$d$-calibers and $d$-tightness in compact spaces

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Abstract

We study the Čech–Stone compactification of the space Seq($u$). We show that its remainder is $F_{\sigma\kappa}$-closed whenever $u$ is a $P(\kappa)$-point and $b \geq \kappa > \omega$. Our results enable us to solve a recent problem by Okunev and Tkachuk regarding a possible relationship between $d$-tightness and $d$-calibers in compact spaces.

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1. Introduction

In their paper [5], Okunev and Tkachuk stated the following problem:

**Problem 1.** Let $X$ be a compact Hausdorff space and suppose that for every dense $Y \subseteq X$, $\omega_1$ is a caliber for $Y$. Is it true that $dt(X) = \omega$?
Here, \( dt \) stands for the \( d \)-tightness and is defined in the following way:

\[
    dt(X) = \min\{ \kappa : \text{every dense subset of } X \text{ is } \kappa \text{-dense} \}.
\]

We shall show that, consistently, their problem has a negative solution. Gearing up towards this goal, let us notice that a space \( Z \) with the following properties will constitute a counterexample.

- \( Z \) is a compact Hausdorff space without isolated points.
- \( Z \) has a dense countable set \( D \) which is a \( P(\kappa) \)-set in \( Z \) and \( \kappa \geq \omega_2 \).

It is obvious that \( dt(Z) \geq \omega_2 \); The fact that for every dense \( Y \subseteq Z \), \( \omega_1 \) is a caliber for \( Y \), follows immediately from our considerations of the concept called \( d \)-caliber (cf. Section 5 and Corollary 1).

A space \( Z \) that possesses the two properties (and many more others) is going to be the Čech–Stone compactification of the space \( \text{Seq}(u) \), where \( u \) is a suitably chosen ultrafilter on \( \omega \). It turns out that such a desirable ultrafilter exists under Martin’s Axiom \(+\) non-CH (see Corollary 3). We do not know of any ZFC example of this sort.

In sections that follow, we study the space \( \text{Seq}(u) \), first for \( u \) being any free ultrafilter on \( \omega \), and then, assuming additionally that \( u \) is a \( P(\kappa) \)-point and that \( b \geq \kappa \). Our main result (Theorem 5) provides a characterization of \( \text{Seq} \) being a \( P(\kappa) \)-set in \( \beta(\text{Seq}(u)) \). We also introduce and study \( d \)-caliber.

For all undefined notions, see [3,4].

2. Some basic properties of the space \( \text{Seq}(u) \)

Let \( \text{Seq} = \bigcup \{ ^n \omega : n \in \omega \} \) denote the set of all finite sequences of non-negative integers; the sequence of length 0 is denoted by \( \emptyset \). If \( s \in \text{Seq} \) is a sequence of length \( n \) and \( k \in \omega \), then \( s^{-}k \) denotes the sequence of length \( n + 1 \) that extends \( s \) and whose last term is \( k \). If \( A \subseteq \omega \), then \( s^{-}A = \{ s^{-}k : k \in A \} \).

For a fixed free filter \( u \) on \( \omega \) we define a topology \( T_u \) on the set \( \text{Seq} \) by the formula:

\[
    U \in T_u \quad \text{if and only if} \quad \forall s \in U \{ n \in \omega : s^{-}n \in U \} \in u.
\]

The topological space \( (\text{Seq}, T_u) \) is denoted by \( \text{Seq}(u) \).

Over the years, there have been many papers published on subjects related to the space \( \text{Seq}(u) \). Vaughan’s paper [6] can be used as a source for a brief historic outline of the origins of the space and of its most elementary properties. In our paper, we are not going to credit (in most cases) anybody with a particular basic fact about the space \( \text{Seq}(u) \). Instead, we will try to give new and elegant proofs (if they are not obvious), or simply, quote results if they have easy or straightforward proofs.

Let \( V \subseteq \text{Seq}(u) \) be open. Then \( \varphi_V \in T_u \) is given by the formula:

\[
    \varphi_V(s) = \{ k \in \omega : s^{-}k \in V \}.
\]

Now let \( \varphi \in \text{Seq}(u) \) and \( s \in \text{Seq} \) be given. We set:
\[ U(s, \varphi, 0) = \{s\} , \]
\[ U(s, \varphi, n + 1) = \bigcup \{ t^\varphi(t) : t \in U(s, \varphi, n) \} , \]
and finally,
\[ U(s, \varphi) = \bigcup \{ U(s, \varphi, n) : n \in \omega \} . \]

It is easy to see that the family
\[ \{ U(s, \varphi) : s \in \text{Seq} \text{ and } \varphi \in \text{Seq}_u \} \]
constitutes a clopen base of the space \( \text{Seq}(u) \).

