On the Neumann problem for elliptic equations involving the $p$-Laplacian

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ABSTRACT

The aim of this paper is to establish the existence of an unbounded sequence of weak solutions to a Neumann problem for elliptic equations involving the $p$-Laplacian.

Here and in the sequel, $\Omega \subset \mathbb{R}^k$ is a bounded open set, with a boundary of class $C^1$, $q \in L^\infty(\Omega)$ with $\text{ess inf}_{\Omega} q > 0$, $p > k$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

In the very interesting paper [11], Ricceri, by using an infinitely many critical points theorem [10, Theorem 2.5], established the existence of an unbounded sequence of weak solutions under an appropriate oscillating behaviour of $f$ for a Neumann problem of the type:

$$
\begin{cases}
-\Delta_p u + q(x)|u|^{p-2}u = f(x, u) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\Delta_p = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian and $\nu$ is the outer unit normal to $\partial \Omega$.

Subsequently, starting from the fruitful papers of Ricceri, cited above, several authors gave further results on infinitely many solutions for Neumann elliptic problems (see, for instance, [3–6,9]).

The aim of this paper is to establish the same conclusion of the result of Ricceri in [11], but under a completely different assumption. For reader’s convenience, Theorem 3 of [11] is here recalled.

**Theorem 1.** Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that the following conditions hold:

(i) let $\{a_n\}$ and $\{b_n\}$ be two sequences in $\mathbb{R}^+$ satisfying

$$
a_n < b_n, \quad \forall n \in \mathbb{N}, \quad \lim_{n \to +\infty} b_n = +\infty, \quad \lim_{n \to +\infty} \frac{a_n}{b_n} = 0,
$$

$$
\max \left\{ \sup_{\xi \in [a_n, b_n]} \int_a^\xi h(t) \, dt, \sup_{\xi \in [-b_n, -a_n]} \int_{-a_n}^\xi h(t) \, dt \right\} \leq 0, \quad \forall n \in \mathbb{N},
$$

and

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(ii) \[
\limsup_{|\xi| \to +\infty} \frac{\int_0^\xi h(t) \, dt}{|\xi|^p} = +\infty.
\]

Then, for every \(\alpha, \beta \in L^1(\Omega)\), with \(\min\{\alpha(x), \beta(x)\} \geq 0\) almost everywhere in \(\Omega\) and with \(\alpha \neq 0\), and for every continuous function \(g : \mathbb{R} \to \mathbb{R}\) satisfying
\[
\sup_{\xi \in \mathbb{R}} \int_0^\xi g(t) \, dt \leq 0
\]
and
\[
\liminf_{|\xi| \to +\infty} \frac{\int_0^\xi g(t) \, dt}{|\xi|^p} > -\infty,
\]
the problem
\[
\begin{aligned}
-\Delta_p u + q(x)|u|^{p-2}u &= \alpha(x)h(u) + \beta(x)g(u) & \text{in } \Omega, \\
\partial u/\partial \nu &= 0 & \text{on } \partial \Omega
\end{aligned}
\]
(\(P_{\alpha, \beta}\))

admits an unbounded sequence of weak solutions in \(W^{1,p}(\Omega)\).

Owing to our principal result (Theorem 3), we can restate Theorem 1, replacing the assumption (i) with the following

(i*) \[
\liminf_{\xi \to +\infty} \frac{\max_{|z| \leq \xi} \int_0^\xi h(t) \, dt}{\xi^p} = 0
\]
(see Theorem 5 and Remark 1).

Clearly, when \(h\) is nonnegative, (i*) becomes \(\liminf_{\xi \to +\infty} \frac{\int_0^\xi h(t) \, dt}{\xi^p} = 0\).

An example of elliptic Neumann problem which cannot be studied by Ricceri’s results, but admits, owing to our result, infinitely many solutions, is pointed out (see Example 1 and Remark 2).

We observe that, when the nonlinear term is independent of \(x\) and nonnegative, Theorem 3 assumes a simpler form (see Corollary 1). Moreover, these results hold again substituting \(\xi \to +\infty\) with \(\xi \to 0^+\) (Theorem 4).

We also remark that infinitely many solutions to elliptic Neumann problems have been obtained by [7] and [8] by using a different approach. It is very easy to verify that the results obtained in these papers (where sign assumptions are made) and ours, are mutually independent.

In order to prove Theorem 3, we use an infinitely many critical points result (see Theorem 2).

