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Lifting Theorems for Tilting Complexes

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INTRODUCTION

Green showed in [4] that a module M for a group algebra kG can be lifted to an $\mathcal{O}G$-lattice, where $\mathcal{O}$ is a complete discrete valuation ring with residue field k, if $\mathrm{Ext}^2_{kG}(M, M)$ vanishes. When we consider equivalences between derived categories of module categories, we are led, as in [7], to consider tilting complexes, which are objects P^* of the derived category for which, among other conditions, $\mathrm{Hom}(P^*, P^*[i])$ vanishes for $i \neq 0$. The case $i = 2$ gives a condition analogous to Green's, and we may ask whether his theorem has an analogy for tilting complexes. The aim of this paper is to show that it does; in fact we shall consider more general situations where we have a quotient ring $\bar{R}$ of a ring R and a tilting complex for $\bar{R}$ that can be lifted to a tilting complex for R.

In Green's original proof he used a standard resolution for group cohomology, so his method will not immediately generalize to algebras other than group algebras; neither does it generalize easily to the case of complexes. The method we use is to lift each term of our complex P^* and each differential separately to obtain a sequence of modules and maps that need no longer be a complex, since the square of the differential need not be zero. However, the amount by which it fails to be a complex can be used to define a map of complexes from P^* to $P^*[2]$—this part of the proof uses a construction that is formally the same as that of Section 1 of Eisenbud's paper [3]—and a homotopy from this map to zero can be used to adjust our lifting of the differential to obtain a genuine complex.

The fact that a tilting complex P^* also satisfies $\mathrm{Hom}(P^*, P^*[1]) = 0$ allows us to deduce that our liftings are essentially unique.

In Section 1 we recall the basic properties of tilting complexes and prove a technical result that we shall need to recognize a tilting complex.

In Section 2 we consider what conditions are needed to lift a tilting complex for an algebra to a tilting complex for an extension of the algebra

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by a bimodule. Our results generalize Corollary 5.4 of [9], where we considered trivial extensions.

In Section 3 we consider algebras over complete local rings and their residue fields. This is the section most closely related to Green's theorem, and our Corollary 3.2 generalizes his result to modules for arbitrary finite rank algebras over a complete Noetherian local ring. By Corollary 2.2 of [9], if two blocks of finite group algebra are derived equivalent over a complete discrete valuation ring $\mathcal{O}$ then they are derived equivalent over the residue field k. Theorem 3.3 of the present paper provides a partial converse; it implies that if B_0 and B_1 are block algebras over $\mathcal{O}$, and if $B_0 \otimes k$ is derived equivalent to $B_1 \otimes k$, then B_0 is derived equivalent to some canonically determined algebra B_1' over $\mathcal{O}$ such that $B_1' \otimes k \cong B_1 \otimes k$. Of course, we need extra information to deduce that $B_1 \cong B_1'$, but this result does suggest that any conjecture such as those of Broué [2] about the existence of derived equivalences for blocks is unlikely to be true over k but false over $\mathcal{O}$. One situation where an equivalence of derived categories over k has been lifted to $\mathcal{O}$ is for blocks of cyclic defect group; Linckelmann [6] showed that when such blocks are derived equivalent over k, which occurs when the associated Brauer trees have the same number of edges and the same multiplicity for the exceptional vertex [8], then they are derived equivalent over $\mathcal{O}$.

Schaps showed in [10] that if we have a flat deformation of a finite dimensional algebra Λ then a projective module for Λ can be lifted to a projective module for the deformation, at least on some étale neighbourhood. Happel and Schaps [5] proved an analogous result for tilting modules of projective dimension one. In Section 4 we use our results to extend this work to general tilting complexes. Our method is essentially the same as that of [10, 5], using the Artin approximation theorem [1], and we refer to [10] for the geometric terminology that we use.

1. Preliminaries

We shall frequently use notation and results from [7, 9]. In particular, recall from [7] that a tilting complex for a ring Λ is an object P^* of $K^b(P_\Lambda)$, the homotopy category of bounded complexes of finitely generated projective Λ-modules, such that

(i) $\operatorname{Hom}(P^*, P^*[i]) = 0$ for $i \neq 0$;

(ii) $\operatorname{add}(P^*)$, the additive category of direct summands of finite direct sums of copies of P^*, generates $K^b(P_\Lambda)$ as a triangulated category.

