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A Generalization of Sum Composition: Self Orthogonal Latin Square Design with Sub Self Orthogonal Latin Square Designs

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A generalization of the theory of sum composition of Latin square designs is given. Via this generalized theory it is shown that a self orthogonal Latin square design of order $(3p^\alpha - 1)/2$ with a subself orthogonal Latin square design of order $(p^\alpha - 1)/2$ can be constructed for any prime $p > 2$ and any positive integer α as long as $p \neq 3, 5, 7$ and 13 if $\alpha = 1$. Additional results concerning sets of orthogonal Latin square designs are also provided.

1. Introduction and Summary

Throughout this paper by an $OL(n, r)$ design we mean a set consisting of r pairwise orthogonal Latin square designs of order n. We also use the notation $L_1 \perp L_2$ to indicate that L_1 and L_2 are orthogonal and each is a Latin square design. We refer to L_2 as an orthogonal mate for L_1. A Latin square design is said to be self orthogonal if $L \perp L'$, where L' denotes the transpose of L in the context of matrices. It is known [1] that a self orthogonal Latin square design exists for every order except 2, 3 and 6.

A sub Latin square design of order t denoted by $SLS(t)$ is a Latin square design of order t embedded in a larger Latin square design. It is known [5] that if L is of order n and if L contains an $SLS(t)$ then $n \geqslant 2t$. By a combination of results given in Hedayat and Seiden [5] and by Mann [7] one can, without much difficulty, argure that if a member of an $OL(n, r)$ design contains an $SLS(t)$ then

$$\begin{aligned}
n &\geqslant 2t && \text{if} \quad n \equiv 0 (\text{mod } 4),\\
n &> 2t + 1 && \text{if} \quad n \equiv 1 (\text{mod } 4),\\
n &> 2t && \text{if} \quad n \equiv 2 (\text{mod } 4),\\
n &\geqslant 2t + 1 && \text{if} \quad n \equiv 3 (\text{mod } 4).
\end{aligned}$$

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By a sub $OL(t, 2)$ design we mean an $OL(t, 2)$ design embedded in an $OL(n, 2)$ design, $n > t$, such that each member of the latter design contains one member of the former design occupying the same t^2 cells in each design. The family of $OL(n, 2)$ designs with sub $OL(t, 2)$ designs is a very useful family of designs in the field of experimental designs and many related fields. In particular, if the $OL(n, 2)$ design and its sub $OL(t, 2)$ design is obtained from a self orthogonal Latin square design then it has many additional useful properties [3, 4]. Note that if L is self orthogonal and if it contains a self orthogonal Latin square design of order t with its main diagonal lying on the main diagonal of L then $\{L, L'\}$ is an $OL(n, 2)$ design with a sub $OL(t, 2)$ design. Therefore, a problem of interest is: for what pairs of integers n and t can one construct a self orthogonal Latin square design L of order n with a sub self orthogonal Latin square design of order t such that its main diagonal lies on the main diagonal of L? It is shown here that if $n = (3p^\alpha - 1)/2$ and $t = (p^\alpha - 1)/2$, p, a prime greater than 13, then such a design can be constructed. Our algorithm is based on an extension of the theory of sum composition of Hedayat and Seiden [6] which is treated briefly in Section 2. Additional results concerning orthogonal Latin square designs based on the theory of sum composition are also given throughout the paper. The reader is referred to Hedayat and Seiden [6] and Dénes and Keedwell [2] for details and definitions of terms used here.

2. Basic Theorems in the Theory of Sum Composition of Latin Square Designs

Consider an $m \times m$ square $A(L)$ with a Latin square design L or order $n < m$ with a transversal in its $n \times n$ top left corner subsquare. By the vertical projection of this transversal in L onto the rth row, $r > n$, of $A(L)$ we mean filling the (r, j) cell of this row by that element of this transversal which appears in the jth column of L, $j = 1, 2,..., n$. Similarly by the horizontal projection of this transversal onto the tth column, $t > n$, of $A(L)$ we mean filling the (i, t) cell of this column by that element of this transversal which appears in the ith row of L, $i = 1, 2,..., n$.

