unique operator $P(\Theta)$.


of $L_2\left(-\infty, \infty\right)$ into itself of norm not exceeding one such that $P(\Theta, f, g) = (P(\Theta) f, g)$ for $f, g$ in $H'$. It follows immediately from the formula for $P(\Theta, f, g)$ that for $f$ in $H'$,

$$P(\Theta) f = \lim_{N_t \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T_j^* \int_{\Theta} e^{i\theta(j-k)} T_k f d\theta$$

the limit being in the mean (the summation over $k$ is finite). Since $P(\Theta)$ is bounded and $f = \lim_{N \to \infty} x_{[-N_1, N_2]}$, $f \in L_2\left(-\infty, \infty\right)$, it follows that for any $f$ in $L_2\left(-\infty, \infty\right)$,

$$P(\Theta) f = \lim_{N_t \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T_j^* \int_{\Theta} e^{i\theta(j-k)} T_k f d\theta.$$

Since $P(\Theta, f, g)$ is additive in $\Theta$ for $f, g$ in $H'$ it follows that $P(\Theta)$ is a finitely additive measure on $R$.

Let $f$ and $g$ be in $H'$. Then $(P^*(\Theta) f, g) = (f, P(\Theta) g)$ and so

$$(P^*(\Theta) f, g) = \frac{1}{2\pi} \sum_j \sum_k \int_0^{2\pi} e^{i\theta(j-k)} (T_j f, x_\Theta(\theta) T_k g) d\theta$$

$$= \frac{1}{2\pi} \sum_j \sum_k \int_{\Theta} e^{i\theta(j-k)} (T_j f, T_k g) d\theta$$

$$= (P(\Theta) f, g)$$

and so $P(\Theta) = P^*(\Theta)$.

Let $\Theta_1$ and $\Theta_2$ be in $R$ and $f$ and $g$ in $H'$. Then

$$(P(\Theta_2) P(\Theta_1) f, g) = (P(\Theta_1) f, P(\Theta_2) g)$$

$$= \lim_{n \to \infty} \frac{1}{4\pi^2} \sum_{j=0}^{n} \sum_{k=0}^{n} \left( \int_0^{2\pi} e^{i\theta(j-k)} x_{\Theta_1}(\theta) T_k f d\theta, \int_0^{2\pi} e^{i\phi(j-k)} x_{\Theta_2}(\phi) T_k g d\phi \right)$$

$$= \lim_{n \to \infty} B_n$$

where

$$B_n = \frac{1}{4\pi^2} \int_{\Theta=0}^{2\pi} \int_{\Phi=0}^{2\pi} \sum_{j=0}^{n} \sum_{k=0}^{n} e^{i(\Theta-\Phi)j} \left( \sum_k e^{-i\theta_k} x_{\Theta_1}(\theta) T_k f, \sum_l e^{-i\phi_l} x_{\Theta_2}(\phi) T_l g \right) d\theta d\phi.$$
and $\sum_i e^{-i\phi} x_{\Theta_i}(\theta)T_i g$ are in $L_\infty((0, 2\pi); L_2(0, 1)) \subset L_2((0, 2\pi); L_2(0, 1))$ and

$$\lim_n B_n$$

exists. Thus it follows from Corollary 7 that

$$(P(\Theta_2)P(\Theta_1)f, g) = \frac{1}{2\pi} \sum_k \sum_{\theta} \int_0^{2\pi} \left( e^{i\theta(l-k)} x_{\Theta_1}(\theta)x_{\Theta_2}(\theta)T_k f, T_2 g \right) d\theta.$$  

Since $x_{\Theta_1}(\theta)x_{\Theta_2}(\theta) = x_{\Theta_1 \cap \Theta_2}(\theta)$ it follows that $P(\Theta_2)P(\Theta_1) = P(\Theta_2 \cap \Theta_1)$. Thus $P(\Theta)$, $\Theta$ in $\mathbb{R}$, defines a self-adjoint, commutative, finitely additive, projection valued measure. That $P([0, 2\pi]) = I$ is readily verified.

By the theorem of Pettis, Theorem 2.12, in order to show that $P(\Theta)$, $\Theta$ in $\mathbb{R}$, is strongly countably additive it is only necessary to show that it is weakly countably additive. To do this (see Halmos, [6, Theorem F, p. 39] which readily generalizes to complex measures) it is sufficient to show that $(P(\Theta)f, g)$ is continuous from above at 0, for arbitrary $f, g \in L_2(-\infty, \infty)$, i.e., if $\{\Theta_i\}$ is a sequence of decreasing sets in $\mathbb{R}$ such that $\lim_i \Theta_i = \emptyset$, the null set, then

$$\lim_i (P(\Theta_i)f, g) = 0.$$  

Let $f$ and $g$ be in $H'$. Then $(P(\Theta_i)f, g)$ is the finite sum

$$(P(\Theta_i)f, g) = \frac{1}{2\pi} \sum_{\theta} \sum_k \int_0^{2\pi} \left( e^{i\theta(l-k)} x_{\Theta_1}(\theta)T_k f, T_2 g \right) d\theta.$$  