By Theorem 6.4 of [7], the derived category $D^b(\text{Mod-}\Gamma)$ of bounded complexes of modules for a ring Γ is equivalent as a triangulated category to $D^b(\text{Mod-}\Lambda)$ if and only if Γ is isomorphic to the endomorphism ring of a tilting complex for Λ.

By Proposition 5.4 of [7], condition (ii) in the definition of a tilting complex can be replaced by

(ii)′ For each non-zero object X^* of $K^-(\text{Proj-}\Lambda)$, the homotopy category of complexes, bounded above, of projective Λ-modules, there is some i for which

$$\text{Hom}_{K^-(\text{Proj-}\Lambda)}(P^*, X^*[i]) \neq 0.$$

In Section 3 we shall need to know that it is sometimes enough to consider only finitely generated projective modules.

LEMMA 1.1. *Let Λ be an R-algebra, finitely generated as an R-module, for some Noetherian commutative ring R. Let P^* be an object of $K^b(P_\Lambda)$ such that*

(i) $\text{Hom}(P^*, P^*[i]) = 0$ *for* $i \neq 0$;

(ii)″ *for each non-zero object X^* of $K^-(P_\Lambda)$ there is some i for which*

$$\text{Hom}(P^*, X^*[i]) \neq 0.$$

Then P^* is a tilting complex.

Proof. This follows from the observation that, for Λ and P^* satisfying our conditions, the "P^*-resolution" functor of Section 4 of [7] maps $K^-(P_\Lambda)$ to $K^-(P_{\text{End}(P^*)})$, since $\text{Hom}(P^*, Y^*)$ is finitely generated as an R-module for any Y^* in $K^-(P_\Lambda)$ (cf. the remark after Proposition 5.4 of [7]). ∎

2. EXTENSIONS OF ALGEBRAS

We shall fix the following notation for this section. The rings Λ and Γ will be derived equivalent algebras over a field k, and P^* will be a tilting complex for Λ with endomorphism algebra Γ. There are two-sided tilting complexes (see Section 3 of [9]) in $D^b(\text{Mod-}(\Gamma^{\text{op}} \otimes_k \Lambda))$ and $D^b(\text{Mod-}(\Lambda^{\text{op}} \otimes_k \Gamma))$ that we shall call X^* and $\tilde{X}^*$, respectively, such that $X^* \otimes^{\mathbf{L}}_\Lambda \tilde{X}^* \cong {}_\Gamma\Gamma_\Gamma$ and $\tilde{X}^* \otimes^{\mathbf{L}}_\Gamma X^* \cong {}_\Lambda\Lambda_\Lambda$, and such that X^* is isomorphic to P^* when considered as an object of $D^b(\text{Mod-}\Lambda)$. We shall choose a Λ-bimodule M and a Γ-bimodule N that correspond under the equivalence of derived categories of bimodule categories given by X^* and $\tilde{X}^*$, i.e., such that ${}_\Gamma N_\Gamma \cong X^* \otimes^{\mathbf{L}}_\Lambda M \otimes^{\mathbf{L}}_\Lambda \tilde{X}^*$. For example, we could take

$M=\Lambda$ and $N=\Gamma$, or for Λ and Γ finite-dimensional algebras we could, by Proposition 5.2 of [9], take M and N to be $\mathrm{Hom}_k(\Lambda, k)$ and $\mathrm{Hom}_k(\Gamma, k)$.

By an *extension* of an algebra Δ by a Δ-bimodule I we shall mean an algebra D with a surjective ring homomorphism $D \to \Delta$ whose kernel has square zero (and so has a Δ-bimodule structure) and is isomorphic to I as a Δ-bimodule.