If L_1 is a Latin square design of order n_1 with n_2 parallel transversals on Σ_1 and if L_2 is a Latin square design of order n_2 on Σ_2, $\Sigma_1 \cap \Sigma_2 = \varnothing$. Then we can construct a Latin square of order $n_1 + n_2$ based on L_1 and L_2. To do this, let $A_1(L_1, L_2)$ be the square of order $n = n_1 + n_2$ with L_1 in its $n_1 \times n_1$ top left corner subsquare and L_2 in its $n_2 \times n_2$ bottom right corner subsquare. Project the n_2 transversals in L_1 vertically and horizontally onto the last n_2 rows and columns of $A_1(L_1, L_2)$ in any order, call the resulting square $A_2(L_1, L_2)$. Now replace the n_1 entries of each transversal in L_1 by a fixed element of Σ_2 such that no two transversals are being replaced by the same element of Σ_2, call the resulting square $A_3(L_1, L_2)$. Clearly $A_3(L_1, L_2)$ is a Latin square design of order n on $\Sigma_1 \cup \Sigma_2$.

It can be shown that if $\Sigma_1 = GF(n_1)$, the Galois field of order n_1, then

$$S = \{B(x_a, y_a), a = 1, 2, 3\}$$

where $B(x_a, y_a)$ is the $n_1 \times n_1$ square with $x_a\alpha_i + y_a\alpha_j$ in its (i, j) entry, $\alpha_i, \alpha_j, x_a, y_a$ in Σ_1, forms an $OL(n_1, 3)$ design if

$$x_a \neq 0, \qquad y_a \neq 0$$

$$y_a x_a^{-1} \neq y_{a'} x_{a'}^{-1}, \qquad a \neq a'.$$

Note that the n_1 cells in $B(x_2, y_2)$ and $B(x_3, y_3)$ corresponding to the n_1 cells in $B(x_1, y_1)$ with a fixed entry, sat t, form a common transversal for

$$S_1 = \{B(x_2, y_2), B(x_3, y_3)\}.$$

We label this transversal in $B(x_2, y_2)$ and $B(x_3, y_3)$ by t. Therefore, as t goes over Σ_1 we can locate and name n_1 parallel transversals for S_1. Now consider $A(B(x_2, y_2))$ and $A(B(x_3, y_3))$ each of size $n \times n$ with $n > n_1$. We are interested to characterize:

(i) The n_1 pairs obtained by the vertical projection of transversal t_1 in $B(x_2, y_2)$ onto say the rth $(r > n_1)$ row of $A(B(x_2, y_2))$ and the vertical projection of transversal t_2 in $B(x_3, y_3)$ onto the rth row of $A(B(x_3, y_3))$.

(ii) The n_1 pairs obtained by the horizontal projection of transversal t_1 in $B(x_2, y_2)$ onto say the cth $(c > n_1)$ column of $A(B(x_2, y_2))$ and the horizontal projection of transversal t_2 in $B(x_3, y_3)$ onto the cth column of $A(B(x_3, y_3))$.

These characterization are needed for further development in the area.

THEOREM 2.1. *The n_1 pairs obtained by the vertical projection of transversal t_1 in $B(x_2, y_2)$ and transversal t_2 in $B(x_3, y_3)$ onto the same row of $A(B(x_2, y_2))$ and $A(B(x_3, y_3))$ is the same as the n_1 pairs obtained by transversal $k_v(t_1, t_2)$ in $B(x_2, y_2)$ and in $B(x_3, y_3)$ where*

$$k_v(t_1, t_2) = (x_3 t_2 - \alpha x_2 t_1)(x_3 - \alpha x_2)^{-1}$$

for

$$\alpha = (y_3 - x_3 x_1^{-1} y_1)(y_2 - x_2 x_1^{-1} y_1)^{-1}.$$

Proof. The n_1 entries of the rth row of $A(B(x_i, y_i))$ upon vertical projection of transversal t_{i-1} onto it are

$$x_i x_1^{-1} t_{i-1} + (y_i - x_i x_1^{-1} y_1)\,\alpha_j, \qquad i = 2, 3; \;\; j = 1, 2, \ldots, n_1.$$

Note that the above expression has nothing to do with the value of r. Therefore, the n_1 pairs obtained by the n_1 entries of the rth row of $A(B(x_2, y_2))$ and $A(B(x_3, y_3))$ are:

$$[x_2 x_1^{-1} t_1 + (y_2 - x_2 x_1^{-1} y_1)\,\alpha_j,\; x_3 x_1^{-1} t_2 + (y_3 - x_3 x_1^{-1} y_1)\,\alpha_j] \quad j = 1, 2, \ldots, n_1. \tag{1}$$

On the other hand, the n_1 pairs obtained from the corresponding cells of transversal t in $B(x_2, y_2)$ and $B(x_3, y_3)$ are:

$$[x_2 x_1^{-1} t + (y_2 - x_2 x_1^{-1} y_1)\,\alpha_{j'},\; x_3 x_1^{-1} t + (y_3 - x_3 x_1^{-1} y_1)\,\alpha_{j'}] \quad j' = 1, 2, \ldots, n_1. \tag{2}$$

Equating (1) to (2) componentwise and solving for t we obtain $t = k_v(t_1, t_2)$ as given in the theorem.

