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Letter to the Editor

On quadrature of Bessel transformations

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Abstract

A method for integral transformations of highly oscillatory functions, Bessel functions, is presented. It is based on the Filon-type method and the decay of the error can be increased as α increases. The effectiveness and accuracy of the quadrature is tested for both large arguments and higher orders of Bessel functions in the case where the orders are nonnegative integers.

1. Introduction

The integration of systems containing Bessel functions (Bessel transformations) is a central point in many practical problems in physics, chemistry and engineering. In most of the cases, these transformations cannot be done analytically and one has to rely on numerical methods. Denote by $J_m(\alpha x)$ the Bessel function of the first kind and of order m, where m and α are arbitrary positive real numbers. For large m or α, the integrand becomes highly oscillatory and thereby presents serious difficulties in obtaining numerical convergence of the integration.

For the Bessel transformation over an infinite interval, several procedures have been described in the literature [1,2,4,8,9]. Here we consider the numerical computation of an integral following Filon [3],

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Iserles [5,6] and Iserles and Nørsett [7] in the case where m is a nonnegative integer

$$I_m(\alpha, f) = \int_0^1 f(x) J_m(\alpha x)\,\mathrm{d}x,$$

where $f(x)$ is suitably smooth on [0,1].

We denote the moments by

$$I_m(\alpha, x^k) = \int_0^1 x^k J_m(\alpha x)\,\mathrm{d}x, \quad k \geqslant 0.$$

We interpolate on $[(j-1)/N, j/N]$ at distinct nodes $c_1 = (j-1)/N$, $c_2 = (2j-1)/2N$, $c_3 = j/N$ by parabolic interpolation $P(x)$ and $c_1 = (j-1)/N$, $c_2 = j/N$ by Hermite interpolation $S(x)$, respectively, and denote by $h = 1/N$, where N is a positive integer, $j = 1, 2, \ldots, N$. Following Filon [3], we approximate

$$I_m(\alpha, f) \approx I^F_{m,1}(f, P) = \int_0^1 P(x) J_m(\alpha x)\,\mathrm{d}x, \quad I_m(\alpha, f) \approx I^F_{m,2}(f, S) = \int_0^1 S(x) J_m(\alpha x)\,\mathrm{d}x.$$

2. Error analysis

Since

$$|J_m(y)| \leqslant 1 \ \forall y \quad \text{and} \quad |J_m(y)| \leqslant B y^{-1/3} \ \forall y \geqslant 1 \ \text{(see [10, p. 357])}, \ \forall m,$$

where B is a constant independent of y and m, then for $x \geqslant 1/\alpha$

$$|J_m(\alpha x)| \leqslant B \frac{1}{\sqrt[3]{\alpha}} x^{-1/3}$$

and

$$\begin{aligned}\int_0^1 |J_m(\alpha x)|\,\mathrm{d}x &= \int_0^{1/\alpha} |J_m(\alpha x)|\,\mathrm{d}x + \int_{1/\alpha}^1 |J_m(\alpha x)|\,\mathrm{d}x \\ &\leqslant \frac{1}{\alpha} + B \frac{1}{\sqrt[3]{\alpha}} \int_0^1 x^{-1/3}\,\mathrm{d}x \\ &\leqslant \frac{A}{\sqrt[3]{\alpha}}\end{aligned}$$

for some positive constant A and $\alpha \geqslant 1$. Note that

$$|f(x) - P(x)| \leqslant \frac{\sqrt{3} h^3}{36} \max_{0 \leqslant x \leqslant 1} |f^{(3)}(x)|$$

and

$$|f(x) - S(x)| \leqslant \frac{h^4}{384} \max_{0 \leqslant x \leqslant 1} |f^{(4)}(x)|.$$

Hence,

$$|I_m(\alpha, f) - I_1^F(f, P)| \leqslant \frac{\sqrt{3}Ah^3}{36\sqrt[3]{\alpha}} \max_{0\leqslant x\leqslant 1} |f^{(3)}(x)|$$

and

$$|I_m(\alpha, f) - I_2^F(f, S)| \leqslant \frac{Ah^4}{384\sqrt[3]{\alpha}} \max_{0\leqslant x\leqslant 1} |f^{(4)}(x)|.$$

The decay of the error can be increased for large α.