Let

$$h_i(\theta) = x_{\Theta_1}(\theta) \sum_k \sum_{\theta} e^{i\theta(l-k)}(T_k f, T_2 g).$$

Then

$$\sup_{i} \sup_{0 \leq \theta < 2\pi} |h_i(\theta)| \leq \sum_{\Theta} \sum_k |T_k f| |T_2 g| < \infty.$$  

Furthermore $\lim_i h_i(\theta) = 0$ for all $\theta$. Thus by the Lebesque dominated convergence theorem

$$\lim_i (P(\Theta_i)f, g) = \lim_i \frac{1}{2\pi} \int_0^{2\pi} h_i(\theta) d\theta = 0$$

for $f$ and $g$ in $H'$. Since $|P(\Theta_i)| \leq 1$, the uniform boundedness principle, Theorem 2.5, implies that $\lim_i (P(\Theta_i)f, g) = 0$ for all $f$ and $g$ in $L_2(-\infty, \infty)$. Thus $P(\Theta)$ is strongly countably additive on $\mathbb{R}$. That $P(\Theta)$ has a unique strongly countably additive, self-adjoint, commutative, and projection valued extension to $P([0, 2\pi]) = S(\mathbb{R})$ follows from Theorems 2.14 and 2.16. The notation $P(\Theta)$ will also be used for this extension.

Given $f$ and $g$ in $H'$, the set of all $\Theta$ in $\beta([0, 2\pi])$ for which the formula

$$(P(\Theta)f, g) = \frac{1}{2\pi} \sum_{\theta} \sum_k \int_0^{2\pi} e^{i\theta(l-k)} x_{\Theta_1}(\theta)(T_k f, T_2 g)d\theta$$

is finite, is a $\sigma$-algebra that contains $\beta([0, 2\pi])$ and therefore $P(\Theta)$ is regular on $\mathbb{R}$. Thus $P(\Theta)$ is normal on $\mathbb{R}$.
is valid is also easily seen to be monotone class by the dominated convergence theorem, and hence is $\beta([0, 2\pi])$. Since $P(\Theta)$ is bounded it then follows that

$$P(\Theta) f = \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T^*_j \int_{\theta=0}^{2\pi} e^{i\theta(j-k)} \chi_\Theta(\theta) d\theta T_k f$$

for all $f$ in $L_2\left(-\infty, \infty\right)$.

If $\Theta$ is a null set all the integrals in the expression for $P(\Theta)f$ are 0 so $P(\Theta)f = 0$. The fourier coefficients of $\chi_\Theta(\theta)$ will all be 0 if and only if $\chi_\Theta(\theta) = 0$, a.a. $\theta$, i.e., if and only if $\Theta$ is a set of measure 0. Thus if $\Theta$ is not a null set, for any $f$ in $L_2(0, 1)$, $f \neq 0$, at least one of the terms in

$$P(\Theta) f = \frac{1}{2\pi} \sum_j T^*_j \int_0^{2\pi} e^{i\theta j} \chi_\Theta(\theta) d\theta f$$

will not be 0. Since these terms all have disjoint carrier it follows that $P(\Theta) f \neq 0$. Hence $P(\Theta) = 0$ if and only if $\Theta$ is a null set. Q.E.D.

5.10 Definition. Let $P(\Theta)$ be the resolution of the identity defined in the above lemma.

It can be shown that, except for a change of variables, $P(\Theta)$ is the resolution of the identity for $S$; more precisely $S = \int_0^{2\pi} e^{i\theta} P(d\theta)$. Thus $P(\Theta)$ is the analogue for $S$ of the resolution of the identity $P_N$ of Section III. Note however that the “natural” square root of $S$, $(S^{1/2} f)(t) = f(t + \frac{1}{2})$ can not be of the form $S^{1/2} = \int_0^{2\pi} (e^{i\theta})^{1/2} P(d\theta)$ for any measurable choice of the function $(e^{i\theta})^{1/2}$ because $S^{1/2}$ does not commute with all operators commuting with $S$.

5.11 Lemma. For $A'$ in the norm closure in $L_\infty\left((0, 2\pi); B(L_2(0, 1))\right)$ of the class of Borel simple functions and $f$ in $L_2\left(-\infty, \infty\right)$ the limit in the $L_2\left(-\infty, \infty\right)$ norm

$$Af = \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T^*_j \int_0^{2\pi} e^{i\theta(j-k)} A'(\theta) T_k f d\theta$$

exists for every $f$ in $L_2\left(-\infty, \infty\right)$ and defines a bounded linear operator $A$ on $L_2\left(-\infty, \infty\right)$ such that $|A| \leq |A'|_\infty$.

