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Boundary-contact problems for domains with conical singularities☆

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Abstract

We study boundary-contact problems for elliptic equations (and systems) with interfaces that have conical singularities. Such problems represent continuous operators between weighted Sobolev spaces and subspaces with asymptotics. Ellipticity is formulated in terms of extra transmission conditions along the interfaces with a control of the conormal symbolic structure near conical singularities. We show regularity and asymptotics of solutions in weighted spaces, and we construct parametrices. The result will be illustrated by a number of explicit examples.

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1. Introduction and formulation of the problems

1.1. Boundary-contact problems

Boundary-contact problems for partial differential equations appear in many areas of physics, elasticity theory and the applied sciences. The domains may consist of different components with some common parts of their boundaries (here called interfaces) with contact conditions for the solutions. It is a natural assumption in such situations that the interfaces S have conical, edge, or other singularities. The simplest case when S is smooth is well known, cf. Picone [16], Lions [14], Schechter [18], Hörmander [8], Kupradze et al. [11], Myshkis [15].

Problems with singularities at the interfaces have been studied by several authors, partly focused on special systems, under extra assumptions on the geometry or the underlying dimensions, cf. Escauriaza et al. [3], Torres and Welland [22], Li and Vogelius [13], Li and Nirenberg [12] (the latter paper studies the case when the interfaces subdivide the medium in a way that there are touching points).

Boundary-contact problems for elliptic systems in a domain $\Omega \subset \mathbb{R}^{n+1}$ refer to a subdivision of the form $\Omega = \Omega_+ \cup \Omega_- \cup S$ for open subdomains $\Omega_\pm$ of Ω such that $\overline{\Omega}_+ \cap \overline{\Omega}_- = S$ is an 'interface' of codimension 1. More precisely, we assume that $\partial\Omega_+ = S$, $S \cap \partial\Omega = \emptyset$, which has the consequence that $\partial\Omega_- = S \cup \partial\Omega$. Starting from a pair of elliptic systems of differential operators $A_\pm$ of order μ in $\Omega_\pm$ (with smooth coefficients up to the respective boundaries) our problems have the form

$$A_\pm u_\pm = f_\pm \quad \text{in} \quad \Omega_\pm, \tag{1}$$

$$Tu_- = h \quad \text{on} \quad \partial\Omega, \tag{2}$$

$$T_+u_+ + T_-u_- = g \quad \text{on} \quad S. \tag{3}$$

Here T is (Shapiro–Lopatinskij) elliptic with respect to the operator A_-, and $T_\pm$ are trace operators of the form $T_\pm = {}^t(T_{\pm,j})_{j=1,\dots,N}$,

$$T_{\pm,j}u_\pm := (B_{\pm,j}u_\pm)|_S \tag{4}$$

for differential operators $B_{\pm,j}$ of order m_j with smooth coefficients, defined in a tubular neighbourhood V of S in Ω. The restriction to S refers to the corresponding plus or minus side. The trace operator $T = {}^t(T_1, \dots, T_{N'})$ is given in an analogous form, i.e., $T_k u_- = B_k u_-|_{\partial\Omega}$ for smooth differential operators of order m'_k in a collar neighbourhood of $\partial\Omega$. The numbers N and N' are known from the context. For instance, if $A_\pm$ are $L \times L$-systems of operators of order $2m$, then we have $N = 2mL$ and $N' = mL$ (under some standard conditions on the principal symbols of the operators near S and $\partial\Omega$, respectively, see Agmon et al. [1]).

The behaviour of solutions far from S is known from the standard theory of elliptic boundary value problems when we assume $\partial\Omega$ to be smooth, and, for instance, Ω bounded.

The main focus of the present paper is the case when S is a manifold with finitely many conical singularities. Also other properties are of interest, e.g., edge singularities. The applications often refer to systems rather than operators $A_\pm$. However, the essential parts of our methods do not depend on that aspect. So, for simplicity, from now on we consider the case $L = 1$.

First, note that when $\Omega_\pm$ are bounded and S is smooth the problem (1)–(3) represents continuous operators

$$\mathcal{A} = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \\ T_+ & T_- \\ 0 & T \end{pmatrix} : \begin{matrix} H^s(\Omega_+) \\ \oplus \\ H^s(\Omega_-) \end{matrix} \to \begin{matrix} H^{s-\mu}(\Omega_+) \\ \oplus \\ H^{s-\mu}(\Omega_-) \\ \oplus \\ \bigoplus_{l=1}^{N} H^{s-m_l-\frac{1}{2}}(S) \\ \oplus \\ \bigoplus_{j=1}^{N'} H^{s-m'_j-\frac{1}{2}}(\partial\Omega) \end{matrix} \tag{5}$$

for arbitrary $s > \max\{m_l + \frac{1}{2}, m'_j + \frac{1}{2}\}$ (in the system case we would have everywhere $\mathbb{C}^L$-valued analogues of the Sobolev spaces).

If S has a conical singularity v (or finitely many, where the considerations are then analogous) it is adequate to replace the standard Sobolev spaces by weighted Sobolev spaces and subspaces with asymptotics in corresponding stretched domains. To have more convenient notation we set $X_\pm = \overline{\Omega}_\pm$; then $S = \partial X_+ (= \partial X_- \setminus \partial\Omega)$ and the stretched regions are defined by replacing conical neighbourhoods of $v \in S$ ($v \in X_\pm$) by cylinders $[0, 1) \times \Sigma$ ($[0, 1) \times \Xi_\pm$), where $r \in [0, 1)$ is the axial variable of the respective cones with Σ ($\Xi_\pm$) as the base manifolds. In the present situation Σ is C^∞, compact and closed, $\Xi_\pm$ are C^∞, compact, and with common boundary Σ. The global stretched 'domains' which include their boundary will be denoted by $\mathbb{X}_\pm$; the stretched 'surface' $\mathbb{S}$ obtained from S by blowing up the singularity near v (similarly as the blowing up the domains as mentioned before) then has the property

$$\partial\mathbb{X}_{+,\mathrm{reg}} = \mathbb{S}_{\mathrm{reg}}, \quad \partial\mathbb{X}_{-,\mathrm{reg}} = \mathbb{S}_{\mathrm{reg}} \cup \partial\Omega,$$

where subscript 'reg' denotes the stretched spaces minus the bottoms $r = 0$ of the corresponding local cylinders (more details will be explained below). There are now weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{X}_\pm)$ and $\mathcal{H}^{s,\gamma}(\mathbb{S})$ of smoothness s and weight γ (and subspaces with asymptotics for $r \to 0$, also to be introduced below). Then our boundary-contact problem in the conical situation represents continuous operators

$$\mathcal{A}:\begin{array}{c}\mathcal{H}^{s,\gamma}(\mathbb{X}_+)\\ \oplus\\ \mathcal{H}^{s,\gamma}(\mathbb{X}_-)\end{array}\to\begin{array}{c}\mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{X}_+)\\ \oplus\\ \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{X}_-)\\ \oplus\\ \bigoplus_{l=1}^{N}\mathcal{H}^{s-m_l-\frac{1}{2},\gamma-m_l-\frac{1}{2}}(\mathbb{S})\\ \oplus\\ \bigoplus_{j=1}^{N'}H^{s-m'_j-\frac{1}{2}}(\partial\Omega)\end{array}\tag{6}$$

for arbitrary $s>\max\{m_l+\frac{1}{2},m'_j+\frac{1}{2}\}$ and $\gamma\in\mathbb{R}$. The operators $A_\pm$ near $r=0$ are assumed to be of Fuchs type. Moreover, the trace operators $T_\pm$ (in (4)) have the form of a composition of a Fuchs-type differential operator with the restriction to int $\mathbb{S}$, cf. the formulas (12), (13) below.

The programme of this paper is to solve problems of the type (1)–(3) in terms of parametrix construction under a natural condition of ellipticity (referring to the weights) and then to obtain regularity and asymptotics of solutions. Our technique from a suitably adapted cone algebra approach is completely general and does not specifically refer to special elliptic operators. The necessary material will be given in Section 3. This will be applied to some categories of examples where we explicitly determine admissible weights and express asymptotics, cf. Section 4.3.

1.2. The symbolic structure

For the case that S is smooth the ellipticity of $\mathcal{A}$ refers to a principal symbolic hierarchy

$$\sigma(\mathcal{A})=(\sigma_\psi(A_+),\sigma_\psi(A_-),\sigma_{\mathrm{tr}}(\mathcal{A}),\sigma_\partial(\mathcal{A})),$$

where $\mathcal{A}$ is regarded as an operator (5). Here, $\sigma_\psi(A_\pm)\in C^\infty(T^*X_\pm\setminus 0)$ are the standard homogeneous principal symbols of the operators $A_\pm$ over int $X_\pm$ (smooth up to respective boundaries). $\sigma_\partial(A_-)(y,\eta)$ is the boundary symbol of A_- with respect to $\partial\Omega$, as is common in the standard calculus of boundary value problem, $(y,\eta)\in T^*(\partial\Omega)\setminus 0$. Recall that when (y,t) is a local splitting of variables in a collar neighbourhood $\partial\Omega\times[0,1)$ of the boundary, with (η,τ) as the covariables, then the boundary symbol of the entry A_- is defined by

$$\sigma_\partial(A_-)(y,\eta):=\sigma_\psi(A_-)(y,0,\eta,D_t),$$

interpreted as an operator family $\sigma_\partial(A_-)(y,\eta):H^s(\mathbb{R}_+)\to H^{s-\mu}(\mathbb{R}_+)$ (or, alternatively, $\mathcal{S}(\overline{\mathbb{R}}_+)\to\mathcal{S}(\overline{\mathbb{R}}_+)$ for $\mathcal{S}(\overline{\mathbb{R}}_+):=\mathcal{S}(\mathbb{R})|_{\mathbb{R}_+}$ with $\mathcal{S}(\mathbb{R})$ being the Schwartz space). Moreover, if $T={}^t(T_1,\dots,T_{N'})$ is given in terms of expressions

$T_k u_- = B_k u_-|_{\partial\Omega}$, see the notation before, we set

$$\sigma_\partial(T_k)(y,\eta)f := (\sigma_\psi(B_k)(y,0,\eta,D_t)f)\big|_{t=0}$$

and

$$\sigma_\partial(T) := {}^t(\sigma_\partial(T_k))_{k=1,\dots,N'}.$$

This gives us altogether a column matrix

$$\sigma_\partial(\mathcal{A})(y,\eta) := \begin{pmatrix} \sigma_\partial(A_-) \\ \sigma_\partial(T) \end{pmatrix}(y,\eta) : H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{N'} \end{matrix}, \tag{7}$$

(which also makes sense for Schwartz spaces rather than Sobolev spaces).

In order to fix notation for the principal transmission symbol we choose a tubular neighbourhood $V \subset \Omega$ of S, i.e., an open submanifold of the form $V \cong S \times (-1,1)$ with a global normal variables $t \in (-1,+1)$ to S with respect to a fixed Riemannian metric (here the metric induced by $\mathbb{R}^{n+1}$). Let $V_\pm := V \cap X_\pm$, and let $\varepsilon : V_- \to V_+$ be defined by $\varepsilon(y,t) = (y,-t)$. Then we can pass to the operator

$$\mathcal{A}_{V_+} := \begin{pmatrix} A_+|_{\mathrm{int}\,V_+} & 0 \\ 0 & \varepsilon_*(A_-|_{\mathrm{int}\,V_-}) \\ T_+ & \varepsilon_* T_- \end{pmatrix}. \tag{8}$$

Here

$$\varepsilon_*(A_-|_{\mathrm{int}\,V_-}) := (\varepsilon^*)^{-1}(A_-|_{\mathrm{int}\,V_-})\varepsilon^* \tag{9}$$

(with ε^* being the function pull back under ε) and

$$(\varepsilon_* T_-)u := (\varepsilon_* B_{-,j}|_{\mathrm{int}\,V_-})u\big|_S \tag{10}$$

for a function u on V_+. The operator $\mathcal{A}_{V_+}$ then represents a boundary value problem on V_+ with a boundary symbol

$$\sigma_\partial(\mathcal{A}_{V_+})(y,\eta) : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ H^s(\mathbb{R}_+) \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^N \end{matrix}$$

which is the analogue (7), here for $(y, \eta) \in T^*S \backslash 0$. This gives rise to our so-called principal transmission symbol of $\mathcal{A}$, namely

$$\sigma_{\mathrm{tr}}(\mathcal{A})(y,\eta): \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ H^s(\mathbb{R}_-) \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ H^{s-\mu}(\mathbb{R}_-) \\ \oplus \\ \mathbb{C}^N \end{matrix}, \tag{11}$$

$(y, \eta) \in T^*S \backslash 0$, obtained from $\sigma_\partial(\mathcal{A}_{V_+})(y, \eta)$ by push forward $(\varepsilon^{-1})_*$ to the operators of the second column from $\mathbb{R}_+$ to $\mathbb{R}_-$, similarly as the relation between the operators (9), (10) (in the scalar case , i.e., $L = 1$, we have $N = \mu$).

The transmission problem (5) is called elliptic if the symbols $\sigma_\psi(A_\pm)(x, \xi)$ are non-vanishing for all $(x, \xi) \in T^*X_\pm \backslash 0$ and if the other components are bijective operators for all sufficiently large s and all $(y, \eta) \in T^*\partial\Omega\backslash 0$ as well as $(y, \eta) \in T^*S\backslash 0$ (if we refer to Schwartz spaces the condition on s disappears). Clearly, the definition of the principal symbols and of ellipticity does not employ the fact that $X_\pm$ (and S) are compact. If, for instance, S has a conical singularity v we can restrict the operator (6) to Sobolev distributions of support disjoint to $\{v\}$. Let $\mathcal{A}_{\mathrm{reg}}$ denote the corresponding operator in this case. Then we have $\sigma(\mathcal{A}_{\mathrm{reg}})$ as before, i.e., the above symbols for the configuration consisting of $X_+\backslash\{v\}$, $X_-\backslash\{v\}$ and the interface $S\backslash\{v\}$. We then have to add the corresponding principal symbols close to the conical singularity v. As noted before we pass to the stretched domains by inserting polar coordinates centered at v. After a translation of Ω we assume $v = 0$. The following considerations are relevant only in a small neighbourhood of 0. Let $B(\varepsilon)$ denote the ball of radius $\varepsilon > 0$ centered at 0 in $\mathbb{R}^{n+1}$. Assume that the intersections $(X_\pm\backslash\{0\}) \cap B(\varepsilon)$ for a sufficiently small $\varepsilon > 0$ are conical in the sense that we have $(X_\pm\backslash\{0\}) \cap B(\varepsilon) = \{\lambda x : 0 < \lambda < \varepsilon, x \in X_\pm \cap \partial B(\varepsilon))\}$. Then also $(S\backslash\{0\}) \cap B(\varepsilon)$ is conical in an analogous sense. This assumption simplifies the formulations though it is completely superfluous; our approach is valid for the general case as well. Let us also fix ε and then omit it in the notation, i.e., $B = B(\varepsilon)$. The operators $A_\pm$ in polar coordinates (r, ϕ) near 0 have the form of operators of Fuchs type which means

$$r^{-\mu} \sum_{j=0}^{\mu} a_{\pm,j}(r)(-r\partial_r)^j \tag{12}$$

with coefficients $a_{\pm,j} \in C^\infty([0, \varepsilon), \mathrm{Diff}^{\mu-j}(\Xi_\pm))$ (here $\mathrm{Diff}^\nu(\cdot)$ denotes the space of all differential operators of order ν on the manifold in the brackets). For our methods it is not essential that (12) comes from a smooth operator in the neighbouring space. Similarly as (4) for the trace operators we assume

$$T_{\pm,l}u_\pm = \mathrm{r}_{\mathrm{int}\,\mathbb{S}}\Big(r^{-m_l} \sum_{j=0}^{m_l} b_{\pm,lj}(r)(-r\partial_r)^j u_\pm\Big). \tag{13}$$

Here $\mathrm{r}_{\mathrm{int}\,\mathbb{S}}$ denotes the operator of the restriction to $\mathrm{int}\,\mathbb{S}$ and the operator-valued coefficients $b_{\pm,lj}$ are elements of $C^\infty([0,\varepsilon), \mathrm{Diff}^{m_l-j}(\Xi_\pm))$. Trace operators of such a form are induced, for instance, by the Neumann condition at S, i.e., $u \to \mathrm{r}_S \frac{\partial}{\partial \nu} u$, with $\frac{\partial}{\partial \nu}$ being the differentiation in outer (with respect to Ξ_+) normal direction to S. Since S has a conical singularity at 0, $\frac{\partial}{\partial \nu}$ is discontinuous near 0, and the transformation in stretched coordinates (r, ϕ) just gives the right expression. To see this we assume, for simplicity, that S is a conical surface. There is then a $\varphi \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$, such that $S \setminus \{0\} = \{x \in \mathbb{R}^{n+1} : \varphi(x) = 0\}$, $\varphi \gtrless 0$ in $\Omega_\pm$ and $\varphi(\delta x) = \delta \varphi(x)$ for all $\delta \in \mathbb{R}_+$, $x \neq 0$. Then $\frac{\partial}{\partial \nu} = \sum_{j=1}^{n+1} \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_j}$ is exactly of the form

$$r^{-1} \sum_{j=0}^{1} b_j (-r\partial_r)^j \tag{14}$$

with coefficients $b_j \in \mathrm{Diff}^{1-j}(\Xi_+)$.