THEOREM 2.1. *Let L be an extension of Λ by M. There is a tilting complex Q^* for L such that $Q^* \otimes_L^{\mathbf{L}} \Lambda \cong P^*$, unique up to isomorphism. The endomorphism algebra of Q^* is an extension of Γ by N.*

Proof. Certainly P^* can be lifted to a graded projective L-module Q^* together with a map $\delta: Q^i \to Q^{i+1}$, so that $(Q^* \otimes_L \Lambda, \ \delta \otimes_L \Lambda)$ is isomorphic to the complex P^*, with δ^2 mapping Q^* to the submodule $Q^{*+2} \otimes_L M$ of Q^{*+2}. The kernel of δ^2 contains $Q^* \otimes_L M$, so δ^2 gives a map

$$\alpha: P^* \to P^{*+2} \otimes_\Lambda M$$

of graded Λ-modules. Since δ^2 and δ commute, α is in fact a map of complexes from P^* to $P^* \otimes_\Lambda M[2]$.

However,

$$\begin{aligned}
&\mathrm{Hom}_{D^b(\text{Mod-}\Lambda)}(P^*, P^* \otimes_\Lambda M[2]) \\
&\quad \cong \mathrm{Hom}_{D^b(\text{Mod-}\Gamma)}(P^* \otimes_\Lambda \tilde{X}^*, P^* \otimes_\Lambda M \otimes_\Lambda^{\mathbf{L}} \tilde{X}^*[2]) \\
&\quad \cong \mathrm{Hom}_{D^b(\text{Mod-}\Gamma)}(\Gamma, N[2]) \cong 0,
\end{aligned}$$

so α is homotopic to zero. A homotopy gives us a map of graded Λ-modules from P^* to $P^{*+1} \otimes_\Lambda M$ and hence a map of graded L-modules

$$h: Q^* \to Q^{*+1}$$

such that $h^2=0$ and $\delta^2 = h\delta + \delta h$, so $(\delta - h)^2 = 0$. Therefore $Q^* = (Q^*, \delta - h)$ is a complex of projective L-modules such that $Q^* \otimes_L \Lambda \cong P^*$.

We have a distinguished triangle

$$P^* \otimes_\Lambda M \to Q^* \to P^* \to$$

in $D^b(\text{Mod-}L)$. Applying the functor $\mathrm{Hom}_{D^b(\text{Mod-}L)}(Q^*, -)$ we obtain a long exact sequence. Using the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{D^b(\text{Mod-}L)}(Q^*, P^*[i]) &\cong \mathrm{Hom}_{D^b(\text{Mod-}\Lambda)}(P^*, P^*[i]) \\
&\cong \begin{cases} \Gamma & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}\operatorname{Hom}_{D^b(\text{Mod-}L)}(Q^*, P^*\otimes_\Lambda M[i]) &\cong \operatorname{Hom}_{D^b(\text{Mod-}\Lambda)}(P^*, P^*\otimes_\Lambda M[i])\\ &\cong \operatorname{Hom}_{D^b(\text{Mod-}\Gamma)}(\Gamma, N[i])\\ &\cong \begin{cases} N & \text{if } i=0\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

we deduce that $\operatorname{Hom}_{D^b(\text{Mod-}L)}(Q^*, Q^*[i])=0$ for $i\neq 0$ and that there is a short exact sequence

$$0\to N\to \operatorname{End}_{D^b(\text{Mod-}L)}(Q^*)\to \Gamma\to 0$$

expressing $\operatorname{End}_{D^b(\text{Mod-}L)}(Q^*)$ as an extension of Γ by N. That the bimodule structure of N is correct follows from the fact (Lemma 3.2 of [9]) that the actions of Γ on X^* and $\tilde{X}^*$ by Λ-endomorphisms and by multiplication coincide.

Let S^* be an object of $K^-(\text{Proj-}L)$ such that $\operatorname{Hom}_{K^-(\text{Proj-}L)}(Q^*, S^*[i])=0$ for all i. To complete the proof that Q^* is a tilting complex we just need to show that $S^*\cong 0$. Applying $\operatorname{Hom}_{D^-(\text{Mod-}L)}(Q^*,-)$ to the distinguished triangle

$$S^*\otimes_L M\to S^*\to S^*\otimes_L \Lambda\to,$$

we deduce that

$$\operatorname{Hom}_{D^-(\text{Mod-}L)}(Q^*, S^*\otimes_L \Lambda[i])\cong \operatorname{Hom}_{D^-(\text{Mod-}L)}(Q^*, S^*\otimes_L M[i+1]).$$