Proof. First let $A'(\theta) \equiv C$ where $C$ is in $B(L_2(0, 1))$. Then

$$Af = \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T^*_j \int_0^{2\pi} e^{i\theta(j-k)} C T_k f d\theta$$

$$= \lim_{N_1 \to \infty} \sum_{j=-N_1}^{N_2} T^*_j C T_j f,$$
i.e., \( A \) is formed from \( C \) by "duplicating" \( C \) in \( L_2(j, j+1) \), \(-\infty < j < \infty\). It follows easily that \( |A| = |C| \), that \( T_kA = AT_k - CT_k \), and that \( AT_j^* = T_j^*A = T_j^*A_{\chi(0,1)} \). Thus

\[
P(\theta)Af = \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_3}^{N_3} T_j^* \int_0^{2\pi} e^{i\theta(j-k)} \chi_\theta(\theta) T_k(Af)
\]

\[
= \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_3}^{N_3} T_j^* \int_0^{2\pi} e^{i\theta(j-k)} C \chi_\theta(\theta) T_k f
\]

and

\[
AP(\theta)f = \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_3}^{N_3} AT_j^* \int_0^{2\pi} e^{i\theta(j-k)} \chi_\theta(\theta) T_k f
\]

\[
= \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_3}^{N_3} T_j^* \int_0^{2\pi} e^{i\theta(j-k)} C \chi_\theta(\theta) T_k f
\]

and so \( AP(\theta) = P(\theta)A \). Since

\[
|P(\theta)| = \text{ess sup } |\chi_\theta(\theta)|,
\]

\[
|AP(\theta)| \leq |A| \text{ ess sup } |\chi_\theta(\theta)| = |C\chi_\theta(\theta)|_\infty.
\]

Now let \( A' \) be a Borel simple function, i.e. \( A'(\theta) = \sum_{i=1}^n C_{\chi_\theta(i)}(\theta) \) where \( \Theta_i \in \beta([0,2\pi]) \), \( 1 \leq i \leq n \), and \( \Theta_i \cap \Theta_j = \emptyset \), \( i \neq j \). Then by the paragraph above the operator \( A \) of the statement of the lemma is nothing but \( A = \sum_{i=1}^n A_i P(\Theta_i) \) where \( A_i f = \sum_{j=-\infty}^{j=\infty} T_j^* C_i T_j f \). It follows from Lemma 3.9 applied to the operators \( A_i P(\Theta_i) \) that

\[
|A| = \max_i |A_i P(\Theta_i)| \leq \max_i |C_i| \text{ ess sup } |\chi_{\Theta_i}(\theta)| = |A'|_\infty
\]

and thus \( |A| \leq |A'|_\infty \).

If \( A' \) is in the norm closure in \( L_\infty((0,2\pi); B(L_2(0,1))) \) of the class of Borel simple functions there is a Cauchy sequence of Borel simple functions \( A_i \) such that \( |A_i - A'|_\infty \to 0 \). Let \( A_j \) be the operator defined by

\[
A_j f = \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_3}^{N_3} T_j^* \int_0^{2\pi} e^{i\theta(j-k)} A_i'(\theta) T_k f d\theta.
\]
For \( f \) and \( g \) in \( H' \) let \( A(f, g) \) be the bilinear form defined by

\[
A(f, g) = \frac{1}{2\pi} \sum_j \sum_k \left( \int_0^{2\pi} e^{i\theta(j-k)} A'(\theta) T_k g d\theta, T_j f \right),
\]

a finite sum. Then

\[
| A(f, g) - (A_i f, g) | \leq \sum_j \sum_k \left| T_k f \right| \left| T_j g \right| | A' - A_i |_{\infty}
\]

and hence \( \lim_{i \to \infty} (A_i f, g) = A(f, g) \). Consequently \( | A(f, g) | \leq \lim_i \sup | A'_i |_{\infty} | f | | g | = | A' |_{\infty} | f | | g | \) and hence there is a bounded linear operator \( A \) on \( L_2(-\infty, \infty) \) of norm \( | A | \leq | A' |_{\infty} \) and such that for \( f \) and \( g \) in \( H' \), \( A(f, g) = (A f, g) \). Since \( A \) is bounded it follows that for \( f \) in \( L_2(-\infty, \infty) \),

\[
Af = \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T_j^* \int_0^{2\pi} e^{i\theta(j-k)} A'(\theta) T_k f d\theta.
\]

Q.E.D.

The simple functions are not dense in the norm topology of \( L_\infty((0, 2\pi); B(L_2(0, 1))) \) but nonetheless it is possible to employ an extension argument to get the above result for all of \( L_\infty((0, 2\pi); B(L_2(0, 1))) \) as follows:

5.12 Lemma. Let \( A' \) be in \( L_\infty((0, 2\pi); B(L_2(0, 1))) \). Then for \( f \) in \( L_2(-\infty, \infty) \) the limit in mean

\[
Af = \lim_{N_1 \to \infty} \frac{1}{2\pi} \sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} T_j^* \int_0^{2\pi} e^{i\theta(j-k)} A'(\theta) T_k f d\theta
\]

exists and defines a bounded operator in \( L_2(-\infty, \infty) \) of norm \( | A | \leq | A' |_{\infty} \). The operator \( A \) commutes with \( S \).