If \( \varphi, \psi \in \text{Seq}_u \) are such that \( \varphi \subseteq \psi \) (i.e., \( \varphi(s) \subseteq \psi(s) \) for each \( s \in \text{Seq} \)), then \( U(s, \varphi) \subseteq U(s, \psi) \) for each \( s \in \text{Seq} \). In the case \( \varphi \in \text{Seq}_u \) is a constant function, say \( \varphi(s) = a \) for each \( s \in \text{Seq} \), then \( U(s, \varphi) \) is denoted by \( U(s, a) \).

For \( \varphi \in \text{Seq}_u \) and \( f \in \text{Seq}_u \), \( \varphi \setminus f \) denotes the element of \( \text{Seq}_u \) defined as follows:
\[ (\varphi \setminus f)(s) = \begin{cases} \varphi(s) & \text{if } s \in L_m \text{ and } m \geq n , \\ \omega & \text{if } s \in T_{n-1} . \end{cases} \]

Let us observe that \( T_n \subseteq U(\emptyset, \varphi|n) \).

**Lemma 1.** Let \( V \subseteq \text{Seq}(u) \) be open and \( T_n \subseteq V , n \geq 1 \). Then
\[ U(\emptyset, \varphi|n) \subseteq V . \]

**Remark 1.** The space \( \text{Seq}(u) \) replicates itself at any point. More formally, if \( \varphi \in \text{Seq}_u \) and \( s \in \text{Seq} \), then there exists a (natural) homeomorphism \( h \) from \( \text{Seq}(u) \) onto \( U(s, \varphi) \) such that \( h(\emptyset) = s \). This fact is going to be utilized in our arguments involving arbitrary point of \( \text{Seq}(u) \) by reducing considerations to the point \( \emptyset \) only.

**Lemma 2.** Let \( V_n , n = 1, 2, \ldots , \) be open neighborhoods of the point \( \emptyset \) such that for each \( n = 1, 2, \ldots , \)
\[ V_{n+1} \supseteq V_n \cap T_n . \]  \( (2.3) \)
Then \( \bigcap \{ V_n : n = 1, 2, \ldots \} \) is an open neighborhood of \( \emptyset \).

In particular, if \( V_n \) is an open subset of the space \( \text{Seq}(u) \) such that \( T_n \subseteq V_n \) for each \( n \in \omega \), then \( \bigcap \{ V_n : n \in \omega \} \) is an open neighborhood of \( \emptyset \).

Dually, if \( F_n \) is a closed subset of the space \( \text{Seq}(u) \) such that \( T_n \cap F_n = \emptyset \) for each \( n \in \omega \), then \( \bigcup \{ F_n : n \in \omega \} \) is a closed subset of the space \( \text{Seq}(u) \) not containing \( \emptyset \).
Proof. Clearly, \( \emptyset \in \bigcap \{ V_n: n \in \omega \} \). To prove that \( \bigcap \{ V_n: n \in \omega \} \) is open, let \( s \in \bigcap \{ V_n: n \in \omega \} \). Then
\[
\begin{align*}
\mathcal{A} &= \bigcap \{ \varphi V_n(s): n \leq m + 1 \} \in \mathcal{U}.
\end{align*}
\]
By 2.3,
\[
\mathcal{A} \subseteq \varphi V_n(s) \quad \text{whenever } n \geq m + 2.
\]
Thus
\[
\mathcal{U} \ni \mathcal{A} = \bigcap \{ \varphi V_n(s): n \in \omega \} = \left\{ k \in \omega: s \models k \in \bigcap \{ V_n: n \in \omega \} \right\}.
\]

Proposition 1. Let \( u \) be a free filter on \( \omega \). Then
\[
\pi \chi (\emptyset, \text{Seq}(u)) > \omega.
\]
Proof. Let \( G_n, n \in \omega \), be non-empty open subsets of the space Seq\((u)\). Take non-empty clopen sets \( V_n, n \in \omega \), such that \( V_n \subseteq G_n \) and \( V_n \cap T_n = \emptyset \). By Lemma 2,
\[
\text{Seq}(u) - \bigcup \{ V_n: n \in \omega \}
\]
is an open neighborhood of \( \emptyset \) witnessing that the sets \( G_n, n \in \omega \), cannot form a local \( \pi \)-base at \( \emptyset \).