3. Calculus of the moments

To calculate the integrals $I^F_{m,1}(f, P) = \int_0^1 P(x)J_m(\alpha x)\,dx$ and $I^F_{m,2}(f, S) = \int_0^1 S(x)J_m(\alpha x)\,dx$, it is only necessary to compute the moments $I_m(\alpha, x^k, a, b) = \int_a^b x^k J_m(\alpha x)\,dx$, $k = 0, 1, 2, 3$, where a and b are nodes of the interpolation schemes $(j-1)/N,\ j/N,\ j = 1, 2, \ldots, N$. The moments for m being a nonnegative integer can be written into

$$\begin{aligned} I_m(\alpha, 1, a, b) = \int_a^b J_m(\alpha x)\,dx &= \frac{1}{\pi}\int_a^b dx \int_0^\pi e^{-im\theta}e^{i\alpha x\sin\theta}\,d\theta \\ &= \frac{1}{\pi}\int_0^\pi e^{-im\theta}\,d\theta \int_a^b e^{i\alpha x\sin\theta}\,dx \\ &= \frac{1}{\pi}\int_0^\pi e^{-im\theta}\frac{e^{i\alpha b\sin(\theta)} - e^{i\alpha a\sin(\theta)}}{i\alpha\sin(\theta)}\,d\theta. \end{aligned}$$

Similarly by repeated integration by parts

$$\begin{aligned} I_m(\alpha, x, a, b) = \int_a^b xJ_m(\alpha x)\,dx = \frac{1}{\pi}\int_0^\pi e^{-im\theta}\Bigg\{&\frac{be^{i\alpha b\sin(\theta)} - ae^{i\alpha a\sin(\theta)} - (b-a)}{i\alpha\sin(\theta)} \\ &- \frac{e^{i\alpha b\sin(\theta)} - e^{i\alpha a\sin(\theta)} - (b-a)i\alpha\sin(\theta)}{(i\alpha\sin(\theta))^2}\Bigg\}\,d\theta, \end{aligned}$$

$$\begin{aligned} I_m(\alpha, x^2, a, b) = \int_a^b x^2J_m(\alpha x)\,dx = \frac{1}{\pi}\int_0^\pi e^{-im\theta}\Bigg\{&\frac{b^2e^{i\alpha b\sin(\theta)} - a^2e^{i\alpha a\sin(\theta)} - (b^2-a^2)}{i\alpha\sin(\theta)} \\ &- \frac{2be^{i\alpha b\sin(\theta)} - 2ae^{i\alpha a\sin(\theta)} - 2(b^2-a^2)i\alpha\sin(\theta) - 2(b-a)}{(i\alpha\sin(\theta))^2} \\ &+ \frac{2e^{i\alpha b\sin(\theta)} - 2e^{i\alpha a\sin(\theta)} - 2(b-a)i\alpha\sin(\theta) - (b^2-a^2)(i\alpha\sin(\theta))^2}{(i\alpha\sin(\theta))^3}\Bigg\}\,d\theta, \end{aligned}$$

and

$$
\begin{aligned}
&I_m(\alpha, x^3, a, b)\\
&\quad = \int_a^b x^3 J_m(\alpha x)\,\mathrm{d}x\\
&\quad = \frac{1}{\pi}\int_0^\pi \mathrm{e}^{-\mathrm{i}m\theta}\left\{\frac{b^3\mathrm{e}^{\mathrm{i}\alpha b\sin(\theta)} - a^3\mathrm{e}^{\mathrm{i}\alpha a\sin(\theta)} - (b^3 - a^3)}{\mathrm{i}\alpha\sin(\theta)}\right.\\
&\qquad - \frac{3b^2\mathrm{e}^{\mathrm{i}\alpha b\sin(\theta)} - 3a^2\mathrm{e}^{\mathrm{i}\alpha a\sin(\theta)} - 3(b^2 - a^2) - 3(b^3 - a^3)\mathrm{i}\alpha\sin(\theta)}{(\mathrm{i}\alpha\sin(\theta))^2}\\
&\qquad + \frac{6b\mathrm{e}^{\mathrm{i}\alpha b\sin(\theta)} - 6a\mathrm{e}^{\mathrm{i}\alpha a\sin(\theta)} - 6(b-a) - 6(b^2 - a^2)\mathrm{i}\alpha\sin(\theta) - 3(b^3 - a^3)(\mathrm{i}\alpha\sin(\theta))^2}{(\mathrm{i}\alpha\sin(\theta))^2}\\
&\qquad \left. - \frac{6\mathrm{e}^{\mathrm{i}\alpha b\sin(\theta)} - 6\mathrm{e}^{\mathrm{i}\alpha a\sin(\theta)} - 2(b-a)\mathrm{i}\alpha\sin(\theta) - 3(b^2-a^2)(\mathrm{i}\alpha\sin(\theta))^2 - (b^3-a^3)(\mathrm{i}\alpha\sin(\theta))^3}{(\mathrm{i}\alpha\sin(\theta))^3}\right\}\mathrm{d}\theta.
\end{aligned}
$$