We have the isomorphisms

$$\operatorname{Hom}_{D^-(\text{Mod-}L)}(Q^*, S^*\otimes_L \Lambda[i])\cong \operatorname{Hom}_{D^-(\text{Mod-}\Gamma)}(\Gamma, S^*\otimes_L \tilde{X}^*[i])$$

and

$$\operatorname{Hom}_{D^-(\text{Mod-}L)}(Q^*, S^*\otimes_L M[i])\cong \operatorname{Hom}_{D^-(\text{Mod-}\Gamma)}(\Gamma, (S^*\otimes_L \tilde{X}^*)\otimes_\Gamma^{\mathbf{L}} N[i]),$$

but since $S^*\otimes_L \Lambda\otimes_\Lambda \tilde{X}^*$ is an object of $D^-(\text{Mod-}\Gamma)$ and N is a Γ-bimodule, the complex $(S^*\otimes_L \tilde{X}^*)\otimes_\Gamma^{\mathbf{L}} N$ has zero homology in degrees greater than the highest degree in which $S^*\otimes_L \tilde{X}^*$ has non-zero homology. Hence

$$S^*\otimes_L \tilde{X}^*\cong (S^*\otimes_L \tilde{X}^*)\otimes_\Gamma^{\mathbf{L}} N\cong 0,$$

and so $S^*\cong 0$, so Q^* is a tilting complex.

To prove the uniqueness of Q^*, let Q'^* be another lifting of P^*. Applying the functor

$$\operatorname{Hom}_{D^b(\text{Mod-}L)}(Q'^*,-)$$

to the distinguished triangle

$$P^* \otimes_\Lambda M \to Q^* \to P^* \to$$

and using the fact that

$$\operatorname{Hom}_{D^b(\text{Mod-}L)}(Q'^*, P^* \otimes_\Lambda M[1]) \cong \operatorname{Hom}_{D^b(\text{Mod-}\Gamma)}(\Gamma, N[1]) \cong 0,$$

we deduce that the map

$$\operatorname{Hom}_{D^b(\text{Mod-}L)}(Q'^*, Q^*) \to \operatorname{Hom}_{D^b(\text{Mod-}\Lambda)}(P^*, P^*)$$

is surjective, so we can lift the identity map to obtain a distinguished triangle

$$Q'^* \to Q^* \to Z^* \to,$$

where $Z^* \otimes_L^{\mathbf{L}} \Lambda \cong 0$. But then $Z^* \cong 0$. Hence $Q'^* \cong Q^*$. ∎

For the case where L is the trivial extension of Λ by M, Theorem 2.1 was proved in Corollary 5.4 of [9], and in this case the endomorphism ring of Q^* is the trivial extension of Γ by N. The extensions of Λ by M and of Γ by N correspond to elements of the Hochschild cohomology groups $HH_k^2(\Lambda, M)$ and $HH_k^2(\Gamma, N)$, which are isomorphic because of the equivalence between $D^b(\text{Mod-}(\Lambda^{\text{op}} \otimes_k \Lambda))$ and $D^b(\text{Mod-}(\Gamma^{\text{op}} \otimes_k \Gamma))$. Presumably the element of $HH_k^2(\Lambda, M)$ corresponding to the extension L is sent by this isomorphism to the element of $HH_k^2(\Gamma, N)$ corresponding to the endomorphism ring of Q^*, but this does not seem to follow easily.

3. Algebras over Local Rings

Throughout this section $\mathcal{O}$ will be a complete commutative Noetherian local ring with maximal ideal $\mathcal{M}$ and with residue field $k = \mathcal{O}/\mathcal{M}$. We shall consider an $\mathcal{O}$-algebra Λ that is free of finite rank as an $\mathcal{O}$-module, and we shall set $\Lambda_n = \Lambda \otimes_{\mathcal{O}} (\mathcal{O}/\mathcal{M}^n)$.

Proposition 3.1. *Let $P^* = (P^*, d_1)$ be an object of $K^-(P_{\Lambda_1})$ such that*

$$\operatorname{Hom}_{K^-(P_{\Lambda_1})}(P^*, P^*[2]) = 0.$$

Then there is an object $Q^* = (Q^*, d)$ of $K^-(P_\Lambda)$ such that $Q^* \otimes_{\mathcal{O}} k \cong P^*$. If also

$$\operatorname{Hom}_{K^-(P_{\Lambda_1})}(P^*, P^*[1]) = 0$$

then Q^* is unique up to isomorphism.