Proof. If \( f \) is in \( H' \) the summation above over \( k \) is finite and that over \( j \) is over functions with disjoint carriers, hence there is no problem in defining \( (Af)(t) \) almost everywhere though we must show it to be in \( L_2(-\infty, \infty) \).

Let \( f \) and \( g \) be fixed and arbitrary elements in \( H' \) and let the integer \( N \) be so large that \( f \) and \( g \) vanish outside \((- N, N)\). Let \( X \) be the finite dimensional subspace of \( L_2(0, 1) \) spanned by \( \{T_j f, T_j g\}, -N \leq j \leq N - 1 \), and let \( E \) be the orthogonal projection of \( L_2(0, 1) \) onto \( X \). By Theorem 4.15, \( EA'(\theta)E \) is the limit in \( L_\infty(((0, 2\pi); B(L_2(0, 1))) \) of a sequence of Borel simple functions \( A'_j(\theta) \).
Thus using the preceding lemma one can define the operator $A_E$ in $L_2(-\infty, \infty)$ by

$$A_E h = \frac{1}{2\pi} \text{LIM} \sum_{j=-N_1}^{N_2} \sum_{k=-N_3}^{N_4} T_j^* \int_0^{2\pi} e^{i\theta(j-k)} E A' E T_k h d\theta$$

and conclude that $|A_E| \leq |E A' E|_\infty \leq |A'|_\infty$. But

$$(E A'(\theta) E T_k f, T_j g) = (A'(\theta) T_k f, T_j g)$$

by the definition of $X$ and $E$ and hence $(A_E f, g) = (A f, g)$. Thus $|A f, g| \leq |A'|_\infty |f| |g|$, $f, g \in H'$, and hence $A$ is uniformly bounded on $H'$. The extension of $A$ to $L_2(-\infty, \infty)$ is easily done by an argument by now familiar.

Since $S T_j^* = T_{j-1}^*$ and $T_k S = T_{k+1}$, it follows that

$$S A f = \text{LIM} \sum_{j=-N_1}^{N_2} \sum_{k=-N_3}^{N_4} S T_j^* \int_0^{2\pi} e^{i\theta(j-k)} A'(\theta) T_k f d\theta$$

and that

$$A S f = \text{LIM} \sum_{j=-N_1}^{N_2} \sum_{k=-N_3}^{N_4} T_j^* \int_0^{2\pi} e^{i\theta(j-k)} A'(\theta) T_{k+1} f d\theta$$

and hence $A S = S A$. Q.E.D.

Theorems 13 and 15 below are analogous to Theorem 3.14.

5.13. **Theorem.** There is a 1 to 1 correspondence between bounded operators $A$ on $L_2(-\infty, \infty)$ which commute with $S$ and elements $A'$ in $L_\infty((0, 2\pi); B(L_2(0, 1)))$ such that

$$A f = \text{LIM} \sum_{j=-N_1}^{N_2} \sum_{k=-N_3}^{N_4} T_j^* \int_0^{2\pi} e^{i\theta(j-k)} A'(\theta) T_k f d\theta,$$

the limit existing in the mean for every $f$ in $L_2(-\infty, \infty)$. Furthermore, $|A| = |A'|_\infty$.

**Proof.** This is just the content of Lemmas 4 and 12. Q.E.D.
5.14 Definition. The 1 to 1 correspondence established in the theorem above is denoted by $A \sim A'$.

5.15 Theorem. Let $A$ and $B$ be bounded operators commuting with $S$, $A \sim A'$, $B \sim B'$. Then $A^\ast \sim A'^\ast$ and $AB \sim A'B'$.

Proof. For $f$ and $g$ in $L_2(-\infty, \infty)$,

$$
(f, A^\ast g) = (Af, g) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_0^{2\pi} (e^{i\theta(j-k)}A'(\theta)T_kf, T_jg) \, d\theta
$$

so $A^\ast \sim A'^\ast$.

For $f$ and $g$ in $H'$,

$$
(ABf, g) = (Bf, A^\ast g) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_0^{2\pi} (T_kf, e^{i\theta(k-j)}A'^\ast(\theta)T_jg) \, d\theta
$$

and hence, by virtue of Corollary 7,

$$
(ABf, g) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_0^{2\pi} e^{i(j-\phi)} \left( \sum_k e^{-i\theta k} B'(\theta)T_k f, \sum_l e^{-i\phi l} A'(\phi)^* T_l g \right) \, d\phi d\theta
$$

and hence $AB \sim A'(\theta)B'(\theta)$. Q.E.D.

The next corollary follows from Theorem 15 using much the same mode of proof as was used in deriving Corollary 3.15 from Theorem 3.14. Compare, however, Corollary 3.15 (e), (f) with Corollary 16 (e), (f).

