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Oscillation of Solutions of Impulsive Delay Differential Equations[1]

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The oscillatory and asymptotic behaviour of first order impulsive delay differential equations and inequalities with oscillatory coefficients are studied. Some new results are obtained. © 2001 Academic Press

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1. INTRODUCTION

Impulsive delay differential equations may express several real-world simulation processes which depend on their prehistory and are subject to short-time disturbances. Such processes occur in the theory of optimal control, theoretical physics, population dynamics, biotechnologies, economics, etc. In the last few years, the qualitative theory of solutions of impulsive ordinary differential equations and the oscillation of delay differential equations have been studied by many mathematicians, respectively. We refer to the monographs [8, 11]. However, not much has been developed in the direction of impulsive delay differential equations. Most of the publications are devoted to oscillation of differential equations with coefficients of definite sign (see [1–7, 12, 13]). The purpose of this paper is

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to study oscillatory properties and asymptotic behaviour of solutions of impulsive delay differential equations and inequalities with oscillatory coefficients. To the best of our knowledge, this paper is probably the first publication on the above mentioned equations and inequalities. In the particular case where equations with coefficients of definite sign, our results improve and generalize some known results in the recent literature.

Let $N = \{1, 2, \dots, \}$. Consider the first order impulsive delay differential inequalities

$$\begin{cases} y'(t) + a(t)y(t) + p(t)y(t-\tau) \le 0, & t \ne t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), & k \in N, \end{cases} \tag{1}$$

$$\begin{cases} y'(t) + a(t)y(t) + p(t)y(t-\tau) \ge 0, & t \ne t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), & k \in N, \end{cases} \tag{2}$$

and the corresponding impulsive delay differential equation

$$\begin{cases} y'(t) + a(t)y(t) + p(t)y(t-\tau) = 0, & t \ne t_k \\ y(t_k^+) - y(t_k) = b_k y(t_k), & k \in N, \end{cases} \tag{3}$$

under the following conditions:

(A_1) $0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, are fixed points with $\lim_{k\to\infty} t_k = \infty$;

(A_2) $a, p \in ([t_0, \infty), R)$ are locally summable functions and τ is a positive constant;

(A_3) $b_k \in (-\infty, -1) \cup (-1, \infty)$ are constants for $k \in N$.

Remark 1. It is obvious that all solutions of (3) are oscillatory if there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $b_{n_k} < -1$ for $k = 1, 2, \dots$. In the sequel we assume $-1 < b_k < \infty$ for $k \in N$.

For any $\sigma \ge t_0$, let PC_σ denote the set of functions ϕ: $[\sigma - \tau, \sigma] \to R$ which are real-valued absolutely continuous in $[t_k, t_{k+1}) \cap (\sigma - \tau, \sigma)$ and at t_k situated in $[\sigma - \tau, \sigma]$ they may have discontinuity of the first kind.

DEFINITION 1. For any $\sigma \ge t_0$ and $\phi \in PC_\sigma$, a function $y \in ([\sigma - r, \infty), R)$ is said to be a solution of (3) on $[\sigma, \infty)$ satisfying the initial value condition

$$y(t) = \phi(t), \qquad t \in [\sigma - \tau, \sigma] \tag{4}$$

if the following conditions are satisfied:

(i) $y(t)$ is absolutely continuous on each interval $[t_k, t_{k+1}) \cap [\sigma, \infty)$;

(ii) for any $t_k \in [\sigma, \infty)$, $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k)$;

(iii) $y(t)$ satisfies (3) a.e. (almost everywhere) in $[\sigma, \infty)$ and at impulsive points t_k situated in $[\sigma, \infty)$ may have discontinuity of the first kind.

DEFINITION 2. A solution of (3) is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

Remark 2. The definition of solution of (1) or (2) is analogous to Definition 1.

We also consider the following delay differential equations and inequalities

$$x'(t) + a(t)x(t) + P(t)x(t-\tau) \leq 0, \qquad \text{a.e. } t \geq t_0 + \tau, \quad (1^*)$$

$$x'(t) + a(t)x(t) + P(t)x(t-\tau) \geq 0, \qquad \text{a.e. } t \geq t_0 + \tau, \quad (2^*)$$

and

$$x'(t) + a(t)x(t) + P(t)x(t-\tau) = 0, \qquad \text{a.e. } t \geq t_0 + \tau, \quad (3^*)$$

where $P(t) = \prod_{t-\tau \leq t_k < t}(1 + b_k)^{-1}p(t)$, $t \geq t_0 + \tau$, and a, p and $\{b_k\}$ satisfy (A_1)–(A_3). Here and in the sequel we assume that a product equals unit if the number of the factor is equal to zero.