3. More on the \( \pi \)-character of Seq\((u)\)

Proposition 2. The space Seq\((u)\) is extremally disconnected if (and only if) \( u \) is an ultrafilter.

Proof. Let \( U \subseteq \text{Seq}(u) \) be open. If \( \text{cl} U \) were not open, then there would be an \( s \in \text{cl} U - U \) such that
\[
A = \{ n \in \omega: s \models n \notin \text{cl} U \} \in \mathcal{U}.
\]
Thus
\[
V = \{ s \} \cup \{ t \in \text{Seq}: \exists n \in A \, s \models n \subseteq t \}
\]
would be an open neighborhood of \( s \) disjoint from \( U \).

From this moment on, the symbol \( u \) stands for a free ultrafilter on \( \omega \). In the sequel, we will provide more detailed (than given in Proposition 1) estimation of the \( \pi \)-character of \( \emptyset \) in the space Seq\((u)\), i.e., estimations of \( \pi \chi (\emptyset, \text{Seq}(u)) \).

Definition 1. A collection \( B \subseteq u \) is a base of ultrafilter \( u \) if for each \( a \in u \) there exists \( b \in B \) such that \( b \subseteq a \). We set
\[
\chi (u) = \inf \{ |B|: B \text{ is a base of } u \}.
\]
Definition 2. A free ultrafilter $u$ on $\omega$ is said to be a $P(\kappa)$-point if for each $A \subseteq [u]^{<\kappa}$ there exists $b \in u$ such that $b - a$ is finite for each $a \in A$, i.e., $b \subseteq^* a$ for each $a \in A$. Usually, $P(\omega_1)$-points are referred to as $P$-points.

Definition 3. For $f, g \in \omega^\omega$ the symbol $f \leq^* g$ means that there exists $m \in \omega$ such that $f(n) \leq g(n)$ for each $n > m$. We set

$$b = \inf \{|A|: A \subseteq^* \omega \text{ and } A \text{ is unbounded with respect to } \leq^*\},$$

and

$$d = \inf \{|A|: A \text{ is cofinal (with respect to } \leq^*) \text{ in } \omega^\omega\}.$$

The cardinals $b$ and $d$ are both uncountable and regular whereas $\chi$ is of uncountable cofinality. It is clear that $b \leq d$ (see [4, 7] for more information).

Theorem 1. $\pi \chi(\emptyset, \text{Seq}(u)) \geq d \cdot \chi(u)$.

Proof. Let $S$ be a family of non-empty clopen subsets of the space $\text{Seq}(u)$ such that $|S| < \chi(u)$. Then there exists $a \in u$ such that

$$\varphi_V(s) - a \neq \emptyset \quad \text{for each } V \in S \text{ and } s \in \text{Seq}.$$

If there were a $V \in S$ such that $V \subseteq U(\emptyset, a)$, then we would have $\varphi_V(s) \subseteq a$ for any $s \in V$, which is impossible. Thus $\pi \chi(\emptyset, \text{Seq}(u)) \geq \chi(u)$.

Let $P$ be a clopen local $\pi$-base of $\emptyset$ in the space $\text{Seq}(u)$. For each $V \in P$ define $f_V \in V^\omega$ by setting

$$f_V(s) = \min\{n: s \upharpoonright n \in V\}.$$

Take any $f \in \text{Seq} \omega$ and consider the thinning of $U(\emptyset, \omega)$ with $f$. If $V \in P$ is contained in the thinning, then

$$f_V(s) \geq f(s) \quad \text{holds for each } s \in V.$$

Using standard techniques, we may then get a cofinal (with respect to $\leq^*$) set $A \subseteq^* \text{Seq} \omega$ of cardinality at most that of $P$. Thus $\pi \chi(\emptyset, \text{Seq}(u)) \geq d$. □

Theorem 2. If $u$ is a $P$-point, then $\chi(\emptyset, \text{Seq}(u)) = \pi \chi(\emptyset, \text{Seq}(u)) = d \cdot \chi(u)$.

Proof. Let $B \subseteq u$ be a base of the ultrafilter $u$ and let $A \subseteq^* \text{Seq} \omega$ be cofinal (with respect to $\leq$) in $\text{Seq} \omega$. We shall show that

$$B = \{U(\emptyset, b \setminus g): b \in B \text{ and } g \in A\}$$

constitutes a neighbourhood base of $\emptyset$ in the space $\text{Seq}(u)$.