The representation of the operators in Fuchs-type form is just the reason for the continuity of (6) in weighted Sobolev spaces. Multiplying the r-dependent coefficients by a cut-off function $\omega(r)$ supported in $[0, \varepsilon)$ and denoting the new coefficients as before, up to the weight factors our operator takes the form of a Mellin operator with operator-valued symbol. Mellin operators are motivated by the identity $-r\partial_r = M^{-1} z M$ with the Mellin transform $Mu(z) = \int_0^\infty r^{z-1} u(r)\, dr$. The complex covariable z is often considered on the 'weight line'

$$\Gamma_\beta = \{z \in \mathbb{C} : \mathrm{Re}\, z = \beta\}$$

for some real β. The Mellin transform will also be applied to vector-valued functions on $\mathbb{R}_+$, first with compact support (which gives us holomorphy in z) and then extended to various function and distribution spaces (which is the moment to pass to $Mu(z)|_{\Gamma_\beta}$). A Mellin pseudo-differential operator with respect to a weight $\gamma \in \mathbb{R}$ is defined by the expression

$$\mathrm{op}_M^\gamma(h)u(r) = (2\pi)^{-1} \int_{\mathbb{R}} \int_0^\infty \left(\frac{r'}{r}\right)^{\frac{1}{2}-\gamma+i\varrho} h\left(r, r', \frac{1}{2} - \gamma + i\varrho\right) u(r') \frac{dr'}{r'}\, d\varrho$$

with $h(r, r', z)$ belonging to Hörmander's class $S^\mu_{(\mathrm{cl})}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_\varrho)$ in the scalar case and otherwise with values in differential (or pseudo-differential) boundary or transmission problems. In the operator-valued case the covariable $\varrho = \mathrm{Im}\, z$ plays the role of a parameter. In our case the Mellin amplitude functions have the form

$$h_\pm(r, z) = \sum_{j=0}^{\mu} a_{\pm,j}(r) z^j \quad \text{or} \quad h'_{\pm,l}(r, z) = \mathrm{r}_\Sigma \sum_{j=0}^{m_l} b_{\pm,lj}(r) z^j .$$

Set $h'_{\pm}(r,z) := {}^t(h'_{\pm,l}(r,z))_{l=1,\dots,N}$. We then have $A_{\pm} = r^{-\mu}\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_{\pm})$ and $T_{\pm} = \mathrm{diag}(r^{-m_l})\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h'_{\pm})$. Thus the operator $\mathcal{A}$ close to 0 has the form

$$\mathcal{A} = m(r)\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h) \tag{15}$$

for a matrix $m(r)$ of weight factors, namely, $m(r) := \mathrm{diag}\big(r^{-\mu}, r^{-\mu}, \mathrm{diag}(r^{-m_l})\big)$, and the matrix of Mellin amplitude functions

$$h(r,z) = \begin{pmatrix} h_+(r,z) & 0 \\ 0 & h_-(r,z) \\ h'_+(r,z) & h'_-(r,z) \end{pmatrix}. \tag{16}$$

The function $h(r,z)$ is smooth in r up to zero and takes values in the space of transmission problems on S^{n-1} with respect to the subdivision $S^{n-1} = \Xi_+ \cup \Xi_-$ with the interface $\Sigma = \Xi_+ \cap \Xi_+$. The covariable z varies on the weight line $\Gamma_{\frac{n+1}{2}-\gamma}$ and is interpreted as a parameter. Adequate choices of γ depend on the behaviour of the so-called conormal symbol $\sigma_M(\mathcal{A})(z) := h(0,z)$ which is also a parameter-dependent family of transmission problems on S^{n-1}, regarded as continuous operators

$$\sigma_M(\mathcal{A})(z) : \begin{matrix} H^s(\mathrm{int}\,\Xi_+) \\ \oplus \\ H^s(\mathrm{int}\,\Xi_-) \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathrm{int}\,\Xi_+) \\ \oplus \\ H^{s-\mu}(\mathrm{int}\,\Xi_-) \\ \oplus \\ \bigoplus_{l=1}^N H^{s-m_l-\frac{1}{2}}(\Sigma) \end{matrix} \tag{17}$$

for real $s > \max\{m_l\} + \frac{1}{2}$.

Summing up a boundary-contact operator (6) with an interface S with conical singularity has a principal symbolic hierarchy

$$\sigma(\mathcal{A}) := (\sigma_\psi(A_+), \sigma_\psi(A_-), \sigma_{\mathrm{tr}}(\mathcal{A}_{\mathrm{reg}}), \sigma_\partial(\mathcal{A}), \sigma_M(\mathcal{A})),$$

where $\sigma_{\mathrm{tr}}(\mathcal{A}_{\mathrm{reg}})$ was defined before.

1.3. Outline of the results

A boundary-contact operator (6) is said to be elliptic with respect to γ if

(i) $\sigma_\psi(A_\pm)$ is elliptic as usual,
(ii) $\sigma_\partial(\mathcal{A})$ is elliptic in the sense of the Shapiro–Lopatinskij condition,
(iii) $\sigma_{\mathrm{tr}}(A_{\mathrm{reg}})$ is elliptic as a transmission condition,
(iv) $\sigma_M(\mathcal{A})$ is elliptic with respect to the weight γ.

More precisely, in (i) the principal symbols $\sigma_\psi(A_\pm)(x,\xi)$ do not vanish for all $x \in X_\pm$, $\xi \in \mathbb{R}^{n+1}\setminus\{0\}$. In (ii) we mean the bijectivity of the family of maps (7) for all $(y,\eta) \in T^*(\partial\Omega)\setminus 0$ (which is also standard). The ellipticity in (iii) first means the bijectivity of $\sigma_{\mathrm{tr}}(\mathcal{A}_{\mathrm{reg}})(y,\eta)$ as a family of maps (11) for all $(y,\eta) \in T^*(S\setminus\{v\})\setminus 0$; in addition, expressing the involved operators in polar coordinates we obtain the transmission symbol in the variables $y=(r,x')$ and covariables $\eta=(\varrho,\xi')$ with (x',ξ') belonging to $T^*\Sigma$, and then

$$\tilde{\sigma}_{\mathrm{tr}}(\mathcal{A})(r,x',\varrho,\xi') := m(r)^{-1}\sigma_{\mathrm{tr}}(\mathcal{A}_{\mathrm{reg}})(r,x',r^{-1}\varrho,\xi')$$

(which is smooth in r up to zero) is required to define a family of isomorphisms

$$\tilde{\sigma}_{\mathrm{tr}}(\mathcal{A})(r,x',\varrho,\xi') : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ H^s(\mathbb{R}_-) \end{array} \to \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ H^{s-\mu}(\mathbb{R}_-) \\ \oplus \\ \mathbb{C}^N \end{array}$$

up to $r=0$, for any $s \in \mathbb{R}$ sufficiently large. In (iv) the ellipticity of $\sigma_M(\mathcal{A})(z)$ means the bijectivity of (17) for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ and any $s \in \mathbb{R}$ sufficiently large.

The conditions (iii) and (iv) are natural; we will return to more details below.

The Section 2 will contain the necessary material on weighted cone Sobolev spaces with asymptotics. In Section 3, we describe a pseudo-differential analogue of boundary-contact operators. In Section 4, we construct parametrices of elliptic elements (cf. Theorem 4.3) and obtain regularity and asymptotics of solutions (cf. Theorem 4.6).

This will be done under some natural weight conditions which ensure elliptic regularity of solutions in weighted spaces and asymptotics in a general qualitative form. Section4.3 is devoted to examples with explicit information on admissible weights and exponents, logarithmic terms and coefficients of asymptotics of solutions.

Our approach to boundary-contact problems for conical singularities can be generalised to a calculus with parameters. Those parameters may be variables and covariables on an edge. In a future we intend to apply the present results to boundary-contact problems when the interface has edges.

2. Cone Sobolev spaces with asymptotics

2.1. The Mellin contribution to parametrices in weighted spaces

The regularity of solutions to our boundary-contact problems refers to a category of weighted Sobolev spaces as they are known from Kondratyev's work [10]. Let us briefly recall the definition. By assumption our subdomains $\Omega_\pm \subset \mathbb{R}^{n+1}$ have a C^∞ boundary, except for the conical singularity 0. Let us omit for the moment subscripts '$\pm$' and simply write Ω.

For the construction we refer to polar coordinates $(r, \phi) \in \mathbb{R}_+ \times S^n$ in $\mathbb{R}^n \setminus \{0\}$. First let $s \in \mathbb{N}$, and let

$$\mathcal{H}^{s,0}(\mathbb{R}_+ \times S^n)$$

denote the subspace of all $u(r, \phi) \in r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times S^n)$ such that

$$(r\partial_r)^k D^\alpha_\phi u(r, \phi) \in r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times S^n)$$

for all $k + |\alpha| \leqslant s$. Here, D^α_ϕ denotes an arbitrary differential operator of order $|\alpha|$ on the sphere S^n. Then, by duality with respect to the scalar product of $r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times S^n)$ and interpolation we obtain $\mathcal{H}^{s,0}(\mathbb{R}_+ \times S^n)$ for arbitrary $s \in \mathbb{R}$. Moreover, we set $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times S^n) := r^\gamma \mathcal{H}^{s,0}(\mathbb{R}_+ \times S^n)$ and

$$\mathcal{H}^{s,\gamma}(\mathbb{R}^{n+1} \setminus \{0\}) := \{u \in H^s_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \{0\}) : u(r, \phi) \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times S^n)\}.$$

This gives us finally

$$\mathcal{H}^{s,\gamma}(\Omega_\pm) := \{u|_{\Omega_\pm} : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}^{n+1} \setminus \{0\})\}.$$

To obtain analogous spaces $\mathcal{H}^{s,\gamma}(S \setminus \{0\})$ on a manifold S (of dimension n) with conical singularity 0 we first modify the former construction for conical subsets in $\mathbb{R}^n \setminus \{0\}$ and then glue together the local spaces, using a suitable partition of unity together with a natural invariance property.

Going back to the notation in Section 1.1 we denote the spaces on the corresponding stretched manifolds also by

$$\mathcal{H}^{s,\gamma}(\mathbb{X}_\pm) \quad \text{and} \quad \mathcal{H}^{s,\gamma}(\mathbb{S}),$$

respectively. Details on constructions of that kind may be found in [9,21]. Note that the continuity (6) is a consequence of the representation of the involved operators in polar coordinates, together with the continuity of the operator of restriction $\mathcal{H}^{s,\gamma}(\mathbb{X}_\pm) \to \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{S})$ for every $s > \frac{1}{2}$. The following simple remark is given for future references:

Remark 2.1. (i) The operator of multiplication by a function $r^\beta \varphi(r)$, $\beta \in \mathbb{R}$, $\varphi \in C_0^\infty(\overline{\mathbb{R}}_+)$ (supported in a small neighbourhood of 0) induces continuous operators $\mathcal{H}^{s,\gamma}(\mathbb{X}_\pm) \to \mathcal{H}^{s,\gamma+\beta}(\mathbb{X}_\pm)$ for all $s, \gamma \in \mathbb{R}$.

(ii) The operators $(r\partial_r)^k D^\alpha_\varphi$ induce continuous operators $\mathcal{H}^{s,\gamma}(\mathbb{X}_\pm) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{X}_\pm)$ for all $s, \gamma \in \mathbb{R}$.

Similar observations are true for the weighted spaces over $\mathbb{S}$. Consider as a simple example the case

$$A_+ = \Delta|_{\Omega_+}, \quad A_- = c\Delta|_{\Omega_-} \tag{18}$$

for a constant $c \neq 0$, with Δ being the Laplace operator in $\mathbb{R}^{n+1}$, and

$$T_\pm = {}^t(T_{\pm,1}\ \ T_{\pm,2}) \quad \text{for} \quad T_{\pm,1}u := \pm u|_{\operatorname{int}\mathbb{S}} \quad \text{and} \quad T_{\pm,2}u := \frac{\partial}{\partial v_\pm} u|_{\operatorname{int}\mathbb{S}}, \tag{19}$$

where $v_\pm$ are the outward normal directions to the boundaries of $\Omega_\pm \backslash \{0\}$. On $\partial\Omega$ we may take any elliptic boundary condition for the Laplacian, e.g., Dirichlet conditions. The corresponding boundary-contact problem then represents a continuous operator

$$\mathcal{A} = \begin{pmatrix} \Delta & 0 \\ 0 & c\Delta \\ T_+ & T_- \\ 0 & T \end{pmatrix} : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s,\gamma}(\mathbb{X}_-) \end{array} \to \begin{array}{c} \mathcal{H}^{s-2,\gamma-2}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s-2,\gamma-2}(\mathbb{X}_-) \\ \oplus \\ \bigoplus_{l=1}^{2} \mathcal{H}^{s-m_l-\frac{1}{2},\gamma-m_l-\frac{1}{2}}(\mathbb{S}) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial\Omega) \end{array} \tag{20}$$

for $m_1 = 0$, $m_2 = 1$ and every $s > \frac{3}{2}$.

As announced before we will obtain the Fredholm property of such operators for all real weights γ except for a discrete set which can be calculated explicitly. Let us illustrate the shape of (20) in connection with the Mellin symbols in (16). For $h_\pm$ we simply have

$$h_\pm(z) = c_\pm(z^2 - (n-1)z + \Delta_{S^n})$$

for $c_+ = 1$, $c_- = c$ with Δ_{S^n} being the Laplace operator on the sphere S^n. Moreover, $h_{\pm,j}$ are defined by

$$h_{\pm,1} : u \to \mathrm{r}'u \quad \text{and} \quad h_{\pm,2} : u \to \pm\mathrm{r}' \sum_{j=0}^{1} b_j z^j u$$

with b_j as in (14) and the operator of restriction r' to $\operatorname{int}\mathbb{S}$ from the $\pm$ side. We then obtain

$$\begin{pmatrix} \Delta & 0 \\ 0 & c\Delta \\ T_+ & T_- \end{pmatrix} = m(r)\,\mathrm{op}^{\gamma-\frac{n}{2}}(h)$$

for $m(r) = \mathrm{diag}(r^{-2}, r^{-2}, 1, r^{-1})$ and the Mellin symbol $h(z)$ as in (16) which has now constant coefficients.

Looking at the sphere S^n which is subdivided by the smooth submanifolds $\Xi_\pm$ with common boundary Σ there is a space $\mathcal{B}^{\mu,d}(\Xi_+, \Xi_-)$ transmission problems (of order μ and so-called type d) with respect to the interface Σ (more details will be given below). The operators of this space are similar to those in the upper 3×2 part of the matrix (5). In the present case we have $\mu = 2$ and $d = 2$. The general definition of $\mathcal{B}^{\mu,d}(\Xi_+, \Xi_-)$ is close to that of the space $\mathcal{B}^{\mu,d}(\Xi)$ of (pseudo-differential) boundary value problems (of order μ and so-called type d) on a (here compact) C^∞ manifold Ξ with boundary Σ which have the transmission property at Σ. Since in that kind of operators also systems are admitted, a local reflection argument near Σ allows us to reduce the definition of $\mathcal{B}^{\mu,d}(\Xi_+, \Xi_-)$ to the case of boundary value problems, similarly as in the beginning in connection with the operators (9), (10). The space $\mathcal{B}^{\mu,d}(\Xi_+, \Xi_-)$ has a natural Fréchet topology. In our example we have

$$h(z) \in \mathcal{A}(\mathbb{C}, \mathcal{B}^{2,2}(\Xi_+, \Xi_-))$$

(here $\mathcal{A}(U, E)$ denotes the space of all holomorphic functions in an open set $U \subseteq \mathbb{C}$ with values in the Frechet space E).

Remark 2.2. $h|_{\Gamma_\beta}$ is a family of transmission problems on the sphere S^n, subdivided into $\Xi_\pm$, with interface Σ, parameter-dependent elliptic with parameter $\mathrm{Im}\, z$ for z varying on Γ_β, for every $\beta \in \mathbb{R}$. If is known from such a situation that

$$h(z) : \begin{array}{c} H^s(\Xi_+) \\ \oplus \\ H^s(\Xi_-) \end{array} \to \begin{array}{c} H^{s-2}(\Xi_+) \\ \oplus \\ H^{s-2}(\Xi_-) \\ \oplus \\ \bigoplus_{l=1}^{2} H^{s-m_l-\frac{1}{2}}(\Sigma) \end{array} \tag{21}$$

for $s > \frac{3}{2}$, $m_0 = 0$, $m_1 = 1$, is invertible for all $z \in \mathbb{C} \setminus D$ where D is a discrete set with the property $D \cap \{c \leqslant \mathrm{Re}\, z \leqslant c'\}$ is finite for every $c \leqslant c'$.

For a number of cases we will explicitly calculate the set D in Section 4.3 below. Now $h^{-1}(z)$ is meromorphic with values in $\mathcal{B}^{-2,0}(\Xi_-, \Xi_+)$ and has poles at the points of D. A first essential step to solve our boundary-contact problem, represented by the operator (20), is to pass to the operator

$$(m(r)\, \mathrm{op}_M^{\gamma-\frac{n}{2}}(h))^{-1} = \mathrm{op}_M^{\gamma-\frac{n}{2}}(h^{-1}) m^{-1}(r) \tag{22}$$

which is well defined for those reals γ where $D \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$. Since h has constant coefficients we have invertibility in the sense that

$$\{\mathrm{op}_M^{\gamma-\frac{n}{2}}(h^{-1})m^{-1}(r)\}\{m(r)\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\} = \mathrm{op}_M^{\gamma-\frac{n}{2}}(h^{-1})\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h) = I$$

and the same from the other side.

Observe that the operator (20) in localised form near 0 with Mellin symbols with constant coefficients is continuous as a map

$$m(r)\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h) : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \Xi_+) \\ \oplus \\ \mathcal{H}^{s,\gamma}(\mathbb{R}_- \times \Xi_-) \end{array} \to \begin{array}{c} \mathcal{H}^{s-2,\gamma-2}(\mathbb{R}_+ \times \Xi_+) \\ \oplus \\ \mathcal{H}^{s-2,\gamma-2}(\mathbb{R}_+ \times \Xi_-) \\ \oplus \\ \bigoplus_{l=1}^{2} \mathcal{H}^{s-m_l-\frac{1}{2},\gamma-m_l-\frac{1}{2}}(\mathbb{R}_+ \times \Sigma) \end{array}$$

for every $s > \frac{3}{2}$ and $\gamma \in \mathbb{R}$ as noted before (in this notation for simplicity we write $\Xi_\pm$ rather than $\operatorname{int}\Xi_\pm$; we hope this will not cause confusion).

