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On coclass and trivial Schur multiplicator

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To John Cannon and Derek Holt in recognition of their distinguished contributions to mathematics

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ABSTRACT

For odd primes p a p-group with coclass 2 and trivial Schur multiplicator has class at most 6.

1. Introduction

This note is an addendum to a recent paper by Eick [2] in which she made a conjecture about p-groups with trivial Schur multiplicator in terms of coclass. In this note her Conjecture 1 is confirmed for coclass 2.

Theorem 1. *For odd primes p a p-group with coclass* 2 *and trivial Schur multiplicator has class at most* 6*.*

The conjecture was based on computations. The proof of Theorem 1 has been derived from a detailed analysis of some related computations.

A significant aspect of Eick's paper is the introduction of Schur towers which generalise lower central quotients of p-groups with trivial Schur multiplicator. The i-th term of the lower central series of a group G will be denoted $\gamma_i(G)$. A p-group G is a *Schur tower* if either it is abelian or it is non-abelian and $G/\gamma_{i+1}(G)$ is a Schur cover of $G/\gamma_i(G)$ for all $i \in \{2, \dots, c\}$ where c is the class of G. Figures in Section 4.3 of [2] give a description of graphs of Schur towers with coclass 2 for odd

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primes; it is conjectural for $p > 19$. The description implies a corresponding conjecture for Schur towers with coclass 2. The proof here also gives the result for Schur towers.

Theorem 2. *For odd primes p a Schur tower p-group with coclass 2 has class at most 6.*

I am indebted to Bettina Eick for a preprint of [2] and to her and Eamonn O'Brien for helpful comments on drafts of this note.

2. Proof

The computations described by Eick show the theorems hold for the odd primes up to 19. These computations show the prime 3 does not fit the general pattern. So from now on p will be a prime greater than 3.

The proof here does not go into the detail given in the figures in [2]. It uses only case distinctions which reflect structural features of the groups.

As usual a *descendant* of a p-group P is a p-group Q such that a lower central factor of Q is isomorphic to P.

There are three non-cyclic abelian p-groups with coclass at most 2. The elementary abelian group E with order p^3 has Schur multiplicator rank 3 and so all the proper Schur tower descendants of E have coclass at least 4. As Eick points out (Section 4.3 of [2]) it is straight-forward to determine the four proper descendants of $C_{p^2} \times C_p$ with coclass 2.

The elementary abelian group with order p^2 has two proper Schur tower descendants with coclass 1 and order p^3. One of these descendants has trivial Schur multiplicator and the other, which will be denoted R, has Schur multiplicator which is elementary abelian with order p^2 (Lemma 9 of [2]).

The rest of the proof deals with the Schur tower descendants of R which have coclass 2. The proof uses commutator computations guided by, but not depending on, details of p-quotient computations which were done with Magma [1] using pQuotientProcess. The identity

$$J(z, y, x): \quad [z, y, x^z][x, z, y^x][y, x, z^y] = 1$$

for $x, y, z \in G$ will be referred to as the Jacobi identity. The Jacobi identity will mostly be used in the simpler form: $[z, y, x][x, z, y][y, x, z] \in \gamma_{i+j+k+1}(G)$ for $x \in \gamma_i(G)$, $y \in \gamma_j(G)$, $z \in \gamma_k(G)$.

The Schur covers of R are Schur towers which have order p^5 and coclass 2. It is known there are $p+7$ groups of this kind; see for example James [3] (p. 621, in isoclinism family Φ_6). For the present proof this detail is not needed. Let G be a Schur cover of R. The argument divides into three cases according to the order of the subgroup $\mho(G)$ of G generated by p-th powers.

Case A. $\mho(G)$ is trivial. The group G is the relatively free group of rank 2 with exponent p and class 3. The relatively free group of rank 2 with exponent p and class 4 has order p^8 and centre elementary abelian with order p^3. So the Schur multiplicator of G has rank at least 3. Hence all proper Schur tower descendants of G have coclass at least 4.

Case B. $\mho(G)$ has order p^2. This case covers $p+3$ of the $p+7$ groups. Of these $p+1$ have trivial Schur multiplicator. When G has non-trivial Schur multiplicator it follows from Theorem 12.2.9 in the book of Leedham-Green and McKay [4] that the Schur covers H of G have order p^6. If the Schur multiplicator of H has rank at least 2, then all the proper Schur tower descendants of H have coclass at least 3. Moreover Theorem 12.2.9 implies that, if the Schur multiplicator of H has rank 1, then the Schur covers of H have trivial Schur multiplicator, and class 5.

