Characters of Finite Groups Having a Self-Normalising Cyclic Subgroup

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1. INTRODUCTION

In the proof of the solubility of groups of odd order, an important role is played by a certain cyclic subgroup which links two nonconjugate maximal subgroups in a minimal simple group of odd order (that is, in a minimal counterexample). Such a subgroup was termed "self-normalising" in a sense which will be made precise below, and a key preliminary lemma (Lemma 13.1 of [1]) showed that its irreducible characters formed a coherent set. The argument given by Feit and Thompson, as also that in an unpublished simplification by Sibley, depends on a careful study of the field of character values and its Galois group: here we shall show that their results may be established using only Suzuki’s original methods of exceptional characters.

Actually, we shall do more. Feit and Thompson needed to assume only that the cyclic subgroup, though not necessarily the whole group, had odd order. We shall start our proof without making even this assumption. The basic argument will handle all but certain small configurations where the subgroup necessarily has even order. In considering these “unnatural” situations, we do consider character values, and we shall eliminate all possibilities except for certain configurations which can occur as solutions of Suzuki’s algorithm only when the cyclic subgroup has four times odd order. We note, in passing, that no assumption about simplicity is ever made: the result is equally valid for soluble groups.

Let $G$ be a finite group. A cyclic subgroup $W$ is said to be a self-normalising cyclic subgroup (with respect to a pair of nonidentity subgroups $W_1$ and $W_2$ of $W$) if $W = W_1 \times W_2$ and $W = N_G(A)$ for each nonempty subset $A$ of $W - (W_1 \cup W_2)$. In particular, $W - (W_1 \cup W_2)$ is a T.I.-set in

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G and will form a closed set of special classes in the sense of Suzuki [2]. This definition depends on $W_1$ and $W_2$: the subgroup $W$ need not necessarily be self-normalising with respect to a different factorisation.

We now fix some notation. Put $V = W - (W_1 \cup W_2)$ and let $\mathcal{M}(V)$ denote the $\mathbb{Z}$-module of generalised characters of $W$ which vanish outside $V$. Then $\mathcal{M}(V)$ is a direct summand of the module of generalised characters of $W$, which we denote by $\mathcal{C}(W)$, and character induction affords an isometry from $\mathcal{M}(V)$ into $\mathcal{C}(G)$. This is the basic situation of exceptional character theory, and we may now state our result.

**Theorem.** Let $G$ be a finite group and let $W$ be a self-normalising cyclic subgroup with respect to subgroups $W_1$ and $W_2$. Assume that, if one of $W_1$ and $W_2$ has order 4, then the other has prime order. Let $V = W - (W_1 \cup W_2)$. Then the isometry from $\mathcal{M}(V)$ into $\mathcal{C}(G)$ afforded by character induction can be extended to a $\mathbb{Z}$-linear isometry $\tau: \mathcal{C}(W) \to \mathcal{C}(G)$.

For $\alpha \in \mathcal{C}(W)$,

$$\tau(\alpha)(v) = \alpha(v)$$

for all $v \in V$, and any irreducible character of $G$ not in the image of $\tau$ vanishes on $V$.

It would seem unlikely that the restriction on the order of $W$ is necessary for the conclusion, but we are unable to eliminate the configurations that may occur in the exceptional cases and which are described in Proposition 9(II). For this reason, we shall not exclude possible orders for $W$ in any of our discussion.

2. **Suzuki's Method**

In the theory of exceptional characters, one usually cannot distinguish between characters and their negatives: so, to avoid carrying unknown signs, we shall use the term "character" for either an irreducible character or its negative, and the term "distinct characters" for a set of such characters which are pairwise orthogonal. Furthermore, the use of distinct symbols will always imply that the characters are distinct in this sense.

Assume that $G$ is a group having a self-normalising cyclic subgroup $W = W_1 \times W_2$. Let $\varphi_1, \ldots, \varphi_s$ be the irreducible characters of $W$ and let $\alpha_1, \ldots, \alpha_t$, ...
be a $\mathbb{Z}$-basis for $\mathcal{M}(V)$. Then, for uniquely determined integer coefficients $\{a_{ij}\}$,

$$\alpha_i = \sum_{k=1}^{s} a_{ik} \phi_k.$$  \hfill (2.1)

Writing $A$ for the matrix $(a_{ij})$ and $A^T$ for its transpose, the matrix whose $(i, j)$-entry is the inner product $(\alpha_i, \alpha_j)$ is just $AA^T$.

Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of $G$. Then there exist integers $b_{ij}$ such that

$$\alpha_i^G = \sum_{j=1}^{r} b_{ij} \chi_j.$$  \hfill (2.2)

Writing $B = (b_{ij})$, character induction, being an isometry from $\mathcal{M}(V)$ to $\mathfrak{S} \mathcal{H}(G)$, yields the matrix equation

$$AA^T = BB^T.$$  \hfill (2.3)

Suppose that the irreducible characters $\chi_1, \ldots, \chi_r$ are ordered so that

$$B = (B_0, 0),$$

where the columns of $B_0$ are nonzero. Then Suzuki's method provides an algorithm in that the size and entries of $B_0$ are bounded by (2.3). Furthermore,

(i) for each matrix solution $B$ to (2.3), there is an integral matrix $X = (x_{ij})$ such that $B = AX$,

(ii) the character values $\chi_i(v)$ for $v \in V$ are uniquely determined by $B$, and

(iii) for any matrix $X$ satisfying $B = AX$, these values are given by

$$\chi_i(v) = \sum_{k=1}^{s} x_{ki} \phi_k(v).$$

These results come from [2].