2.2. Asymptotics

Asymptotics of solutions (also to be expressed explicitly for specific examples) will be formulated in terms of suitable subspaces of the weighted Sobolev spaces. The information is of the following kind. Given a weight $\gamma \in \mathbb{R}$ such that our operator is Fredholm, there are sequences of triples

$$\{(p_{\pm,j}, n_{\pm,j}, L_{\pm,j})\}_{j\in\mathbb{N}} \tag{23}$$

consisting of $p_{\pm,j} \in \mathbb{C}$, $n_{\pm,j} \in \mathbb{N}$, and finite-dimensional subspaces $L_{\pm,j} \subset C^\infty(\Xi_\pm)$, where

$$\mathrm{Re}\, p_{\pm,j} < \frac{n+1}{2} - \gamma \quad \text{for all} \quad j$$

and $\mathrm{Re}\, p_{\pm,j} \to -\infty$ as $j \to \infty$, such that the components of a solution

$$u(r,\phi) = {}^t(u_+(r,\phi), u_-(r,\phi)) \in \mathcal{H}^{s,\gamma}(\mathbb{X}_+) \oplus \mathcal{H}^{s,\gamma}(\mathbb{X}_-)$$

can be written as

$$u_\pm(r,\phi) = \omega(r) \sum_{j=0}^{J} \sum_{k=0}^{n_{\pm,j}} c_{\pm,jk}(\phi) r^{-p_{\pm,j}} \log^k r + u_{\pm,J}(r,\phi),$$

for coefficients $c_{\pm,jk} \in L_{\pm,j}$. Here ω is a cut-off function, i.e., an element of $C_0^\infty(\overline{\mathbb{R}}_+)$ which is equal to 1, in a neighbourhood of 0, and the remainders $u_{\pm,J}(r,\phi)$ belong to $\mathcal{H}^{s,\gamma+\beta}(\mathbb{X}_\pm)$ for arbitrary given $\beta \in \mathbb{R}_+$ and a resulting length $J = J(\beta)$ of the sums over j. The so-called asymptotic types (23) depend on the operator and the boundary conditions as well as on the data $f_\pm$ and g that are assumed to be given with similar asymptotics, cf. the formulas (1), (3).

Let us first give the definition of subspaces $\mathcal{H}_P^{s,\gamma}(\cdot)$ of $\mathcal{H}^{s,\gamma}(\cdot)$ of elements with asymptotic type P when the space in the brackets is one of our stretched domains $\mathbb{X}_\pm$, briefly denoted by $\mathbb{X}$. It will be convenient to admit asymptotic types of finite or infinite length

$$P = \{(p_j, n_j, L_j)\}_{j=0,\dots,N}, \quad N \in \mathbb{N} \cup \{\infty\}$$

with $p_j \in \mathbb{C}$,

$$\frac{n+1}{2} - \gamma - \vartheta < \operatorname{Re} p_j < \frac{n+1}{2} - \gamma \tag{24}$$

for some $\vartheta \in \mathbb{R}_+ \cup \{\infty\}$ and N finite as soon as ϑ is finite, $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ for $N = \infty$. As before we assume $n_j \in \mathbb{N}$, and $L_j \subset C^\infty(\Xi)$ is a subspace of finite dimension; Ξ is of analogous meaning as $\Xi_\pm$ before. Denoting by $\Theta = [0, \vartheta)$ the weight interval occurring in (24) we set

$$\mathcal{H}_\Theta^{s,\gamma}(\mathbb{X}) := \bigcap_{\varepsilon>0} \mathcal{H}^{s,\gamma+\vartheta-\varepsilon}(\mathbb{X}) \tag{25}$$

in the topology of the projective limit. Thus (25) is a Fréchet space. Let N be finite and fix any cut-off function ω. We then form the space of singular functions belonging to the finite asymptotic type P

$$\mathcal{E}_P(\mathbb{X}) := \left\{ \omega(r) \sum_{j=0}^{N} \sum_{k=0}^{n_j} c_{jk} r^{-p_j} \log^k r : c_{jk} \in L_j \right. \\ \left. \text{for all} \quad 0 \leqslant k \leqslant n_j, 0 \leqslant j \leqslant N \right\}. \tag{26}$$

The space $\mathcal{E}_P(\mathbb{X})$ is finite-dimensional, and we have $\mathcal{H}_\Theta^{s,\gamma}(\mathbb{X}) \cap \mathcal{E}_P(\mathbb{X}) = \{0\}$. The sum

$$\mathcal{H}_P^{s,\gamma}(\mathbb{X}) := \mathcal{H}_\Theta^{s,\gamma}(\mathbb{X}) + \mathcal{E}_P(\mathbb{X})$$

is a Fréchet subspace of $\mathcal{H}^{s,\gamma}(\mathbb{X})$, independent of the choice of ω. If P is infinite we can pass to the sequence

$$P_k := \left\{(p, n, L) \in P : \frac{n+1}{2} - \gamma - (k+1) < \mathrm{Re}\, p < \frac{n+1}{2} - \gamma\right\}$$

of finite asymptotic types, $k \in \mathbb{N}$, and set $\mathcal{H}^{s,\gamma}_P(\mathbb{X}) := \bigcap_{k\in\mathbb{N}} \mathcal{H}^{s,\gamma}_{P_k}(\mathbb{X})$ in the Fréchet topology of the projective limit.

In a similar manner, we can define Fréchet subspaces $\mathcal{H}^{s,\gamma}_Q(\mathbb{S})$ of $\mathcal{H}^{s,\gamma}(\mathbb{S})$ for asymptotic types $Q = (q_j, l_j, H_j)_{j=0,\ldots,N}$ for finite or infinite N, with $q_j \in \mathbb{C}$, $l_j \in \mathbb{N}$ and finite-dimensional subspaces $H_j \subset C^\infty(\Sigma)$. Concerning q_j we assume $\mathrm{Re}\, q_j < \frac{n+1}{2} - \gamma$ (according to $n-1 = \dim S$) and $\mathrm{Re}\, q_j \to -\infty$ as $j \to \infty$ when $N = \infty$. For finite N we have analogues of the spaces (26), namely, $\mathcal{E}_Q(\mathbb{S})$, and the definition of the spaces $\mathcal{H}^{s,\gamma}_Q(\mathbb{S})$ is practically as before.

Proposition 2.3. *Given a boundary-contact problem of the form* (1)–(3) *for the case that the interface S has a singularity* (*at zero*) *the associated operators* (6) *restrict to continuous operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{H}^{s,\gamma}_{P_+}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s,\gamma}_{P_-}(\mathbb{X}_-) \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu,\gamma-\mu}_{R_+}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma-\mu}_{R_-}(\mathbb{X}_-) \\ \oplus \\ \bigoplus_{l=1}^{N} \mathcal{H}_Q^{s-m_l-\frac{1}{2},\gamma-m_l-\frac{1}{2}}(\mathbb{S}) \\ \oplus \\ \bigoplus_{j=1}^{N'} H^{s-m'_j-\frac{1}{2}}(\partial\Omega) \end{array}$$

for every pair (P_+, P_-) *of asymptotic types, associated with the weight* γ *as described before, with some resulting asymptotic types* (R_+, R_-, Q) *associated with the weights in the respective spaces, for every* $s > \max\{m_l, m'_j\} + \frac{1}{2}$ *and* $\gamma \in \mathbb{R}$.

Proof. It suffices to consider the entries of $\mathcal{A}$ separately. Let us take, for instance, the upper left corner which can be written in the form

$$\omega(r) r^{-\mu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_+)\tilde{\omega} + (1-\omega)A_+(1-\tilde{\tilde{\omega}})$$

for cut-off functions ω, $\tilde{\omega}$, $\tilde{\tilde{\omega}}$ such that $\tilde{\omega} \equiv 1$ on $\mathrm{supp}\,\omega$, $\omega \equiv 1$ on $\mathrm{supp}\,\tilde{\tilde{\omega}}$. The Mellin symbol h_+ was defined in Section 1.2. The continuity of the second summand is a finite linear combination of expressions

$$H_+ := \omega a_{+,j}(r)(-r\partial_r)^j \tilde{\omega}.$$

Consider, for instance, the case of finite P_+ (the infinite case then follows by letting the length of asymptotic expansions going to infinity). We then have $\mathcal{H}^{s,\gamma}_{P_+}(\mathbb{X}_+) = \mathcal{H}^{s,\gamma}_{\Theta}(\mathbb{X}_+) + \mathcal{E}_{P_+}(\mathbb{X}_+)$. The continuity of $H_+ : \mathcal{H}^{s,\gamma}_{\Theta}(\mathbb{X}_+) \to \mathcal{H}^{s,\gamma}_{\Theta}(\mathbb{X}_+)$ is a direct consequence of the definition, together with the relation $(r\partial_r)^k r^\beta = r^\beta(r\partial_r + \beta)^k$ for every $\beta \in \mathbb{R}$ which shows why the flatness is preserved under the action. The application of H_+ to $\mathcal{E}_{P_+}(\mathbb{X}_+)$ shows how the singular functions together with the spaces of coefficients are transformed to a space $\mathcal{E}_{R_+}(\mathbb{X}_+)$, modulo flat remainders belonging to $\mathcal{H}^{\infty,\gamma-\mu}_{\Theta}(\mathbb{X}_+)$. The other entries can be treated in an analogous manner. □

Remark 2.4. The asymptotic types in Proposition 2.3 may be infinite or finite (in the latter case the weight intervals $\Theta = [0, \vartheta)$ are the same for all of them).

Set $\mathcal{H}^{\infty,\gamma}_P(\mathbb{X}) := \bigcap_{s\in\mathbb{R}} \mathcal{H}^{s,\gamma}_P(\mathbb{X})$, and define $\mathcal{H}^{\infty,\gamma}_Q(\mathbb{S})$ in an analogous manner. An application of these spaces will be that parametrices $\mathcal{P}$ to our operators (6) have the property

$$\mathcal{P}\mathcal{A} = \mathcal{I} - \mathcal{K}_l, \quad \mathcal{A}\mathcal{P} = \mathcal{I} - \mathcal{K}_r, \tag{27}$$

where $\mathcal{K}_l$ and $\mathcal{K}_r$ are smooth in the sense that they induce continuous operators

$$\mathcal{K}_l : \begin{matrix} \mathcal{H}^{s,\gamma}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s,\gamma}(\mathbb{X}_-) \end{matrix} \to \begin{matrix} \mathcal{H}^{\infty,\gamma}_{U_+}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{\infty,\gamma}_{U_-}(\mathbb{X}_-) \end{matrix} \tag{28}$$

and

$$\mathcal{K}_r : \begin{matrix} \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{X}_-) \\ \oplus \\ \bigoplus_{l=1}^{N} \mathcal{H}^{s-m_l-\frac{1}{2},\gamma-m_l-\frac{1}{2}}(\mathbb{S}) \\ \oplus \\ \bigoplus_{j=1}^{N'} H^{s-m'_j-\frac{1}{2}}(\partial\Omega) \end{matrix} \to \begin{matrix} \mathcal{H}^{\infty,\gamma-\mu}_{V_+}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{\infty,\gamma-\mu}_{V_-}(\mathbb{X}_-) \\ \oplus \\ \bigoplus_{l=1}^{N} \mathcal{H}^{\infty,\gamma-m_l-\frac{1}{2}}_{S}(\mathbb{S}) \\ \oplus \\ \bigoplus_{j=1}^{N'} H^{\infty}(\partial\Omega) \end{matrix} \tag{29}$$

for all sufficient large reals s (as described before), plus analogous properties of adjoint for the so-called type zero ingredients cf. the notation in Section 3.4 below. The asymptotic types U_+, U_-, etc. in the relations (28) and (29) depend on $\mathcal{K}_l$ and $\mathcal{K}_r$, respectively. The final classes of smoothing operators of type $d \in \mathbb{N}$ will be denoted by $\mathcal{C}^d(\mathbb{X}_+, \mathbb{X}_-)$ while $\mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ will denote the space of all boundary-contact problems on our configuration characterised by $\overline{\Omega}$ together with the subdivision into $X_\pm$ (the explicit definitions will be given below).

3. Cone transmission operators

3.1. The structure of parametrices

We now consider the parametrices of elliptic boundary-contact problems of the form (6) and explain the meaning of the various contributions.

Let us restrict the operators to a neighbourhood of $\overline{\Omega}_+$ and ignore for a while the elliptic boundary value problem $\begin{pmatrix} A_- \\ T \end{pmatrix}$ (the parametrix for the latter operator is standard and can be added afterwards). The remaining operator then has the form $\mathcal{A} := \begin{pmatrix} A_+ & 0 \\ 0 & A_- \\ T_+ & T_- \end{pmatrix}$. Our calculus will show that there are parametrices of $\mathcal{A}$ from both sides; then left and right parametrices coincide modulo smoothing operators (that have as the corresponding mapping properties as at the end of Sections 2.2). Let us concentrate on a parametrix $\mathcal{P}$ from the left (the technique from the right is analogous and left to the reader). Write $\mathcal{P}$ in the form

$$\begin{pmatrix} P_+ & G_+ & K_+ \\ G_- & P_- & K_- \end{pmatrix}.$$

In order to describe the entries we choose cut-off functions ω, $\tilde{\omega}$, $\tilde{\tilde{\omega}}$ on the r-half axis that are equal to 1 in a neighbourhood of 0 (the conical point) such that $\tilde{\omega} \equiv 1$ on $\operatorname{supp} \omega$, $\omega \equiv 1$ on $\operatorname{supp} \tilde{\tilde{\omega}}$. Then for $P_\pm$ we obtain

$$P_\pm = \omega\{H_\pm + M_\pm\}\tilde{\omega} + C_\pm + (1-\omega)B_\pm(1-\tilde{\tilde{\omega}}),$$

where

(i) $H_\pm$ and $M_\pm$ are Mellin pseudo-differential operators ($H_\pm$ of order $-\mu$ with holomorphic symbols, $M_\pm$ smoothing with meromorphic symbols);
(ii) $C_\pm$ are smoothing operators of a similar kind as the first two diagonal entries of $\mathcal{K}_r$ in the formula (29);
(iii) $B_\pm$ are pseudo-differential operators with the transmission property at $S\backslash\{0\}$ plus Green operators in Boutet de Monvel's calculus on $X_\pm\backslash\{0\}$.

Moreover, the operators $K_\pm$ are potential operators from the cone algebra of boundary value problems on the respective sides. Finally, $G_\pm$ are transmission operators of a similar structure as smoothing Mellin and Green operators in Boutet de Monvel's calculus in a domain with conical singularities.

Let us now have a look at the example from Section 2.1. Choose a covering of $\overline{\Omega}$ by (relatively) open sets $\{U_j\}_{j=1,2,3}$ (say, with smooth boundaries) such that $U_1 = B(\varepsilon)$ for some $\varepsilon > 0$ (cf. the notation in Section 1.2), U_2 open, $U_2 \cap \{0\} = \emptyset$, $X_+ \subset U_1 \cup U_2$, $(U_1 \cup U_2) \cap \partial\Omega = \emptyset$. Let $\{\varphi_j\}_{j=1,2,3}$ be a subordinate partition of unity, and let

$\{\psi_j\}_{j=1,2,3}$ functions $\psi_j \in C_0^\infty(U_j)$ such that $\psi_j \equiv 1$ on $\operatorname{supp}\varphi_j$ for all j. Then the operator (20) can be decomposed into

$$\mathcal{A} = \sum_{j=1}^{3} \mathcal{M}_{\varphi_j} \mathcal{A} \mathcal{M}_{\psi_j}$$

for $\mathcal{M}_{\psi_j} := \operatorname{diag}(\psi_j|_{\mathbb{X}_+}, \psi_j|_{\mathbb{X}_-})$, $\mathcal{M}_{\varphi_j} := \operatorname{diag}(\varphi_j|_{\mathbb{X}_+}, \varphi_j|_{\mathbb{X}_-}, \varphi_j|_{\mathbb{S}}, \varphi_j|_{\mathbb{S}}, \varphi_j|_{\partial\Omega})$ with entries being interpreted as operators of multiplication by the corresponding functions. To express a parametrix of $\mathcal{A}$ we first form local parametrices of

$$\begin{pmatrix} \Delta & 0 \\ 0 & c\Delta \\ T_+ & T_- \end{pmatrix} \quad \text{separately in } U_1 \text{ and } U_2 \tag{30}$$

and of $\begin{pmatrix} c\Delta \\ T \end{pmatrix}$ in U_3 and fill them up by corresponding zeros to 2×3 matrices of operators. Let us denote these enlarged matrices referring to U_j by $\mathcal{P}_j$, such that,

$$\mathcal{P}_1 = \left(\operatorname{op}_M^{\gamma-\frac{n}{2}}(h^{-1})m^{-1} \begin{matrix} \vdots & 0 \\ \vdots & 0 \end{matrix} \right),$$

cf. the formula (22),

$$\mathcal{P}_2 = \left(\mathcal{S}_2 \begin{matrix} \vdots & 0 \\ \vdots & 0 \end{matrix} \right)$$

with $\mathcal{S}_2$ having the meaning of a parametrix of the elliptic transmission problem (30) over U_2, and

$$\mathcal{P}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P_3 & 0 & K_3 \end{pmatrix},$$

where (P_3, K_3) is a parametrix of the standard elliptic boundary value problem $\begin{pmatrix} c\Delta \\ T \end{pmatrix}$ in U_3 in Boutet de Monvel's calculus, cf. [2,17], or [6]. In this way we obtain the following result:

Theorem 3.1. *Let $\mathcal{A}$ be our boundary-contact operator* (20)*. Then*

$$\mathcal{P} = \sum_{j=1}^{3} \mathcal{M}_{\psi_j} \mathcal{P}_j \mathcal{M}_{\varphi_j}$$

is a parametrix of $\mathcal{A}$ in the sense of the relations (23) *and remainders*

$$\mathcal{K}_l \in \mathcal{C}^2(\mathbb{X}_+, \mathbb{X}_-), \quad \mathcal{K}_r \in \mathcal{C}^0(\mathbb{X}_+, \mathbb{X}_-).$$

Theorem 3.2. *Consider the boundary-contact operator* (20) *in dimension* 2, *and let $u_\pm \in \mathcal{H}^{-\infty,\gamma}(\mathbb{X}_\pm)$ be a solution of*

$$\mathcal{A}\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = {}^t(f_+, f_-, g, b)$$

for $f_\pm \in \mathcal{H}^{s-2,\gamma-2}_{(R_\pm)}(\mathbb{X}_\pm)$, $g \in \oplus_{l=1}^2 \mathcal{H}^{s-m_l-\frac{1}{2},\gamma-m_l-\frac{1}{2}}_{(Q)}(\mathbb{S})$, $b \in H^{s-\frac{1}{2}}(\partial\Omega)$, for $m_1 = 0$, $m_2 = 1$, $s > \frac{3}{2}$, $\gamma \in \mathbb{R} \backslash \{1 - \frac{\pi k}{\pi - \alpha} : k \in \mathbb{Z}\}$ (*subscripts with asymptotic types in brackets mean either Sobolev spaces without asymptotics or subspaces with asymptotics of the corresponding types*). *Then it follow that $u_\pm \in \mathcal{H}^{s,\gamma}_{(P_\pm)}(\mathbb{X}_\pm)$* (*with asymptotic types $P_\pm$ that depend on $R_\pm$ and Q*).