Case C. $\mho(G)$ has order p. There are three groups of this kind. For the present purpose it is enough to observe that such a group G has a maximal subgroup M with exponent p and that G has a generating pair $\{g_1, g_2\}$ with $g_2 \in M$ and $g_1 \notin M$. Therefore G has a presentation

$$\{g_1, g_2 \mid g_1^p = u,\ g_2^p = 1,\ \text{class } 3\}$$

with $u \in \gamma_3(G) \setminus \gamma_4(G)$.

Consider G as F/S where F is the free group generated by $\{a, b\}$. Then S is generated by $\{a^p u^{-1}, b^p, \gamma_4(F)\}$ with $u \in \gamma_3(F) \setminus \gamma_4(F)$. Hence $[F, S]$ contains $\gamma_5(F)\gamma_2(F)^p$ and is generated by $\{[u, a], [u, b], \gamma_5(F)\gamma_2(F)^p\}$. Thus $(\gamma_2(F) \cap S)/[F, S]$ has order p, and the Schur multiplicator of G has order p.

Every Schur cover H of G has order p^6 and $H/\gamma_4(H)$ is isomorphic to G. Let h_1, h_2 denote preimages in H of g_1, g_2 in G. Then H has a presentation

$$\{h_1, h_2 \mid h_1^p = u,\ h_2^p = v,\ \text{class } 4\} \qquad (*)$$

with $u \in \gamma_3(H) \setminus \gamma_4(H)$ and $v \in \gamma_4(H)$.

The order of the subgroup $\mho(H)$ of H generated by p-th powers gives a further case distinction.

Case C1. $\mho(H)$ has order p. There is a presentation $(*)$ for H with $v = 1$. Consider H as F/T where F is the relatively free group generated by $\{a, b\}$ with class 5 and commutator subgroup with exponent p. The subgroup $\gamma_5(F)$ has order p^6 and $[F, T]$ is generated by $\{[u, a], [u, b], [u, a, a], [u, a, b], [u, b, a], [u, b, b]\}$. Hence the Schur multiplicator of H has rank at least 2 and all proper Schur tower descendants of H have coclass at least 3.

Case C2. $\mho(H)$ has order p^2. There is a presentation $(*)$ for H with $v \notin \gamma_5(H)$. Schur tower descendants of H will be shown to have class at most 6. In other words the group K presented by

$$\{a, b \mid a^p = [b, a, a]^j [b, a, b]^k u,\ b^p = v,\ \text{class } 7\} \qquad (**)$$

with $u \in \gamma_4(K)$ and with $v \in \gamma_4(K) \setminus \gamma_5(K)$ will be shown to have class at most 6. From now on $\gamma_i = \gamma_i(K)$.

Commutator calculations are used to build partial power-commutator presentations for K. Put $a_1 = a$, $a_2 = b$, $a_3 = [a_2, a_1]$, $a_4 = [a_3, a_1]$, $a_5 = [a_3, a_2]$. Since $v \in \gamma_4$ it follows from the second relation of K that $a_3^p \in \gamma_5$. The Jacobi instance $J(a_3, a_2, a_1)$ gives $[a_5, a_1][a_4, a_2]^{-1} \in \gamma_5$. There are now two cases according as j is prime to p or not.

Case C2a. j is not prime to p. Here take class 6 in $(**)$ and it will be shown that K has class at most 5. The first relation of K gives $[a_4, a_2]^j [a_5, a_2]^k \in \gamma_5$ and $[a_4, a_1]^j [a_5, a_1]^k \in \gamma_5$. Therefore γ_4 is generated by $\{[a_5, a_2], \gamma_5\}$. Put $a_6 = [a_5, a_2]$. Write $v = a_6^m v^*$ with m prime to p and $v^* \in \gamma_5$. It follows that $[a_6, a_2] \in \gamma_6$ and γ_5 is generated by $\{[a_6, a_1], \gamma_6\}$. Put $a_7 = [a_6, a_1]$, $a_8 = [a_7, a_1]$ and $a_9 = [a_7, a_2]$. From the Jacobi identity:

$$\begin{aligned}
J(a_6, a_2, a_1):&\quad [a_6, a_3] = [a_6, a_2, a_1][a_6, a_1, a_2]^{-1} = a_9^{-1},\\
J(a_5, a_2, a_1):&\quad [a_5, a_3]a_7^{-1} \in \gamma_6,\\
J(a_5, a_3, a_2):&\quad 1 = [a_5, a_3, a_2][a_2, a_5, a_3] = a_9^2.
\end{aligned}$$