By a solution $x(t)$ of (3*) on $[\sigma, \infty)$ we mean an absolutely continuous function on $[\sigma, \infty)$ which satisfies (3*) on $[\sigma, \infty)$ and satisfies condition $x(t) = \phi(t)$, $t \in [\sigma - \tau, \tau]$. Similarly the solutions of (1*) and (2*) can be defined respectively. A solution of (3*) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

Recently, for the first order impulsive delay differential equation (3), under appropriate hypotheses, the oscillation criteria are established in [1], but some [1, Theorem 3, Corollary 2, and Theorem 4] of the results are incorrect. Now, let us introduce the following

Assertion [1, Theorem 3]. Let the following conditions (H_1)–(H_3) hold:

(H_1) $a, p \in C([0, \infty), (0, \infty))$, $\tau > 0$ is a constant;

(H_2) $0 < t_1 < t_2 < \cdots < t_k < \cdots$, are fixed points with $\lim_{k\to\infty} t_k = \infty$ and there exists a positive constant T such that $t_{k+1} - t_k \geq T > \tau$, $k = 1, 2, \ldots$;

(H_3) $\{b_k\}$ is a sequence of real numbers and $b_k \neq -1$ for all $k \in N$.

Suppose that

$$a(t) + \frac{1}{1 + b_k} p(t) > 0, \qquad t \in [0, \infty), \ k \in N. \quad (5)$$

Then all solutions of

$$\begin{cases} x'(t) + a(t)x(t) + p(t)x(t-\tau) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in N \end{cases}$$

with the initial condition $x(t) = \phi(t)$, $-\tau \leq t \leq 0$, where $\phi \in C([-\tau, 0], R)$, are oscillatory.

Consider the counterexample

$$\begin{cases} x'(t) + \alpha x(t) + \beta x(t-\tau) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in N, \end{cases} \tag{6}$$

where $\{t_k\}$ and $\{b_k\}$ satisfy (H_2) and (H_3) with $b_k \geq 0$, $k \in N$, and α, β, τ are positive constants satisfying $\beta\tau e^{\alpha\tau} \leq \frac{1}{e}$. Clearly (6) satisfies (H_1)–(H_3) and (5). The transformation $x(t) = e^{-\alpha t}y(t)$, which is oscillation-invariant, reduces (6) to

$$\begin{cases} y'(t) + \beta e^{\alpha\tau} y(t-\tau) = 0, & t \neq t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), & k \in N. \end{cases} \tag{6'}$$

By Corollary 3 in [5] the first order impulsive delay equation (6′) has a nonoscillatory solution $y(t)$, that is, (6) has a nonoscillatory solution $x(t)$, which implies that the Assertion is incorrect.

Remark 3. Reference [1, Theorem 4] is also an incorrect result. Its counterexample is given in [13, p. 462].

2. MAIN RESULTS

In this section, first we establish a fundamental theorem that enables us to reduce some properties of solutions of (1) (or (2) and (3)) respectively to corresponding properties of (1*) (or (2*) and (3*)).

THEOREM 1. *Assume that* (A_1)–(A_3) *hold. Then*

(i) *Inequality* (1) *has no eventually positive solution if and only if* (1*) *has no eventually positive solutions.*

(ii) *Inequality* (2) *has no eventually negative solution if and only if* (2*) *has no eventually negative solutions.*

(iii) *All solutions of* (3) *are oscillatory if and only if all solutions of* (3*) *are oscillatory.*

Proof. Clearly, it is sufficient to prove (i), since (ii) and (iii) follow from (i).

Let $y(t)$ be an eventually positive solution of (1). Then there exists a $T \geq 0$ such that $y(t) > 0$ and $y(t-\tau) > 0$ for $t \geq T$. From Remark 1, $b_k > -1$, $k \in N$. Set $x(t) = \prod_{T \leq t_k < t}(1+b_k)^{-1} y(t)$. Hence $x(t) > 0$ and $x(t-\tau) > 0$ for $t \geq T$.