This result is a special case of Theorem 4.6 that we prove together with the characterisation of admitted weights, given in Section 4.3.

3.2. Transmission operators on the sphere

As we saw in the preceding section, the main contribution to parametrices of boundary-contact problems come from Mellin operators with values in a space of transmission operators on the sphere S^n. In order to make the information more explicit we now outline the basic features of the so-called transmission algebra. Transmission operators with smooth interface in a pseudo-differential set-up have been considered also by Myshkis [15]. Here we establish a parameter-dependent calculus and give more details, since we employ transmission operators as values of operator-valued Mellin symbols. The sphere S^n is subdivided into compact C^∞ submanifolds $\Xi_\pm$ with common boundary Σ. We want to introduce a space of parameter-dependent transmission operators $\mathcal{B}^{\mu,d}(\Xi_+, \Xi_-; \mathbb{R}^l)$ of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}$, with the parameter $\lambda \in \mathbb{R}^l$, $l \in \mathbb{N}$. For the case $l = 0$ we simply write $\mathcal{B}^{\mu,d}(\Xi_+, \Xi_-)$. By assumption $\Xi_\pm$ are smooth compact submanifolds of S^n with boundary Σ. We will explain our transmission operators in terms of the pseudo-differential formalism of 'standard' boundary value problems on a smooth manifold Ξ with boundary Σ. Of course, it is not essential that Σ is embedded in a sphere or in another smooth and closed manifold M, but for convenience we assume that. In addition let Σ be compact which is the case in our application.

There is then the space $L^{\mu}_{\mathrm{cl}}(M;\mathbb{R}^l)_{\mathrm{tr}}$ of parameter-dependent pseudo-differential operators of order $\mu\in\mathbb{R}$ that have the transmission property at Σ. Let $(y,t)\in\mathbb{R}^{n-1}\times\mathbb{R}$ be local coordinates on M near Σ such that $y\in\mathbb{R}^{n-1}$ are local coordinates on Σ and t the normal variable with $t\geqslant 0$ on the Ξ-side. Recall that a symbol $a(y,t,\eta,\tau,\lambda)$ in $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{n-1}_y\times\mathbb{R}_t\times\mathbb{R}^{n+l}_{\eta,\tau,\lambda})$ is said to have the transmission property at $t=0$ if the homogeneous components $a_{(\mu-j)}(y,t,\eta,\tau,\lambda)$ of order $\mu-j$ in $(\eta,\tau,\lambda)\neq 0$, $j\in\mathbb{N}$, satisfy the condition

$$[D_t^k D^{\alpha}_{\eta,\lambda}a_{(\mu-j)}(y,t,\eta,\tau,\lambda)-(-1)^{\mu-j}a_{(\mu-j)}(y,t,-\eta,-\tau,-\lambda)]|_{t=0,(\eta,\lambda)=0}=0$$

for all $y\in\mathbb{R}^{n-1}$, $\tau\in\mathbb{R}\backslash\{0\}$, for all $k\in\mathbb{N}$, $\alpha\in\mathbb{N}^{n-1+l}$ and all j. Let $L^{\mu}_{\mathrm{cl}}(M;\mathbb{R}^l)_{\mathrm{tr}}$ denote the set of all $A(\lambda)$ in the space $L^{\mu}_{\mathrm{cl}}(M;\mathbb{R}^l)$ of parameter-dependent pseudo-differential operators on M with local amplitude functions $a(y,t,\eta,\tau,\lambda)$ (near Σ) having the transmission property at $t=0$. In the following definition we denote by D any differential operator of first order on M which has the form ∂_t in a tubular neighbourhood of Σ. Moreover, let e^+ denote the operator of extension by zero from $\operatorname{int}\Xi$ to M and r^+ the restriction to $\operatorname{int}\Xi$.

Definition 3.3. The space of $\mathcal{B}^{\mu,d}(\Xi;\mathbb{R}^l)$ (pseudo-differential) parameter-dependent boundary value problems on Ξ of order $\mu\in\mathbb{Z}$ and type $d\in\mathbb{N}$ is defined to be the set of all families of block matrix operators

$$\mathcal{A}(\lambda):=\begin{pmatrix} A_+ + G & K\\ T & Q\end{pmatrix}:\begin{matrix}C^{\infty}(\Xi)\\ \oplus\\ C^{\infty}(\Sigma,\mathbb{C}^I)\end{matrix}\to\begin{matrix}C^{\infty}(\Xi)\\ \oplus\\ C^{\infty}(\Sigma,\mathbb{C}^J)\end{matrix}\tag{31}$$

of the form

$$\mathcal{A}(\lambda)=\begin{pmatrix} A_+ & 0\\ 0 & 0\end{pmatrix}+\mathcal{G}(\lambda)+\mathcal{C}(\lambda)\tag{32}$$

for $A_+(\lambda):=\mathrm{r}^+A(\lambda)\mathrm{e}^+$ with dimensions I,J depending on the operator and entries as follows:

(i) $A(\lambda)\in L^{\mu}_{\mathrm{cl}}(M;\mathbb{R}^l)_{\mathrm{tr}}$.

(ii) The operator $\mathcal{C}(\lambda)$ belongs to $\mathcal{B}^{-\infty,d}(\Xi;\mathbb{R}^l)$, i.e., there is a representation

$$\mathcal{C}=\mathcal{C}_0+\sum_{j=1}^{d}\mathcal{C}_j\begin{pmatrix} D^j & 0\\ 0 & 0\end{pmatrix}$$

for elements $\mathcal{C}_j\in\mathcal{B}^{-\infty,0}(\Xi;\mathbb{R}^l)$; here $\mathcal{B}^{-\infty,0}(\Xi;\mathbb{R}^l):=\mathcal{S}(\mathbb{R}^l;\mathcal{B}^{-\infty,0}(\Xi))$ where $\mathcal{B}^{-\infty,0}(\Xi)$ denotes the space of all operators $\mathcal{C}=(C_{ij})_{i,j=1,2}$ between the spaces as in (31), with C^∞ kernels over $\Xi \times \Xi$, $\Xi \times \Sigma$, $\Sigma \times \Xi$ and $\Sigma \times \Sigma$, respectively (with smoothness up to the boundary).

(iii) The operator $\mathcal{G}(\lambda)$ is a block matrix $(G_{ij}(\lambda))_{i,j=1,2}$ where $G_{i,j}(\lambda)$ for $i, j = 1, 2$, $i + j < 4$, are Schwartz functions in $\lambda \in \mathbb{R}^l$ with values in the smoothing operators on $\operatorname{int}\Xi \times \operatorname{int}\Xi$, $\operatorname{int}\Xi \times \Sigma$, and $\Sigma \times \operatorname{int}\Xi$, respectively, while $G_{22} \in L^\mu_{\mathrm{cl}}(\Sigma; \mathbb{R}^l)$, and $\mathcal{G}(\lambda)$ is locally near Σ in the coordinates $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ a pseudo-differential operator along $\mathbb{R}^{n-1}$ with operator-valued symbol $g(y, \eta)$, namely,

$$\mathrm{Op}(g)(\lambda)u(y) = \int\int e^{i(y-y')\eta} g(y, \eta, \lambda)u(y')\, dy'\, đ\eta,$$

$u \in C_0^\infty(\mathbb{R}^q, \mathcal{S}(\overline{\mathbb{R}}_+))$, with $g(y, \eta, \lambda)$ being a Green symbol of order μ and type d, cf. the material at the end of this section.

Remark 3.4. The space $\mathcal{B}^{\mu,d}(\Xi; \mathbb{R}^l)$ for every $\mu \in \mathbb{Z}$ and $d \in \mathbb{N}$ is Fréchet in a natural way, cf., e.g., Schrohe [19].

The operators $\mathcal{A}(\lambda) \in \mathcal{B}^{\mu,d}(\Xi; \mathbb{R}^l)$ have a parameter-dependent principal symbolic structure

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$$

consisting of the (usual) parameter-dependent principal symbol $\sigma_\psi(A)(x, \xi, \lambda)$ of the operator in the upper left corner of (32), restricted to Ξ, and the parameter-dependent principal boundary symbol $\sigma_\partial(\mathcal{A})(y, \eta, \lambda) \in C^\infty(T^*\Xi \times \mathbb{R}^l \setminus 0)$. The boundary symbol is operator-valued in the sense of a family of operators

$$\sigma_\partial(\mathcal{A})(y, \eta, \lambda) : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^I \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^J \end{matrix} \tag{33}$$

depending on $(y, \eta, \lambda) \in (T^*\Sigma \times \mathbb{R}^l)\setminus 0$ with 0 indicating $(\eta, \lambda) = 0$ (in (33) we assume $s \in \mathbb{R}$ to be sufficiently large). The definition is as follows:

$$\sigma_\partial(\mathcal{A})(y, \eta, \lambda) = \begin{pmatrix} \mathrm{r}^+\mathrm{op}(a'_{(\mu)})(y, \eta, \lambda)\mathrm{e}^+ & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\partial(\mathcal{G})(y, \eta, \lambda). \tag{34}$$

Here $a'_{(\mu)}(y, \eta, \tau, \lambda) := \sigma_\psi(\mathcal{A})(y, t, \eta, \tau, \lambda)|_{t=0}$ in local coordinates (y, t) near the boundary Σ, and $\sigma_\partial(\mathcal{G})(y, \eta, \lambda)$ is the homogeneous principal operator-valued symbol in the sense of twisted homogeneity, cf. the explicit expressions at the end of this section.

To see the structure in some important cases, consider, for example

$$\mathcal{A}(\lambda) = \begin{pmatrix} \mathrm{r}^{+}(\Delta - \lambda^2)\mathrm{e}^{+} \\ T \end{pmatrix}$$

where Δ is the Laplacian on M and T the operator that expresses Dirichlet or Neumann conditions. In the Dirichlet case we have

$$\sigma_{\partial}(T)u = u|_{t=0}$$

and in the Neumann case

$$\sigma_{\partial}(T)u = \left.\frac{\partial u}{\partial t}\right|_{t=0}$$

for $u \in H^s(\mathbb{R}_+)$, $s > \frac{3}{2}$.

In the general calculus it makes sense to unify the orders of the operators referring to the boundary. This will produce families of continuous operators

$$\mathcal{A}(\lambda) : \begin{matrix} H^s(\Xi) \\ \oplus \\ H^{s-\frac{1}{2}}(\Sigma, \mathbb{C}^I) \end{matrix} \to \begin{matrix} H^{s-\mu}(\Xi) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}(\Sigma, \mathbb{C}^J) \end{matrix} \tag{35}$$

for $s > \max(\mu, d) - \frac{1}{2}$. For 'realistic' boundary value problems this is not necessarily the case as we saw in the example, but we can compose the operators by order reductions on the boundary. To be more precise, if we have first continuity in the sense

$$\mathcal{A}(\lambda) : H^s(\Xi) \oplus \bigoplus_{i=1}^{I} H^{s-n_i-\frac{1}{2}}(\Sigma) \to H^s(\Xi) \oplus \bigoplus_{j=1}^{J} H^{s-m_j-\frac{1}{2}}(\Sigma)$$

for certain n_j, m_j, then we can pass to the operator

$$\mathrm{diag}(1, (R^{\mu_j}(\lambda))_{j=1,\dots,J})\mathcal{A}(\lambda)\,\mathrm{diag}(1, (R^{\nu_i}(\lambda))_{i=1,\dots,I}), \tag{36}$$

$\nu_i = n_j$, $\mu_j = \mu - m_j$ with parameter-dependent elliptic operators $R^{\nu}(\lambda) \in L^{\nu}_{\mathrm{cl}}(\Sigma; \mathbb{R}^l)$ of the corresponding order ν which induce isomorphisms

$$R^{\nu}(\lambda) : H^s(\Sigma) \to H^{s-\nu}(\Sigma)$$

for all $s \in \mathbb{R}$, $\lambda \in \mathbb{R}^l$.

Nevertheless in the application of the calculus below we employ the realistic orders from the concrete problems.

The class of operators $\mathcal{B}^{\mu,d}(\Xi;\mathbb{R}^l)$ is nothing other than Boutet de Monvel's space of (pseudo-differential) boundary value problems of order μ (in the sense (35)) and type d, with parameters $\lambda \in \mathbb{R}^l$. More details may be found in [2,9].

As noted before the calculus may also be formulated in the context of systems in the upper left corners. Moreover, instead of the space $\mathbb{C}^I$, $\mathbb{C}^J$ we could also speak about vector bundles and distributional sections in those bundles. However, this is not necessary for our application.

Let us now formulate the related spaces $\mathcal{B}^{\mu,d}(\Xi_+,\Xi_-;\mathbb{R}^l)$ of transmission operators (with parameters $\lambda \in \mathbb{R}^l$, $l \in \mathbb{N}$; in the case $l = 0$ we simply write $\mathcal{B}^{\mu,d}(\Xi_+,\Xi_-)$)

$$\mathcal{A}(\lambda) = \begin{pmatrix} A_+(\lambda) & G_+(\lambda) & K_+(\lambda) \\ G_-(\lambda) & A_-(\lambda) & K_-(\lambda) \\ T_+(\lambda) & T_-(\lambda) & Q(\lambda) \end{pmatrix} : \begin{matrix} H^s(\Xi_+) \\ \oplus \\ H^s(\Xi_-) \\ \oplus \\ H^{s-\frac{1}{2}}(\Sigma,\mathbb{C}^I) \end{matrix} \to \begin{matrix} H^{s-\mu}(\Xi_+) \\ \oplus \\ H^{s-\mu}(\Xi_-) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}(\Sigma,\mathbb{C}^J) \end{matrix}. \tag{37}$$

First, the operator Q is an $J \times I$ matrix of elements of $L^{\mu}_{\mathrm{cl}}(\Sigma;\mathbb{R}^l)$. Moreover, the submatrices $\begin{pmatrix} A_\pm & K_\pm \\ T_\pm & 0 \end{pmatrix}$ consist of operators in $\mathcal{B}^{\mu,d}(\Xi_\pm;\mathbb{R}^l)$ as defined at the end of this section. The most typical contribution are the Green transmission operators $G_\pm$ of order μ and type d. Let us define G_+; the minus case is similar. As before we fix a first-order differential operator D which is equal to ∂_t in a tubular neighbourhood $V \cong \Sigma \times (-1,1) \ni (y,t)$ of Σ. Let $(U_j)_{j=1,\dots,L}$ be coordinate neighbourhoods on M such that $U_j \cap \Sigma \neq \emptyset$ for all j and $\Sigma \subset \bigcup U_j$. Choose functions $\varphi_j,\ \psi_j \in C_0^\infty(U_j)$, $j = 1,\dots,L$ such that $\psi_j \equiv 1$ on $\operatorname{supp}\varphi_j$ for all j and $\sum_{j=1}^L \varphi_j = 1$ in a neighbourhood of Σ. Then G_+ has the form

$$G_+ = \sum_{j=1}^{L} \varphi_j G_{+,j} \psi_j + C_+, \tag{38}$$

where the ingredients are as follows:

$$C_+(\lambda)u = \sum_{k=0}^{d} C_{+,k}(\lambda) D^k u,$$

where

$$C_{+,k}(\lambda)v(x) = \int_{\Xi_-} c_{+,k}(x,\tilde{x},\lambda)v(\tilde{x})\,d\tilde{x}$$

for some kernel $c_{+,k}(x, \tilde{x}, \lambda) \in \mathcal{S}(\mathbb{R}^l, C^\infty(\Xi_+ \times \Xi_-))$. Moreover, the operators, $G_{+,j}(\lambda)$ are defined as

$$G_{+,j}(\lambda) = \sum_{k=0}^{d} G_{+,jk}(\lambda) D^k, \tag{39}$$

where $G_{+,jk}(\lambda)$ is the operator pull back of a Green transmission operator of type 0 in $\mathbb{R}^n$ with respect to the interface $\mathbb{R}^{n-1}$ under a chart $\chi_j : U_j \to \mathbb{R}^n$ which induces a chart $\chi'_j : U_j \cap \Sigma \to \mathbb{R}^{n-1}$ and $\chi_j(U_j \cap \Xi_\pm) = \overline{\mathbb{R}}^n_\pm$. It remains to explain the latter notion. For simplicity we also denote the variables in $\mathbb{R}^n$ by (y, t), where $y \in \mathbb{R}^{n-1}$, and $\overline{\mathbb{R}}^n_\pm$ is the closed half-space for $t \geqslant 0$ and $t \leqslant 0$, respectively. For the definition we need a construction from the general (edge-) analysis, namely, operator-valued symbols with twisted homogeneity.

Let H be a Hilbert space in which we have fixed a strongly continuous group $\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ of isomorphisms

$$\kappa_\delta : H \to H, \quad \delta \in \mathbb{R}_+$$

(that is, $\kappa_\delta h \in C(\mathbb{R}_+, H)$ for every $h \in H$).

In our application we have $H = H^s(\mathbb{R}_+)$, $s \in \mathbb{R}$, with $(\kappa_\delta u)(t) = \delta^{\frac{1}{2}} u(\delta t)$, $\delta \in \mathbb{R}_+$. In particular, for $s = 0$ the operators $\kappa_\delta : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ are unitary. Moreover, if $E \subset H$ is a Fréchet subspace, written as a projective limit $E = \varprojlim_{k \in \mathbb{N}} E^k$ of Hilbert spaces E^k with continuous embeddings $\ldots \hookrightarrow E^{k+1} \hookrightarrow E^k \hookrightarrow \ldots \hookrightarrow E^0 = H$, such that $\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ induces by restriction a strongly continuous group of isomorphisms on E^k for every k, we talk about an action of $\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ on E. An example for the latter situation is $H = L^2(\mathbb{R}_+)$, $E = \mathcal{S}(\overline{\mathbb{R}}_+)$ $(= \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+})$, with $E^k := \langle t \rangle^{-k} H^k(\mathbb{R}_+)$, $k \in \mathbb{N}$.