5.16 Corollary. Let $A$ and $B$ be bounded operators commuting with $S$. Let $A \sim A'$, $B \sim B'$. Then

(a) $A$ is a projection if and only if $A'(\theta)$ is a projection, a.a. $\theta$,
(b) $A$ is normal if and only if $A'(\theta)$ is normal, a.a. $\theta$,
(c) $A$ is self-adjoint if and only if $A'(\theta)$ is self-adjoint, a.a. $\theta$, 
(d) $AB = BA$ if and only if $A'(\theta)B'(\theta) = B'(\theta)A'(\theta)$, a.a. $\theta$.

(e) The scalar $\lambda \in \rho(A)$ if and only if $\lambda \in \rho(A'(\theta))$, a.a. $\theta$, and $R(\lambda, A'(\theta)) \in L_\infty((0, 2\pi); B(L_2(0, 1)))$, in which case $R(\lambda, \Lambda) \sim R(\lambda, A'(\theta))$, and

(f) $A$ is quasi-nilpotent if and only if $\lim_{n\to\infty} |A^n|^{1/n}_\infty = 0$ (which implies but is not implied by the condition that $A'(\theta)$ be quasi-nilpotent for a.a. $\theta$).

**Proof.** Assertions (a) through (d) are immediate consequences of the uniqueness almost everywhere of $A'(\theta)$ and Theorem 15. To prove (e) suppose $\lambda \in \rho(A)$. Then $SR(\lambda, A) = R(\lambda, A)S$ so $R(\lambda, A) \sim C'$ for some $C'$ in $L_{\infty}((0, 2\pi); B(L_2(0, 1)))$. Now $R(\lambda, A) (\lambda I - A) = I$ and hence by Theorem 15, $C'(\theta) (\lambda I - A'(\theta)) = I$, a.a. $\theta$. Thus $C'(\theta) \sim R(\lambda, A'(\theta))$, a.a. $\theta$. Conversely, suppose $R(\lambda, A'(\theta)) \in L_{\infty}((0, 2\pi); B(L_2(0, 1)))$. Let $C \sim R(\lambda, A'(\theta))$. Then $C(\lambda I - A) \sim R(\lambda, A'(\theta)) (\lambda I - A'(\theta)) = I$ and hence

$$C(\lambda I - A) = I.$$ 

The operator $A$ is quasi-nilpotent if and only if $\lim_n |A^n|^{1/n}_\infty = 0$, and since $A^n \sim A'(\theta)_n$ and $|A^n| = |A^n|_\infty$ it follows that $A$ is quasi-nilpotent if and only if $\lim_n |A^n|^{1/n}_\infty = 0$. Q.E.D.

Theorem 15 and Corollary 16 establish the basic representations of operators commuting with $S$. The remainder of this part will develop the applications to spectral operator theory. Problems of finding suitable representations of large classes of measurable operator valued functions makes it necessary to use a result of Wermer [8] which is modified for the job at hand in Lemma 18.

5.17 Lemma. Let $\{A_i\}$ be a uniformly bounded sequence of operators commuting with $S$ and let $A_i \sim A'_i$. If for every $f$ and $g$ in $L_2(0, 1)$ $\lim_i (A'_i(\theta)f, g) = 0$, a.a. $\theta$, then for every $f$ and $g$ in $L_2((-\infty, \infty))$, $(A_i f, g) \to 0$.

**Proof.** For $f$ and $g$ in $H'$, $(A_i f, g)$ is the finite sum

$$(A_i f, g) = \frac{1}{2\pi} \int_0^{2\pi} \sum_j \sum_k e^{i\theta(j-k)} (A'_i(\theta)T_k f, T_j g) d\theta.$$ 

Let

$$h_i(\theta) = \frac{1}{2\pi} \sum_j \sum_k e^{i\theta(j-k)} (A'_i(\theta)T_k f, T_j g).$$ 

By hypothesis, $\lim_i h_i(\theta) = 0$, a.a. $\theta$. Also

$$\text{ess sup}_\theta |h_i(\theta)| \leq \frac{1}{2\pi} \sum_j \sum_k |T_k f| \cdot |T_j g| \cdot |A'_i|_2$$ 

$$\leq M \sum_j \sum_k |T_k f| \cdot |T_j g| < \infty.$$
where $M = \sup |A_i|$. Thus by the Lebesgue dominated convergence theorem, $\lim (A_i f, g) = \lim \int_0^{\infty} h_i(\theta) d\theta = 0$. The uniform boundedness principle, Theorem 2.5, then yields the same result for all $f$ and $g$ in $L_2 (-\infty, \infty)$. Q.E.D.

Wermer in [8, p. 355] has proved that if $E(\sigma)$ and $F(\delta)$ are two commuting resolutions of the identity on a Hilbert space, i.e., $E(\sigma)F(\delta) = F(\delta)E(\sigma)$ for all Borel sets $\sigma$, $\delta$, then there exists a bicontinuous operator $A$ such that $A^{-1}E(\sigma)A$ and $A^{-1}F(\delta)A$ are self-adjoint for every $\sigma$ and $\delta$. It follows easily from this result, and in fact is implicit in Wermer’s proof, that if $F(\delta)$ is already self-adjoint for all $\delta$, then $A$ may be picked so that $A^{-1}F(\delta)A = F(\delta)$, as is shown in the next lemma. In the proof in Volume 2 of Dunford and Schwartz of Wermer’s result [5, Lemma XV.6.21] the operator $A$ that is found already has the property.