Proof. Projective Λ_1-modules can be lifted to Λ, so we may choose a graded projective Λ-module Q^* such that $P^* \cong Q^* \otimes_{\mathcal{O}} k$. Suppose that for some n we can lift the differential d_1 to a differential d_n of $Q^* \otimes_\Lambda \Lambda_n$; this is trivially true for $n=1$. We can lift d_n to a degree one graded endomorphism δ_{n+1} of $Q^* \otimes_\Lambda \Lambda_{n+1}$, where δ_{n+1}^2 factors as

$$Q^* \otimes_\Lambda \Lambda_{n+1} \to Q^* \otimes_{\mathcal{O}} k \to Q^{*+2} \otimes_{\mathcal{O}} (\mathcal{M}^n/\mathcal{M}^{n+1}) \to Q^{*+2}.$$

Since δ_{n+1}^2 commutes with δ_{n+1}, the second map is a map of complexes from a complex isomorphic to P^* to a finite direct sum of complexes isomorphic to $P^*[2]$. By assumption, this is homotopic to zero. Composing a homotopy with the projection $Q^* \otimes_\Lambda \Lambda_{n+1} \to Q^* \otimes_{\mathcal{O}} k$ and the embedding $Q^* \otimes_{\mathcal{O}} (\mathcal{M}^n/\mathcal{M}^{n+1}) \to Q^* \otimes_\Lambda \Lambda_{n+1}$, we obtain a degree one graded endomorphism h_{n+1} of the graded module $Q^* \otimes_\Lambda \Lambda_{n+1}$ such that $\delta_{n+1}^2 = h_{n+1}\delta_{n+1} + \delta_{n+1}h_{n+1}$. We can then set $d_{n+1} = \delta_{n+1} - h_{n+1}$, so that d_{n+1} is a lifting of d_n to a differential. Since $\mathcal{O}$ is complete, we may take the limit of the d_n to obtain a complex Q^* as required.

Let $Q'^* = (Q'^*, d')$ be another lifting of P^*. Suppose we can lift an isomorphism $b_1: Q'^* \otimes_{\mathcal{O}} k \to Q^* \otimes_{\mathcal{O}} k$ to a map $b_n: Q'^* \otimes_\Lambda \Lambda_n \to Q^* \otimes_\Lambda \Lambda_n$ of complexes for some n. We can lift b_n to a map β_{n+1} of graded modules $Q'^* \otimes_\Lambda \Lambda_{n+1} \to Q^* \otimes_\Lambda \Lambda_{n+1}$, and $d'\beta_{n+1} - \beta_{n+1}d$ factors as

$$Q'^* \otimes_\Lambda \Lambda_{n+1} \to Q'^* \otimes_\Lambda \Lambda_1 \to Q^{*+1} \otimes_{\mathcal{O}} (\mathcal{M}^n/\mathcal{M}^{n+1}) \to Q^{*+1} \otimes_\Lambda \Lambda_{n+1}.$$

Since

$$\begin{aligned}(d' \otimes_\Lambda \Lambda_n)&((d' \otimes_\Lambda \Lambda_n)b_n - b_n(d \otimes_\Lambda \Lambda_n)) \\ &= ((d' \otimes_\Lambda \Lambda_n)b_n - b_n(d \otimes_\Lambda \Lambda_n))(-d \otimes_\Lambda \Lambda_n),\end{aligned}$$

the second map is a map of complexes from a complex isomorphic (in $K^-(P_{\Lambda_1})$) to P^* to a finite direct sum of complexes isomorphic to $P^*[1]$. If $\operatorname{Hom}_{K^-(P_{\Lambda_1})}(P^*, P^*[1]) = 0$, then this is homotopic to zero, and the composition of a homotopy with the projection $Q'^* \otimes_\Lambda \Lambda_{n+1} \to Q'^* \otimes_\Lambda \Lambda_1$ and the embedding $Q^* \otimes_{\mathcal{O}} (\mathcal{M}^n/\mathcal{M}^{n+1}) \to Q^* \otimes_\Lambda \Lambda_{n+1}$ gives a map H_{n+1} such that

$$\begin{aligned}(d' \otimes_\Lambda \Lambda_{n+1})&H_{n+1} - H_{n+1}(d \otimes_\Lambda \Lambda_{n+1}) \\ &= (d' \otimes_\Lambda \Lambda_{n+1})\beta_{n+1} - \beta_{n+1}(d \otimes_\Lambda \Lambda_{n+1}).\end{aligned}$$