Here we consider the composite two-points Gauss–Legendre quadrature to compute the moments and denote by n the number of subintervals, where n satisfies $n \geqslant \max\{\alpha, 10m\}$ (see Fig. 1). For example, let $N = 1$, $a = 0$ and $b = 1$:

Quadrature of $\int_0^1 x^k J_1(\alpha x)\,\mathrm{d}x$, $n = 1000$, $k = 0, 1, 2, 3$

α	1	10	100	1000
$Q\left[\int_0^1 J_1(\alpha x)\,\mathrm{d}x\right]$	0.23480231344203	0.12459357644513	0.00980014149696	0.00097521331385
Exact	0.23480231344203	0.12459357644513	0.00980014149696	0.00097521331385
$Q\left[\int_0^1 x J_1(\alpha x)\,\mathrm{d}x\right]$	0.15453272353179	0.03526368948470	−0.00010759224735	−2.37819826324e−5
Exact	0.15453272353179	0.03526368948470	−0.00010759224735	−2.37819826318e−5
$Q\left[\int_0^1 x^2 J_1(\alpha x)\,\mathrm{d}x\right]$	0.11490348493212	0.02546303136851	−2.15287573445e−4	−2.47772295286e−5
Exact	0.11490348493190	0.02546303136851	−2.15287573445e−4	−2.47772295286e−5
$Q\left[\int_0^1 x^3 J_1(\alpha x)\,\mathrm{d}x\right]$	0.09135590264862	0.02483984817202	−2.22969830969e−4	−2.47724298708e−5
Exact	0.09135590008145	0.02483984814566	−2.22969830972e−4	−2.47724298707e−5

Quadrature of $\int_0^1 x^k J_{10}(\alpha x)\,\mathrm{d}x$, $n = 1000$, $k = 0, 1, 2, 3$

α	1	10	100	1000
$Q\left[\int_0^1 J_{10}(\alpha x)\right]\mathrm{d}x$	0.0000000000240	0.03142969034819	0.01058904472314	0.00099406297232
Exact	0.2399869325432e−10	0.03142969034819	0.01058904472314	0.00099406297232
$Q\left[\int_0^1 x J_{10}(\alpha x)\,\mathrm{d}x\right]$	0.0000000000221	0.02772342545019	0.00158357930607	4.03844335624e−6
Exact	0.2199328902289e−10	0.02772342545019	0.00158357930607	4.03844335626e−6
$Q\left[\int_0^1 x^2 J_{10}(\alpha x)\,\mathrm{d}x\right]$	0.0000000000206	0.02475329664786	0.00067699376219	−5.8870737831e−6
Exact	0.2029708832215e−10	0.02475329664786	0.00067699376219	−5.8870737831e−6
$Q\left[\int_0^1 x^3 J_{10}(\alpha x)\,\mathrm{d}x\right]$	0.0000013755767	0.02232725260929	0.00058189136307	−6.0096189491e−6
Exact	0.1884371471050e−10	0.02232725270167	0.00058189136308	−6.0096189491e−6