5.18 Lemma. Let $E(\sigma)$ and $F(\delta)$ be resolutions of the identity on a Hilbert space $H$ such that $E(\sigma)F(\delta) = F(\delta)E(\sigma)$, $\sigma, \delta \in \beta$, and such that $F(\delta)$ is self-adjoint, $\delta \in \beta$. Then there exists a bicontinuous operator $A$ such that $A^{-1}E(\sigma)A$ is self-adjoint, $\sigma \in \beta$, and

$$A^{-1}F(\delta)A = F(\delta), \quad \delta \in \beta.$$  

Proof. Wermer’s result establishes the existence of a bicontinuous operator $A_0$ such that $A_0^{-1}E(\sigma)A_0$ and $A_0^{-1}F(\delta)A_0$ are self-adjoint. Let $B = A_0A_0^*$. Then $B$ is a positive definite operator with spectral representation $B = \int_0^{\infty} \lambda G(d\lambda)$ where $G(\delta)$ is self-adjoint with carrier $[a, b]$, $0 < a \leq b$. Let $A$ be the positive definite square root of $B$ defined by $A = \int_0^{\infty} \lambda^{1/2} G(d\lambda)$, where $\lambda^{1/2}$ is the positive square root of $\lambda$. Then $A^{-1}$ is bicontinuous, $A = A^*$, and $A$ commutes with every operator that commutes with $B$.

Let $T$ be either $E(\sigma)$ or $F(\sigma)$ for some $\sigma$. Since $A_0^{-1}T A_0$ is self-adjoint, $A_0^*T^*A_0^{-1} = A_0^{-1}T A_0$ and thus $T = (A_0 A_0^*)T^*(A_0 A_0^*)^{-1} = BTB^{-1} = A^*TA^{-1}$ and thus $A^{-1}TA = AT^*A^{-1}$, i.e., $(A^1TA)^* = A^1TA$. For $T = E(\sigma)$ this implies that $A^{-1}E(\sigma)A$ is self-adjoint for every $\sigma \in \beta$. The relation $T = BTB^{-1}$ implies that $F(\delta) = BF(\delta)B^{-1}$, i.e., $B$ commutes with $F(\delta)$ and thus $A$ commutes with $F(\delta)$. Thus $A$ has the desired properties. Q.E.D.

5.19 Corollary. Let $B$ be a scalar type spectral operator which commutes with $S$. Then there exists a bicontinuous operator $A$ which commutes with $S$ such that $A^{-1}BA$ is normal.

Proof. Let $E(\delta)$ be the self-adjoint resolution of the identity of $S$ and let $F(\sigma)$ be the resolution of the identity of $B$. Since $BS = SB$ it follows from Theorem 2.23 that $E(\delta)F(\sigma) = F(\sigma)E(\delta)$. Hence by the preceding
lemma there exists a bicontinuous operator $A$ which commutes with $E(\delta)$ such that $A^{-1}F(\sigma)A$ is self-adjoint. Hence $A$ commutes with $S$. Since a scalar type operator with self-adjoint resolution of the identity is normal, $A^{-1}BA$ is normal. Q.E.D.

5.20 Lemma. Let $B'$ in $L_\infty((0, 2\pi); B(L_2(0, 1)))$ be such that $B'(\theta)$ is normal, a.a. $\theta$. Let the representation $B'(\theta)$ of $B'$ be fixed. For those $\theta$ such that $B'(\theta)$ is normal define $E'(\theta, \delta)$ to be the resolution of the identity of $B'(\theta)$. Then for $\delta \in \beta$, $E'(\cdot, \delta)$ is in $L_\infty((0, 2\pi); B(L_2(0, 1)))$.

Proof. Since $B'(\theta)$ is normal, a.a. $\theta$, $E'(\theta, \delta)$ is a self-adjoint projection for a.a. $\theta$ and hence $|E'(\theta, \delta)| \leq 1$, a.a. $\theta$. Thus it is only necessary to show that $E'(\theta, \delta)f$ is in $M(L_2(0, 1))$ for every $f$ in $L_2(0, 1)$ and $\delta \in \beta$. Let $\delta$ be a half-closed rectangle, i.e., a set of the form $\{z = x + iy | a \leq x < b, c \leq y < d, -\infty < a, b, c, d < \infty\}$, and let $D = \{z | |z| \leq |B'|_{\infty}\}$ so that $\sigma(B'(\theta)) \subseteq D$, a.a. $\theta$. It is not hard to find a sequence of functions, $\{C_i\}$ continuous on $D$, uniformly bounded on $D$, and such that the pointwise limit