Let \(X\) be a reflexive real Banach space, \(\Phi : X \to \mathbb{R}\) is a (strongly) continuous, coercive sequentially weakly lower semicontinuous and Gâteaux differentiable functional, \(\Psi : X \to \mathbb{R}\) is a sequentially weakly upper semicontinuous and Gâteaux differentiable functional.

For all \(r > \inf_X \Phi\), put
\[
\varphi(r) = \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{(\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)}
\]
and
\[
\gamma = \liminf_{r \to +\infty} \varphi(r), \quad \delta = \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).
\]

The following result, as given in [2], is a more precise version of [10, Theorem 2.5].

**Theorem 2.** Under the above assumptions on \(X, \Phi, \) and \(\Psi\), the following conditions hold:

1. If \(\gamma < +\infty\) then, for each \(\lambda \in [0, \frac{1}{\gamma}]\), the following alternative holds: either the functional \(\Phi - \lambda \Psi\) has a global minimum, or there exists a sequence \(\{u_n\}\) of critical points (local minima) of \(\Phi - \lambda \Psi\) such that \(\lim_{n \to +\infty} \Phi(u_n) = +\infty\).
2. If \(\delta > +\infty\) then, for each \(\lambda \in \left[0, \frac{1}{\delta}\right]\), the following alternative holds: either there exists a global minimum of \(\Phi\) which is a local minimum of \(\Phi - \lambda \Psi\), or there exists a sequence \(\{u_n\}\) of pairwise distinct critical points (local minima) of \(\Phi - \lambda \Psi\), with \(\lim_{n \to +\infty} \Phi(u_n) = \inf_X \Phi\), which weakly converges to a global minimum of \(\Phi\).

After introducing the questions we are interested in and the approach we intend to follow, we consider the following problem.
\[
\begin{aligned}
-\Delta u + q(x)|u|^{p-2}u &= \lambda f(x, u) \quad \text{in } \Omega, \\
\partial u / \partial v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] 

(P\_\lambda)

where \( \lambda \) is a real positive parameter.

We recall that a weak solution of the problem (P\_\lambda) is any \( u \in W^{1,p}(\Omega) \), such that

\[
\int_\Omega (|\nabla u(x)|^p - \lambda^n u(x)) \, dx + \int_\Omega f(x, u(x)) u(x) \, dx = 0, \quad \forall v \in W^{1,p}(\Omega).
\]

Put

\[
c = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\sup_{x \in \Omega} |u(x)|}{(\int_\Omega |\nabla u(x)|^p \, dx + \int_\Omega q(x) u(x)^p \, dx)^{\frac{1}{p}}}.
\]

(1)

If \( \Omega \) is convex, an explicit upper bound for the constant \( c \) is

\[
c \leq 2^{\frac{p-1}{p}} \max \left\{ \frac{1}{\int_\Omega q(x) \, dx}, \frac{d}{n^\frac{1}{p}} \left( \frac{p-1}{p} \right) \right\} \frac{\|u\|}{\int_\Omega q(x) \, dx},
\]

where \( d = \text{diam}(\Omega) \) (see \cite[Remark 1]{1}).

Put \( F(x, t) = \int_0^t f(x, \xi) \, d\xi \) for all \( (x, t) \in \Omega \times \mathbb{R} \),

\[
A = \liminf_{\xi \to +\infty} \int_\Omega \max_{|t| \leq \xi} F(x, t) \, dx, \quad B = \limsup_{\xi \to +\infty} \int_\Omega F(x, \xi) \, dx,
\]

where, as usual, \( \|q\|_1 = \int_\Omega q(x) \, dx \), moreover

\[
\lambda_1 = \frac{\|q\|_1}{pB}, \quad \lambda_2 = \frac{1}{pc^p A}.
\]

(2)

Our main result is the following.

**Theorem 3.** Assume that

\[
\liminf_{\xi \to +\infty} \int_\Omega \max_{|t| \leq \xi} F(x, t) \, dx \frac{1}{\xi^p} < \frac{1}{c^p \|q\|_1} \limsup_{\xi \to +\infty} \int_\Omega F(x, \xi) \, dx \frac{1}{\xi^p}.
\]

(3)

Then, for each \( \lambda \in ]\lambda_1, \lambda_2[ \), where \( \lambda_1, \lambda_2 \) are given in (2), problem (P\_\lambda) possesses an unbounded sequence of weak solutions in \( W^{1,p}(\Omega) \).