Take an arbitrary $\varphi \in \text{Seq} u$. Since $u$ is a $P$-point, there exists a $b \in B$ such that

$$b \subseteq^* \varphi(s) \quad \text{for each } s \in \text{Seq}.$$
Let \( f \in \text{Seq} \omega \) be such that
\[
b - f(s) \subseteq \varphi(s) \quad \text{for each } s \in \text{Seq}.
\]
Now if \( g \in A \) is such that \( f \leq g \), then
\[
U(\emptyset, b \setminus g) \subseteq U(\emptyset, \varphi).
\]
Since clearly \( |B| \leq |A| \cdot |B| \), this implies
\[
\chi(\emptyset, \text{Seq}(u)) \leq d \cdot \chi(u).
\]
Moreover, by Theorem 1,
\[
\chi(\emptyset, \text{Seq}(u)) \geq \pi \chi(\emptyset, \text{Seq}(u)) \geq d \cdot \chi(u).
\]

4. \( F_{\sigma \lambda} \)-closedness

- In our discussion that follows, free ultrafilters on \( \omega \) are identified with points of the space \( \beta \omega - \omega \), where \( \beta \omega \) is the Ščech–Stone compactification of \( \omega \). Let us notice that for each \( s \in \text{Seq}(u) \), the subspace \( \{s\} \cup \{s \upharpoonright n : n \in \omega\} \) of \( \text{Seq}(u) \) is homeomorphic to the subspace \( \{u\} \cup \omega \) of \( \beta \omega \).

By Lemma 2, for each free ultrafilter \( u \) on \( \omega \), the space \( \beta \text{Seq}(u) \) is a Hausdorff extremally disconnected compact space. We set \( \text{Seq}^*(u) = \beta \text{Seq}(u) - \text{Seq}(u) \).

A subset \( E \) of a topological space \( X \) is said to be \( F_{\sigma \lambda} \), \( \lambda \geq \omega \), if \( E \) is the union of \( \leq \lambda \) many closed subsets of \( X \).

**Theorem 3.** Let \( u \) be a \( P(\kappa) \)-point, \( \kappa > \omega \), and let \( E \) be an \( F_{\sigma \lambda} \) subset of the space \( \beta \text{Seq}(u) \) such that \( E \cap \text{cl} T_n = \emptyset \) for each \( n = 1, 2, \ldots \). If \( \lambda < \kappa \) and \( b \geq \kappa \), then there exists a clopen neighborhood \( W \) of \( \emptyset \) such that \( W \cap E = \emptyset \).

**Proof.** Let \( E = \bigcup \{F_\xi : \xi < \lambda\} \), where each \( F_\xi \) is a closed subset of \( \beta \text{Seq}(u) \). For each \( \xi < \lambda \) and \( 1 \leq n \leq \omega \) let \( V(\xi, n) \) be any clopen subset of the space \( \beta \text{Seq}(u) \) such that
\[
T_n \subseteq V(\xi, n) \quad \text{and} \quad F_\xi \cap V(\xi, n) = \emptyset.
\]
Let \( \varphi_{\xi n} \) stand for \( \varphi_{V(\xi, n)} \) (see (2.1)). Take any \( \varphi \in \text{Seq} u \) such that
\[
\varphi(s) \subseteq \varphi_{\xi n}(s) \quad \text{for each } \xi < \lambda \; \text{and} \; 1 \leq n < \omega;
\]
such a \( \varphi \) exists since \( u \) is a \( P(\kappa) \)-point.

Let \( \xi < \lambda \) be fixed. If \( s \in \omega \), where \( 1 \leq n < \omega \), then by (4.1), \( f_\xi(s) \in \omega \) exists so that the following holds:
\[
\varphi(s) - f_\xi(s) \subseteq \bigcap \{\varphi_{\xi m}(s) : 1 \leq m \leq n\}.
\]