Since $y(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}]$, and in view of $y(t_k^+) = (1+b_k)y(t_k)$, it follows that for $t \geq T$

$$x(t_k^+) = \prod_{T \leq t_j \leq t_k} (1+b_j)^{-1} y(t_k^+) = \prod_{T \leq t_j < t_k} (1+b_j)^{-1} y(t_k) = x(t_k),$$

and for all $t_k \geq T$,

$$x(t_k^-) = \prod_{T \leq t_j \leq t_{k-1}} (1+b_j)^{-1} y(t_k^-) = \prod_{T \leq t_j < t_k} (1+b_j)^{-1} y(t_k) = x(t_k),$$

which implies that $x(t)$ is continuous on $[T, \infty)$ and it is easy to prove that $x(t)$ is also absolutely continuous on $[T, \infty)$. Moreover, we obtain that for almost everywhere $t \in [\sigma, \infty)$

$$\begin{aligned} x'(t) &+ a(t)x(t) + P(t)x(t-\tau) \\ &= \prod_{T \leq t_k < t} (1+b_k)^{-1} y'(t) + a(t) \prod_{T \leq t_k < t} (1+b_k)^{-1} y(t) \\ &\quad + P(t) \prod_{T \leq t_k < t-\tau} (1+b_k)^{-1} y(t-\tau) \\ &= \prod_{T \leq t_k < t} (1+b_k)^{-1} (y'(t) + a(t)y(t) + p(t)y(t-\tau)) \leq 0 \end{aligned}$$

which implies that $x(t)$ is a positive solution of (1*).

Conversely, let $x(t)$ be an eventually positive solution of (1*) and $x(t) > 0$ and $x(t-\tau) > 0$ for $t \geq T \geq t_0$. Set $y(t) = \prod_{T \leq t_k < t}(1+b_k)x(t)$ where $b_k > -1$. As $x(t)$ is absolutely continuous on $[T, \infty)$, $y(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}]$, $t_k \geq T$ and for almost everywhere $t \in [\sigma, \infty)$,

$$\begin{aligned} y'(t) &+ a(t)y(t) + p(t)y(t-\tau) \\ &= \prod_{T \leq t_k < t} (1+b_k)x'(t) + a(t) \prod_{T \leq t_k < t} (1+b_k)x(t) \\ &\quad + p(t) \prod_{T \leq t_k < t-\tau} (1+b_k)x(t-\tau) \\ &\leq \prod_{T \leq t_k < t} (1+b_k)(x'(t) + a(t)x(t) + P(t)x(t-\tau)) \leq 0. \quad (7) \end{aligned}$$

On the other hand for every $t_k \geq T$

$$y(t_k^+) = \lim_{t \to t_k^+} \prod_{T \leq t_j < t} (1 + b_j) x(t) = \prod_{T \leq t_j \leq t_k} (1 + b_j) x(t_k)$$

and

$$y(t_k) = \prod_{T \leq t_j < t_k} (1 + b_j) x(t_k).$$

Thus for every $t_k \geq T$, $k \in N$, we have

$$y(t_k^+) = (1 + b_k) y(t_k)$$

which together with (7) implies that $y(t)$ is a positive solution of (1). The proof of Theorem 1 is complete.

Remark 4. It is obvious that the conclusion (iii) of Theorem 1 improves and generalizes noticeably Theorem 1 in [5].

The following results provide several explicit sufficient conditions for the oscillation of all solutions of (3). For delay differential equations without impulses similar results have been established (see [9, 10]). Furthermore, by Theorem 1 we note that Theorems 2, 3 and Corollaries 1, 2 below of this paper can be formulated in a more general form as follows.

If a set of conditions holds, then

(i) Inequality (1) has no positive solutions.

(ii) Inequality (2) has no negative solutions.

(iii) All solutions of (3) are oscillatory.

THEOREM 2. *Assume that* (A_1)–(A_3) *hold and that there exists a sequence of intervals* $\{(\xi_n, \eta_n)\}$ *with* $\lim_{n \to \infty}(\xi_n - \eta_n) = \infty$. *Suppose that*

$$p(t) \geq 0 \qquad \text{for all } t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n), \text{ where } N \geq 1$$

and for all $t \in \bigcup_{n=N}^{\infty}(\xi_n + \tau, \eta_n)$

$$\liminf_{t \to \infty} \int_{t-\tau}^{t} \prod_{s-\tau \leq t_k < s} (1 + b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma \right) ds > \frac{1}{e}. \quad (8)$$