Now, if H and $\widetilde{H}$ are Hilbert spaces with groups $\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ and $\{\tilde{\kappa}_\delta\}_{\delta \in \mathbb{R}_+}$, respectively, we have the space of symbols

$$S^\mu(\mathbb{R}^q \times \mathbb{R}^p; H, \widetilde{H})$$

of all $a(y, \eta) \in C^\infty(\mathbb{R}^q \times \mathbb{R}^p; \mathcal{L}(H, \widetilde{H}))$ such that

$$\sup_{y \in \mathbb{R}^q, \eta \in \mathbb{R}^p} \langle \eta \rangle^{-\mu + |\beta|} ||\tilde{\kappa}^{-1}_{\langle \eta \rangle} \{D^\alpha_y D^\beta_\eta a(y, \eta)\} \kappa_{\langle \eta \rangle}||_{\mathcal{L}(H, \widetilde{H})} < \infty$$

for all multi-indices $\alpha \in \mathbb{N}^q$, $\beta \in \mathbb{N}^p$; $\langle \eta \rangle = (1 + |\eta|^2)^{\frac{1}{2}}$. The subspace $S^\mu_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^p; H, \widetilde{H})$ of classical symbols $a(y, \eta)$ is defined by the condition that there is a sequence of homogeneous components $a_{(\mu - j)}(y, \eta) \in C^\infty(\mathbb{R}^q \times (\mathbb{R}^p \setminus \{0\}), \mathcal{L}(H, \widetilde{H}))$

in the sense

$$a_{(\mu-j)}(y,\delta\eta)=\delta^{\mu-j}\tilde{\kappa}_\delta a_{(\mu-j)}(y,\eta)\kappa_\delta^{-1}$$

for all $\delta\in\mathbb{R}_+$, $(y,\eta)\in\mathbb{R}^q\times(\mathbb{R}^p\setminus\{0\})$, such that

$$a(y,\eta)-\chi(\eta)\sum_{j=0}^{N}a_{(\mu-j)}(y,\eta)\in S^{\mu-(N+1)}(\mathbb{R}^q\times\mathbb{R}^p;H,\widetilde{H})$$

for all $N\in\mathbb{N}$. This easily extends to the case of Fréchet spaces endowed with groups as mentioned before, in particular, when H is a Hilbert space and $\widetilde{E}\subset\widetilde{H}$ a Fréchet space, $\widetilde{E}=\varprojlim_{k\in\mathbb{N}}\widetilde{E}^k$. In the latter case the space $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q\times\mathbb{R}^p;H,\widetilde{E})$ is defined as the intersection over k of the spaces referring to H and $\widetilde{E}^k$. The notation '(cl)' means that we admit both the classical and the general case. In the classical case we set

$$\sigma_\partial(a)(y,\eta):=a_{(\mu)}(y,\eta).$$

Let us apply this concept for the case $H=L^2(\mathbb{R}_-)$, $\widetilde{E}=\mathcal{S}(\overline{\mathbb{R}}_+)$ or $\widetilde{H}=L^2(\mathbb{R}_+)$, $E=\mathcal{S}(\overline{\mathbb{R}}_-)$. An element $g(y,\eta)\in C^\infty(\mathbb{R}^p\times\mathbb{R}^q;\mathcal{L}(L^2(\mathbb{R}_-),L^2(\mathbb{R}_+)))$ is called a Green transmission symbol (from the minus to the plus side) of order $\mu\in\mathbb{R}$ and type 0 if it has the properties

$$g(y,\eta)\in S^\mu_{\mathrm{cl}}(\mathbb{R}^q\times\mathbb{R}^p;L^2(\mathbb{R}_-),\mathcal{S}(\overline{\mathbb{R}}_+)),\quad g^*(y,\eta)\in S^\mu_{\mathrm{cl}}(\mathbb{R}^q\times\mathbb{R}^p;L^2(\mathbb{R}_+),\mathcal{S}(\overline{\mathbb{R}}_-)).$$

Here g^* means the pointwise adjoint of g as an operator $L^2(\mathbb{R}_-)\to L^2(\mathbb{R}_+)$. An operator function $g(y,\eta):L^2(\mathbb{R}_-)\to L^2(\mathbb{R}_+)$ is said to be a Green transmission symbol (from the minus to the plus side) of order $\mu\in\mathbb{R}$ and type $d\in\mathbb{N}$ if

$$g(y,\eta)=\sum_{k=0}^{d}g_k(y,\eta)\frac{\partial^k}{\partial t^k}$$

for Green transmission symbols g_k of order $\mu-k$ and type 0.

In the present parameter-dependent situation we now replace η by $(\eta,\lambda)\in\mathbb{R}^{n-1+l}$ and obtain in this way parameter-dependent Green transmission symbols. A local Green transmission operator $G(\lambda)$ of order μ and type d is nothing other than a corresponding pseudo-differential operator

$$G(\lambda)u(y):=\mathrm{Op}_y(g)(\lambda)u(y)=\int\int e^{i(y-y')\eta}g(y,\eta,\lambda)u(y')\,dy'\,đ\eta$$

for a Green transmission symbol $g(y,\eta,\lambda)$, $u(y)\in C_0^\infty(\mathbb{R}^{n-1},L^2(\mathbb{R}_-))$.

Now the operators $G_{+,jk}(\lambda)$ in the formula (39) are Green transmission operators of order $\mu - k$ and type 0. This completes the definition of (38).

It remains to define the Green, trace and potential operators in boundary value problems of Boutet de Monvel's class. As in the beginning of this section we consider the manifold Ξ with boundary Σ. Let $(y, t) \in \mathbb{R}^{n-1} \times \overline{\mathbb{R}}_+$ be local coordinates on Ξ near Σ. A (parameter-dependent) Green symbol $g(y, \eta, \lambda)$ is defined by the properties

$$g(y, \eta, \lambda) = \operatorname{diag}(1, \langle \eta, \lambda \rangle^{\frac{1}{2}}) \tilde{g}(y, \eta, \lambda) \operatorname{diag}(1, \langle \eta, \lambda \rangle^{-\frac{1}{2}})$$

for a $\tilde{g}(y, \eta, \lambda)$ such that

$$\tilde{g}(y, \eta, \lambda) \in S^{\nu}_{\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1+l}; L^2(\mathbb{R}_+) \oplus \mathbb{C}^I, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^J),$$

$$\tilde{g}^*(y, \eta, \lambda) \in S^{\nu}_{\mathrm{cl}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1+l}; L^2(\mathbb{R}_+) \oplus \mathbb{C}^J, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^I).$$

Writing g as a block matrix $g = (g_{ij})_{i,j=1,2}$ we also call g_{21} a trace symbol of order $\mu + \frac{1}{2}$ and type 0 and g_{12} a potential symbol of order $\mu - \frac{1}{2}$. Despite of this order convention we call g a Green symbol of order μ and type 0. Note that the definition implies that the lower right corner is a $J \times I$ matrix of classical (parameter-dependent) symbols of order μ.

A Green symbols $g(y, \eta, \lambda)$ of order μ and type $d \in \mathbb{N}$ is defined by

$$g(y, \eta, \lambda) = g_0(y, \eta, \lambda) + \sum_{k=1}^{d} g_k(y, \eta, \lambda) \operatorname{diag}\left(\frac{\partial^k}{\partial t^k}, 0 \right) \tag{40}$$

for Green symbols g_k of order $\mu - k$ and type 0. From the definition of classical symbols we see that (parameter-dependent) Green symbols $g(y, \eta, \lambda)$ have homogeneous components in $(\eta, \lambda) \neq 0$ of order $\mu - j$, $j \in \mathbb{N}$. More precisely, the diagonal entries are homogeneous of order $\mu - j$, while the trace and potential entries are homogeneous of order $\mu - j + \frac{1}{2}$ and $\mu - j - \frac{1}{2}$, respectively. In particular, there is a homogeneous principal symbol

$$g_{(\mu)}(y, \eta, \lambda) \in C^{\infty}(\mathbb{R}^{n-1} \times (\mathbb{R}^{n-1+l} \setminus \{0\}), \mathcal{L}(L^2(\mathbb{R}_+) \oplus \mathbb{C}^I, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^J))$$

in that sense, such that $g(y, \eta, \lambda) - \chi(\eta, \lambda) g_{(\mu)}(y, \eta, \lambda)$ is of order $\mu - 1$. If $\mathcal{G}(\lambda) := \mathrm{Op}(g)(\lambda)$ is the associated pseudo-differential operator we set

$$\sigma_{\partial}(\mathcal{G})(y, \eta, \lambda) = g_{(\mu)}(y, \eta, \lambda).$$

Observe that then

$$\sigma_\partial(g)(y,\delta\eta,\delta\lambda)=\delta^\mu\begin{pmatrix}\kappa_\delta & 0\\ 0 & \delta^{\frac12}\end{pmatrix}\sigma_\partial(g)(y,\eta,\lambda)\begin{pmatrix}\kappa_\delta^{-1} & 0\\ 0 & \delta^{-\frac12}\end{pmatrix}.$$

We now complete Definition 3.3(iii) as follows: The operator $\mathcal{G}(\lambda)$ in (32) has the form

$$\mathcal{G}(\lambda)=\sum_{j=1}^{L}\mathcal{M}_{\varphi_j}\mathcal{G}_j(\lambda)\mathcal{M}_{\psi_j}.$$

Here $(\phi_j)_{j=1,\dots,L}$, $(\psi_j)_{j=1,\dots,L}$ are functions in $C_0^\infty(U_j)$ for coordinate neighbourhoods $(U_j)_{j=1,\dots,L}$ on Ξ near Σ, $U_j\cap\Sigma\neq 0$, $\Sigma\subset\bigcup_{j=1}^L U_j$, such that $\sum_{j=1}^L\varphi_j=1$ in a neighbourhood of Σ, $\psi_j\equiv 1$ on $\operatorname{supp}\varphi_j$ for all j and $\mathcal{M}_{\varphi_j}$ and $\mathcal{M}_{\psi_j}$ are operators of multiplication by $\operatorname{diag}(\varphi_j,\varphi_j|_\Sigma)$ and $\operatorname{diag}(\psi_j,\psi_j|_\Sigma)$, respectively. Moreover, $\mathcal{G}_j(\lambda)$ is the operator pull back under a chart $U_j\to\mathbb{R}^{n-1}\times\mathbb{R}_+$ of an operator $\operatorname{Op}_y(g_j)(\lambda)$ for some Green symbol $g_j(y,\eta,\lambda)$ of the form (40). The elements $\mathcal{G}(\lambda)$ have an invariantly defined homogeneous principal boundary symbol $\sigma_\partial(\mathcal{G})(y,\eta,\lambda)$ for $(y,\eta,\lambda)\in(T^*\Sigma\times\mathbb{R}^l)\backslash 0$. By this we have completed (34).

The space $\mathcal{B}^{\mu,d}(\Xi;\mathbb{R}^l)$ is nothing other than the set of all (pseudo-differential) boundary value problems of order $\mu\in\mathbb{Z}$ and type $d\in\mathbb{N}$ of Boutet de Monvel's calculus, here in parameter-dependent form, with the parameter $\lambda\in\mathbb{R}^l$. We also admit the case $l=0$; then we have such operators in the form from [2]. Every $\mathcal{A}(\lambda)\in\mathcal{B}^{\mu,d}(\Xi;\mathbb{R}^l)$ extends from (31) to a family of continuous operators (35) for every real $s>d-\frac12$ and $\lambda\in\mathbb{R}_+$. This is a known result from the general calculus. Let us also mention the composition property which means that

$$\mathcal{C}(\lambda)\in\mathcal{B}^{\mu,d}(\Xi;\mathbb{R}^l),\quad \mathcal{D}(\lambda)\in\mathcal{B}^{\nu,e}(\Xi;\mathbb{R}^l)$$

entails $\mathcal{C}(\lambda)\mathcal{D}(\lambda)\in\mathcal{B}^{\mu+\nu,h}(\Xi;\mathbb{R}^l)$ for $h=\max(\nu+d,e)$, where $\sigma(\mathcal{CD})=\sigma(\mathcal{C})\sigma(\mathcal{D})$ with componentwise composition. Clearly we assume that rows and columns in the middle fit together.

An element $\mathcal{A}(\lambda)\in\mathcal{B}^{\mu,d}(\Xi;\mathbb{R}^l)$ is said to be parameter-dependent elliptic (of order μ) if $\sigma_\psi(\mathcal{A})$ does not vanish on $(T^*\Xi\times\mathbb{R}^l)\backslash 0$ and if the operators (33) are bijective for all $(y,\eta,\lambda)\in(T^*\Sigma\times\mathbb{R}^l)\backslash 0$.

Let us now recall the following result. Set $\nu^+:=\max(\nu,0)$ for any $\nu\in\mathbb{R}$.

Theorem 3.5. *Every elliptic* $\mathcal{A}\in\mathcal{B}^{\mu,d}(\Xi;\mathbb{R}^l)$ *has a parametrix* $\mathcal{P}\in\mathcal{B}^{-\mu,(d-\mu)^+}(\Xi;\mathbb{R}^l)$ *in the sense that*

$$\mathcal{I}-\mathcal{PA}=:\mathcal{K}_l\quad\textit{and}\quad\mathcal{I}-\mathcal{AP}=:\mathcal{K}_r$$

belong to $\mathcal{B}^{-\infty,\max(d,\mu)}(\Xi;\mathbb{R}^l)$ *and* $\mathcal{B}^{-\infty,(d-\mu)^+}(\Xi;\mathbb{R}^l)$, *respectively* ($\mathcal{I}$ *is the identity operator in the corresponding spaces*).

Results of that kind may be found in [2,6], or [17] under different assumptions on the types. The present version with arbitrary types may be found in [9].

Remark 3.6. The definitions and results on $\mathcal{B}^{\mu,d}(\Xi;\mathbb{R}^l)$ have a straightforward extension to the case of a non-compact C^∞ manifold Ξ with boundary Σ. The principal symbolic structure is as before, while the continuity refers to 'comp' and 'loc' versions of Sobolev spaces. There is also an analogue of the composition property between operators combined with a localisation in the middle and of Theorem 3.5.

Remark 3.7. The construction of the space of (parameter-dependent) transmission operators $\mathcal{B}^{\mu,d}(\Xi_+,\Xi_-;\mathbb{R}^l)$ refers to the sphere S^n, subdivided into Ξ_+,Ξ_-. A slight modification allows us to define such spaces of transmission operators on our domain $\Omega\backslash\{0\}$ in $\mathbb{R}^3$ with respect to the decomposition

$$\Omega\backslash\{0\}=\{(\Omega\backslash\{0\})\cap X_+\}\cup\{(\Omega\backslash\{0\})\cap X_-\}.$$

The interface then consists of $S\backslash\{0\}$ and is non-compact. In other words there is a straightforward definition of corresponding spaces $\mathcal{B}^{\mu,d}(X_+\backslash\{0\},X_-\backslash\{0\})$ of transmission operators (which we need here without parameters) where the former S^n is replaced by Ω, furthermore, $\Xi_\pm$ by $X_\pm\backslash\{0\}$, and Σ by $S\backslash\{0\}$.

In other words in the definition of $\mathcal{B}^{\mu,d}(X_+\backslash\{0\},X_-\backslash\{0\})$ we ignore the presence of $\partial\Omega$ as well as of the origin, just as in the analogous case of boundary value problems on a (not necessarily compact) C^∞ manifold with boundary. The global smoothing operators are defined in terms of kernels in a similar manner as in the compact case. The non-smoothing ingredients are defined by local expressions and a partition of unity. The continuity then refers to 'comp' and 'loc'-analogues of Sobolev spaces.

Let us now complete the information on parameter-dependent transmission operators $\mathcal{A}\in\mathcal{B}^{\mu,d}(\Xi_+,\Xi_-;\mathbb{R}^l)$ by formulating the principal symbolic structure and basic statements about ellipticity. The parameter-dependent homogeneous principal symbol is defined as a pair

$$\sigma(\mathcal{A}):=(\sigma_\psi(\mathcal{A}),\sigma_{\mathrm{tr}}(\mathcal{A}))$$

with interior and transmission symbols $\sigma_\psi(\mathcal{A})$ and $\sigma_{\mathrm{tr}}(\mathcal{A})$, respectively. If $\mathcal{A}$ is written in the form (37) we set $\sigma_\psi(\mathcal{A})=(\sigma_\psi(A_+),\sigma_\psi(A_-))$. In order to define $\sigma_{\mathrm{tr}}(\mathcal{A})$ we proceed in a similar manner as for (11). Analogously as (8) we consider a localised operator

$$\mathcal{A}_{V_+}=\begin{pmatrix} A_+|_{\mathrm{int}\,V_+} & (G_+|_{\mathrm{int}\,V_-})\varepsilon^* & K_+ \\ (\varepsilon^*)^{-1}G_-|_{\mathrm{int}\,V_+} & (\varepsilon^*)^{-1}(A_-|_{\mathrm{int}\,V_-})\varepsilon^* & (\varepsilon^*)^{-1}K_- \\ T_+|_{\mathrm{int}\,V_+} & (T_-|_{\mathrm{int}\,V_-})\varepsilon^* & Q \end{pmatrix},$$

which obviously belongs to $\mathcal{B}^{\mu,d}(V_+;\mathbb{R}^l)$, cf. Remark 3.6. We then have the boundary symbol $\sigma_\partial(\mathcal{A}_{V_+})(y,\eta,\lambda)$. By transforming back the entries in the second row and column via the reflection map $\varepsilon : \mathbb{R}_- \to \mathbb{R}_+$ we obtain the principal transmission symbol of $\mathcal{A}$ itself, namely

$$\sigma_{\mathrm{tr}}(\mathcal{A})(y,\eta,\lambda) : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ H^s(\mathbb{R}_-) \\ \oplus \\ \mathbb{C}^I \end{array} \to \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ H^{s-\mu}(\mathbb{R}_-) \\ \oplus \\ \mathbb{C}^J \end{array}, \tag{41}$$

$(y,\eta,\lambda) \in (T^*\Sigma \times \mathbb{R}^l)\setminus 0$.