Now these statements relate, really, only to the submatrix $B_0$, all characters not indexed by the columns of $B_0$ then vanishing on $V$. If we allow $\chi_1, \ldots, \chi_r$ to be "characters" in our more general sense, the same results hold, with possible changes of sign in the columns of $B_0$. So we can regard a "natural situation" as occurring if Eqs. (2.1) and (2.2) are essentially the same: namely if, after permuting columns and changing signs, $B_0 = A$. In this case, we can take $X = I$, though there is no assertion about uniqueness: any solution to $B_0 = AX$ will suffice. On the other hand, if we can establish
that \( B_0 = A \), then we will have established the conclusion of the theorem, which is what is meant by coherence.

This, then, is our goal, and we shall regard it as the generic case: we shall need to apply the precise computation of (iii) to eliminate some "unnatural" configurations that occur as solutions to (2.3) in certain small cases.

3. THE GENERIC CASE

In this section we will establish coherence except for when one of \( W_1 \) and \( W_2 \) has order 2 or 4, and we shall obtain some precise information about the unnatural configurations that may occur in these cases. In particular, coherence will be established if \( W \) has odd order.

We fix the following notation which will better suit our approach than the notation of [1]. Let \( |W_1| = m + 1 \) and \( |W_2| = n + 1 \). Let \( \varphi_1, \ldots, \varphi_m \) be the nonprincipal characters of \( W \) which have \( W_2 \) in their kernels, and let \( \psi_1, \ldots, \psi_n \) be those with \( W_1 \) in their kernels. For each pair \( i, j \), let \( \theta_{ij} = \varphi_i \psi_j \); then all the irreducible characters of \( W \) have been given. Put

\[
\alpha_{ij} = 1_W - \varphi_i - \psi_j + \theta_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq n.
\]

**Lemma 1.** The generalised characters \( \alpha_{ij} \) form a \( \mathbb{Z} \)-basis for \( \mathcal{M}(V) \).

**Proof.** \( \alpha_{ij} = (1_W - \varphi_i)(1_W - \psi_j) \); hence \( \alpha_{ij} \in \mathcal{M}(V) \). Now \( \mathcal{M}(V) \) has \( \mathbb{Z} \)-rank \( mn \). By suitably reordering characters, the relation matrix given by (2.1) for the \( \alpha_{ij} \)'s in terms of the irreducible characters of \( W \) has the form

\[
A = (A_0 I);
\]

in particular, the elementary divisors are all 1 so that the \( \alpha_{ij} \)'s generate a direct summand of \( \mathcal{H}(W) \) of rank \( mn \). Thus they form a basis for \( \mathcal{M}(V) \).

Let \( \sigma \) denote the isometry from \( \mathcal{M}(V) \) to \( \mathcal{H}(G) \) afforded by character induction. We shall write \( \tilde{\alpha}_{ij} \) for the image of \( \alpha_{ij} \) under \( \sigma \) throughout. The following is well known, using the Frobenius reciprocity theorem to establish part (i).

**Lemma 2.** (i) \( \tilde{\alpha}_{ij} \) involves \( 1_G \) with multiplicity 1.

(ii) \( (\tilde{\alpha}_{ij}, \tilde{\alpha}_{ij}) = 1 + \delta_{ik} + \delta_{jl} + \delta_{ik} \delta_{jl} \); in particular, \( \|\tilde{\alpha}_{ij}\|^2 = 4 \).

(iii) \( \tilde{\alpha}_{ij} \) is multiplicity-free.

(iv) If \( g \in G \) and \( g \) is not conjugate to any element of \( V \), then \( \tilde{\alpha}_{ij}(g) = 0 \).
We now attempt to determine decompositions of the type (2.2) for the generalised characters $\tilde{\chi}_{ij}$. To simplify notation, we shall use distinct natural numbers just to distinguish distinct suffices $i$ and similarly (and independently) for $j$, rather than to denote fixed values. Also, any statement for suffices $i$ will have its counterpart for the suffices $j$, unless explicit assumptions have been made about $m$ and $n$ which remove symmetry.

**Lemma 3.** If $m \geq 2$, then $\tilde{\chi}_{11}$ and $\tilde{\chi}_{21}$ have exactly one nonprincipal constituent in common, with the same sign.

**Proof.** If not, then Lemma 2(ii) forces the existence of characters $\chi_1$, $\chi_2$, and $\chi_3$ such that

$$\tilde{\chi}_{11} = 1_G + \chi_1 + \chi_2 + \chi_3$$

and

$$\tilde{\chi}_{21} = 1_G + \chi_1 + \chi_2 - \chi_3.$$ 

But then $\chi_3(1) = 0$, which is impossible.