Then all solutions of (3) *are oscillatory.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (3). By Theorem 1 there exists a nonoscillatory solution $x(t)$ of (3*) and suppose that $x(t) > 0$ and $x(t - \tau) > 0$ for all $t \geq T \geq t_0$. Set

$$z(t) = e^{\int_T^t a(s)\, ds} x(t), \qquad t \geq T.$$

Thus (3*) reduces

$$z'(t) + P_1(t)z(t-\tau) = 0, \qquad \text{a.e. for } t \geq T \tag{9}$$

where

$$P_1(t) = P(t)e^{\int_{t-\tau}^{t} a(s)\,ds} = \prod_{t-\tau \leq t_k < t} (1+b_k)^{-1} p(t) e^{\int_{t-\tau}^{t} a(s)\,ds}. \tag{10}$$

From (8) and (10) there exists $\gamma > \frac{1}{e}$ and $N_1 \geq T$ such that

$$\int_{t-\tau}^{t} P_1(s)\,ds \geq \gamma > \frac{1}{e}, \qquad t \in \bigcup_{n=N_1}^{\infty} (\xi_n + \tau, \eta_n). \tag{11}$$

Moreover, since $\lim_{n\to\infty}(\eta_n - \xi_n) = \infty$, we can choose $N_2 \geq N_1$ and an integer m such that $\eta_n - \tau \geq \xi_n + (m+2)\tau$ and

$$\left(\frac{2}{\gamma}\right)^2 < (e\gamma)^m. \tag{12}$$

From (9) we find that $z'(t) \leq 0$ a.e. for $t \in \bigcup_{n=n^*}^{\infty}(\xi_n + \tau, \eta_n)$ and that $z(t) \leq z(t-\tau)$ for $t \in \bigcup_{n=n^*}^{\infty}(\xi_n + \tau, \eta_n)$ where $\xi_{n^*} + \tau \geq T$. Hence

$$z'(t) + P_1(t)z(t) \leq 0, \qquad \text{a.e. for } t \in \bigcup_{n=N_3}^{\infty} (\xi_n + \tau, \eta_n),$$

where $N_3 = \max\{N_2, n^*\}$. Thus from (11) we have

$$\ln \frac{z(t)}{z(t-\tau)} + \gamma \leq 0 \qquad \text{for } t \in \bigcup_{n=N_3}^{\infty} (\xi_n + \tau, \eta_n)$$

or

$$e\gamma z(t) \leq e^{\gamma} z(t) \leq z(t-\tau) \qquad \text{for } t \in \bigcup_{n=N_3}^{\infty} (\xi_n + \tau, \eta_n).$$

Repeating the above procedure, it follows by induction that

$$(e\gamma)^m z(t) \leq z(t-\tau) \qquad \text{for } t \in \bigcup_{n=N_3}^{\infty} (\xi_n + (m+1)\tau, \eta_n). \tag{13}$$

Now fix $\bar{t} \in \bigcup_{n=N_3}^{\infty}(\xi_n + (m+1)\tau, \eta_n)$. Then because of (11) there exists a $t^* \in (\bar{t}, \bar{t}+\tau) \subset \bigcup_{n=N_3}^{\infty}(\xi_n + (m+1)\tau, \eta_n)$ such that

$$\int_{\bar{t}}^{t^*} P_1(s)\,ds \geq \frac{\gamma}{2} \qquad \text{and} \qquad \int_{t^*}^{\bar{t}+\tau} P_1(s)\,d \geq \frac{\gamma}{2}. \tag{14}$$

By integrating (9) on intervals $[\bar{t}, t^*]$ and $[t^*, \bar{t} + \tau]$ we find

$$z(t^*) - z(\bar{t}) + \int_{\bar{t}}^{t^*} P_1(s) z(s - \tau)\, ds = 0, \qquad \text{for } t \in \bigcup_{n=N_3}^{\infty} (\xi_n + (m+1)\tau, \eta_n) \tag{15}$$

and

$$z(\bar{t} + \tau) - z(t^*) + \int_{t^*}^{\bar{t}+\tau} P_1(s) z(s - \tau)\, ds = 0 \qquad \text{for } t \in \bigcup_{n=N_3}^{\infty} (\xi_n + (m+1)\tau, \eta_n). \tag{16}$$

But using the decreasing nature of $z(t)$ on $\bigcup_{n=N_3}^{\infty}(\xi_n + \tau, \eta_n)$ we have from (15) and (16) that

$$-z(\bar{t}) + z(t^* - \tau)\frac{\gamma}{2} < 0 \qquad \text{and} \qquad -z(t^*) + z(\bar{t})\frac{\gamma}{2} < 0$$

or

$$z(t^*) > \frac{\gamma}{2} z(\bar{t}) > \frac{\gamma^2}{4} z(t^* - \tau).$$

This implies that

$$(e\gamma)^m \le \frac{z(t^* - \tau)}{z(t^*)} \le \frac{4}{\gamma^2}$$

which contradicts (12) and completes the proof of Theorem 2.