Proposition 3.8. *Let $\mathcal{C} \in \mathcal{B}^{\mu,d}(\Xi_+,\Xi_-;\mathbb{R}^l)$, $\mathcal{D} \in \mathcal{B}^{\nu,e}(\Xi_+,\Xi_-;\mathbb{R}^l)$, and assume that weight data and dimensions in the spaces that are of $\mathcal{D}$ fit to corresponding data in the domain of $\mathcal{C}$. Then $\mathcal{CD} \in \mathcal{B}^{\mu+\nu,h}(\Xi_+,\Xi_-;\mathbb{R}^l)$ for $h = \max(\nu+d,e)$, and we have $\sigma(\mathcal{CD}) = \sigma(\mathcal{C})\sigma(\mathcal{D})$ with componentwise composition.*

A transmission operator (37) is called parameter-dependent elliptic if $\sigma(\mathcal{A}_\pm)$ are elliptic as usual and if (41) is a family of isomorphisms for every $s > \max(\mu,d) - \frac{1}{2}$.

Theorem 3.9. *Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(\Xi_+,\Xi_-;\mathbb{R}^l)$ be elliptic. Then there is a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu,(d-\mu)^+}(\Xi_+,\Xi_-;\mathbb{R}^l)$ in the sense that*

$$\mathcal{I} - \mathcal{PA} =: \mathcal{K}_l \quad \textit{and} \quad \mathcal{I} - \mathcal{AP} =: \mathcal{K}_r$$

belong to $\mathcal{B}^{-\infty,\max(d,\mu)}(\Xi_+,\Xi_-;\mathbb{R}^l)$ and $\mathcal{B}^{-\infty,(d-\mu)^+}(\Xi_+,\Xi_-;\mathbb{R}^l)$, respectively.

Proposition 3.8 as well as Theorem 3.9 can be proved in a similar manner as the corresponding results in boundary value problems.

3.3. Mellin operators with transmission operator-valued symbols

We now formulate a crucial contribution to the pseudo-differential analogue of boundary-contact operators in Ω near the conical singularity of the interface S (the origin in $\mathbb{R}^{n+1}$). As noted in the beginning the original (differential) boundary-contact operators have the form (15) with a $\mathcal{B}^{\mu,d}(\Xi_+,\Xi_-)$-valued symbol (16) (in this notation μ represents the tuple of all involved orders and $d = \max\{m_l + 1 : l = 1,\dots,N\}$). The entries of h are holomorphic in z and behave like parameter-dependent transmission operators on (Ξ_+,Ξ_-) with parameters $\operatorname{Im} z$ on every line Γ_β, $\beta \in \mathbb{R}$.

For the general pseudo-differential scenario we employ unified orders on the interface (for concrete operators a simple reduction of orders allows us to return to the original orders). In the following construction we use the fact that $\mathcal{B}^{\mu,d}(\Xi_+,\Xi_-)$ is a Fréchet space in a natural way, cf., analogously, Remark 3.4. Thus we can talk about $\mathcal{A}(\mathbb{C}, \mathcal{B}^{\mu,d}(\Xi_+, \Xi_-))$, the space of entire functions in $z \in \mathbb{C}$ with values in $\mathcal{B}^{\mu,d}(\Xi_+, \Xi_-)$. The symbols of Mellin operators contain $\operatorname{Im} z$ as a parameter, as we saw in the case of differential boundary-contact operators. This is typical also in the general case, although there are not only holomorphic but also meromorphic ingredients. In the present section we concentrate on the holomorphic part. Let $\mathcal{M}_{\mathcal{O}}^{\mu,d}(\Xi_+, \Xi_-)$ denote the subspace of all $h(z) \in \mathcal{A}(\mathbb{C}, \mathcal{B}^{\mu,d}(\Xi_+, \Xi_-))$ such that $h(\beta + i\varrho) \in \mathcal{B}^{\mu,d}(\Xi_+, \Xi_-; \mathbb{R}^l)$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals. For the parametrix construction below we employ the following Mellin quantisation result:

Theorem 3.10. *For every $f(z) \in \mathcal{B}^{\mu,d}(\Xi_+, \Xi_-; \Gamma_\beta)$ for any fixed $\beta \in \mathbb{R}$ (where $\operatorname{Im} z$ is interpreted as the parameter for $z \in \Gamma_\beta$) there exists an $h(z) \in \mathcal{M}_{\mathcal{O}}^{\mu,d}(\Xi_+, \Xi_-)$ such that*

$$h(z)|_{\Gamma_\beta} - f(z) \in \mathcal{B}^{-\infty,d}(\Xi_+, \Xi_-; \Gamma_\beta)$$

and h is unique $\operatorname{mod} \mathcal{M}_{\mathcal{O}}^{-\infty,d}(\Xi_+, \Xi_-)$.

An analogous result holds for $f(r,z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{\mu,d}(\Xi_-, \Xi_+; \Gamma_\beta))$ with a corresponding $h(r,z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{\mu,d}(\Xi_+, \Xi_-))$.

The proof of Theorem 3.10 is based on a kernel cut-off construction with respect to the parameter $z \in \Gamma_\beta$ as is mentioned in a similar situation, for instance, in [9, Section 1.3.2]. Since the ideas are completely analogous, we omit the details. In Section 1.2 we defined Mellin operators $\operatorname{op}_M^\beta(h)$ for scalar amplitude functions $h(r, r', z)$ given for $z \in \Gamma_{\frac{1}{2}-\beta}$. In a similar manner we proceed in the operator-valued case. For $h(r,z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{\mu,d}(\Xi_+, \Xi_-))$ we have operators

$$r^{-\mu} \operatorname{op}_M^{\gamma-\frac{n}{2}}(h) : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s,\gamma}(\mathbb{X}_-) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{S}, \mathbb{C}^I) \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{X}_-) \\ \oplus \\ \mathcal{H}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(\mathbb{S}, \mathbb{C}^J) \end{array}$$

which are continuous for all $s > d - \frac{1}{2}$; because of the holomorphy of h in z the weight γ is arbitrary. Observe that $\operatorname{op}_M^{\gamma-\frac{n}{2}}(h) \in \mathcal{B}^{\mu,d}(\Xi_+^\wedge, \Xi_-^\wedge)$, cf. Remark 3.7. Operators of the kind $\omega r^{-\mu} \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega}$ for cut-off functions $\omega, \tilde{\omega}$ will also be interpreted as operators between Sobolev spaces defined on $\mathbb{X}_\pm$ and $\mathbb{S}$, respectively (when the support of $\omega, \tilde{\omega}$ is sufficiently close to 0). Let us set $\sigma_M(\omega r^{-\mu} \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega}) := h(0,z)$ interpreted as z-dependent operators in $\mathcal{B}^{\mu,d}(\Xi_+, \Xi_-)$, z varying on the weight line $\Gamma_{\frac{n+1}{2}-\gamma}$.

3.4. Smoothing operators with asymptotic data

The parametrices of elliptic boundary-contact problems (localised near the conical point 0) will belong to a kind of transmission cone algebra, containing operators with smoothing meromorphic Mellin symbols. In fact, if we start from an elliptic operator with holomorphic Mellin symbols (say, with constant coefficients in r) $h(z)$, we have to pass to $h^{-1}(z)$ which is, in general, a meromorphic operator function. Applying Theorem 3.10 to $h^{-1}(z)|_{\Gamma_\beta}$ for some β such that Γ_β does not contain a pole of $h^{-1}(z)$ we obtain an element $k(z) \in \mathcal{B}^{-\mu,(d-\mu)^+}(\Xi_+, \Xi_-; \mathbb{C})$, and $l(z) := k(z) - h^{-1}(z)$, first given on Γ_β and of order $-\infty$ has an extension to a meromorphic Mellin symbol of the following class.

Definition 3.11. Let $R = \{(r_j, n_j, N_j)\}_{j\in\mathbb{Z}}$ be a sequence of triples with $(r_j, n_j) \in \mathbb{C}\times\mathbb{N}$, $|\mathrm{Re}\, r_j| \to \infty$ as $|j| \to \infty$ and finite-dimensional spaces N_j of operators of finite rank belonging to $\mathcal{B}^{-\infty,d}(\Xi_+, \Xi_-)$ for some given $d \in \mathbb{N}$. Then $\mathcal{M}_R^{-\infty,d}(\Xi_+, \Xi_-)$ denotes the space of all elements $f \in \mathcal{A}(\mathbb{C}\backslash\pi_{\mathbb{C}} R, \mathcal{B}^{-\infty,d}(\Xi_+, \Xi_-))$ for $\pi_{\mathbb{C}} R := \bigcup_{j\in\mathbb{Z}}\{r_j\}$ with the following properties:

(i) f extends to a meromorphic function with poles at r_j of multiplicities $n_j + 1$ and Laurent coefficients at $(z - r_j)^{-(k+1)}$ in N_j for all $0 \leqslant k \leqslant m_j$, $j \in \mathbb{Z}$;
(ii) if $\chi(z) \in C^\infty(\mathbb{C})$ is any function with $\chi(z) = 0$ for $\mathrm{dist}(z, \pi_{\mathbb{C}} R) < c_0$, $\chi(z) = 1$ for $\mathrm{dist}(z, \pi_{\mathbb{C}} R) > c_1$ for certain $0 < c_0 < c_1$ we have

$$\chi(z) f(z)|_{\Gamma_\beta} \in \mathcal{B}^{-\infty,d}(\Xi_+, \Xi_-; \Gamma_\beta)$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.

Every $f \in \mathcal{M}_R^{-\infty,d}(\Xi_+, \Xi_-)$ (given together with dimensions I, J) induces continuous operators

$$\omega r^{-\nu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega} : \begin{array}{c} \mathcal{H}^{s,\gamma}_{(P_+)}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s,\gamma}_{(P_-)}(\mathbb{X}_-) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{(S)}(\mathbb{S}, \mathbb{C}^I) \end{array} \to \begin{array}{c} \mathcal{H}^{\infty,\gamma-\nu}_{(Q_+)}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{\infty,\gamma-\nu}_{(Q_-)}(\mathbb{X}_-) \\ \oplus \\ \mathcal{H}^{\infty,\gamma-\nu-\frac{1}{2}}_{(T)}(\mathbb{S}, \mathbb{C}^J) \end{array} \tag{42}$$

for arbitrary $s > d - \frac{1}{2}$ and $\nu \in \mathbb{R}$; here ω, $\tilde{\omega}$ are cut-off functions supported in a sufficiently small neighbourhood of 0. The subscripts $(P_\pm)$, (S), etc. indicate spaces with (or without) the corresponding asymptotic types. The weight γ in (42) is chosen in such a way that $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$. The transformation of asymptotic types $(P_+, P_-, S) \to (Q_+, Q_-, T)$ comes from the application of the meromorphic operator function f (with poles and Laurent expansions encoded by R) to the meromorphic function $M(\tilde{\omega}u)$ (having poles and Laurent expansions according to (P_+, P_-, S)).

Let $\mathcal{M}_{\mathrm{as}}^{-\infty,d}(\Xi_+,\Xi_-)$ denote the union of spaces $\mathcal{M}_R^{-\infty,d}(\Xi_+,\Xi_-)$ over all R as mentioned in Definition 3.11. Fix cut-off functions ω, $\tilde{\omega}$ and consider operators of the form

$$\mathcal{M} := \omega r^{-\mu} \sum_{j=0}^{k} r^j \, \mathrm{op}_M^{\gamma_j - \frac{n}{2}}(f_j)\tilde{\omega} \tag{43}$$

for $f_j \in \mathcal{M}_{\mathrm{as}}^{-\infty,d}(\Xi_+,\Xi_-)$ with weight $\gamma_j \in \mathbb{R}$ such that f_j is holomorphic in a strip around $\Gamma_{\frac{n+1}{2}-\gamma_j}$ (such γ_j may always be found, because the distribution of poles is discrete), and we assume

$$\gamma - j \leqslant \gamma_j \leqslant \gamma \quad \text{for all} \quad j = 0, \ldots, k. \tag{44}$$

Then $\mathcal{M}$ induces continuous operators between the spaces as in (42) for $\mu = \nu$. The asymptotic types in this case refer to the weight interval $\Theta = [0, k+1)$, $k \in \mathbb{N}$.

Given an operator $\mathcal{M}$ of the kind (43) we set

$$\sigma_M(\mathcal{M})(z) := f_0(z), \quad z \in \Gamma_{\frac{n+1}{2}-\gamma}. \tag{45}$$

The operators (43) belong to the structure of parametrices of elliptic boundary-contact problems and contribute to the asymptotic properties of solutions, as we shall see below. The choice of the cut-off functions ω, $\tilde{\omega}$ as well as of the weights γ_j (under the condition (44)) only affects such operators modulo so-called Green operators (of type d). These form another important class of smoothing operators, the so-called Green operators of type d. Such an operator induces (by definition) continuous maps

$$\mathcal{G} : \begin{array}{c} \mathcal{H}^{s,\gamma}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s,\gamma}(\mathbb{X}_-) \\ \oplus \\ \mathcal{H}^{\tilde{s}-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{S}, \mathbb{C}^I) \\ \oplus \\ H^{s'-\frac{1}{2}}(\partial\Omega, \mathbb{C}^{I'}) \end{array} \to \begin{array}{c} \mathcal{H}_{Q_+}^{\infty,\gamma-\mu}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}_{Q_-}^{\infty,\gamma-\mu}(\mathbb{X}_-) \\ \oplus \\ \mathcal{H}_T^{\infty,\gamma-\mu-\frac{1}{2}}(\mathbb{S}, \mathbb{C}^J) \\ \oplus \\ H^{\infty}(\partial\Omega, \mathbb{C}^{J'}) \end{array} \tag{46}$$

for $s, \tilde{s}, s' \in \mathbb{R}$, $s > d - \frac{1}{2}$, with asymptotic types $Q_\pm$ and T, depending on $\mathcal{G}$. The operators $\mathcal{G}$ are assumed to be of the form $\mathcal{G} = \mathcal{G}_0 + \sum_{k=1}^{d} \mathcal{G}_k \,\mathrm{diag}(D^k, 0, 0, 0)$ for an operator D as in Section 3.2 and Green operators $\mathcal{G}_k$ of type 0 which are characterised by the mapping properties (46), in this case for all $s > -\frac{1}{2}$, and by analogous properties of the formal adjoint $\mathcal{G}^*$ (with other asymptotic types).

Let $\mathcal{C}_G^d(\mathbb{X}_+, \mathbb{X}_-)$ denote the space of all Green operators of type d and $\mathcal{C}_{M+G}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ the space of all operators of the form $\operatorname{diag}(\mathcal{M}, 0) + \mathcal{G}$ for arbitrary $\mathcal{G} \in \mathcal{C}_G^d(\mathbb{X}_+, \mathbb{X}_-)$ and $\mathcal{M}$ of the kind (43).

Note that we also have the spaces $\mathcal{C}_{M+G}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ for the infinite weight interval $\Theta = [0, \infty)$, defined as the intersection of the corresponding spaces for $\Theta_k = [0, k+1)$, $k \in \mathbb{N}$.

3.5. Boundary-contact operators

The category of operators $\mathcal{A}$ that we study in this section are a pseudo-differential analogue of the boundary-contact problems of Section 1.1. Because of the expected shape of parametrices of elliptic elements, cf. Section 3.1, and in order to have the freedom to carry out compositions within our class of operators we start with 4×4 block matrices $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,\ldots,4}$ which contain trace and potential operators with respect to $\mathbb{S}$ and $\partial\Omega$ at the same time. In other words our operators will define maps of operators

$$\mathcal{A} : \begin{array}{c} \mathcal{H}^{s,\gamma}_{(P_+)}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s,\gamma}_{(P_-)}(\mathbb{X}_-) \\ \oplus \\ \mathcal{H}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{(S)}(\mathbb{S}, \mathbb{C}^I) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial\Omega, \mathbb{C}^{I'}) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{s-\mu,\gamma-\mu}_{(Q_+)}(\mathbb{X}_+) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma-\mu}_{(Q_-)}(\mathbb{X}_-) \\ \oplus \\ \mathcal{H}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}_{(T)}(\mathbb{S}, \mathbb{C}^J) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}(\partial\Omega, \mathbb{C}^{J'}) \end{array} \tag{47}$$

with entries A_{ij} acting between corresponding components of the involved spaces; $s \in \mathbb{R}$, $s > d - \frac{1}{2}$.

As in Section 3.1 we choose a covering of $\overline{\Omega}$ by (relatively) open sets U_1, U_2, U_3 and fix a subordinate partition of unity $\{\varphi_j\}_{j=1,2,3}$ and functions $\{\psi_j\}_{j=1,2,3}$ with the properties mentioned before. Let us form diagonal matrices $\mathcal{M}_{\varphi_j}$ of operators of multiplication by functions $\mathcal{M}_{\varphi_j} := \operatorname{diag}(\varphi_j|_{\mathbb{X}_+}, \varphi_j|_{\mathbb{X}_-}, \varphi_j|_{\mathbb{S}}, \varphi_j|_{\partial\Omega})$ and, similarly, $\mathcal{M}_{\psi_j}$. Here, $\varphi_j|_{\mathbb{S}}$, $\varphi_j|_{\partial\Omega}$ are combined with the identity operators in spaces $\mathbb{S}$ and $\partial\Omega$, respectively, and we omit these entries as soon as a corresponding dimension $I, J, \ldots,$ is zero.

Definition 3.12. Let $\mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ for $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$ denote the space of all operators of the form

$$\mathcal{A} = \sum_{j=1}^{3} \mathcal{M}_{\varphi_j} \mathcal{A}_j \mathcal{M}_{\psi_j} + \mathcal{G} \tag{48}$$

for arbitrary elements $\mathcal{G} \in \mathcal{C}_G^d(\mathbb{X}_+, \mathbb{X}_-)$ and operators $\mathcal{A}_j$ as follows:

(i) $\mathcal{A}_1 := \mathrm{diag}(r^{-\mu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h), 0) + \mathrm{diag}(\mathcal{M}, 0)$ for an $h(r,z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{\mu,d}(\Xi_+, \Xi_-))$, cf. Section 3.3 and an operator $\mathcal{M}$ of the kind (43);

(ii) $\mathcal{A}_2 := \mathrm{diag}(\mathcal{A}_{2,\mathrm{int}}, 0)$ for an element $\mathcal{A}_{2,\mathrm{int}} \in \mathcal{B}^{\mu,d}(X_+ \setminus \{0\}, X_- \setminus \{0\})$, cf. Remark 3.7;

(iii) $\mathcal{A}_3 := (\mathcal{A}_{3,ij})_{i,j=1,\ldots,4}$ for an $(\mathcal{A}_{3,ij})_{i,j=2,4} \in \mathcal{B}^{\mu,d}(\overline{\Omega})$, cf. Definition 3.3 for $l = 0$ and $\mathcal{A}_{3,ij} = 0$ for $(i,j) \neq (2,4)$.