If we now set $b_{n+1} = \beta_{n+1} - H_{n+1}$ then b_{n+1} is a lifting of b_n to a map of complexes $Q'^* \otimes_\Lambda \Lambda_{n+1} \to Q^* \otimes_\Lambda \Lambda_{n+1}$. The limit b of the b_n gives a distinguished triangle

$$Q'^* \to Q^* \to Z^* \to$$

in $K^-(P_\Lambda)$, with $Z^* \otimes_{\mathcal{O}} k$ acyclic. Since all the $\mathcal{O}$-modules involved are finitely generated, Z^* is therefore acyclic, so b is an isomorphism. ∎

To prove that P^* could be lifted to Λ, we did not need Λ to be of finite $\mathcal{O}$-rank, and P^* could have been any complex of (possibly infinitely generated) projective modules. However, the proof of uniqueness did use all the finiteness assumptions.

The following corollary, generalizing Green's theorem to more general algebras, seems to be quite well known; proofs have been found by both Kroll and Stammbach; however, I know of no published version.

COROLLARY 3.2. *Let M be a finitely generated Λ_1-module such that $\mathrm{Ext}^2_{\Lambda_1}(M, M) = 0$. There is a Λ-module N, free as an $\mathcal{O}$-module, such that $M \cong N \otimes_{\mathcal{O}} k$. If also $\mathrm{Ext}^1_{\Lambda_1}(M, M) = 0$ then N is unique up to isomorphism.*

Proof. Applying Proposition 3.1 to a projective resolution P^* of M, we obtain a complex Q^* of finitely generated projective Λ-modules. Since the homology of $P^* \cong Q^* \otimes_{\mathcal{O}} k$ is concentrated in degree zero, Q^* is isomorphic in $K^-(P_{\mathcal{O}})$ to a free $\mathcal{O}$-module in degree zero. Thus Q^* is a projective resolution of an Λ-module N with the required properties. Uniqueness follows immediately from Proposition 3.1. ∎

THEOREM 3.3. *Let P^* be a tilting complex for Λ_1. There is a tilting complex Q^* for Λ, unique up to isomorphism, such that $P^* \cong Q^* \otimes_{\mathcal{O}} k$. The endomorphism $\mathcal{O}$-algebra of Q^* is free as an $\mathcal{O}$-module and*

$$\mathrm{End}_{D^b(\mathrm{Mod}\text{-}\Lambda_1)}(P^*) \cong \mathrm{End}_{D^b(\mathrm{Mod}\text{-}\Lambda)}(Q^*) \otimes_{\mathcal{O}} k.$$

Proof. By Proposition 3.1 there is a unique (up to isomorphism) complex Q^* of finitely generated projective Λ-modules such that $P^* \cong Q^* \otimes_{\mathcal{O}} k$. Consider

$$\mathrm{Hom}^*_{\Lambda_1}(P^*, P^*) \cong \mathrm{Hom}^*_{\Lambda}(Q^*, Q^*) \otimes_{\mathcal{O}} k.$$

The left-hand side has homology concentrated in degree zero, since P^* is a tilting complex, so, as in the proof of Corollary 3.2, we deduce that $\mathrm{Hom}^*_{\Lambda}(Q^*, Q^*)$ has homology that is concentrated in degree zero and is $\mathcal{O}$-free. Hence $\mathrm{Hom}_{D^b(\mathrm{Mod}\text{-}\Lambda)}(Q^*, Q^*[i])$ vanishes for $i \neq 0$, $\mathrm{End}_{D^b(\mathrm{Mod}\text{-}\Lambda)}(Q^*)$ is $\mathcal{O}$-free, and $\mathrm{End}_{D^b(\mathrm{Mod}\text{-}\Lambda_1)}(P^*)$ is isomorphic to $\mathrm{End}_{D^b(\mathrm{Mod}\text{-}\Lambda)}(Q^*) \otimes_{\mathcal{O}} k$.