So $a_9 = [a_7, a_2] = [a_6, a_3] = 1$. Then

$$\begin{aligned}
J(a_5, a_3, a_1):&\quad [a_5, a_4] = [a_7, a_1] = a_8,\\
J(a_4, a_3, a_2):&\quad 1 = [a_4, a_3, a_2][a_2, a_4, a_3][a_3, a_2, a_4] = a_8.
\end{aligned}$$

Therefore K has class at most 5.

Case 2Cb. $j = 0$ and k is not prime to p. The first relation of K gives $[a_5, a_1], [a_5, a_2] \in \gamma_5$. So γ_4 is generated by $\{[a_4, a_1], \gamma_5\}$. Put $a_6 = [a_4, a_1]$. Write $v = a_6^m v^*$ with m prime to p and $v^* \in \gamma_5$. Hence $[a_6, a_2] \in \gamma_6$ and γ_5 is generated by $\{[a_6, a_1], \gamma_6\}$. Put $a_7 = [a_6, a_1]$. From the Jacobi identity:

$$\begin{aligned}
J(a_4, a_2, a_1):&\quad [a_4, a_3] \in \gamma_6,\\
J(a_4, a_3, a_1):&\quad [a_6, a_3] \in \gamma_7,\\
J(a_6, a_2, a_1):&\quad [a_7, a_2] \in \gamma_7.
\end{aligned}$$

Put $a_8 = [a_7, a_1]$, $a_9 = [a_8, a_1]$ and $a_{10} = [a_8, a_2]$. Hence

$$J(a_7, a_2, a_1): \quad [a_7, a_3] = a_{10}^{-1}.$$

It follows from $a_3^p \in \gamma_5$ that $\gamma_4^p \leqslant \gamma_6$. Hence $[a^p, b][b^p, a] \in \gamma_6$. So the relations give $[a_5, a_2]^k a_7^m \in \gamma_6$ and hence $[a_5, a_2, a_1]^k a_8^m \in \gamma_7$.

Since $[a_5.a_2] \in \gamma_5$ and $[a_5, a_2, a_2] \in \gamma_7$,

$$J\big([a_5, a_2], a_2, a_1\big): \quad [a_5, a_2, a_3]^k a_{10}^{-m} = 1. \quad \text{Then}$$

$$J(a_5, a_2, a_1): \quad [a_5, a_3]^k a_8^m \in \gamma_7.$$

Hence $[a_5, a_3, a_1]^k a_9^m = 1$ and $[a_5, a_3, a_2]^k a_{10}^m = 1$. Now, using $[a_5, a_2] \in \gamma_5$,

$$J(a_5, a_3, a_2): \quad [a_5, a_3, a_2][a_2, a_5, a_3] = 1.$$

Therefore $a_{10}^{2m} = 1$ and $a_{10} = 1$. Hence

$$J(a_4, a_3, a_2): \quad [a_5, a_4] = 1$$

since $[a_4, a_2] \in \gamma_5$. Then

$$J(a_5, a_3, a_1): \quad 1 = [a_5, a_3, a_1][a_5, a_1, a_3]^{-1}.$$

Since $[a_5, a_1, a_1, a_2] = [a_5, a_1, a_2, a_1] = 1$,

$$J\big([a_5, a_1], a_2, a_1\big): \quad [a_5, a_1, a_3] = 1.$$

Hence $[a_5, a_3, a_1] = 1$ and $a_9 = 1$. Therefore K has class at most 6.

References

[1] J. Cannon, W. Bosma, C. Fieker, A. Steel, Handbook of Magma Functions (provisional), Version 2.15, 2008.
[2] B. Eick, Computing p-groups with trivial Schur multiplicator, J. Algebra 322 (2009) 741–751.
[3] R. James, The groups of order p^6 (p an odd prime), Math. Comp. 34 (1980) 613–637.
[4] C.R. Leedham-Green, S. McKay, The Structure of Groups of Prime Power Order, London Math. Soc. Monogr. New Ser., vol. 27, Oxford Science Publications, Oxford University Press, Oxford, 2002.