**Corollary 4.** The conclusion of the Theorem holds if $|W| = 6$.

**Proof.** This is the case $m = 2, n = 1$. We may choose signs to write

$$\tilde{\chi}_{11} = 1_G - \Phi_1 - \Psi_1 + \Theta_{11}$$

$$\tilde{\chi}_{21} = 1_G - \Phi_2 - \Psi_1 + \Phi_{21}$$

to see that a natural induction occurs.

For the remainder of this section, we shall assume that neither $m$ nor $n$ is 1, i.e., that neither $W_1$ nor $W_2$ has order 2. In the next lemma, we examine a certain subconfiguration which is contrary to what happens in the natural situation (the first conclusion), and show that the exception can arise only if $|W| = 12$; we shall identify the configurations precisely, and eliminate them in the next section, in Section 4.

We remark, before proceeding, that it is useful to think of the suffices $i$ and $j$ as labelling the generalised characters $\chi_{ij}$ presented in an $m \times n$ array.

**Lemma 5.** Either (i) or (ii) holds.

(i) $\tilde{\chi}_{11}$, $\tilde{\chi}_{21}$, and $\tilde{\chi}_{12}$ do not share a common nonprincipal constituent.

(ii) $|W| = 12$ and notation may be chosen so that one of the following four character decompositions occurs:
Proof. We attempt to establish (i). By symmetry, we may suppose that $m > n$. Then either $(m, n) = (3, 2)$, in which case we shall find the configurations in (ii) are possible obstructions, or else $m \geq 4$.

Assume then that (i) is false and that $\chi_1$ is a common constituent for $\tilde{\alpha}_{11}$, $\tilde{\alpha}_{21}$, and $\tilde{\alpha}_{12}$, necessarily with the same sign by Lemma 3. Then we may choose notation so that we have decompositions

\[ \tilde{\alpha}_{11} = 1_G + \chi_1 + \chi_2 + \chi_3 \quad \tilde{\alpha}_{12} = 1_G + \chi_1 - \chi_2 + \chi_3'' \]
\[ \tilde{\alpha}_{21} = 1_G + \chi_1 + \chi_2' + \chi_3' \quad \tilde{\alpha}_{22} = 1_G + \chi_1 - \chi_2 + \chi_3'' \]
\[ \tilde{\alpha}_{31} = 1_G + \chi_1 - \chi_3'' - \chi_3''' \quad \tilde{\alpha}_{32} = 1_G + \chi_1 - \chi_3 - \chi_3'. \]

(Note. It is useful to think of this as a "hook" in the array of $\tilde{\alpha}_{ij}$'s.)

Case 1. $\tilde{\alpha}_{22}$ involves $\chi_1$ ("square" with a common constituent).

We may choose $\chi_2$ and $\chi_3$ in (3.1) so that

\[ \tilde{\alpha}_{22} = 1_G + \chi_1 - \chi_2 + \chi_3' \] (3.2)

Notice that there is complete symmetry between $i = 1, 2$ and $j = 1, 2$.

Suppose, first, that some generalised character $\tilde{\alpha}_{31}$ involves $\chi_1$. Then Lemma 2(ii) forces the decomposition

\[ \tilde{\alpha}_{31} = 1_G + \chi_1 - \chi_3'' - \chi_3''' \]

and, if $m \geq 4$, it is then impossible to find a decomposition for $\tilde{\alpha}_{41}$. Thus $m = 3$, and a decomposition

\[ \tilde{\alpha}_{32} = 1_G + \chi_1 - \chi_3 - \chi_3'. \]
is forced. This is (a): we notice in passing that it would follow that 
\(\chi_1(1) = -1\), so that this situation could not occur in any case in a simple 
group.

We may therefore assume, given (3.1) and (3.2), that no \(\tilde{\alpha}_{ij}\), for \(i \geq 3\) and 
\(j \leq 2\), can involve \(\chi_1\). Suppose, next, that \(\tilde{\alpha}_{31}\) involves \(\chi_2\) (which is 
distinguished from \(\chi_3\) by \((\tilde{\alpha}_{22}, \chi_2) \neq 0\)). Then we are forced into decom-
positions

\[
\tilde{\alpha}_{31} = l_G + \chi_2 + \chi'_3 + \chi''_3 \\
\tilde{\alpha}_{32} = l_G + \chi''_3 + \chi''_3 + \chi_4.
\]

If \(m = 3\), this is (b). If \(m > 4\) then, since \(\tilde{\alpha}_{41} \neq \tilde{\alpha}_{31}\), a decomposition of the form

\[
\tilde{\alpha}_{41} = l_G + \chi_3 + \chi'_3 + \chi_5
\]
is forced, and it is impossible to find a decomposition for \(\tilde{\alpha}_{42}\): this follows 
from exhaustive analysis of Lemma 2(ii). Now, by symmetry, we may 
suppose that no \(\tilde{\alpha}_{ij}\), for \(i > 3\) and \(j \leq 2\), involves either \(\chi_2\) or \(\chi'_2\). Then

\[
\tilde{\alpha}_{31} = l_G + \chi_3 + \chi'_3 + \chi_4 \\
\tilde{\alpha}_{32} = l_G + \chi''_3 + \chi''_3 + \chi_4.
\]

This forces \(m = 3\), and so gives (c).