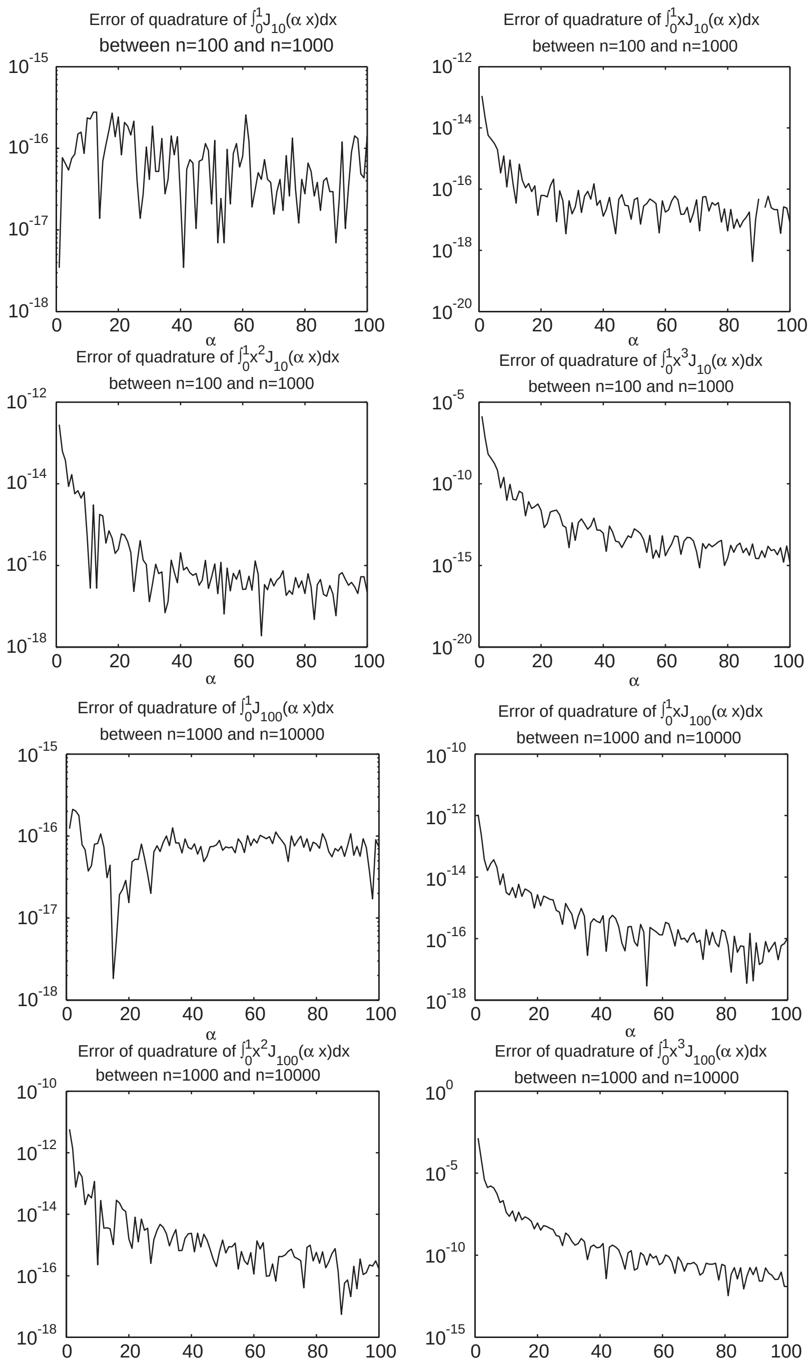

Fig. 1. Error analysis of calculus of the moments by the composite two-points Gauss–Legendre quadrature with different n.