$$\lim_{i} C_i(z) = \chi_{D}(z), \quad z \in D.$$  

By the Stone-Weierstrass Theorem, for each $i$ there exists a polynomial $P_i$ in $z$ and $\bar{z}$ such that

$$\sup_{z \in D} |P_i(z, \bar{z}) - C_i(z)| \leq i^{-1}$$

and hence $\{P_i\}$ is uniformly bounded on $D$ and $\lim_{i} P_i(z, \bar{z}) = \chi_{D}(z), \quad z \in D$. Thus by a result in the theory of the operational calculus of normal operators ([5, Corollary X.2.8]),

$$\lim_{i} P_i(B'(\theta), B'^{*}(\theta))f = E'(\theta, \delta)f$$

for every $\theta$ such that $B'(\theta)$ is normal. By Corollary 4.6, $P_i(B'(\theta), B'^{*}(\theta))f$ is measurable and hence by Lemma 4.7 $E'(\theta, \delta)f$ is measurable.

Let $R$ be the ring generated by half-rectangles. It follows that $E'(\theta, \delta)f$ is measurable, $\delta \in R$. Let $M$ be the set of $\delta \in S(R) = \beta$ such that $E'(\theta, \delta)f$ is measurable. Let $\{\delta_i\}$ be a monotone sequence in $M$ and let $\delta = \lim_i \delta_i$. Then $E'(\theta, \delta)f = \lim_i E'(\theta, \delta_i)f$, all $\theta$ such that $B'(\theta)$ is normal, i.e., a.a. $\theta$, and hence $E'(\theta, \delta)f$ is measurable and thus $\delta \in M$. Thus $M$ is a monotone class, hence $M = \beta$. Q.E.D.

It is an open question whether the above result is true if one only assumes that $B'(\theta) \in L_\infty((0, 2\pi); B(L_2(0, 1)))$ and is scalar for a.a. $\theta$ with essentially uniformly bounded resolution of the identity, to say nothing of the case where $B'(\theta)$ has a quasi-nilpotent part.

We now prove the analogue of Theorem 3.16.
5.21 Theorem. Let $A$ be a bounded linear operator in $L_2(-\infty, \infty)$ such that $AS = SA$ and let $A \sim A'(\theta)$. Then $A$ is a spectral operator if and only if $A'(\theta)$ is a spectral operator for a.a. $\theta$ with resolution of the identity $E'(\theta, \delta)$, scalar part $B'(\theta)$ and radical part $N'(\theta)$ such that $E'(\theta, \delta)$, $B'(\theta)$, and $N'(\theta) \in L_\infty((0, 2\pi); B(L_2(0, 1)))$.

$$\sup_{\delta \in \delta} |E'(\theta, \delta)|_\infty < \infty, \quad \text{and} \quad \lim_{n \to \infty} (|N'(\theta)^n|_\infty)^{1/n} = 0.$$  

If this is the case and $A = B + N$ where $B$ is the scalar part of $A$ and $N$ is the radical part, and $A$ has the resolution of the identity $E(\delta)$, then $B \sim B'(\theta)$, $N \sim N'(\theta)$, $E(\delta) \sim E'(\theta, \delta)$.

Proof. Suppose first that $A'(\theta)$ is spectral, a.a. $\theta$ with resolution of the identity $E'(\theta, \delta)$, scalar part $B'(\theta)$ and radical part $N'(\theta)$ such that $B'(\theta)$, $N'(\theta)$ are in $L_\infty((0, 2\pi); B(L_2(0, 1)))$.

$$\sup_{\delta \in \delta} |E'(\theta, \delta)|_\infty = K < \infty \quad \text{and} \quad \lim_{n \to \infty} |N'(\theta)^n|_\infty^{1/n} = 0.$$  

Let $B \sim B'$, $N \sim N'$, $E(\delta) \sim E'(\theta, \delta)$. It follows from Theorem 14 and Corollary 16 that $A = B + N$, $BN = NB$, $N$ is quasi-nilpotent, and that $E(\delta)$ is a finitely additive, commutative, projection valued measure defined for $\delta$ in $\beta$ such that $|E(\delta)| = |E'(\theta, \delta)|_\infty \leq K$. To show that $E(\delta)$ is strongly countably additive it is sufficient to show that it is weakly continuous from above at 0, i.e., given any decreasing sequence $\{\delta_i\}$ in $\beta$ such that $\delta_i \to 0$, the null set and any $f$ and $g$ in $L_2(-\infty, \infty)$ that $\lim_i (E(\delta_i)f, g) = 0$. Since $E'(\theta, \delta)$ is countably additive for a.a. $\theta$, $\lim_i (E'(\theta, \delta_i)f, g) = 0$, a.a. $\theta$, $f$, $g$ in $L_2(0, 1)$. Thus Lemma 17 implies $(E(\delta_i)f, g)$ approaches 0 for $f$ and $g$ in $L_2(-\infty, \infty)$, so $E(\delta)$ is strongly countably additive.