THEOREM 3. *Assume that* (A_1)–(A_3) *hold and there exists a sequence of intervals* $\{(\xi_n, \eta_n)\}$ *such that* $\lim_{n\to\infty} \xi_n = \infty$ *and* $\eta_n - \xi_n > \tau$ *for all* $n \ge N > 1$. *If* $p(t) \ge 0$ *for all* $t \in \bigcup_{n=N}^{\infty}(\xi_n, \eta_n)$ *and for all* $t \in \bigcup_{n=N}^{\infty}(\xi_n + \tau, \eta_n)$

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} \prod_{s-\tau \le t_k < s} (1 + b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma\right) ds > 1, \tag{17}$$

then all solutions of (3) *are oscillatory.*

Proof. From Theorem 1 we only prove that (3*) has no nonoscillatory solutions. Let $x(t)$ be a nonoscillatory solution of (3*). Without loss of generality we suppose that $x(t) > 0$ and $x(t - \tau) > 0$ for $t \ge T$. Set $z(t) = \exp(-\int_T^t a(s)\, ds) x(t) > 0$ for $t \ge T$. Thus we obtain (9). Hence

$z'(t) \le 0$ a.e. in $\bigcup_{n=N}^{\infty}(\xi_n + \tau, \eta_n)$ which implies $z(t)$ is nonincreasing in $\bigcup_{n+N}^{\infty}(\xi_n + \tau, \eta_n)$. Integrating (9) from $t - \tau$ to t we have that

$$z(t) - z(t-\tau) + \int_{t-\tau}^{t} P_1(s) z(s-\tau)\, ds \le 0, \qquad \text{for all } t \in \bigcup_{n=N}^{\infty} (\xi_n + \tau, \eta_n).$$

From the nonincreasing character of $z(t)$, we derive that

$$z(t) + z(t-\tau)\left(\int_{t-\tau}^{t} P_1(s)\, ds - 1\right) \le 0 \qquad \text{for all } t \in \bigcup_{n=N}^{\infty} (\xi_n + \tau, \eta_n)$$

which contradicts (17). The proof of Theorem 3 is complete.

In particular, when $p(t) \ge 0$, by Theorem 2 and Theorem 3, we obtain the following results.

THEOREM 2′. *Assume that* (A_1)–(A_3) *hold and* $p(t) \ge 0$ *for* $t \ge t_0$. *If*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \prod_{s-\tau \le t_k < s} (1 + b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma\right) ds > \frac{1}{e},$$

then all solutions of (3) *are oscillatory.*

Remark 5. It is obvious that Theorem 2′ improves and generalizes Theorem 3.2 in [7] and Theorem 5 in [1]. Moreover, since

$$\int_{t-\tau}^{t} \prod_{s-\tau \le t_k < s} (1 + b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma\right) ds \ge \inf_{t-\tau \le s < t} \left\{ \tau \prod_{s-\tau \le t_k < s} (1 + b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma\right)\right\}$$

and

$$\int_{t-\tau}^{t} \prod_{s-\tau \le t_k < s} (1 + b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma\right) ds \ge \inf_{t-\tau \le s < t} \prod_{s-\tau \le t_k < s} (1 + b_k)^{-1} \int_{t-\tau}^{t} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma\right) ds,$$

Theorem 2′ improves and generalizes respectively Corollary 7 in [3] and Theorem 2 in [13].

THEOREM 3′. *Assume that* (A_1)–(A_3) *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *If*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} \prod_{s-\tau \leq t_k < s} (1+b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma \right) ds > 1$$

then all solutions of (3) *are oscillatory.*

Remark 6. Theorem 3′ improves and generalizes Corollary 2 and Corollary 3 in [13].

We introduce the following conditions.

(A_1') $0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, are fixed points with $\lim_{k\to\infty} t_k = \infty$ and there exists an integer m such that $m(t_{k+1} - t_k) \geq \tau$ for all $k \in N$.

(A_3') There exists a constant $M > 0$ such that $0 \leq b_k \leq M$ for all $k \in N$.

The following two results are respectively immediate corollaries of Theorem 2′ and Theorem 3′.

COROLLARY 1. *Assume that* (A_1'), (A_2), (A_3') *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *If*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma \right) ds > \frac{(1+M)^m}{e}$$

then all solutions of (3) *are oscillatory.*

Remark 7. Corollary 1 improves and generalizes respectively Theorem 5 in [1], Theorem 3.2 in [7], and Theorem 2 in [13].