It follows from formula (4.2) that
\[
\varphi \setminus f_\xi \subseteq \varphi_{\xi n}
\]
(4.3)
for each \( n = 1, 2, \ldots \). Let \( g \in \text{Seq}_{\omega} \) be such that \( f_{\xi} \leq g \) for each \( \xi < \lambda \). Then, in particular, 
\[
(\psi \setminus f_{\xi})(s) \supseteq (\psi \setminus g)(s)
\]
for each \( s \notin T_{k} \) and for some sufficiently large \( k \in \omega \). Since \( T_{k} \subseteq V(\xi, k) \),
\[
(\psi \setminus g)[k] \subseteq \varphi_{\xi k}.
\]
Thus \( \text{cl} U(\emptyset, (\psi \setminus g)[k]) \subseteq \text{cl} U(\emptyset, \varphi_{\xi k}) \subseteq V(\xi, k) \). Since \( T_{k} \subseteq V(\xi, k) \),
\[
(\psi \setminus g)[k] \subseteq \varphi_{\xi k}.
\]
Hence \( U(\emptyset, (\psi \setminus g)[k]) \subseteq U(\emptyset, \varphi_{\xi k}) \subseteq V(\xi, k) \).

It follows from Lemma 2 that
\[
W = \bigcap \{ \text{cl} U(\emptyset, (\psi \setminus g)[k]) : k \in \omega - \{0\} \}
\]
is a required neighborhood of \( \emptyset \). \( \square \)

It is a known fact that the closure of the union of a countable family of closed \( P(\kappa) \)-subsets of \( \omega^{\ast} \) is a \( P(\kappa) \)-subset of \( \omega^{\ast} \) provided that \( b \geq \kappa \) (cf. [7]). We will need the following obvious consequence thereof.

**Lemma 3.** Let \( \{ u_{n} : n \in \omega \} \) be a subset of the space \( \beta \omega \) consisting of \( P(\kappa) \)-points and let \( E \) be an \( F_{\sigma \lambda} \) subset of the space \( \omega^{\ast} \) such that
\[
(\text{cl} \{ u_{n} : n \in \omega \}) \cap E = \emptyset.
\]
If \( \lambda < \kappa \) and \( b \geq \kappa \), then there exists a clopen subset \( V \) of the space \( \beta \omega \) such that
\[
\{ u_{n} : n \in \omega \} \subseteq V \quad \text{and} \quad E \cap V = \emptyset.
\]

**Theorem 4.** Let \( u \) be a \( P(\kappa) \)-point, \( \kappa > \omega \), and let \( E \) be an \( F_{\sigma \lambda} \) subset of the space \( \beta \text{Seq}(u) \) such that \( E \subseteq \text{Seq}^{\ast}(u) \). If \( \lambda < \kappa \) and \( b \geq \kappa \), then there exists a clopen neighborhood \( V \) of \( \emptyset \) such that
\[
V \cap \left( \bigcup \{ E \cap \text{cl} T_{n} : n \in \omega \} \right) = \emptyset.
\]

**Proof.** The subspace \( \text{cl} T_{1} \) of the space \( \beta \text{Seq}(u) \) is a copy of the space \( \beta \omega \) with \( \emptyset \) being a \( P(\kappa) \)-point. Therefore there exists a clopen neighborhood \( V_{1} \) of \( \emptyset \) (in the space \( \beta \text{Seq}(u) \)) such that
\[
V_{1} \cap E \cap \text{cl} T_{1} = \emptyset.
\]
Suppose that for each \( i = 1, 2, \ldots, n \) we have defined clopen (in the space \( \beta \text{Seq}(u) \)) neighborhoods \( V_{i} \) of the point \( \emptyset \) such that
\[
V_{i} \supseteq V_{i-1} \cap T_{i-1}
\]
holds true if \( i = 2, 3, \ldots, n \), and
\[
V_{1} \cap E \cap \text{cl} T_{i} = \emptyset
\]
holds true if \( i = 1, 2, \ldots, n \).

Consider the subspace \( \text{cl} T_{n+1} \) of the space \( \beta \text{Seq}(u) \). It is a copy of the space \( \beta \omega \) with \( L_{n+1} \) being its countable dense discrete part. Let us also notice that \( L_{n} \) is a discrete countable collection of \( P(\kappa) \)-points in the space \( \text{cl} T_{n+1} \). Applying Lemma 3 to the collection
$V_n \cap L_n$ of $P(\kappa)$-points and to the $F_{\sigma\lambda}$ set $E \cap \text{cl} \, T_{n+1}$ we get clopen (in the space $\beta \text{Seq}(u)$) neighborhood $V_{n+1}$ of the point $\emptyset$ such that
\[ V_{n+1} \supseteq V_n \cap L_n \]
and
\[ V_{n+1} \cap E \cap \text{cl} \, T_{n+1} = \emptyset. \]
Since $L_n$ is a dense subset of $T_n$, $V_n \cap T_n \subseteq \text{cl}(V_n \cap L_n) \subseteq V_{n+1}$. So the induction carries through.