**Proof.** Our aim is to apply part (1) of Theorem 2. Take as \( X \) the Sobolev space \( W^{1,p}(\Omega) \) endowed with the norm

\[
\|u\| = \left( \int_\Omega |\nabla u(x)|^p \, dx + \int_\Omega q(x)|u(x)|^p \, dx \right)^{\frac{1}{p}}.
\]

We observe that the above norm is equivalent to the usual one.

On the space \( C^0(\overline{\Omega}) \), we consider the norm \( \|u\|_\infty := \sup_{x \in \Omega} |u(x)| \). Since \( p > k \), \( X \) is compactly embedded in \( C^0(\overline{\Omega}) \), and taking into account (1), we have

\[
\|u\|_\infty \leq c \|u\|.
\]

(4)

For each \( u \in X \), put

\[
\Phi(u) := \frac{1}{p} \|u\|^p, \quad \Psi(u) := \int_\Omega F(x, u(x)) \, dx.
\]

It is well known that the critical points in \( X \) of the functional \( \Phi - \lambda \Psi \) are exactly the weak solutions of the problem (P\_\lambda).

Pick \( \lambda \in ]\lambda_1, \lambda_2[ \). Let \( \{\rho_n\} \) be a real sequence such that \( \lim_{n \to \infty} \rho_n = +\infty \) and

\[
\lim_{n \to \infty} \int_\Omega \max_{|t| \leq \rho_n^p} F(x, t) \, dx = A.
\]

Put \( r_n = \left( \frac{\rho_n}{c} \right)^p \) for all \( n \in \mathbb{N} \). Taking into account \( \|v\|^p < pr_n \) and \( \|v\|_\infty \leq c \|v\| \), one has \( |v(x)| \leq \rho_n \), for every \( x \in \Omega \). Therefore,

\[
\varphi(r_n) = \inf_{\|v\|^p < pr_n} \frac{\sup_{\|v\|^p < pr_n} \int_\Omega F(x, v(x)) \, dx - \int_\Omega F(x, u(x)) \, dx}{r_n - \frac{\|u\|^p}{p}} \leq \frac{\sup_{\|v\|^p < pr_n} \int_\Omega F(x, v(x)) \, dx}{r_n}.
\]
Hence,

\[
\varphi(r_n) \leq \frac{\int_\Omega \max_{|t| \leq \rho_n} F(x, t) \, dx}{r_n} = \frac{pc^p \int_\Omega \max_{|t| \leq \rho_n} F(x, t) \, dx}{\rho_n^p}, \quad \forall n \in \mathbb{N}.
\]

Then,

\[
\gamma \leq \liminf_{n \to +\infty} \varphi(r_n) \leq pc^p A < +\infty.
\]

Now, we claim that the functional \( \Phi - \lambda \Psi \) is unbounded from below. Let \( \{d_n\} \) be a real sequence such that \( \lim_{n \to +\infty} d_n = +\infty \) and

\[
\lim_{n \to +\infty} \int_{\Omega} F(x, d_n) \, dx = B. \tag{5}
\]

For each \( n \in \mathbb{N} \), put \( w_n(x) = d_n \), for all \( x \in \Omega \). Clearly \( w_n \in W^{1,p}(\Omega) \) for each \( n \in \mathbb{N} \). Hence,

\[
\|w_n\|^p = d_n^p \|q\|_1
\]

and

\[
\Phi(w_n) - \lambda \Psi(w_n) = \frac{\|w_n\|^p}{p} - \lambda \int_{\Omega} F(x, w_n(x)) \, dx = \frac{d_n^p \|q\|_1}{p} - \lambda \int_{\Omega} F(x, d_n) \, dx.
\]

Now, if \( B < +\infty \), let \( \varepsilon \in [0, B - \frac{\|q\|_1}{pc^p}] \). From (5) there exists \( \nu_{\varepsilon} \) such that

\[
\int_{\Omega} F(x, d_n) \, dx > (B - \varepsilon) d_n^p, \quad \forall n > \nu_{\varepsilon}.
\]

Therefore,

\[
\Phi(w_n) - \lambda \Psi(w_n) = \frac{d_n^p \|q\|_1}{p} - \lambda \int_{\Omega} F(x, d_n) \, dx < \frac{d_n^p \|q\|_1}{p} - \lambda d_n^p (B - \varepsilon) = d_n^p \left( \frac{\|q\|_1}{p} - \lambda (B - \varepsilon) \right).
\]

From the choice of \( \varepsilon \), one has

\[
\lim_{n \to +\infty} \left[ \Phi(w_n) - \lambda \Psi(w_n) \right] = -\infty.
\]