Let S^* be an object of $K^-(P_\Lambda)$ such that $\mathrm{Hom}_{D^-(\mathrm{Mod}\text{-}\Lambda)}(Q^*, S[i])$ vanishes for all i. Then $\mathrm{Hom}_{D^-(\mathrm{Mod}\text{-}\Lambda_1)}(P^*, S^* \otimes_{\mathcal{O}} k[i])$ also vanishes. Since P^* is a tilting complex, this means that $S^* \otimes_{\mathcal{O}} k$ is acyclic and so S^* is acyclic. Hence, by Lemma 1.1, Q^* is a tilting complex. ∎

4. Deformations of Tilting Complexes

In this section (χ, x_0) will be a (pointed) affine scheme of finite type over a field k, where $\chi = \mathrm{Spec}(R)$ and x_0 corresponds to the prime ideal m_0 of R. The local ring of χ at x_0 will be denoted by $\mathcal{O}$, its maximal ideal by $\mathcal{M}$, and its completion by $\hat{\mathcal{O}}$. We shall let Λ be an R-algebra, free and of finite rank as an R-module, and Λ_0 will be the finite-dimensional $(\mathcal{O}/\mathcal{M})$-algebra $\Lambda \otimes_R (\mathcal{O}/\mathcal{M})$, so Λ is a flat deformation of Λ_0 over χ. By Theorem 3.3 we know that we can lift a tilting complex for Λ_0 to $\Lambda \otimes_R \hat{\mathcal{O}}$, and we shall use the Artin approximation theorem to show that we can in fact lift it to a tilting complex in some étale neighbourhood of x_0.

Theorem 4.1. *Let P^* be a tilting complex for Λ_0. There is an étale neighbourhood (χ', x_0') of x_0, with $\chi' = \mathrm{Spec}(R')$, and a tilting complex P'^* for $\Lambda \otimes_R R'$ such that $P^* \cong P'^* \otimes_{R'} (\mathcal{O}'/\mathcal{M}')$, where $\mathcal{O}'$ and $\mathcal{M}'$ are the local ring of χ' at x_0' and its maximal ideal.*

Proof. By Theorem 3.3, there is a tilting complex $\hat{P}^*$ for $\hat{\Lambda} = \Lambda \otimes_R \hat{\mathcal{O}}$ such that $P^* \cong \hat{P}^* \otimes_{\hat{\mathcal{O}}} \hat{\mathcal{O}}/\hat{\mathcal{M}}$. Let

$$\hat{P}^* \cong \hat{P}^*(1) \oplus \cdots \oplus \hat{P}^*(m),$$

where the $\hat{P}^*(\mu)$ are the indecomposable direct summands of $\hat{P}^*$. It will be convenient to regard the $\hat{P}^*(\mu)$ as complexes of free $\hat{\Lambda}$-modules. Of course, they are not in general isomorphic to bounded complexes of finitely generated free modules, but up to homotopy they are isomorphic to complexes, bounded above, that are eventually periodic to the left. In other words, we may choose a complex

$$\hat{L}^*(\mu) = \cdots \to \hat{\Lambda}^{p_{-1}(\mu)} \xrightarrow{d_{-1}(\mu)} \hat{\Lambda}^{p_0(\mu)} \xrightarrow{d_0(\mu)} \cdots \xrightarrow{d_{n-1}(\mu)} \hat{\Lambda}^{p_n(\mu)} \to 0 \to \cdots$$

for each μ, where $p_i(\mu) = p_1(\mu)$ and $d_{i-2}(\mu) = d_i(\mu)$ for $i \leqslant 0$, that is isomorphic up to homotopy to $\hat{P}^*(\mu)$. These complexes are determined by the sequences of integers $(p_1(\mu), ..., p_n(\mu))$ and of matrices $(d_{-1}(\mu), d_0(\mu), ..., d_{n-1}(\mu))$, whose entries lie in $\hat{\Lambda}$. The fact that $\hat{L}^*(\mu)$ is a complex means by definition that