By Lemma 2, $V = \bigcap \{V_i: i = 1, 2, \ldots\}$ is a required clopen neighborhood of $\emptyset$. □

**Definition 4.** A subspace $Y$ of space $X$ is said to be $F_{\sigma\lambda}$-closed if for any $F_{\sigma\lambda}$ subset $F$ of $X$,
\[ \text{if } F \subseteq Y, \quad \text{then } \text{cl} \, F \subseteq Y. \]

**Lemma 4.** A subspace $Y$ of space $X$ is $F_{\sigma\lambda}$-closed for each $\lambda < \kappa$ if and only if $X - Y$ is a $P(\kappa)$-subset of $X$.

**Theorem 5 (Main).** Let $\kappa > \omega$. $\text{Seq}^*(u)$ is an $F_{\sigma\lambda}$-closed subspace of the space $\beta \text{Seq}(u)$ for each $\lambda < \kappa$ if and only if $u$ is a $P(\kappa)$-point and $b \geq \kappa$.

**Proof.** ($\Rightarrow$). Let $\lambda < \kappa$ and let $E$ be an $F_{\sigma\lambda}$ subset of the space $\beta \text{Seq}(u)$ such that $E \subseteq \text{Seq}^*(u)$. By Theorem 4, there exists a clopen neighborhood $V$ of $\emptyset$ such that
\[ V \cap \left( \bigcup \{E \cap \text{cl} \, T_n: n \in \omega\} \right) = \emptyset. \]
We can apply Theorem 3 to the $F_{\sigma\lambda}$ set $E \cap V$ and we get a clopen neighborhood $W$ of $\emptyset$ such that $W \cap (E \cap V) = \emptyset$. Hence $V \cap W$ is a clopen neighborhood of $\emptyset$ disjoint from the set $E$.

By the argument of Remark 1, the implication $\Leftarrow$ is proved.

($\Rightarrow$). If $u$ were not a $P(\kappa)$-point, then there would be a $\lambda < \kappa$ and an $F_{\sigma\lambda}$ subset $E$ of the space $\text{cl} \, L_1 - L_1$ such that $\emptyset \in \text{cl} \, E - E$. Hence $E$ would be an $F_{\sigma\lambda}$ subset of the space $\beta \text{Seq}(u)$ such that $E \subseteq \text{Seq}^*(u)$ but $\text{cl} \, E \not\subseteq \text{Seq}^*(u)$.

If $b \geq \kappa$ were not true, then there would be an $A \subseteq \text{Seq}$ such that $|A| = \lambda < \kappa$ and $A$ is unbounded with respect to $\preceq^*$. Without loss of generality we may assume that $A = \{f_\xi: \xi < \lambda\}$ is a collection of strictly increasing functions (with respect to some labeling of $\text{Seq}$ by $\omega$) and is also increasing with respect to $\preceq^*$, i.e., $f_\xi \preceq^* f_\zeta$ whenever $\xi \leq \zeta < \lambda$. Let us point out that for any infinite subset $S$ of $\text{Seq}$, the collection $\{f_\xi \downarrow S: \xi < \lambda\}$ is also unbounded with respect to $\preceq^*$.

For each $\xi < \lambda$, let $A_\xi = \{s \upharpoonright m: s \in \text{Seq} \text{ and } m \preceq f_\xi(s)\}$.  

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Then \( F_\xi = \text{cl} A_\xi - A_\xi \subseteq \text{Seq}^*(u) \) for each \( \xi < \lambda \). Set \( E = \bigcup \{ F_\xi : \xi < \lambda \} \). Suppose that there is a clopen neighborhood \( U \) of \( \emptyset \) disjoint from \( E \). Then \( U \cap A_\xi \) is finite for each \( \xi < \lambda \). The function \( g \), defined on the set \( U \cap \text{Seq} \) by the rule
\[
g(s) = \inf \{ n : s \upharpoonright n \in U \},
\]
satisfies the following:
\[
\left| \{ s \in U \cap \text{Seq} : g(s) < f_\xi(s) \} \right| < \omega,
\]
i.e., \( f_\xi \leq g \) on the infinite set \( U \cap \text{Seq} \). This is impossible; the implication \( \Rightarrow \) is proved.