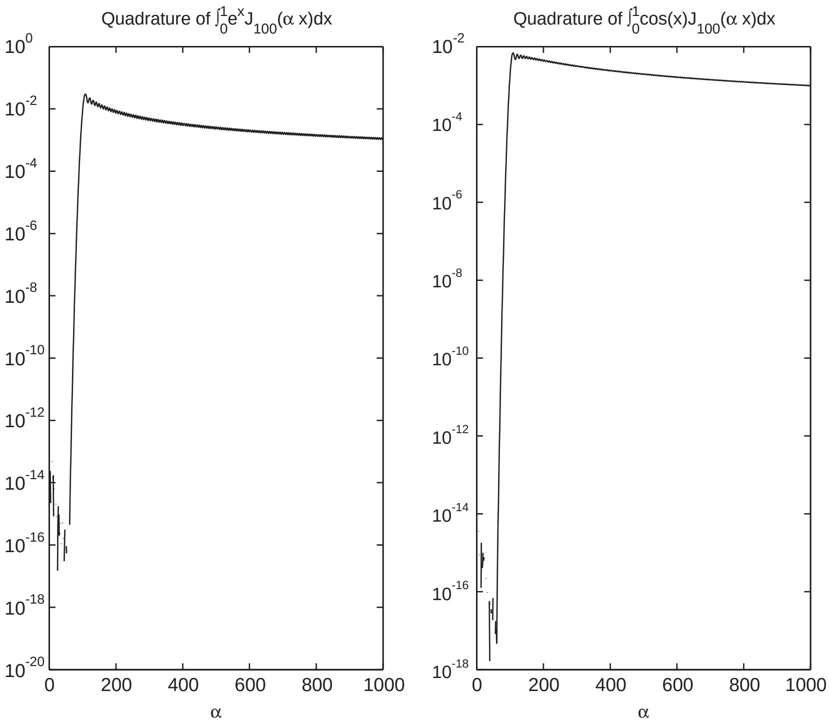

Fig. 2. Numerical quadrature by the Filon-type method with parabolic interpolation and the composite two-points Gauss–Legendre quadrature. $N=10,\ n=1000$.

Quadrature of $\int_0^1 x^k J_{100}(\alpha x)\,\mathrm{d}x,\ n=1000,\ k=0,1,2,3$

α	1	10	100	1000
$Q\left[\int_0^1 J_{100}(\alpha x)\,\mathrm{d}x\right]$	$-1.16817\mathrm{e}{-16}$	$-9.02829\mathrm{e}{-17}$	0.00329456068425	0.00097744221569
Exact	0	0	0.00329456068425	0.00097744221569
$Q\left[\int_0^1 x J_{100}(\alpha x)\,\mathrm{d}x\right]$	$1.0880983\mathrm{e}{-13}$	$6.26814464\mathrm{e}{-16}$	0.0032025835101	$7.745398603355\mathrm{e}{-5}$
Exact	0	0	0.0032025835101	$7.745398603352\mathrm{e}{-5}$
$\int_0^1 x^2 J_{100}(\alpha x)\,\mathrm{d}x$	$2.27909288\mathrm{e}{-13}$	$-3.1807405\mathrm{e}{-15}$	0.00311509537523	$-1.253519802691\mathrm{e}{-5}$
Exact	0	0	0.00311509537523	$-1.253519802693\mathrm{e}{-5}$
$Q\left[\int_0^1 x^3 J_{100}(\alpha x)\,\mathrm{d}x\right]$	$-1.3723295\mathrm{e}{-6}$	$-9.2181694\mathrm{e}{-11}$	0.00303180213246	$-2.152273656221\mathrm{e}{-5}$
Exact	0	0	0.00303180213246	$-2.152273656222\mathrm{e}{-5}$

From the computation of the moments, we know that parabolic interpolation is more stable than Hermite interpolation. In the following, we present some numerical examples using the Filon-type method to compute Bessel transformations with parabolic interpolation and Hermite interpolation, respectively (Figs. 2 and 3).