If \( B = +\infty \), fix \( M > \frac{\|q\|_1}{pc^p} \). From (5) there exists \( \nu_M \) such that

\[
\int_{\Omega} F(x, d_n) \, dx > Md_n^p, \quad \forall n > \nu_M.
\]

Moreover,

\[
\Phi(w_n) - \lambda \Psi(w_n) = \frac{d_n^p \|q\|_1}{p} - \lambda \int_{\Omega} F(x, d_n) \, dx < \frac{d_n^p \|q\|_1}{p} - \lambda Md_n^p = d_n^p \left( \frac{\|q\|_1}{p} - \lambda M \right).
\]

Taking into account the choice of \( M \), also in this case, one has

\[
\lim_{n \to +\infty} \left[ \Phi(w_n) - \lambda \Psi(w_n) \right] = -\infty.
\]

From part (1) of Theorem 2, the functional \( \Phi - \lambda \Psi \) admits a sequence \( u_n \) of critical points, and the conclusion is proven.

Now, we point out the following consequence of Theorem 3.

**Corollary 1.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a nonnegative continuous function. Moreover assume that

\[
\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} < \frac{1}{c^p \|q\|_1} \quad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p}.
\]

Then, for each \( \lambda \in \left[ \frac{\|q\|_1}{pc^p |\Omega| \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} \right], \) the problem

\[
\begin{align*}
-\Delta_p u + q(x)|u|^{p-2} u &= \lambda f(u) & \text{in } \Omega, \\
\partial u / \partial \nu &= 0 & \text{on } \partial \Omega
\end{align*}
\]

possesses an unbounded sequence of weak solutions in \( W^{1,p}(\Omega) \).
Furthermore, by using (2) of Theorem 2 and arguing as in the proof of Theorem 3, put
\[
\lambda_1^* := \frac{\|q\|_1}{\|p\| \limsup_{\xi \to 0^+} \frac{\int_0^\xi F(x,\xi) \, dx}{\xi^p}} \quad \text{and} \quad \lambda_2^* := \frac{1}{pc^p \liminf_{\xi \to 0^+} \frac{\int_{\Omega} \max_{|\xi| \leq \xi} F(x,\xi) \, dx}{\xi^p}},
\]
we obtain the following result.

**Theorem 4.** Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be a continuous function.

Assume that
\[
\liminf_{\xi \to 0^+} \frac{\int_{\Omega} \max_{|\xi| \leq \xi} F(x,\xi) \, dx}{\xi^p} \leq \frac{1}{pc^p \liminf_{\xi \to 0^+} \frac{\int_{\Omega} F(x,\xi) \, dx}{\xi^p}}.
\]

Then, for each \( \lambda \in ]\lambda_1^*, \lambda_2^*[ \), problem (\( P_\lambda \)) possesses a sequence of non-zero weak solutions which strongly converges uniformly to 0 in \( W^{1,p}(\Omega) \).

The following ensures what was claimed at the beginning.

**Theorem 5.** Let \( h : \mathbb{R} \to \mathbb{R} \) be a continuous function, such that
\[
(i') \quad \liminf_{\xi \to +\infty} \frac{\int_0^\xi h(z) \, dz}{\xi^p} < +\infty \quad \text{and} \quad (ii') \quad \limsup_{\xi \to +\infty} \frac{\int_0^\xi h(t) \, dt}{\xi^p} = +\infty.
\]

Then, for every \( \alpha, \beta \in L^1(\Omega) \), with \( \min\{|\alpha(x)|, |\beta(x)|\} \geq 0 \) almost everywhere in \( \Omega \) and with \( \alpha \neq 0 \), and for every continuous function \( g : \mathbb{R} \to \mathbb{R} \) satisfying
\[
\sup_{t \in \mathbb{R}} \int_0^\xi g(t) \, dt \leq 0 \tag{6}
\]
and
\[
\liminf_{\xi \to +\infty} \frac{\int_0^\xi g(t) \, dt}{\xi^p} > -\infty, \tag{7}
\]
the problem
\[
\begin{cases}
-\Delta_p u + g(x)|u|^{p-2}u = \lambda (\alpha(x)h(u) + \beta(x)g(u)) & \text{in } \Omega, \\
\frac{\partial u}{\partial v} = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{P_{\alpha, \beta, \lambda}}
\]
for every \( \lambda \in ]0, \frac{1}{pc^p \liminf_{\xi \to +\infty} \frac{\max_{|\xi| \leq \xi} h(z) \, dz}{\xi^p}} \), admits an unbounded sequence of weak solutions in \( W^{1,p}(\Omega) \).