\textit{Case 2.} \(\tilde{\alpha}_{22}\) does not involve \(\chi_1\).

Lemmas 2(ii) and 3 force a decomposition

\[
\tilde{\alpha}_{22} = l_G + \chi_3 + \chi''_3 + \chi''_3.
\]

In view of the completion of Case 1, we may assume that no “squares” can 
share a common nonprincipal constituent. If \(\tilde{\alpha}_{31}\) involves \(\chi_1\), then we can 
force

\[
\tilde{\alpha}_{31} = l_G + \chi_1 - \chi''_3 + \chi''_3,
\]

whereupon no decomposition can be found for \(\tilde{\alpha}_{32}\). So we may suppose 
that no \(\tilde{\alpha}_{ii}\) involves \(\chi_1\) for \(i \geq 3\), and we can force

\[
\tilde{\alpha}_{31} = l_G + \chi_2 + \chi'_3 - \chi''_3
\]
(distinguishing \(\chi_2\) from \(\chi_3\) only by \((\tilde{\alpha}_{31}, \chi_2) \neq 0\)). Now the argument above, 
applied to \(\tilde{\alpha}_{21}, \tilde{\alpha}_{31}, \tilde{\alpha}_{41}, \) and \(\tilde{\alpha}_{32}\) as a “long hook” if \(m \geq 4\), shows that 
\(m \leq 3\), and then the only possibility for \(\tilde{\alpha}_{32}\) is that

\[
\tilde{\alpha}_{32} = l_G + \chi_2 - \chi_3 + \chi''_3.
\]

This is (d).
For the remainder of the section, we assume that conclusion (i) holds.

**Lemma 6.** \(\tilde{\alpha}_{11}\) and \(\tilde{\alpha}_{22}\) have no common nonprincipal constituent.

**Proof.** Since we are assuming the absence of hooks with a common nonprincipal constituent, \(\tilde{\alpha}_{22}\) cannot have in common with \(\tilde{\alpha}_{11}\) either of its common nonprincipal constituents with \(\tilde{\alpha}_{12}\) or \(\tilde{\alpha}_{21}\). But then the existence of any common constituent will imply that \(\langle \tilde{\alpha}_{11}, \tilde{\alpha}_{22} \rangle = 0\) or 2, or that \(\|\tilde{\alpha}_{22}\|^2 > 4\), contrary to Lemma 2(ii).

**Lemma 7.** Assume that \(m \geq 4\). Then there exist characters \(\{\Phi_i, \Theta_{i1}, \Theta_{i2}: i = 1, \ldots, m\}\), \(\Psi_1\) and \(\Psi_2\) of \(G\) such that, for \(1 < i < m\),

\[\tilde{\alpha}_{ii} = 1_G + \Phi_i + \Psi_i + \Theta_{i1} \quad \text{and} \quad \tilde{\alpha}_{i2} = 1_G + \Phi_i + \Psi_2 + \Theta_{i2}.\]

**Proof.** By Lemma 6, we may start with decompositions

\[
\begin{align*}
\tilde{\alpha}_{11} &= 1_G + \Phi_1 + \Psi_1 + \Theta_{11} \quad \tilde{\alpha}_{12} = 1_G + \Phi_1 + \Psi_2 + \Theta_{12}, \\
\tilde{\alpha}_{21} &= 1_G + \Phi_2 + \Psi_1 + \Theta_{21} \quad \tilde{\alpha}_{22} = 1_G + \Phi_2 + \Psi_2 + \Theta_{22}. 
\end{align*}
\]

(3.3)

If \(\langle \tilde{\alpha}_{31}, \Psi_1 \rangle = 0\), then there exists a character \(\Phi_3\) such that

\[\tilde{\alpha}_{31} = 1_G + \Phi_3 + \Theta_{11} + \Theta_{21}.\]

(3.4)

Since there are no hooks, \(\langle \tilde{\alpha}_{32}, \Phi_3 \rangle = 1\), but no decomposition can then be found for \(\tilde{\alpha}_{41}\).

Similarly, we may suppose that \(\langle \tilde{\alpha}_{32}, \Psi_2 \rangle \neq 0\), and we may label characters so that

\[\tilde{\alpha}_{31} = 1_G + \Phi_3 + \Psi_1 + \Theta_{31} \quad \tilde{\alpha}_{32} = 1_G + \Phi_3 + \Psi_2 + \Theta_{32}.\]

**Note.** We may always suppose that \(m > n\) after Lemma 7. Then the only obstruction raised by (3.4) occurs when \((m, n) = (3, 2)\).

**Lemma 8.** Suppose that \((m, n) = (3, 2)\) and that (3.3) and (3.4) hold. Then one of the following holds:

(c) \(\tilde{\alpha}_{32} = 1_G + \Phi_3 + \Psi_2 + \Theta_{32}\); or

(f) \(\tilde{\alpha}_{32} = 1_G + \Phi_3 + \Theta_{12} + \Theta_{22}\).