Similarly we can prove that:

THEOREM 2.2. *The n_1 pairs obtained by the horizontal projection of transversal t_1 in $B(x_2, y_2)$ and transversal t_2 in $B(x_3, y_3)$ is the same as the n_1 pairs obtained by the transversal $k_h(t_1, t_2)$ where*

$$k_h(t_1, t_2) = (y_3 t_2 - \beta y_2 t_1)(y_3 - \beta y_2)^{-1}$$

for

$$\beta = (x_3 - y_3 y_1^{-1} x_1)(x_2 - y_2 y_1^{-1} x_1)^{-1}.$$

COROLLARY 2.1. (i) *$k_v(t_1, t_2) = t_1$ or t_2 if and only if $t_1 = t_2$,*

(ii) *$k_h(t_1, t_2) = t_1$ or t_2 if and only if $t_1 = t_2$.*

COROLLARY 2.2. *For $x_1 = 1$, $y_1 = -1$, $x_2 = x$, $y_2 = y$, $x_3 = y$, $y_3 = x$*

(i) $k_v(t_1, t_2) = (y t_2 - x t_1)(y - x)^{-1}$,

(ii) $k_h(t_1, t_2) = (y t_1 - x t_2)(y - x)^{-1}$,

(iii) $k_v(t_1, t_2) + k_h(t_1, t_2) = t_1 + t_2$.

This later corollary is needed for the development of the results given in Section 4.

3. Six Steps for the Construction of r Mutually Orthogonal Latin Squares of Order n via Sum Composition Method

Successful implementation of all the following steps produces a set of r mutually orthogonal Latin square of order n via sum composition technique.

Step One. Decompose n as $n = n_1 + n_2$ with $n_1 = p^\alpha$, p a prime, α a positive integer and $n_1 \geqslant rn_2$.

Step Two. Construct a set of r mutually orthogonal Latin squares of order n_2. Denote it by $\{H_1, H_2, \ldots, H_r\}$.

Step Three. Select x_j, y_j, $j = 0, 1, \ldots, r$ in $GF(n_1)$ such that $x_j \neq 0, y_j \neq 0$ and $y_j x_j^{-1} \neq y_{j'} x_{j'}^{-1}$, $j \neq j'$.

Step Four. Construct $\{B(x_j, y_j), j = 0, 1, \ldots, r\}$.

Step Five. Select a set of n_2 parallel transversals determined by the elements of $B(x_0, y_0)$ in $B(x_j, y_j)$. Say $T_j = \{t_{j1}, t_{j2}, \ldots, t_{jn_2}\}$ such that $T_j \cap_{j \neq j'} T_{j'} = \varnothing, j = 1, 2, \ldots, r$.

Step Six. Construct $L_j = A_3(B(x_j, y_j), H_j)$, i.e., a Latin square of order n based on $B(x_j, y_j)$ and H_j (see Section 2), $j = 1, 2, \ldots, r$, such that

$$K_v(j, j') \cup K_h(j, j') = T_j \cup T_{j'}, j \neq j', j, j' = 1, 2, \ldots, r,$$

where $K_v(j, j')$ and $K_h(j, j')$ are the sets of transversals $k_v(\,,\,)$ and $k_h(\,,\,)$ as characterized in Theorems 2.1 and 2.2 while generating L_j and $L_{j'}$ respectively. Then $\{L_1, L_2, \ldots, L_r\}$ is a set of r mutually orthogonal Latin squares of order n.

It is clear that if step one and two can be completed then one can automatically finish steps three, four and five. However, it is quite possible that by no means one can finish step six for some cases. Failure to finish step six usually come from bad choices of x_j, y_j and T_j in steps three and five. Therefore, one should go back to these steps and modify the choices. It is also possible that for some decomposition of n it is impossible to finish step six even though step two is possible. To mention one example, it is impossible to go through step six, if $n_2 = 1$ as it is evident from the results in Section 2.