**Proof.** In order to prove this result, we apply Theorem 3.

To this end, put \( f(x,z) = \alpha(x)h(z) + \beta(x)g(z) \), for all \((x,z) \in \Omega \times \mathbb{R}\).

One has \( F(x,\xi) = \int_0^\xi f(x,z) \, dz = \alpha(x)\xi \xi + \beta(x)\xi \xi \), for all \((x,\xi) \in \Omega \times \mathbb{R}\), where \( G(\xi) = \int_0^\xi g(z) \, dz \) and \( H(\xi) = \int_0^\xi h(z) \, dz \), for all \( \xi \in \mathbb{R} \).

From (ii') and (7), one has
\[
\limsup_{\xi \to +\infty} \frac{\int_{\Omega} F(x,\xi) \, dx}{\xi^p} = \limsup_{\xi \to +\infty} \frac{\|\alpha\|_1 \xi \xi + \|\beta\|_1 \xi \xi}{\xi^p} = +\infty.
\]

On the other hand, from (i') and (6)
\[
\liminf_{\xi \to +\infty} \frac{\int_{\Omega} \max_{|\xi| \leq \xi} F(x,\xi) \, dx}{\xi^p} \leq \liminf_{\xi \to +\infty} \frac{\|\alpha\|_1 \max_{|\xi| \leq \xi} H(t) \, dx}{\xi^p} < +\infty.
\]

Hence, (3) of Theorem 3 is satisfied. Hence, since
\[
\lambda_2 = \frac{1}{pc^p \liminf_{\xi \to +\infty} \frac{\int_{\Omega} \max_{|\xi| \leq \xi} F(x,\xi) \, dx}{\xi^p}} \geq \frac{1}{pc^p \liminf_{\xi \to +\infty} \frac{\int_{\Omega} \max_{|\xi| \leq \xi} H(t) \, dx}{\xi^p}},
\]
the conclusion is achieved. \( \square \)
Remark 3. Clearly, if \( \liminf_{\xi \to +\infty} \frac{\max_{\xi \leq \xi} \int_{\Omega} h(z) dz}{\xi^p} = 0 \), (i') is verified and problem \((P_{\alpha, \beta, \lambda})\) admits an unbounded sequence of weak solutions in \( W^{1, p}(\Omega) \) for each \( \lambda > 0 \). Hence, we obtain what we stated at the beginning.

Example 1. Put
\[
a_n := \frac{2n!(n+2)! - 1}{4(n+1)!}, \quad b_n := \frac{2n!(n+2)! + 1}{4(n+1)!}
\]
for every \( n \in \mathbb{N} \) and define the positive continuous function \( h : \mathbb{R} \to \mathbb{R} \) as follows
\[
h(\xi) = \begin{cases} \frac{32(n+1)^2|p-1|\xi^{p-1}(n+1)^{p-2}-(n-1)|p|\xi^{p-2}}{\pi} & \sqrt{1-\frac{1}{16(n+1)^2}} - (\xi - \frac{n(n+2)}{2})^2 + 1 \\
1 & \text{otherwise.}
\end{cases}
\]

One has \( \lim_{n \to +\infty} \frac{h(\alpha_n)}{\alpha_n^p} = +\infty \) and \( \lim_{n \to +\infty} \frac{h(\alpha_n)}{\alpha_n^2} = 0 \). Therefore, we obtain \( \liminf_{\xi \to +\infty} \frac{h(\xi)}{\xi^p} = 0 \) and \( \limsup_{\xi \to +\infty} \frac{h(\xi)}{\xi^p} = +\infty \).

So, from Theorem 5, for each \( \lambda > 0 \), the problem
\[
\begin{cases}
-\Delta_p u + q(x)|u|^{p-2}u = \lambda h(u) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]

admits an unbounded sequence of weak solutions in \( W^{1, p}(\Omega) \).

Remark 2. We observe that Theorem 3 of [11] requires a kind of sign hypothesis on the nonlinear term. More precisely, the function cannot be positive only. Therefore, we note the problem in Example 1, given above, cannot be investigated using results in [11], seeing that the function is positive.

Remark 3. We also observe that in Theorem 1 a nonlinear term behaviour at \( -\infty \) is requested (see assumption (ii)). On the contrary, in Theorem 3 and its consequences, the nonlinear term behaviour at \( -\infty \) is completely arbitrary.

References