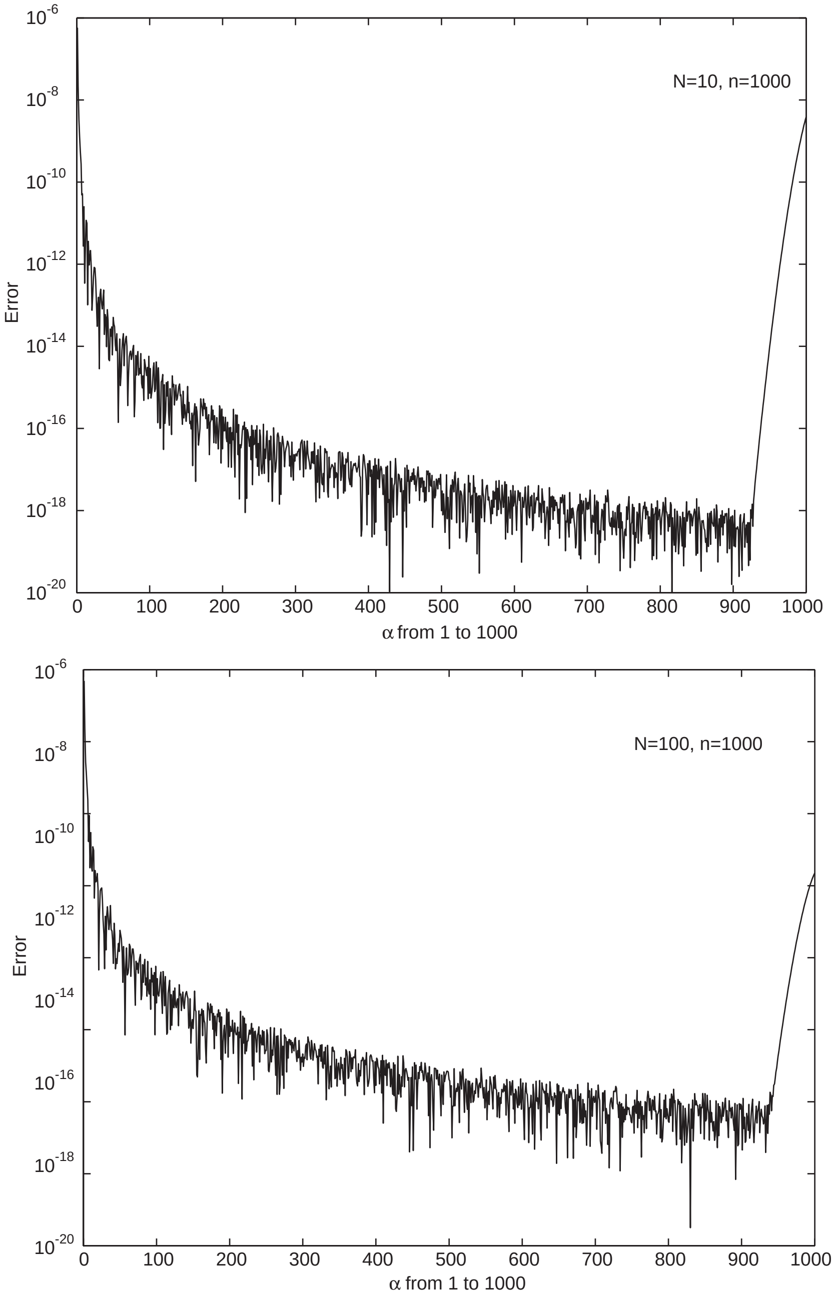

Fig. 3. Error analysis of computation $\int_0^1 e^x J_{1000}(\alpha x)\,dx$ between the two Filon-type methods with parabolic interpolation and the composite two-points Gauss–Legendre quadrature. $N = 10$, $n = 1000$ and $N = 100$, $n = 1000$.

Quadrature of $\int_0^1 e^x J_1(\alpha x)\,dx$, $\int_0^1 \cos(x) J_1(\alpha x)\,dx$, $n=1000$, $k=0,1,2,3$

α	1	10	100	1000
Exact	0.465799029519	0.177967132924	0.009535755084	0.0009336356538
$I_{1,1}^F(\alpha, e^x, P)(N=10)$	0.465799147141	0.177967131923	0.009535700364	0.0009336347905
$I_{1,1}^F(\alpha, e^x, P)(N=100)$	0.465799029531	0.177967132924	0.009535754886	0.0009336356537
$I_{1,2}^F(\alpha, e^x, S)(N=10)$	0.465798965759	0.177967107825	0.009535754885	0.0009336356373
$I_{1,2}^F(\alpha, e^x, S)(N=100)$	0.465799030385	0.177967132942	0.009535755091	0.0009336356539
Exact	0.180430620804	0.112845126701	0.009898513984	0.0009866037250
$I_{1,1}^F(\alpha, \cos(x), P)(N=10)$	0.180430653472	0.112845120598	0.009898515885	0.0009866036878
$I_{1,1}^F(\alpha, \cos(x), P)(N=100)$	0.180430620807	0.112845126700	0.009898513983	0.0009866037250
$I_{1,2}^F(\alpha, \cos(x), S)(N=10)$	0.180430601073	0.112845110880	0.009898513818	0.0009866037250
$I_{1,2}^F(\alpha, \cos(x), S)(N=100)$	0.180430621064	0.112845126706	0.009898513985	0.0009866037250

Quadrature of $\int_0^1 e^x J_{100}(\alpha x)\,dx$, $\int_0^1 \cos(x) J_{100}(\alpha x)\,dx$, $n=1000$, $k=0,1,2,3$