The hidden data such as weight data, asymptotic types in the smoothing Mellin and Green operators and the dimensions I, J and I', J' are assumed to be given for every concrete operator.

Remark 3.13. Writing $\mathcal{A}$ in the form (48) there is a $\mathcal{K} \in \mathcal{C}_G^d(\mathbb{X}_+, \mathbb{X}_-)$ such that

$$\mathcal{A} = \sum_{j=1}^{3} \mathcal{M}_{\varphi_j} \mathcal{A} \mathcal{M}_{\psi_j} + \mathcal{K}.$$

Theorem 3.14. *Every $\mathcal{A} \in \mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ induces continuous operators* (47) *for all $s > \max(\mu, d) - \frac{1}{2}$ and* (*in the case of spaces with asymptotics*) *for every triple (P_+, P_-, S) of asymptotic types with some resulting (Q_+, Q_-, T)* (*depending on $\mathcal{A}$, not on s*).

Remark 3.15. A slight modification of Definition 3.12 allows us to define the space $\mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ also for the case of boundary-contact configuration with non-compact interfaces or non-compact $\overline{\Omega}$, cf. also Remark 3.6. As a generalisation of Theorem 3.14 we then have continuous maps between ‘comp’ and ‘loc’-versions of the corresponding spaces.

The operators $\mathcal{A} \in \mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ have a principal symbolic hierarchy which is responsible for ellipticity and parametrices

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_{\mathrm{tr}}(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_M(\mathcal{A})). \tag{49}$$

Here $\sigma_\psi(\mathcal{A}) := (\sigma_\psi(A_+), \sigma_\psi(A_-))$ is the pair of homogeneous principal symbols of order μ of the operators $A_\pm$ on $X_\pm \setminus \{0\}$. Incidentally we also write $\sigma_\psi(\mathcal{A})_\pm$ instead of $\sigma_\psi(A_\pm)$. Moreover, $\sigma_{\mathrm{tr}}(\mathcal{A})$ is the transmission symbol of $\mathcal{A}$ with respect to the interface $S \setminus \{0\}$, defined in the same way as in Section 3.2 as an operator function

$$\sigma_{\mathrm{tr}}(\mathcal{A})(y,\eta) : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ H^s(\mathbb{R}_-) \\ \oplus \\ \mathbb{C}^I \end{array} \to \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ H^{s-\mu}(\mathbb{R}_-) \\ \oplus \\ \mathbb{C}^J \end{array}, \quad (y,\eta) \in T^*(S \setminus \{0\}) \setminus 0 \tag{50}$$

(recall that this does not employ the compactness of the interface). $\sigma_\partial(\mathcal{A})$ is nothing other than the principal boundary symbol of the restriction of the operator $(\mathcal{A}_{ij})_{i,j=2,4}$ to a collar neighbourhood of $\partial\Omega$, namely,

$$\sigma_\partial(\mathcal{A})(y,\eta): \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{I'} \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{J'} \end{matrix}, \quad (y,\eta) \in T^*(\partial\Omega)\setminus 0. \tag{51}$$

Finally, $\sigma_M(\mathcal{A})$ is the principal conormal symbol referring only to $(\mathcal{A}_{ij})_{i,j=1,2,3}$, defined by

$$\sigma_M(\mathcal{A})(z) := h(0,z) + \sigma_M(\mathcal{M})(z),$$

cf. the expression (45) and Definition 3.12 (ii). The conormal symbol $\sigma_M(\mathcal{A})(z)$ represents a family of continuous operators

$$\sigma_M(\mathcal{A})(z): \begin{matrix} H^s(\Xi_+) \\ \oplus \\ H^s(\Xi_-) \\ \oplus \\ H^{s-\frac{1}{2}}(\Sigma, \mathbb{C}^I) \end{matrix} \to \begin{matrix} H^{s-\mu}(\Xi_+) \\ \oplus \\ H^{s-\mu}(\Xi_-) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}(\Sigma, \mathbb{C}^J) \end{matrix}, \quad z \in \Gamma_{\frac{n+1}{2}-\gamma}, \tag{52}$$

$s > d - \frac{1}{2}$, with values in $\mathcal{B}^{\mu,d}(\Xi_+, \Xi_-)$. Interpreting the components of $\sigma_\psi(\mathcal{A})$ and $\sigma_{\mathrm{tr}}(\mathcal{A})$ in a neighbourhood of 0 in polar coordinates with variables and covariables $(r,\phi,\varrho,\varepsilon) \in T^*(\mathbb{R}_+ \times \Xi_\pm)\setminus 0$ and $(r,\phi',\varrho,\varepsilon') \in T^*(\mathbb{R}_+ \times \Sigma)\setminus 0$, respectively, the expressions

$$\tilde{\sigma}_\psi(\mathcal{A})_\pm(r,\phi,\varrho,\varepsilon) := r^\mu \sigma_\psi(\mathcal{A})_\pm(r,\phi,r^{-1}\varrho,\varepsilon) \tag{53}$$

and

$$\tilde{\sigma}_{\mathrm{tr}}(\mathcal{A})(r,\phi',\varrho,\varepsilon') := r^\mu \sigma_{\mathrm{tr}}(\mathcal{A})(r,\phi',r^{-1}\varrho,\varepsilon') \tag{54}$$

are smooth up to $r = 0$.

Remark 3.16. Let $\mathcal{A}, \mathcal{B} \in \mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ and assume that $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$. Then $\mathcal{A}$–$\mathcal{B}$ is compact as an operator between the spaces in (47) (with or without asymptotics).

Remark 3.17. Let $\mathcal{A} \in \mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$, $\mathcal{B} \in \mathcal{C}^{\nu,e}(\mathbb{X}_+, \mathbb{X}_-)$, and assume that weight data and dimensions in the image of $\mathcal{B}$ fit to corresponding data in the domain of $\mathcal{A}$. Then $\mathcal{AB} \in \mathcal{C}^{\mu+\nu,h}(\mathbb{X}_+, \mathbb{X}_-)$ for $h = \max(\nu + d, e)$, and we have $\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ with componentwise composition, where $\sigma_M(\mathcal{AB})(z) = \sigma_M(\mathcal{A})(z-\nu)\sigma_M(\mathcal{B})(z)$. If $\mathcal{A}$ or

$\mathcal{B}$ belong to the subspace of Green or smoothing Mellin plus Green operators (indicated by subscripts G or $M+G$) then the same is true of the composition.

In the non-compact case, cf. Remark 3.15, we have a similar composition result when we pass to $\mathcal{A}\varphi\mathcal{B}$ for a localising factor φ in the middle, or when $\mathcal{A}$ or $\mathcal{B}$ are properly supported in an analogous sense as for standard pseudo-differential operators.

Comparing (47) with operators of the kind (6) which represent boundary value problems for differential operators we should have a generalisation of Definition 3.12 to the case of arbitrary order of the operators referring to $\mathbb{S}$ (and also to $\partial\Omega$). What concerns the smooth boundary component $\partial\Omega$ we simply apply standard reductions of orders, represented by diagonal matrices of elliptic operators of the kind $R^{\varrho} \in L^{\varrho}_{\mathrm{cl}}(\partial\Omega)$, $\varrho \in \mathbb{R}$, which induce isomorphisms $R^{\varrho} : H^{s}(\partial\Omega) \to H^{s-\mu}(\partial\Omega)$ for all $s \in \mathbb{R}$, cf., analogously, the formula (36).

Similar reductions of orders should be applied with respect to $\mathbb{S}$. Since $\mathbb{S}$ is the stretched manifold of a (compact) manifold with conical singularity 0 we need a corresponding special result such that asymptotic data near the tip of the cone are not influenced. Such an order reduction result exists, indeed, and the corresponding theorem is based on the (pseudo-differential) cone calculus on $\mathbb{S}$ which consists of those continuous operators

$$R^{\varrho} : \mathcal{H}^{s,\gamma-\frac{1}{2}}(\mathbb{S}) \to \mathcal{H}^{s-\varrho,\gamma-\varrho-\frac{1}{2}}(\mathbb{S}) \tag{55}$$

for some given $\gamma \in \mathbb{R}$, that are contained in the operators $\mathcal{A}$ from Definition 3.12 (for $I = J = 1$ and $\varrho = \mu$) as the space of entries $\mathcal{A}_{33}$ (when we write $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,\dots,4}$).

Theorem 3.18. *For every $\gamma, \varrho \in \mathbb{R}$ there exists an elliptic element $\mathcal{R}^{\varrho}$ in the above mentioned cone algebra on $\mathbb{S}$ which induces isomorphisms* (55) *for all $s \in \mathbb{R}$. For every $\vartheta > 0$ we can choose $\mathcal{R}^{\varrho}$ in such a way that its principal conormal symbol*

$$\sigma_M(\mathcal{R}^{\varrho})(z) : H^{s}(\Sigma) \to H^{s-\varrho}(\Sigma)$$

represents a holomorphic family of isomorphisms in the strip $\frac{n}{2}-\gamma-\vartheta < \operatorname{Re} z < \frac{n}{2}-\gamma+\vartheta$ and that the inverse $\mathcal{R}^{-\varrho}$ (which exists in the cone algebra with respect to $\gamma-\varrho$) has an analogous property.

Theorem 3.18 is a direct consequence of order reducing results from [7], based on certain specific symbols from [5] with the transmission property.

Remark 3.19. Applying from both sides diagonal matrices of order reductions on $\partial\Omega$ and $\mathbb{S}$, respectively, for a suitable choice of orders we can modify the space $\mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ to a space of operators where the entries referring to $\partial\Omega$ and $\mathbb{S}$ are of different orders (analogously as in Douglis–Nirenberg systems). Because of the holomorphy of conormal symbols of the operators referring to $\mathbb{S}$ in a strip of arbitrary finite width we do not lose the essential mapping properties in subspaces with asymptotics.

4. Asymptotics of solutions

We now pass to the ellipticity of boundary-contact operators which is based on the principal symbolic hierarchy (49).

4.1. Ellipticity

Definition 4.1. An operator $\mathcal{A} \in \mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ is said to be elliptic (of order μ and with respect to the weight γ) if

(i) $\sigma_\psi(\mathcal{A})_\pm(x,\xi) \neq 0$ for all $(x,\xi) \in T^*(X_\pm \backslash \{0\})$, and $\tilde{\sigma}_\psi(A_\pm)(r,\phi,\varrho,\varepsilon) \neq 0$ for all $(r,\phi,\varrho,\varepsilon)$, $(\varrho,\varepsilon) \neq 0$, up to $r=0$;
(ii) $\sigma_{\rm tr}(\mathcal{A})(y,\eta)$ is a family of isomorphisms (50) and $\tilde{\sigma}_{\rm tr}(\mathcal{A})(r,\phi',\varrho,\varepsilon')$ is bijective in an analogous sense for all $(r,\phi',\varrho,\varepsilon')$, $(\varrho,\varepsilon') \neq 0$, up to $r=0$;
(iii) $\sigma_M(\mathcal{A})(z)$ is a family of isomorphisms (52) and all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$;
(iv) $\sigma_\partial(\mathcal{A})(y,\eta)$ is a family of isomorphisms (51) for all $(y,\eta) \in (T^*(\partial\Omega)\backslash 0)$.

The conditions (ii)–(iv) are required for any sufficiently large s.

Example 4.2. The operator $\mathcal{A}$ defined by (20) belongs to $\mathcal{C}^{2,2}(\mathbb{X}_+, \mathbb{X}_-)$ (the notation of orders is to be interpreted here in a corresponding generalised sense when we do not reduce the orders on $\mathbb{S}$). The weight data follow immediately from the nature of the problem: in (20) there are no smoothing Mellin plus Green terms; so the weight intervals in asymptotic considerations are admitted to be infinite. Concerning the dimensions we have $I=0$, $J=2$, $I'=0$, $J'=1$. As we shall see below there is a discrete set D of reals such that $\mathcal{A}$ is elliptic for every weight $\gamma \in \mathbb{R}\backslash D$.

Theorem 4.3. *Let* $\mathcal{A} \in \mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ *be elliptic; there is then a parametrix* $\mathcal{P} \in \mathcal{C}^{-\mu,(d-\mu)^+}(\mathbb{X}_+, \mathbb{X}_-)$ *such that*

$$\mathcal{P}\mathcal{A} = \mathcal{I} - \mathcal{K}_l, \quad \mathcal{A}\mathcal{P} = \mathcal{I} - \mathcal{K}_r$$

for elements $\mathcal{K}_l \in \mathcal{C}_G^{\max(\mu,d)}(\mathbb{X}_+, \mathbb{X}_-)$ *and* $\mathcal{K}_r \in \mathcal{C}_G^{(d-\mu)^+}(\mathbb{X}_+, \mathbb{X}_-)$.

According to Remark 3.17 we then have $\sigma(\mathcal{P}) = \sigma^{-1}(\mathcal{A})$ with componentwise inversion (in particular, $\sigma_M(\mathcal{A})^{-1}(z-\mu) = \sigma_M(\mathcal{B})(z)$).

Corollary 4.4. Let $\mathcal{A} \in \mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ be elliptic. Then $\mathcal{A}$ induces a Fredholm operator (47) between the corresponding spaces without asymptotic types, for all $s > \max(\mu,d) - \frac{1}{2}$, where the weight γ is determined by Definition 4.1 (iii).

Proof of Theorem 4.3. Similarly as in Remark 3.13 we localise the operator $\mathcal{A}$ in neighbourhoods U_j, $j = 1, 2, 3$, and write

$$\mathcal{A} = \sum_{j=1}^{3} \mathcal{M}_{\varphi_j} \mathcal{A} \mathcal{M}_{\psi_j} + \mathcal{G}.$$

The construction of a parametrix can be done locally in U_j for $j = 1, 2, 3$, and we may ignore $\mathcal{G}$ which belongs to the ideal of smoothing operators (cf. the last statement of Remark 3.17). Then, since there is a compatibility in intersections of the open sets we may set

$$\mathcal{P} = \sum_{j=1}^{3} \mathcal{M}_{\varphi_j} \mathcal{P}_j \mathcal{M}_{\psi_j} \tag{56}$$

with $\mathcal{P}_j$ being a local parametrix of $\mathcal{A}$ on U_j. The operator $\mathcal{A}$ in the neighbourhood U_3 represents an elliptic operator in Boutet de Monvel's calculus $\mathcal{B}^{\mu,d}(U_3)$. We identify it with $(\mathcal{A}_{ij})_{i,j=2,4}$, ignoring smoothing operators in other entries. This gives us a parametrix $\mathcal{P}_3 \in \mathcal{B}^{-\mu,(d-\mu)^+}(U_3)$ (we will be tacitly identify $\mathcal{P}_3$ with a corresponding 4×4 block matrix by filling up the 2×2 matrix by zeros at the remaining places). In a similar manner, we can proceed with $(\mathcal{A}_{ij})_{i,j=1,2,3}$ over U_2 which is elliptic in the transmission algebra $\mathcal{B}^{\mu,d}(X_+ \cap U_2, X_- \cap U_2)$ with smooth interface $S \cap U_2$. The parametrix construction in that calculus gives us an element $\mathcal{P}_2 \in \mathcal{B}^{\mu,(d-\mu)^+}(X_+ \cap U_2, X_- \cap U_2)$ (which is also filled up by zeros to the 4×4 size). Thus it remains to consider $(\mathcal{A}_{ij})_{i,j=1,2,3}$ over U_1 which is the part referring to the conical singularity. The shape of U_1 is still arbitrary, and it suffices to take $U_1 = \{x \in \mathbb{R}^{n+1} : |x| < \varepsilon\}$ for some $\varepsilon > 0$. Near 0 the operator has the form

$$r^{-\mu} \omega \operatorname{op}_M^{\gamma - \frac{n}{2}}(h) \tilde{\omega} + \omega r^{-\mu} \sum_{j=0}^{k} r^j \operatorname{op}_M^{\gamma_j - \frac{n}{2}}(h_j) \tilde{\omega}$$

for cut-off functions ω and $\tilde{\omega}$ near 0 and Mellin symbols

$$h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{\mu,d}(\Xi_+, \Xi_-)) \quad \text{and} \quad h_j(z) \in \mathcal{M}_{\mathrm{as}}^{-\infty,d}(\Xi_+, \Xi_-)$$

for $j = 0, \ldots, k$. From the ellipticity of $\mathcal{A}$ it follows that there is a Mellin–Leibniz inverse $f^{(-1)}(r, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{-\mu,(d-\mu)^+}(\Xi_+, \Xi_-; \Gamma_{\frac{n+1}{2}-\gamma}))$. Applying Theorem 3.10 we can replace $f^{(-1)}(r, z)$ by an element $h^{(-1)}(r, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{-\mu,(d-\mu)^+}(\Xi_+, \Xi_-))$ which is also a Mellin–Leibniz inverse. For the associated conormal symbols we know in this construction that

$$h^{(-1)}(0,z)h(0,z) = 1 + l(z) \tag{57}$$

for some $l \in \mathcal{M}_{\mathrm{as}}^{-\infty,\max(\mu,d)}(\Xi_+,\Xi_-)$. This implies

$$h^{(-1)}(0,z)\{h(0,z) + h_0(z)\} = 1 + m(z) \tag{58}$$

for another $m \in \mathcal{M}_{\mathrm{as}}^{-\infty,\max(\mu,d)}(\Xi_+,\Xi_-)$. As is known from the nature of smoothing Mellin symbols with asymptotics there is an $n \in \mathcal{M}_{\mathrm{as}}^{-\infty,d}(\Xi_+,\Xi_-)$ such that

$$(1+n(z))(1+m(z)) = 1$$

in the sense of the composition of meromorphic operator functions for all $z \in \mathbb{C}$. From (58) we obtain $(1+n(z))h^{(-1)}(0,z) = (h(0,z)+h_0(z))^{-1}$ in the same sense. The left-hand side has the form $h^{(-1)}(0,z) + l_0(z)$ for some $l_0 \in \mathcal{M}_{\mathrm{as}}^{-\infty,(d-\mu)^+}(\Xi_+,\Xi_-)$. By virtue of the invertibility of

$$\sigma_M(\mathcal{A})(z) = h(0,z) + h_0(z)$$

for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ it follows that

$$\sigma_M(\mathcal{A})^{-1}|_{\Gamma_{\frac{n+1}{2}-\gamma}}(z) = (h^{(-1)}(0,z) + l_0(z))|_{\Gamma_{\frac{n+1}{2}-\gamma}}.$$