**Proof.** This follows from Lemma 2(ii).

**Proposition 9.** Suppose that \(m, n \geq 2\) and that \(m > n\). Then one of the following holds:
(I) \(|W| = 12, and one of the unnatural situations given by Lemmas 5 and 8 holds;

(II) \(n = 3\) and, for some nonempty subset \(I\) of \(\{1, \ldots, m\}\), there exist decompositions of the form

\[ \tilde{x}_{i1} = 1_G + \Phi_i + \Psi_1 + \Theta_{i1} \quad \text{and} \quad \tilde{x}_{i2} = 1_G + \Phi_i + \Psi_2 + \Theta_{i2} \]

and

\[ \tilde{x}_{i3} = \begin{cases} 1_G + \Theta_{i1} + \Theta_{i2} + \Psi_3 & \text{if } i \in I \\ 1_G + \Phi_i + \Psi_3 + \Theta_{i3} & \text{if } i \notin I; \text{ or} \end{cases} \]

(III) a natural induction takes place.

Remark. With a suitable renaming of the characters, the two possibilities of Lemma 8 correspond to conclusion (II) with \(m = 2\) and \(I\) of order 1 or 2, respectively. For this reason, we shall treat those cases in the discussion of the case \(|W| = 4p\), where \(p\) is prime, in Section 6; we shall also see there that we can deduce that, in general, \(I\) must be a proper subset of \(\{1, \ldots, m\}\) in (II). We remark also that, if \(I\) were empty, then we would be in case (III).

Proof. By Lemmas 5, 7, and 8, if (I) does not hold then we certainly have decompositions of the form

\[ \tilde{x}_{i1} = 1_G + \Phi_i + \Psi_1 + \Theta_{i1} \quad \text{and} \quad \tilde{x}_{i2} = 1_G + \Phi_i + \Psi_2 + \Theta_{i2} \]

for \(i = 1, \ldots, m\). If (III) does not hold, then we may suppose that

\[ \tilde{x}_{i3} = 1_G + \Theta_{i1} + \Theta_{i2} + \Psi_3 \]

for some suitable ordering of the characters, following the argument of Lemma 7 (with \(m\) and \(n\) interchanged). But now Lemma 5 implies that \((\tilde{x}_{i3}, \Psi_3) = 1\). By Lemma 7, \(n \leq 3\). Then only the possibilities listed in (II) can occur as decompositions.

The conclusion of the Theorem now holds for the generic case (III), by writing

\[ \tilde{x}_{ij} = 1_G + \Phi_i + \Psi_j + \Theta_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq n, \]

and putting

\[ \phi_i^* = -\Phi_i, \psi_j^* = -\Psi_j, \theta_{ij}^* = \Theta_{ij}. \]
4. **The Exceptional Cases (a)–(d) of Proposition 9(I)**

In this section, we shall eliminate the unnatural situations which arise out of Lemma 5: we shall describe them as Cases (a)–(d), as in that lemma. The methods described in this section will also be applied in the subsequent sections.

We shall apply the method described in Section 2 in order to determine all the nonzero character values for a fixed element $g$ of order 12. These values are, of course, determined only up to sign, so that they will not necessarily form a closed set when taking all conjugates in an algebraic number field: however, for any irreducible character $\chi$ of $G$ and any automorphism $\gamma$ of such a splitting field, either $\chi(g)^{\gamma}$ or $-\chi(g)^{\gamma}$ must appear in our set of values and, if $\gamma$ does not fix $\chi(g)$, this is for a different character. In particular, if we let $\omega$ be a primitive cube root of unity, we may choose $\gamma$ to map $i \to -i$ and to fix $\omega$, or to map $\omega \to \omega^{-1}$ and to fix $i$, as well as complex conjugation. From the character values that we shall determine, a contradiction will be clear in all but two cases.

From the way in which the generalised characters $\{x_{ij}\}$ were defined, it is evident that the characters of $W$ may be ordered so that the matrix $A$ has a block decomposition

$$A = (A_0 \ I).$$

Then a solution to the matrix equation $B_0 = AX$ is given by

$$X = \begin{pmatrix} 0 \\ B_0 \end{pmatrix}.$$

Thus, given the character values $\theta_{ij}(g)$, we may determine (up to sign) the values $\chi(g)$ for the irreducible characters of $G$. We simply state the results and note when a character (under a specified field automorphism) fails the algebraic conjugacy test; these are given in Table I. The matrix $B_0$ is, of course, given by the decompositions (a)–(d).

There is a minor complication in assigning values to particular characters. The decompositions (a)–(d) simply list the distinct "patterns": $x_{ij}$ is determined by a pair $\varphi_i$, $\psi_j$ and in theory all possible choices for these need analysis. Fortunately, however, two types of symmetry exist. First, as we cannot distinguish algebraically between $i$ and $-i$, only the real-valued character among $\varphi_1$, $\varphi_2$, and $\varphi_3$ need be identified, and similarly $\psi_1$ and $\psi_2$ are indistinguishable. Then symmetry may occur within the decompositions (a)–(d). For example, by construction there is no initial distinction between taking $i = 1$ and $i = 2$ in all but case (d), and this remains so after the decompositions for $\delta_{31}$ and $\delta_{32}$ have been determined: on the other hand, inspection will reveal no distinguished $i$ in either case (a) or (d)—this can
also be seen from the proof of Lemma 5 which implicitly characterises the various cases. So in Table I, we describe only genuine subcases.