COROLLARY 2. *Assume that* (A_1'), (A_2), (A_3') *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *If*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma \right) ds > (1+M)^m$$

then all solutions of (3) *are oscillatory.*

The following result provides a sufficient condition for the existence of a nonoscillatory solution of (3).

THEOREM 4. *Let* (A_1)–(A_3) *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *Assume that there exists* $T \geq t_0$ *such that for all* $t \geq T$

$$\int_{t-\tau}^{t} \prod_{s-\tau \leq t_k < s} (1+b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma \right) ds \leq \frac{1}{e}. \tag{18}$$

Then (3) *has a nonoscillatory solution on* $[T, \infty)$.

Proof. By Theorem 1, we only need to prove that

$$z'(t) + P_1(t)z(t-\tau) = 0$$

has a nonoscillatory solution $z(t)$ on $[T,\infty)$, where $P_1(t)$ is defined by (10).

We define a sequence of functions as

$$u_1(t) = \begin{cases} P_1(t), & t \geq T, \\ 0, & T-\tau \leq t < T, \end{cases}$$

$$u_{k+1}(t) = \begin{cases} P_1(t)\exp\left(\int_{t-\tau}^{t} u_k(s)\,ds\right), & t \geq T, \\ 0, & T-\tau \leq t < T,\ k \in N. \end{cases} \tag{19}$$

It is easy to show that for all $t \geq T \geq \tau$ and $k \in N$

$$0 \leq u_k(t) \leq u_{k+1}(t). \tag{20}$$

For $u_1(t)$ we have $u_1(t) \leq P_1(t)e$, $t \geq T-\tau$. Suppose that for some k

$$u_k(t) \leq P_1(t)e, t \geq T-\tau.$$

Thus, by (18) and (19) we obtain that for $t \geq T$

$$u_{k+1}(t) = P_1(t)\exp\left(\int_{t-\tau}^{t} u_k(s)\,ds\right) \leq P_1(t)\exp\left(e\int_{t-\tau}^{t} P_1(s)\,ds\right) \leq P_1(t)e.$$

Hence by induction we prove

$$u_k(t) \leq eP_1(t), \qquad t \geq T-\tau,\ k \in N \tag{21}$$

which with (20) implies $\{u_k(t)\}$ is a nondecreasing bounded sequence of functions on $[T-\tau,\infty)$. Thus $\{u_k(t)\}$, $t \geq T-\tau$, has a pointwise limiting function $u(t)$, that is, $\lim_{k\to\infty} u_k(t) = u(t)$. Furthermore, (21) implies $\{u_k(t)\}$ is uniformly bounded on $[t-\tau, t]$ for all $t \geq T$. Consequently, by applying the Lebesgue's dominated convergence theorem, we obtain

$$u(t) = \begin{cases} P_1(t)\exp\left(\int_{t-\tau}^{t} u(s)\,ds\right), & t \geq T, \\ 0, & T-\tau \leq t < T. \end{cases}$$

Set

$$z(t) = \exp\left(-\int_T^t u(s)\,ds\right), \ t \geq T-\tau.$$

It is easy to check that $z(t)$ is a positive solution of (9) on $[T,\infty)$. The proof of Theorem 4 is complete.

The following corollary is an immediate result of Theorem 4.

COROLLARY 3. *Let* (A_1'), (A_2), (A_3) *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *Assume that there exist* $T \geq t_0$ *and a constant* δ *with* $-1 < \delta \leq b_k$ *for* $t_k \geq T$ *such that*

$$\int_{t-\tau}^{t} p(s)\exp\left(\int_{s-\tau}^{s} a(\sigma)\, d\sigma\right) ds \leq \frac{(1+\delta)^m}{e} \qquad \text{for } t \geq T. \tag{22}$$

Then (3) *has a nonoscillatory solution.*

Remark 8. In particular, by applying Theorem 4 to the differential equation,

$$\begin{cases} x'(t) + px(t-\tau) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in N, \end{cases} \tag{23}$$

where $p > 0$, $\tau > 0$, and $b_k \geq 0$ for $k \in N$. It is easy from (18) to see that if

$$p\tau e \leq 1$$

then (23) has a nonoscillatory solution. From which we find that Theorem 4 is a substantial improvement of Theorem 3.3 in [7] and it generalizes and improves Theorem 3 and Theorem 4 in [13].

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