Choose cut-off functions ω_1, $\tilde{\omega}_1$ such that $\omega \equiv 1$ on $\operatorname{supp}\tilde{\omega}_1$ and $\tilde{\omega}_1 \equiv 1$ on $\operatorname{supp}\omega_1$. Then, composing the operator $\omega_1 \operatorname{op}_M^{\gamma-\frac{n}{2}}(h^{(-1)}(r,z) + l_0(z))\tilde{\omega}_1$ from the right with $\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega} + \omega \sum_{j=0}^{k} r^j \operatorname{op}_M^{\gamma_j-\frac{n}{2}}(h_j)\tilde{\omega}$ and taking into account the rules of evaluating compositions of Mellin operators of that shape in the cone algebra we obtain an expression of the form

$$\omega\left(1 + \sum_{j=1}^{k} r^j \operatorname{op}_M^{\delta_j-\frac{n}{2}}(m_j)\right)\tilde{\omega} + G$$

for elements $m_j \in \mathcal{M}_{\mathrm{as}}^{-\infty,\max(\mu,d)}(\Xi_+,\Xi_-)$, with suitable weights δ_j (in order to avoid possible poles on the corresponding weight lines involved in the Mellin actions), and a Green operator G. Now the Ansatz

$$\omega\left(1 + \sum_{j=1}^{k} r^j \operatorname{op}_M^{\beta_j-\frac{n}{2}}(n_j)\right)\tilde{\omega}\left(1 + \sum_{j=1}^{k} r^j \operatorname{op}_M^{\delta_j-\frac{n}{2}}(m_j)\right)\omega = \omega \cdot \mathrm{Id}$$

modulo a Green operator allows us to calculate the elements $n_j \in \mathcal{M}_{\mathrm{as}}^{-\infty,d}(\Xi_+, \Xi_-)$. This gives us finally

$$\omega\Big(1+\sum_{j=1}^{k} r^j \operatorname{op}_M^{\beta_j-\frac{n}{2}}(n_j)\Big)\tilde{\omega}\omega_1 \operatorname{op}_M^{\gamma-\frac{n}{2}}(h^{(-1)}(r,z)+l_0(z))\tilde{\omega}_1$$

$$= \omega_1 \operatorname{op}_M^{\gamma-\frac{n}{2}}(h^{(-1)}(r,z))\tilde{\omega}_1 + \omega_1 \sum_{j=0}^{k} r^j \operatorname{op}_M^{\varepsilon_j}(l_j(z))\tilde{\omega}_1$$

modulo a Green operator, where $l_0, \ldots, l_k \in \mathcal{M}_{\mathrm{as}}^{-\infty,(d-\mu)^+}(\Xi_+, \Xi_-)$ are resulting smoothing Mellin symbols. The right-hand side, composed from the right by r^μ is just the desired local (left) parametrix of $\mathcal{A}$ in U_1 (which is finally filled up by zeros at the appropriate places to a 4×4 matrix $\mathcal{P}_1$).

Summing up, we have calculated all ingredients of the expression (56) and thus obtained a left parametrix of $\mathcal{A}$. Analogous constructions give us a right parametrix; hence (56) is a two-sided parametrix as desired. □

Remark 4.5. Note that (in the notation of the preceding proof) we also have

$$(h(0,z)+h_0(z))^{-1} = h^{-1}(0,z) + f_0(z) \tag{59}$$

for some $f_0 \in \mathcal{M}_{\mathrm{as}}^{-\infty,(d-\mu)^+}(\Xi_+, \Xi_-)$, where $h^{-1}(0,z)$ denotes the inverse of $h(0,z)$ as a meromorphic operator function in the complex plane which may have poles on the reference weight line. The relation (59) easily follows from (57) by composing both sides from the right by $h^{-1}(0,z)$.

4.2. Regularity with asymptotics

We now turn to the regularity of solutions in weighted spaces with and without asymptotics. The interpretation of the subscripts in spaces of the following theorem is as in (47).

Theorem 4.6. *Let $\mathcal{A} \in \mathcal{C}^{\mu,d}(\mathbb{X}_+, \mathbb{X}_-)$ be elliptic, and let*

$$u \in \mathcal{H}^{r,\gamma}(\mathbb{X}_+) \oplus \mathcal{H}^{r,\gamma}(\mathbb{X}_-) \oplus \mathcal{H}^{-\infty,\gamma-\frac{1}{2}}(\mathbb{S}, \mathbb{C}^I) \oplus H^{-\infty}(\partial\Omega, \mathbb{C}^{I'}), \tag{60}$$

$r > \max(\mu, d) - \frac{1}{2}$, *be a solution of the equation* $\mathcal{A}u = f$,

$$f \in \mathcal{H}_{Q_+}^{s-\mu,\gamma-\mu}(\mathbb{X}_+) \oplus \mathcal{H}_{Q_-}^{s-\mu,\gamma-\mu}(\mathbb{X}_-) \oplus \mathcal{H}_T^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(\mathbb{S}, \mathbb{C}^J) \oplus H^{s-\mu-\frac{1}{2}}(\partial\Omega, \mathbb{C}^{J'})$$

for an $s > (d-\mu)^+ - \frac{1}{2}$. *Then we have*

$$u \in \mathcal{H}_{P_+}^{s,\gamma}(\mathbb{X}_+) \oplus \mathcal{H}_{P_-}^{s,\gamma}(\mathbb{X}_-) \oplus \mathcal{H}_S^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{S}, \mathbb{C}^I) \oplus H^{s-\frac{1}{2}}(\partial\Omega, \mathbb{C}^{I'})$$

for every prescribed (Q_+, Q_-, T) *with resulting asymptotic types* (P_+, P_-, S) *that also depend on the operator* $\mathcal{A}$ (*not on* s).

An analogous regularity result holds in weighted spaces without asymptotics.

Proof. From Theorem 4.3 we have a parametrix $\mathcal{P}$ of $\mathcal{A}$. Then, because of the assumption on f from $\mathcal{A}u = f$ we can pass to $\mathcal{P}\mathcal{A}u = \mathcal{P}f$. Let us discuss, for instance, the case with asymptotics, the assertion without asymptotics follows in an analogous manner. By virtue of the continuity of $\mathcal{P}$ between spaces with asymptotics, cf. Theorem 3.14, it follows that $\mathcal{P}f$ is as on the left-hand side of (47) for certain resulting asymptotic types. Moreover, we have $\mathcal{P}\mathcal{A}u = \mathcal{I}u - \mathcal{K}_l u$ for some $\mathcal{K}_l \in \mathcal{C}_G^{\max(\mu,d)}(\mathbb{X}_+, \mathbb{X}_-)$. Therefore, from (60) and (46) we obtain that $\mathcal{K}_l u$ also belongs to a space as on the left-hand side of (47), with some asymptotic types and $s = \infty$. Thus $u = \mathcal{P}f + \mathcal{K}_l u$ is as desired. $\square$

Remark 4.7. Definition 4.1 also makes sense in the non-compact case, cf. Remark 3.15. There is then an analogue of Theorem 4.3, for instance, when $\mathcal{A}$ or $\mathcal{P}$ are properly supported in a suitable sense (or when we localise the notion of a parametrix). This allows us to conclude elliptic regularity of solutions with (or without) asymptotics also in the non-compact case. In particular, if $\mathcal{A}$ is an elliptic problem of the form (20), where all entries are local operators, then we have an analogue of Theorem 4.6 locally near the conical points of our interface S also in the case of non-compact S.

4.3. Examples and remarks

We now specify our results for concrete boundary-contact problems, first for (1)–(3). The corresponding operator $\mathcal{A}$ is continuous in the sense (20) for every $s > \frac{3}{2}$ and $\gamma \in \mathbb{R}$. The main contribution comes from a neighbourhood of the conical point 0; the first task is to study the principal conormal symbol. In our example we assume that the interface S is conical in a neighbourhood of 0. A simple delation of variables allows us to pass to the case $S \cap \{|x| < 1\} = \{\lambda x : 0 < \lambda < 1, x \in S^n \cap S\}$. We consider the case $n = 1$; then $\Sigma = S^1 \cap S$ consists of two points, and after a rotation we can set

$$\Xi_+ = \{\phi \in S^1 : 0 \leqslant \phi \leqslant \alpha\}, \quad \Xi_- = \{\phi \in S^1 : \alpha \leqslant \phi \leqslant 2\pi\}.$$

The Laplace operator Δ in polar coordinates $r^{-2}((-r\partial_r)^2 + \partial_\phi)$ gives rise to the conormal symbol $\sigma_M(\Delta)(z) = z^2 + \partial_\phi^2$. The map (21) then has the form

$$h(z) = \begin{pmatrix} z^2 + \partial_\phi^2 & 0 \\ 0 & c(z^2 + \partial_\phi^2) \\ \mathrm{r}'_0 & -\mathrm{r}'_0 \\ \mathrm{r}'_0\partial_\phi & \mathrm{r}'_0\partial_\phi \\ \mathrm{r}'_\alpha & -\mathrm{r}'_\alpha \\ \mathrm{r}'_\alpha\partial_\phi & \mathrm{r}'_\alpha\partial_\phi \end{pmatrix} : \begin{matrix} H^s(\Xi_+) \\ \oplus \\ H^s(\Xi_-) \end{matrix} \longrightarrow \begin{matrix} H^{s-2}(\Xi_+) \\ \oplus \\ H^{s-2}(\Xi_-) \\ \oplus \\ \mathbb{C}^2 \\ \oplus \\ \mathbb{C}^2 \end{matrix},$$

where r'_β denotes the operator of restriction to $\{0 < r < 1, \phi = \beta\}$, here for $\beta = 0, \alpha$.

As noted in Remark 2.2 before, $h(z)$ is a parameter-dependent elliptic family of transmission problems on S^1. The admissible weight γ for our boundary-value problem follow from the set D of those points $z \in \mathbb{C}$ where $h(z)$ in not bijective. That means we ask those $z \in \mathbb{C}$ where the problem

$$\begin{cases} v''_+ + z^2 v_+ = 0 & v_+(0) = v_-(2\pi) \quad v_+(\alpha) = v_-(\alpha), \\ c(v''_- + z^2 v_-) = 0 & v'_+(0) = -v'_-(2\pi) \; v'_+(\alpha) = -v'_-(\alpha) \end{cases} \tag{61}$$

has non-trivial solutions. Let us first verify that $0 \in D$. In this case (61) has the solutions $v_+ = v_- = C$, for an arbitrary constant C. For the case $z = a + ib \neq 0$ the solutions of the equations $v''_\pm + z^2 v_\pm = 0$ may be represented as $v_\pm = C_{1\pm} e^{-b\phi} e^{ia\phi} + C_{2\pm} e^{b\phi} e^{-ia\phi}$, respectively. Moreover, we have $v'_\pm = (b - ia)(-C_{1\pm} e^{-b\phi} e^{ia\phi} + C_{2\pm} e^{b\phi} e^{-ia\phi})$. From the boundary conditions in (61) it follows that

$$\begin{cases} C_{1+} + C_{2+} = C_{1-} e^{-2\pi b} e^{2ia\pi} + C_{2-} e^{2\pi b} e^{-2ia\pi}, \\ -C_{1+} + C_{2+} = C_{1-} e^{-2\pi b} e^{2ia\pi} - C_{2-} e^{2\pi b} e^{-2ia\pi}, \\ C_{1+} e^{-b\alpha} e^{ia\alpha} + C_{2+} e^{b\alpha} e^{-ia\alpha} = C_{1-} e^{-b\alpha} e^{ia\alpha} + C_{2-} e^{b\alpha} e^{-ia\alpha}, \\ -C_{1+} e^{-b\alpha} e^{ia\alpha} + C_{2+} e^{b\alpha} e^{-ia\alpha} = C_{1-} e^{-b\alpha} e^{ia\alpha} - C_{2-} e^{b\alpha} e^{-ia\alpha}. \end{cases}$$

An easy calculation gives us

$$\begin{cases} C_{2+} = C_{1-} e^{-2\pi b} e^{2ia\pi} \;\; C_{2+} = C_{1-} e^{-2\alpha b} e^{2ia\alpha}, \\ C_{1+} = C_{2-} e^{2\pi b} e^{-2ia\pi} \;\; C_{1+} = C_{2-} e^{2\alpha b} e^{-2ia\alpha}. \end{cases}$$

Since we are looking for non-trivial solutions let us assume $C_{1-} \neq 0$ (the case $C_{1-} = 0$, $C_{2-} \neq 0$ is analogous). Then we obtain $e^{-2\pi b} e^{2ia\pi} = e^{-2\alpha b} e^{2ia\alpha}$. Because of $\alpha \neq \pi$ we have $b = 0$ and $e^{2i(\pi-\alpha)a} = 0$, i.e., $a = \dfrac{\pi}{\pi - \alpha} k$, for $k \in \mathbb{Z} \backslash \{0\}$. Summing up we finally obtain $D = \left\{ \dfrac{\pi}{\pi - \alpha} k \right\}_{k \in \mathbb{Z}}$.

Theorem 4.8. *The boundary-contact problem* (1)–(3) *for the operators* (18), (19) (*in dimension* 2) *defines a Fredholm operator* (20) *for all* $s > \frac{3}{2}$ *and all* $\gamma \in \mathbb{R} \backslash \{1 - \frac{\pi k}{\pi - \alpha} : k \in \mathbb{Z}\}$.

Proof. Let $\mathcal{A}$ denote the operator represented by the problem (1)–(3). The ellipticity conditions (i), (ii) and (iv) of Definition 4.1 are obviously satisfied for our problem. Moreover, we saw that the principal conormal symbol $\sigma_M(\mathcal{A}) = h(z)$ is bijective for all $z \in \mathbb{C} \backslash D$. Applying Corollary 4.4 we obtain the Fredholm property for all weights γ such that $D \cap \Gamma_{1-\gamma} = \emptyset$. $\square$

The set

$$D \cap \{\mathrm{Re}\, z < 1 - \gamma\} \tag{62}$$

gives us more precise information about the asymptotics of solutions in the sense of Theorem 3.2. In fact, the main contribution comes from the poles of $h^{-1}(z)$ which just constitute the set (62).

Theorem 4.6 can be specialised to the present situation. In particular, let $u \in \mathcal{H}^{s,\gamma}(\mathbb{X}_+) \oplus \mathcal{H}^{s,\gamma}(\mathbb{X}_-)$ be a solution of $\mathcal{A}u = 0$. Then we have $u \in \mathcal{H}^{\infty,\gamma}(\mathbb{X}_+) \oplus \mathcal{H}^{\infty,\gamma}(\mathbb{X}_-)$. Near the origin, in the splitting of variables into $(r, \phi) \in \mathbb{R}_+ \times S$, we obtain asymptotics of $u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ of the form

$$u_\pm(r,\phi) \sim \sum_{k \in \mathbb{Z}\setminus\{0\},\ \frac{\pi}{\pi-\alpha}k < 1-\gamma} c_{\pm,k}(\phi) r^{-\frac{\pi k}{\pi - \alpha}} + c_{\pm,00}(\phi) + c_{\pm,01}(\phi)\log r$$

with coefficients $c_{\pm,k}, c_{\pm,00}, c_{\pm,01} \in C^\infty(\Xi_\pm)$. The second two terms only occur in the case $\gamma < 1$.

Remark 4.9. Our approach can also be applied to boundary-contact problems, with different composites meeting, for instance, in conical points. This would be modelled in terms of decompositions of S^n into subregions $\Xi_1, \dots, \Xi_N$ with smooth (and partly common) boundaries, $S^n = \bigcup_{j=1}^N \Xi_j$.

Another modification of our method also allows us to treat plane crack problems locally near a conical singularity of the crack S which is represented, for instance, by two intervals $S_0 := \{(r,0) : 0 \leqslant r \leqslant 1\}$, $S_\alpha := \{(r,\alpha) : 0 \leqslant r \leqslant 1\}$, i.e., $S = S_0 \cup S_\alpha$ contained in Ω. Since S does not decompose Ω in this case we can consider the Laplacian in $\Omega\setminus S$ without any factor c. To determine asymptotics of solutions locally near 0 we apply Remark 4.7. Analogously as in the example in the beginning we calculate the non-bijectivity points of the conormal symbols including multiplicities (that are 1 in this case) as follows:

The map (21) has the form

$$h(z) = \begin{pmatrix} z^2 + \partial_\phi^2 & 0 \\ 0 & z^2 + \partial_\phi^2 \\ \mathrm{r}_0'\partial_\phi & 0 \\ 0 & \mathrm{r}_0'\partial_\phi \\ \mathrm{r}_\alpha'\partial_\phi & 0 \\ 0 & \mathrm{r}_\alpha'\partial_\phi \end{pmatrix} : \begin{matrix} H^s(\Xi_+) \\ \oplus \\ H^s(\Xi_-) \end{matrix} \longrightarrow \begin{matrix} H^{s-2}(\Xi_+) \\ \oplus \\ H^{s-2}(\Xi_-) \\ \oplus \\ \mathbb{C}^2 \\ \oplus \\ \mathbb{C}^2 \end{matrix}.$$

Similarly as above (see also [9, Section 5.5.3]) we now solve the following problem:

$$\begin{cases} v_+'' + z^2 v_+ = 0 \;\; v_+'(0) = v_+'(\alpha) = 0, \\ v_-'' + z^2 v_- = 0 \;\; v_-'(2\pi) = v_-'(\alpha) = 0 \end{cases}$$

and for the non-bijectivity points we obtain $\left\{\frac{\pi}{\alpha}k\right\}_{k\in\mathbb{Z}}$ and $\left\{\frac{\pi}{2\pi-\alpha}k\right\}_{k\in\mathbb{Z}}$, i.e., for the admissible weights γ we have

$$\gamma \in \mathbb{R} \Big\backslash \left(\left\{1 - \frac{\pi}{\alpha}k\right\}_{k\in\mathbb{Z}} \bigcup \left\{1 - \frac{\pi}{2\pi-\alpha}k\right\}_{k\in\mathbb{Z}} \right).$$

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