In view of the algebraic conjugacy test, two cases remain. First we shall consider the case \(\{(a), \varphi_2\}\). We observe from the computed character values that the characters \(\chi_2, \chi_2', \chi_3,\) and \(\chi_3''\) are, up to signs, algebraically conjugate and thus have the same degree with signs given by

\[
\chi_2(1) = -\chi_2'(1) = -\chi_3'(1) = -\chi_3''(1).
\]

Similarly, \(\chi_3\) and \(\chi_3''\) are algebraically conjugate and hence

\[
\chi_3(1) = -\chi_3''(1).
\]

From the degree equations \(\bar{\sigma}_{11}(1) = \bar{\sigma}_{12}(1) = 0\), we deduce that \(\chi_3(1) = 0\), a contradiction.

This leaves the case \(\{(c), \varphi_3\}\), and this may be eliminated by a class algebra constant argument. Let \(g\) be a fixed element of order 12 in \(W\) as before, and let \(t\) be an involution in \(G\). Then no conjugate of \(t\) can invert \(g\): consequently (cf. [2]), we have the equation

\[
\sum \frac{\chi(t)^2 \chi(g)}{\chi(1)} = 0,
\]

where the summation is carried over all the irreducible characters \(\chi\) of \(G\). We notice that the character values of \(g\) actually appear as algebraic conjugates, rather than involving negatives, so that algebraically conjugate "characters" take the same values on each of 1 and \(t\).

Let

\[
\chi_2(1) = \chi_2'(1) = x, \quad \chi_3(1) = \chi_3'(1) = \chi_3''(1) = \chi_3'''(1) = y, \quad \chi_4(1) = z
\]

and

\[
\chi_2(t) = \chi_2'(t) = u, \quad \chi_3(t) = \chi_3'(t) = \chi_3''(t) = \chi_3'''(t) = v, \quad \chi_4(t) = w.
\]

Then Eq. (4.1) becomes

\[
1 + \frac{w^2}{z} + \frac{2v^2}{y} = 0.
\]

Since \(\bar{\sigma}_{31}(1) = \bar{\sigma}_{31}(t) = 0\), we obtain relations

\[
1 + 2y + z = 0 \quad (4.3)
\]

and

\[
1 + 2v + w = 0. \quad (4.4)
\]
<table>
<thead>
<tr>
<th>Case</th>
<th>Real $\varphi_i$</th>
<th>$\chi_1$</th>
<th>$\chi_2$</th>
<th>$\chi_2'$</th>
<th>$\chi_3$</th>
<th>$\chi_3'$</th>
<th>$\chi_3''$</th>
<th>$\chi_4$</th>
<th>Failing $\chi$</th>
<th>Failing $\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$\varphi_2$</td>
<td>1</td>
<td>$i\omega + \omega^{-1}$</td>
<td>$-\omega - i\omega^{-1}$</td>
<td>$-i$</td>
<td>$-\omega + i\omega^{-1}$</td>
<td>$-i$</td>
<td>$i\omega - \omega^{-1}$</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>$\varphi_2$</td>
<td>$1-i$</td>
<td>$\omega^{-1}$</td>
<td>$-\omega - i\omega^{-1}$</td>
<td>$i\omega$</td>
<td>$-(1+i)\omega$</td>
<td>0</td>
<td>$i-\omega^{-1}$</td>
<td>$-i\omega^{-1}$</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>$\varphi_3$</td>
<td>0</td>
<td>$-i-\omega$</td>
<td>$i$</td>
<td>$i\omega$</td>
<td>$-(1+i)\omega$</td>
<td>$(i-1)\omega^{-1}$</td>
<td>$1-i\omega^{-1}$</td>
<td>$-\omega^{-1}$</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>$\varphi_2$</td>
<td>$1-i$</td>
<td>$i\omega + \omega^{-1}$</td>
<td>$-\omega - i\omega^{-1}$</td>
<td>0</td>
<td>$-(1+i)\omega$</td>
<td>0</td>
<td>$-1+(1+i)\omega^{-1}$</td>
<td>$i$</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>$\varphi_3$</td>
<td>0</td>
<td>$-i$</td>
<td>$i$</td>
<td>$(-1+i)\omega$</td>
<td>$(1+i)\omega^{-1}$</td>
<td>$(-1+i)\omega^{-1}$</td>
<td>$1$</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>$\varphi_2$</td>
<td>$-i-\omega$</td>
<td>$-i\omega^{-1}$</td>
<td>$-\omega - i\omega^{-1}$</td>
<td>$-i$</td>
<td>$1-i\omega$</td>
<td>$-\omega^{-1}$</td>
<td>$-i\omega - \omega^{-1}$</td>
<td>$\chi_3$</td>
<td>$i \rightarrow -i$</td>
</tr>
</tbody>
</table>
Substituting (4.3) and (4.4) into Eq. (4.2), we obtain the equation
\[
\frac{(y - v)^2}{y(1 + 2y)} = 0,
\]
whence
\[v = y.\]

We deduce that \(t \in \ker \chi_3\) and, in particular, that \(g^6 \in \ker \chi_3\). On the other hand, \(\chi_3(g) \notin \mathbb{Q}(\sqrt[6]{1})\). This contradiction eliminates the case \(\{(c), \varphi_3\}\).