Remark 3.1. As apparent the construction of an $OL(n, r)$ design via the method of sum composition is not a trivial problem and the major confronting questions are:

(i) How to choose x_j and y_j, $j = 0, 1, \ldots, r$? If $r = 2$ then $x_2 = x_1^{-1}$ and $x_0 = y_0 = y_1 = y_2 = 1$ simplifies the calculation considerably.

(ii) How to choose T_j, $j = 1, 2, \ldots, r$? If n_2 divides $n_1 - 1$ then our experience indicates that we are computationally better off if we let T_j be a coset of a subgroup of order n_2 in $GF(n_1) - \{0\}$. This point will be supported shortly via an example.

(iii) How to project vertically and horizontally the members of T_j, so that

$$K_v(j, j') \cup K_h(j, j') = T_j \cup T_{j'} .$$

EXAMPLE 3.1. One can utilize a result of Hedayat and Seiden [3] and construct an $OL(46, 2)$ design via sum composition of either an $OL(43, 2)$ design and an $OL(3, 2)$ design or an $OL(41, 2)$ design and an $OL(5, 2)$ design. In the sequel we show, explicitly, how to construct an $OL(46, 3)$ design via sum composition of an $OL(41, 3)$ design and an $OL(5, 3)$ design. To do this we have to find x_j, y_j, $j = 0, 1, 2, 3$ together with three non-intersecting subsets each of size 5 in $GF(41)$. Also we should exhibit a successful projection rule mentioned in Step six.

Let $x_0 = -1$, $x_1 = 33$, $x_2 = 37$, $x_3 = 38$ and $y_j = 1$, $j = 0, 1, 2, 3$. Since $\{1, 10, 18, 16, 37\}$ is a subgroup of order 5 in $GF(41)$–$\{0\}$, we take the the following three costs as our T_1, T_2, and T_3.

$$T_1 = \{30, 29, 13, 7, 3\} \text{ generated by } 3,$$
$$T_2 = \{26, 19, 17, 14, 6\} \text{ generated by } 6,$$
$$T_3 = \{40, 31, 23, 25, 4\} \text{ generated by } 4.$$

Construct $\{B(-1, 1), B(33, 1), B(37, 1), B(38, 1)\}$. The table below summarizes the rules of projections.

Vert. and horiz. projections	-th column or row				
	42	43	44	45	46
$B(33, 1)$	30	13	7	29	3
$B(37, 1)$	6	19	26	14	17
$B(38, 1)$	40	31	23	25	4

For example, in forming L_1 based on $B(33, 1)$, we should project transversals 30, 13, 7, 29 and 3 onto rows (columns) 42, 43, 44, 45, 46 respectively Now the reader can check for himself that

$$K_v(j, j') \cup K_h(j, j') = T_j \cup T_{j'}, j \neq j', j, j' = 1, 2, 3.$$

4. A Family of Self Orthogonal Latin Square Designs with Subself Orthogonal Latin Square Designs

Throughout this section it is assumed that $x,\ y \in GF(n_1)$ with $x \neq 0$ and $y \neq 0$.

THEOREM 4.1. *$L_1 = B(x, y)$ is a self orthogonal Latin square design of order n_1 if and only if $x \neq y \neq 0$.*

Proof. $B(x, y)$ is self orthogonal if it is orthogonal to its transpose $B(y, x)$. Consider all cells of $B(x, y)$ which contain a fixed element of $GF(n_1)$ say t. Such entries are representable as

$$x\alpha_i + y\alpha_j = t.$$

The corresponding entries in $B(y, x)$ are

$$y\alpha_i + x\alpha_j \qquad \text{with} \quad x\alpha_i + y\alpha_j = t.$$

This implies that these n_1 entries are representable by

$$tyx^{-1} + (x^2 - y^2)x^{-1}\alpha_j$$

for n_1 different choices of α_j. Thus these entries exhaust $GF(n_1)$ if and only if $(x^2 - y^2)x^{-1} \neq 0$ or equivalently $x \neq y \neq 0$.

Before we proceed further we need the following lemma.

LEMMA 4.1. *Let $S = \{B(1, -1),\ B(x, y),\ B(y, x)\}$ such that $x \neq y$. If $B(x, y)$ is trnasposed then the transversal t occupies the same cells and has the same values as the transversal $-t$ in $B(y, x)$.*

Proof. It follows directly from the definition of the transversal t and the fact that $B(y, x)$ is the transpose of $B(x, y)$.