**Remark 2.** Our Main Theorem is known for \( \kappa = \omega_1 \): \( P(\omega_1) \)-points are \( P \)-points and \( b \geq \omega_1 \) is a ZFC theorem (see [2,1]). However the methods applied there do not generalize to higher cardinals.

5. Dense calibers

For a topological space \( X \) and an uncountable regular cardinal \( \kappa \), we say that \( \kappa \) is a \( d \)-caliber of \( X \), in symbols:
\[
\kappa \in d \text{-cal}(X),
\]
if \( \kappa \in \text{cal}(Y) \), i.e., \( \kappa \) is a caliber of \( Y \) for each dense subset \( Y \) of \( X \).

**Theorem 6.** Let \( \kappa \) be a regular uncountable cardinal. Then for an arbitrary topological space \( X \), the following conditions are equivalent:

(i) \( \kappa \in d \text{-cal}(X) \);

(ii) If \( \{ U_\xi : \xi < \kappa \} \) is a decreasing (with respect to inclusion) collection of non-empty open subsets of \( X \), then \( \text{Int}(\bigcap \{ U_\xi : \xi < \kappa \}) \neq \emptyset \); \n
(iii) \( \hat{c}(X) \leq \kappa \) and if \( \{ G_\xi : \xi < \kappa \} \) is a decreasing (with respect to inclusion) collection of dense open subsets of \( X \), then \( \text{Int}(\bigcap \{ G_\xi : \xi < \kappa \}) \) is dense;

(iv) \( \hat{c}(X) \leq \kappa \) and if \( \{ E_\xi : \xi < \kappa \} \) is an increasing (with respect to inclusion) collection of nowhere dense subsets of \( X \), then \( \bigcup \{ E_\xi : \xi < \kappa \} \) is nowhere dense.

**Proof.** (i) \( \Rightarrow \) (ii). If \( \{ U_\xi : \xi < \kappa \} \) is a decreasing (with respect to inclusion) collection of non-empty open subsets of \( X \) such that \( \text{Int}(\bigcap \{ U_\xi : \xi < \kappa \}) = \emptyset \), then \( \kappa \) is not a caliber of the dense subset \( X - \bigcap \{ U_\xi : \xi < \kappa \} \).

(ii) \( \Rightarrow \) (i). Let \( Y \) be a dense subset of \( X \) and let \( \{ V_\xi : \xi < \kappa \} \) be a collection of non-empty open subsets of \( Y \). Take a collection \( \{ W_\xi : \xi < \kappa \} \) of open subsets of \( X \) such that
\[
W_\xi \cap Y = V_\xi \quad \text{for each } \xi < \kappa.
\]
Set
\[
U_\alpha = \bigcup \{ W_\xi : \alpha \leq \xi < \kappa \}.
\]
Since \( \{ U_\xi : \xi < \kappa \} \) is a decreasing (with respect to inclusion) collection of non-empty open subsets of \( X \), Int(\( \bigcap \{ U_\xi : \xi < \kappa \} \)) \( \neq \emptyset \). Clearly, if \( p \in Y \cap \text{Int}(\bigcap \{ U_\xi : \xi < \kappa \}) \), then \( p \) belongs to \( \kappa \) many sets \( V_\xi \). This proves that \( \kappa \in d\, \text{cal}(Y) \), hence \( \kappa \in d\, \text{cal}(X) \).

It is clear that the equivalence (iii) \( \Leftrightarrow \) (iv) holds true (even if we drop the assumption \( \hat{c}(X) \leq \kappa \)).

(ii) \( \Rightarrow \) (iii). Since \( \kappa \in d\, \text{cal}(X) \) implies that \( \kappa \in \text{cal}(X) \), we clearly have \( \hat{c}(X) \leq \kappa \).

If \( \{ G_\xi : \xi < \kappa \} \) were a decreasing (with respect to inclusion) collection of dense open subsets of \( X \) and Int(\( \bigcap \{ G_\xi : \xi < \kappa \} \)) were not dense, say \( V \) is non-empty open and \( V \cap \text{Int}(\bigcap \{ G_\xi : \xi < \kappa \}) = \emptyset \), then setting \( G_\xi \cap V = U_\xi \) we would get a violation of (ii) by the collection \( \{ U_\xi : \xi < \kappa \} \).