$$\begin{aligned} d_{i-1}(\mu)\, d_i(\mu) &= 0, \qquad \text{for} \quad 0 \leqslant i < n \\ d_0(\mu)\, d_{-1}(\mu) &= 0, \end{aligned} \tag{1}$$

and the fact that it is homotopy equivalent to a bounded complex of projective modules is equivalent to the existence of $p_1(\mu)\times p_1(\mu)$ matrices $h_0(\mu)$ and $h_{-1}(\mu)$ such that

$$\begin{aligned} d_0(\mu)\,h_0(\mu)+h_{-1}(\mu)\,d_{-1}(\mu)&=1\\ h_0(\mu)\,d_0(\mu)+d_{-1}(\mu)\,h_{-1}(\mu)&=1. \end{aligned} \tag{2}$$

The fact that $\mathrm{add}(\hat{P}^*)$ generates $K^b(P_{\hat{\Lambda}})$ as a triangulated category is equivalent to the existence of a complex F^* of free $\hat{\Lambda}$-modules, isomorphic to $\hat{\Lambda}$ up to homotopy, with a filtration

$$0=F_0^*\subset F_1^*\subset\cdots\subset F_n^*=F^*,$$

where F_{j+1}^*/F_j^* is isomorphic to $\hat{L}^*(\mu_j)[t_j]$ for some μ_j and t_j. The differential δ_* of F^* can be chosen to be eventually periodic so is given by finitely many matrices a_{ij}^k, corresponding to maps

$$F_{i+1}^k/F_i^k\to F_{j+1}^{k+1}/F_j^{k+1}\qquad\text{for}\quad i>j.$$

These matrices are related to the $d_i(\mu)$ by quadratic equations stating that

$$\delta_i\delta_{i+1}=0. \tag{3}$$

The fact that F^* is isomorphic to $\hat{\Lambda}$ means that there are maps $b\colon F^0\to\hat{\Lambda}$ and $c\colon\hat{\Lambda}\to F^0$ and maps $H_i\colon F^i\to F^{i-1}$, which are given by various matrices over $\hat{\Lambda}$ (and, as before, we can choose H_i to be eventually periodic, so there are only finitely many matrices involved) satisfying equations

$$\begin{aligned} cb&=1\\ bc&=1+H_0\delta_{-1}+\delta_0H_1\\ 0&=1+H_i\delta_{i-1}+\delta_iH_{i+1}\qquad\text{for}\quad i\neq 0. \end{aligned} \tag{4}$$

If Λ is of rank r as a free R-module, then it determines an r-dimensional vector bundle V over $\mathrm{Spec}(R)$. We can take the product of s copies of V, one for each entry of each of the matrices

$$d_{-1}(\mu),\ldots,d_{n-1}(\mu),h_{-1}(\mu),h_0(\mu),a_{ij}^k,b,c, \tag{$*$}$$

to obtain an rs-dimensional vector bundle

$$V^s\to\mathrm{Spec}(R).$$

Equations (1)–(4) are polynomial equations with integer coefficients in the entries of the matrices $(*)$ and the structure constants of Λ, so they determine a closed subscheme U of V^s.

By construction, $U \to \mathrm{Spec}(R)$ has a formal section

$$\mathrm{Spec}(\hat{\mathcal{O}}) \to U \times_{\chi} \mathrm{Spec}(\hat{\mathcal{O}}),$$

and so the Artin approximation theorem [1] gives us an étale neighbourhood $\chi' = \mathrm{Spec}(R')$ of x_0 and a section

$$\chi' \to U \times_{\chi} \chi'.$$

Equations (1)–(4) tell us that this section describes an object

$$P'^* = P'^*(1) \oplus \cdots \oplus P'^*(m)$$

of $K^b(P_{\Lambda \otimes_R R'})$ such that $\mathrm{add}(P'^*)$ generates a triangulated category containing $\Lambda \otimes_R R'$.

The complex $\mathrm{Hom}^*_{\Lambda \otimes R'}(P'^*, P'^*)$ is a complex of free R'-modules whose reduction modulo $\mathcal{M}'$ has homology concentrated in degree zero. Hence, with $\mathrm{Spec}(R')$ replaced by some open neighbourhood of x'_0 if necessary, P'^* is a tilting complex, as required. ▮

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