5. THE CASE \(n = 1\)

By Corollary 4, we can suppose that \(m \geq 3\) and then, since \(m\) and \(n\) cannot both be odd, that \(m \geq 4\) and \(m \neq 5\). The arguments are similar to those of the previous sections and we merely sketch the consequences. We drop the \(j\)-suffix.

Case 1. Four of the \(\tilde{\alpha}_i\)'s have a common nonprincipal constituent.

Let them be \(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3,\) and \(\tilde{\alpha}_4,\) with common constituent \(\Psi.\) If \((\tilde{\alpha}_5, \Psi) = 0,\) then \(\|\tilde{\alpha}_5\|^2 \geq 5,\) contrary to Lemma 2(ii). We now write
\[\tilde{\alpha}_i = 1_G + \Phi_i + \Psi + \Theta_i,\]
and see that a natural induction takes place.

Case 2. Some three, but no four, of the \(\tilde{\alpha}_i\)'s have a common nonprincipal constituent.

Write
\[
\begin{align*}
\tilde{\alpha}_1 &= 1_G + \Phi_1 + \Psi + \Theta_1 \\
\tilde{\alpha}_2 &= 1_G + \Phi_2 + \Psi + \Theta_2 \\
\tilde{\alpha}_3 &= 1_G + \Phi_3 + \Psi + \Theta_3 \\
\tilde{\alpha}_4 &= 1_G + \Phi_1 + \Phi_2 + \Phi_3.
\end{align*}
\]
If \(m \geq 5,\) then we may take
\[\tilde{\alpha}_5 = 1_G + \Phi_1 + \Theta_2 + \Theta_3\]
and
\[\tilde{\alpha}_6 = 1_G + \Theta_1 + \Phi_2 + \Phi_3,\]
but cannot continue to \(\tilde{\alpha}_7.\) So \(|W| = 10\) or \(14\) in this case. If \(|W| = 10,\) we cannot distinguish subcases and the character values on an element of
order 10 fail the algebraic conjugacy test. If \(|W| = 14\), there is no such symmetry, and we argue as follows.

There are seven nonprincipal characters that appear in the decompositions, and on a fixed element of order 14, three take values of the form \(\pm (\eta_1 + \eta_2)\) and four take values of the form \(\pm (\eta_1 + \eta_2 + \eta_3)\), where \(\eta_1, \eta_2,\) and \(\eta_3\) are three distinct seventh roots of unity. Since there is a field automorphism of the type \(\eta \rightarrow \eta^3\), the four of the latter type (which are necessarily nonreal) must fall into two orbits of length 2 under such an automorphism, so that the character values are of the form \(\pm (\eta + \eta^2 + \eta^4)\).

This has no root in common with its algebraic conjugate (namely, its complex conjugate), while a calculation using the methods of Section 2 shows that any two character values of the form \(\pm (\eta_1 + \eta_2 + \eta_3)\) do have a root in common. So this situation cannot arise.

Case 3. No three of the \(\tilde{\alpha}_i\)'s have a common constituent. This forces

\[
\begin{align*}
\tilde{\alpha}_1 &= 1_G + \Phi_1 + \Psi + \Theta_1 \\
\tilde{\alpha}_2 &= 1_G + \Phi_2 + \Psi + \Theta_2 \\
\tilde{\alpha}_3 &= 1_G + \Phi_1 + \Phi_2 + \Theta_3 \\
\tilde{\alpha}_4 &= 1_G + \Theta_1 + \Theta_2 + \Theta_3
\end{align*}
\]

and \(|W| = 10\). Again, there are no distinguishable subcases and, for a fixed element of \(W\) of order 10, of the six nonzero nonprincipal character values, two are of the form \(\pm (\eta + \eta^{-1})\) and four are of the form \(\pm (\eta + \eta^2)\), where \(\eta^2 = 1\). These must form two orbits under algebraic conjugacy, and we may proceed to obtain equations of the form (4.2), (4.3), and (4.4) exactly as in Section 4. Let \(t = g^5\). Then we may deduce, as there, that \(g^5 \in \ker \Psi\) (or, indeed, any other of these characters). Let \(\bar{G} = G/\ker \Psi\) and let \(\bar{g}\) be the image of \(g\) in \(\bar{G}\). It follows that \(\bar{g}\) is a nonreal element of order 5 in \(\bar{G}\) with \(|C_{\bar{G}}(\bar{g})| \leq 10\). But then \(\langle \bar{g} \rangle\) is a Sylow 5-subgroup of \(\bar{G}\) and, by Burnside's transfer theorem, \(\bar{G}\) has a normal 5-complement. However, this is inconsistent with the character values determined above.