α	1	10	100	1000
Exact	0	0	0.008711493656	0.001043883989
$I_{100,1}^F(\alpha, e^x, P)(N=10)$	−6.041783e−12	−3.920122e−14	0.008711526704	0.001043883638
$I_{100,1}^F(\alpha, e^x, P)(N=100)$	−2.348340e−13	−2.961851e−15	0.008711493660	0.001043883999
$I_{100,2}^F(\alpha, e^x, S)(N=10)$	6.072326e−4	−2.6185757e−8	0.008711492664	0.001043883989
$I_{100,2}^F(\alpha, e^x, S)(N=100)$	−6.2230110e−7	−3.864815e−11	0.008711493657	0.001043883989
Exact	0	0	0.001856201310	0.000982806722
$I_{100,1}^F(\alpha, \cos(x), P)(N=10)$	1.489514e−12	8.943060e−15	0.001856211677	0.000982806631
$I_{100,1}^F(\alpha, \cos(x), P)(N=100)$	−2.860181e−14	−2.484192e−16	0.001856201311	0.000982806723
$I_{100,2}^F(\alpha, \cos(x), S)(N=10)$	2.058497e−4	−9.800843e−9	0.001856201087	0.000982806722
$I_{100,2}^F(\alpha, \cos(x), S)(N=100)$	−1.891940e−7	−1.306837e−11	0.001856201310	0.000982806721

Remark. The numerical examples show that the number of correct digits increases faster than $\lg(\sqrt[3]{\alpha})$, where $\lg(x)$ is the logarithm function and the base is 10. Since m is fixed, by [1,10] for large y,

$$J_m(y) = \left(\frac{2}{\pi}\right)^{1/2} y^{-1/2} \cos\left(y - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O(y^{-3/2}), \quad |J_m(y)| \leqslant 1,$$

then

$$J_m(\alpha x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{\sqrt{\alpha}} x^{-1/2} \cos\left(\alpha x - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{x^{-3/2}}{\alpha\sqrt{\alpha}}\right),$$

and

$$\int_0^1 |J_m(\alpha x)|\,\mathrm{d}x = \int_0^{1/\sqrt{\alpha}} |J_m(\alpha x)|\,\mathrm{d}x + \int_{1/\sqrt{\alpha}}^1 |J_m(\alpha x)|\,\mathrm{d}x$$
$$\leqslant \frac{1}{\sqrt{\alpha}} + \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{\sqrt{\alpha}} \int_0^1 x^{-1/2}\,\mathrm{d}x + B\left|\int_{1/\sqrt{\alpha}}^1 \frac{x^{-3/2}}{\alpha\sqrt{\alpha}}\,\mathrm{d}x\right|$$
$$\leqslant \frac{A}{\sqrt{\alpha}}$$

for some positive constant A, B and $\alpha \geqslant 1$. Hence

$$|I_m(\alpha, f) - I_1^F(f, P)| \leqslant \frac{\sqrt{3}Ah^3}{36\sqrt{\alpha}} \max_{0\leqslant x\leqslant 1} |f^{(3)}(x)|$$

and

$$|I_m(\alpha, f) - I_2^F(f, S)| \leqslant \frac{Ah^4}{384\sqrt{\alpha}} \max_{0\leqslant x\leqslant 1} |f^{(4)}(x)|.$$

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References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables, Dover Publications Inc., New York, 1965.
[2] P.I. Davis, P. Rabinowitz, Methods of Numerical Integral Integration, Second ed., Academic Press, New York, 1984.
[3] L.N.G. Filon, On a quadrature formula for trigonometric integrals, Proc. Roy. Soc. Edinburgh 49 (1928) 38–47.
[4] Q. Haider, L.C. Liu, Fourier or Bessel transformations of highly oscillatory functions, J. Phys. A: Math. Gen. 25 (1992) 6755–6760.
[5] A. Iserles, On the numerical quadrature of highly-oscillating integrals I: Fourier transforms, Report No. NA2003/05, DAMTP, University of Cambridge.
[6] A. Iserles, On the numerical quadrature of highly-oscillating integrals II: irregular oscillators, Report No. NA2003/09, DAMTP, University of Cambridge.
[7] A. Iserles, S.P. Nørsett, Efficient quadrature of highly-oscillatory integrals using derivatives, Report No. NA2004/03, DAMTP, University of Cambridge.
[8] R. Piessens, Automatic computation of Bessel function integrals, Comput. Phys. Comm. 25 (1982) 289–295.
[9] M. Puoskari, A method for computing Bessel function integrals, J. Comput. Phys. 75 (1988) 334–344.
[10] E. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, NJ, 1993.