(iii) \( \Rightarrow \) (ii). Let \( \{ U_\xi : \xi < \kappa \} \) be a decreasing (with respect to inclusion) collection of non-empty open subsets of \( X \). Since \( \hat{c}(X) \leq \kappa \), there exists \( \alpha < \kappa \) such that

\[
\text{cl}(X - \text{cl}(U_\alpha)) \cup U_\xi : \alpha \leq \xi < \kappa
\]

is a decreasing (with respect to inclusion) collection of dense open subsets of \( X \). Since \( \bigcap \{ \text{cl}(X - \text{cl}(U_\alpha)) \cup U_\xi : \alpha \leq \xi < \kappa \} = (X - \text{cl}(U_\alpha)) \cup \bigcap \{ U_\xi : \alpha \leq \xi < \kappa \} \),

\[
\text{Int}(\bigcap \{ U_\xi : \xi < \kappa \}) = \text{Int}(\bigcap \{ U_\xi : \alpha \leq \xi < \kappa \}) \neq \emptyset.
\]

This finishes the proof of our theorem.

**Proposition 3.** Let \( X \) be a regular space such that \( \hat{c}(X) \leq \rho \), where \( \rho \) is a regular uncountable cardinal. Suppose that \( X \) possesses a dense subspace \( S \) with \( \rho \in d\, \text{cal}(S) \) and a \( \pi \)-base \( \mathcal{P} \) which satisfy the following condition:

\((\otimes)\) If \( V \in \mathcal{P} \) and if \( \{ F_\xi : \xi < \rho \} \) is an increasing (with respect to inclusion) collection of closed subsets of \( X \) such that \( F_\xi \subseteq \text{cl}(V - S) \) for each \( \xi < \rho \), then \( \bigcup \{ F_\xi : \xi < \rho \} \) is nowhere dense.

Then \( \rho \in d\, \text{cal}(X) \).

**Proof.** We will verify condition (iv) of Theorem 6. To this end, let \( \{ E_\xi : \xi < \rho \} \) be an increasing (with respect to inclusion) collection of nowhere dense subsets of \( X \). To show that \( \bigcup \{ E_\xi : \xi < \rho \} \) is nowhere dense, take an open non-empty set \( U \). Since \( \rho \in d\, \text{cal}(S) \), we can apply (iv) of Theorem 6 to conclude that

\[
\bigcup \{ \text{cl}(E_\xi \cap S) : \xi < \rho \}
\]

is nowhere dense in \( S \) and hence in \( X \). By the regularity of \( X \) we may pick a \( V \in \mathcal{P} \) such that

\[
\text{cl}(V) \subseteq U - \left( \bigcup \{ \text{cl}(E_\xi \cap S) : \xi < \rho \} \right).
\]
We can then apply \((\otimes)\) to the collection \(\{ F_\xi : \xi < \rho \}\), where \(F_\xi = \text{cl} V \cap \text{cl} E_\xi\). Thus there exists a non-empty open set \(W\) such that
\[
W \subseteq V - \left( \bigcup \{ F_\xi : \xi < \rho \} \right).
\]
But then \(W \subseteq U\) and \(W \cap \left( \bigcup \{ E_\xi : \xi < \rho \} \right) = \emptyset\).

**Corollary 1.** Let \(X\) be a regular space. Suppose that \(X\) possesses a countable dense and co-dense subset \(D\) that is a \(P(\kappa)\)-set in \(X\). If \(\lambda < \kappa\) and \(\lambda\) is a regular uncountable cardinal, then \(\lambda \in d_\text{cal}(X)\).

**Proof.** Proposition 3 applies to the space \(X\) with \(\lambda\) as \(\rho\), the countable dense set \(D\) as \(S\), and with any \(\pi\)-base \(P\) of \(X\).

**Corollary 2.** Let \(u\) be a \(P(\kappa)\)-point and let \(b \geq \kappa\). If \(\lambda < \kappa\) is an uncountable regular cardinal, then \(\lambda \in d_\text{cal}(\beta \text{Seq}(u))\).

**Proof.** By Theorem 5, Corollary 1 applies with \(X = \beta \text{Seq}(u)\) and \(D = \text{Seq}(u)\).

**Corollary 3.** Assume MA and \(2^\omega \geq \omega_2\). Then there exists a compact Hausdorff space \(X\) such that
\[
\omega_1 \in d_\text{cal}(X)
\]
and
\[
dt(X) \geq \omega_2.
\]

**Proof.** It is well known that MA implies both the existence of a \(P(2^\omega)\)-point and that \(b = 2^\omega\). Thus, if \(u\) is a \(P(2^\omega)\)-point, then the space \(\beta \text{Seq}(u)\) is as required.

**References**