Our goal here is to compose $L_1 = B(x, y)$ with a self orthogonal Latin square design of order $n_2 \leqslant [n_1/2]$ such that the resulting design $A_3(L_1, L_2)$ [see Section 2] is self orthogonal. This is equivalent to composing $L_1' = B(y, x)$ with L_2' such that $A_3(L_1', L_2')$ is orthogonal to $A_3(L_1, L_2)$ and $A_3(L_1', L_2')$ is the transpose of $A_3(L_1, L_2)$. These requirements put heavy restrictions on the choice of transversals to be removed for projections in $B(y, x)$.

The patterns of projections will be also controlled by those given in the formation of $A_3(L_1, L_2)$ as it is evident by the following lemmas whose proofs are straight forward and thus omitted.

LEMMA 4.2. *A necessary condition for $A_3(L_1', L_2')$ to be the transpose and an orthogonal mate of $A_3(L_1, L_2)$ is: If $T_1 = \{t_{1i}, i = 1, 2,..., n_2\}$ is selected for projections in $B(x, y)$ then the set of selected transversals for projections in $B(y, x)$ should be $T_2 = -T_1$.*

Since $T_1 \cap T_2$ should be empty and since $T_2 = -T_1$ this means that $t_i + t_j \neq 0$ for any choice of i and j. This leads us to the following definition.

DEFINITION 4.1. By an inverse free set of transversals we mean a set T of transversals such that $T \cap -T = \varnothing$.

LEMMA 4.3. *A necessary condition for $A_3(L_1', L_2')$ to be the transpose and an orthogonal mate of $A_3(L_1, L_2)$ is: If Transversal $t_i \in T_i$ is projected vertically and horizontally on the* pth *row and* qth *column in the formation of $A_3(L_1, L_2)$ then Transversal $-t_i \in T_2$ should be projected vertically and horizontally on the* pth *row and* qth *column in the formation of $A_3(L_1', L_2')$.*

EXAMPLE 4.1. Let $n_1 = 11$ and $n_2 = 4$, $x = 2$ and $y = -1$. Then form $L = A_3(L_1, L_2)$ by the sum composition of a self orthogonal Latin square L_2 of order 4 and $L_1 = B(2, -1)$ using $T_1 = \{7, 8, 9, 10\}$ for projections. If we project vertically on the 12th row,..., and 15th row transversals 10, 9, 8 and 7 respectively and if we project horizontally on the 12th column,..., and 15th column transversals 8, 10, 7 and 9 respectively then L is a self orthogonal Latin square of order 15 with a sub-self orthogonal Latin square of order 4.

Wallis [5] has also produced a self orthogonal Latin square design of order 15 which resembles the above design. However, Wallis does not explain how his design has been obtained. The above design is non-isomorphic to the one given by Brayton, Coppersmith and Hoffman [1].

THEOREM 4.2. *Let p be any prime except* 2, 3, 5, 7 *and* 13. *Then there exists a self orthogonal Latin square design of order $(3p - 1)/2$ with a subself orthogonal Latin square design of order $(p - 1)/2$.*

Proof. By construction. It is well known [1] that there exists a self orthogonal Latin square design for any order except 2, 3, and 6. So let L_2 be any self orthogonal Latin square design of order $(p - 2)/2$. Let also $L_1 = B(x, y)$ be a Latin square design with x, y in $GF(p)$ as long as $x \neq y \neq 0$ Project the elements of the inverse free set $T_1 = \{1, 2,..., (p - 1)/2\}$ in the following fashions:

Horizontal projection: project the ith transversal on the $(n_1 + i)$th column.

Vertical projection: project the ith transversal on the $(n_1 + i)$th row. It can be easily verified that $k_v(t, -t) = t(x + y)(x - y)^{-1}$ and $k_v(t_1, -t_1) \neq$

$k_v(t_2, -t_2)$ for $t_1 \neq t_2$. Therefore $K_v(1, 2)$ generates half of $GF(p) - \{0\}$. On the other hand $K_h(1, 2) = -K_v(1, 2)$ and thus

$$K_v(1, 2) \cup K_h(1, 2) = T_1 \cup T_2 \quad \text{with } T_2 = -T_1 .$$

COROLLARY 4.1. *Let p be any prime except 2, 3, 5, 7 and 13. Then there exists a self orthogonal Latin square design of order* $(3p^\alpha - 1)/2$ *with a sub self orthogonal Latin square design of order* $(P^\alpha - 1)/2$ *for any positive integer* α.

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