6. Case (II)

We first determine the nature of the matrix \(B_0\) simply under the hypothesis that (II) of Proposition 9 holds. Then we shall show that \(I = \{1, \ldots, m\}\) if \(|W| = 4p\), for \(p\) prime. We recall that this is to be taken as including the possibilities from Lemma 8 where \(m = 2\) and \(p = 3\). Under this conclusion, we shall derive a contradiction, thus justifying the remark made after the statement of Proposition 9.
Suppose that the generalised characters of \( W \) are ordered \( \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \ldots \), and similarly the \( \tilde{\alpha}_{ij} \). Then the nonzero entries corresponding to the three rows for a given value of the index \( i \) are as in Tables II and III, according to whether \( i \in I \) or \( i \notin I \). In these tables, the value of the character \( \theta_j \) is given for a fixed element \( g \) of \( W \) of order \( 4k \) for \( k > 1 \), where \( \omega \) is some \( k \)th root of unity. By the symmetry of the decompositions, it is immaterial which of the characters \( \psi_j \) is taken to be real.

For \( i \in I \), we may compute that
\[
\Phi_i(g) = (i - 1)\omega \quad \Theta_{i1}(g) = 0 \quad \Theta_{i2}(g) = (-i - 1)\omega
\]
and, for \( i' \notin I \), that
\[
\Phi_i(g) = -\omega \quad \Theta_{i'1}(g) = i\omega \quad \Theta_{i'2}(g) = -\omega \quad \Theta_{i'3}(g) = -i\omega,
\]
while
\[
\psi_i(g) = -i \quad \psi_{i'}(g) = 1 \quad \psi_{i3}(g) = i.
\]

Suppose now that \( |W| = 4p \), where \( p \) is prime. In this case, by taking \( \omega \) a nonidentity \( p \)th root of unity and observing that every algebraic conjugate (up to sign) of a character \( \Phi_i \) for \( i \in I \) or of \( \Theta_{i'1} \) for \( i' \notin I \) must appear amongst the characters described, we can see that either \( I = \{1, \ldots, m\} \) or \( I = \emptyset \). So \( I = \{1, \ldots, m\} \). In addition, all the characters \( \Phi_i \) are algebraically conjugate and thus take the same values on all nonspecial classes, and similarly for the characters \( \Theta_{i2} \).

From the conclusion that \( I = \{1, \ldots, m\} \) alone, we shall obtain a contradiction. The effect of having the decompositions of Table II throughout

**TABLE II**

<table>
<thead>
<tr>
<th>( 1_G )</th>
<th>( \psi_1 )</th>
<th>( \psi_2 )</th>
<th>( \psi_3 )</th>
<th>( \Phi_i )</th>
<th>( \Theta_{i1} )</th>
<th>( \Theta_{i2} )</th>
<th>( \Theta_{i3} )</th>
<th>( \theta_{ij} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\alpha}_{i1} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( i\omega )</td>
</tr>
<tr>
<td>( \tilde{\alpha}_{i2} )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( -\omega )</td>
</tr>
<tr>
<td>( \tilde{\alpha}_{i3} )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( -i\omega )</td>
</tr>
</tbody>
</table>

**TABLE III**

<table>
<thead>
<tr>
<th>( 1_G )</th>
<th>( \psi_1 )</th>
<th>( \psi_2 )</th>
<th>( \psi_3 )</th>
<th>( \Phi_i )</th>
<th>( \Theta_{i1} )</th>
<th>( \Theta_{i2} )</th>
<th>( \Theta_{i3} )</th>
<th>( \theta_{ij} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\alpha}_{i1} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( i\omega )</td>
</tr>
<tr>
<td>( \tilde{\alpha}_{i2} )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( -\omega )</td>
</tr>
<tr>
<td>( \tilde{\alpha}_{i3} )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( -i\omega )</td>
</tr>
</tbody>
</table>
is, as in the case \{(c), \varphi_3\} of Section 4, to ensure additional relations between degrees and between the character values on involutions; these do not have analogues under the decompositions of Table III. These lead to equations identical to (4.2), (4.3), and (4.4), as may be easily verified. In particular, we deduce that \(\Phi_i(t) = \Phi_i(1)\) for any involution \(t\) of \(G\). Now suppose that the characters are numbered so that \(\Phi_i(G) = (i-1)\omega\) where \(\omega\) is a primitive \(r\)th root of unity with \(|W| = 4r\). Then \(\Phi_i(g)\) lies in \(Q(i\omega)\) but not in \(Q(\omega)\). Hence \(W\) is represented faithfully by the representation affording \(\Phi_1\), contrary to the assertion above (with \(t = g^{2r}\)) that \(\Phi_1(g^{2r}) = \Phi_1(1)\).

We conclude by remarking that, if \(|W| = 4r\) where \(r\) is not prime, then the characters of \(G\) that arise from this method can be split into blocks (in a nontechnical sense) under algebraic conjugacy, each block containing characters for which either Table II or Table III gives the character values. We cannot expect the above argument to apply if Table III ever occurs since this is precisely what happens if the conclusion of the main theorem (i.e., coherence) holds.

Note added in proof. Nowhere in Section 3 do we use the hypothesis that \(W\) is cyclic. Hence, a result analogous to the Theorem holds for an Abelian subgroup \(W = W_1 \times W_2\) where \(W_1\) and \(W_2\) have coprime orders, provided that \(W\) is cyclic if \(|W| = 4p\).

REFERENCES