The proof that $B$ is a scalar type spectral operator with resolution of the identity $E(\delta)$ will consist of a proof that $B = \int \lambda E(d\lambda)$. Let $D$ be the disc $\{z \mid |z| \leq |A|\}$. Then $\sigma(A'(\theta)) \subseteq D$, a.a. $\theta$. Let $\pi_i$ be any partition of $D$ of norm less that $i^{-1}$, i.e., $\pi_i = \{\sigma_{i(j)}\}$ is a finite collection of pairs $(\sigma_{i(j)}, \lambda_{i(j)})$ where $\sigma_{i(j)}$ is a Borel subset of $D$, $\lambda_{i(j)} \in \sigma_{i(j)}$. $\cup_{j} \sigma_{i(j)} = D$, $\sigma_{i(j)} \cap \sigma_{i(k)} = \Phi$, $j \neq k$, and $\max_j \sup_{u \in \sigma_{i(j)}} |u - v| < i^{-1}$. Let $B_i'(\theta) = \sum_j \lambda_{i(j)}E_i'(\theta, \sigma_{i(j)})$ and let $B_i = \sum_j \lambda_{i(j)}E_i(\sigma_{i(j)})$. Then $B_i \sim B_i'$. Now by definition of the Riemann integral and Theorem 2.24 $\lim_{i \to \infty} B_i - \int \lambda E(d\lambda) = 0$, a.a. $\eta$. Hence by Lemma 17, for $f \in L_2(-\infty, \infty)$, $\lim_{i \to \infty} (B - B_i)f \to 0$, i.e., $B = \int \lambda E(d\lambda)$ and thus $B$ is a scalar type spectral operator. Hence by Theorem 2.26 $A$ is a spectral operator.

Suppose now that $A$ is an arbitrary spectral operator such that $AS = SA$. Let $A = B + N$ where $B$ and $N$ are the scalar and radical parts of $A$, res-
pectively. It follows from Theorem 2.23 that $SB = BS$ and $SN = NS$. It follows from Corollary 19 that there is a bicontinuous operator $C$ such that $CS = SC$ and $B_1 = C^{-1}BC$ is normal. Let $C \sim C'$. Now $B_1 \sim B'_1$ where $B'_1(\theta)$ is normal, a.a. $\theta$. Let $B'_1(\theta)$ be a fixed representation of $B'_1$. Let $E'_1(\theta, \delta)$ be the resolution of the identity for $B'_1(\theta)$, defined for all $\theta$ such that $B'_1(\theta)$ is normal. Then $E'_1(\theta, \delta)$ is in $L_\infty((0, 2\pi); B(L_2(0, 1)))$ by Lemma 20. Let $C(\theta)$ be a fixed representation of $C'$, defined for a.a. $\theta$. Let $E'(\theta, \delta) = C'(\theta)E'_1(\theta, \delta)C'^{-1}(\theta)$. Then for all $\theta$ for which $E'(\theta, \delta)$ is defined (which is independent of $\delta$), $E'(\theta, \delta)$ is the resolution of the identity for the scalar type operator $C'(\theta)B'_1(\theta)C'^{-1}$ which is, for a.a. $\theta$, just $B'(\theta)$. Let $E(\delta) \sim E'(\theta, \delta)$. Applying the preceding paragraph it follows that $E(\delta)$ is the resolution of the identity for $B$. Since Corollary 16 implies that $N'(\theta)$ is quasi-nilpotent, a.a. $\theta$, and $B'(\theta)N'(\theta) = N'(\theta)B'(\theta)$, a.a. $\theta$, it follows that for a.a. $\theta$ $A'(\theta)$ is a spectral operator with resolution of the identity $E'(\theta, \delta)$, scalar part $B'(\theta)$, and radical part $N'(\theta)$; and that $B \sim B'(\theta)$, $N \sim N'(\theta)$, $E(\delta) \sim E'(\theta, \delta)$ where $E(\delta)$ is the resolution of the identity of $A$. Q.E.D.

The use of Wermer's result in the above proof is more than a convenience. To be sure, it is possible to define $E'(\cdot, \delta)$ as an element in $L_\infty((0, 2\pi); B(L_2(0, 1)))$ and show directly that $E'(\cdot, \delta)$ is countably additive in the sense that, given any decreasing sequence $\{\delta_i\}$ of Borel sets such that $\lim_i \delta_i = \Phi$, there is a set $\Theta$ of measure 0 such that $E'(\theta, \delta_i) f \rightarrow 0$, $f \in L_2(0, 1)$, $\theta \notin \Theta$. However, it is still necessary to show that for each $\delta$ a representation $E'(\theta, \delta)$ of $E'(\cdot, \delta)$ can be picked so that the set $\Theta$ can be made independent of the choice of the $\{\delta_i\}$. It seems that to do this one must invoke the axiom of choice in one of its many forms. This has been done here by using Wermer's result which employs Tychonoff's theorem in its proof.

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