Some properties of equivalence soft set relations

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\textbf{ABSTRACT}

The soft set theory is a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Babitha and Sunil [Soft set relations and function, Computers and Mathematics with Applications 60 (2010) 1840–1849] introduced the notion of soft set relations as a soft subset of the Cartesian product of soft sets and discussed many related concepts such as equivalence soft set relations, partitions and functions. In this paper, we further study the equivalence soft set relations and obtain soft analogues of many results concerning ordinary equivalence relations and partitions. Furthermore, we introduce and discuss the transitive closure of a soft set relation and prove that the poset of the equivalence soft set relations on a given soft set is a complete lattice with the least element and greatest element.

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1. Introduction

In the study for solving complicated problems in economics, engineering, environmental sciences and medical sciences, we cannot successfully use classical methods because of different kinds of incomplete knowledge. The wide scope of theories such as probability theory [1], fuzzy set theory [2–6], intuitionistic fuzzy set theory [7,8], rough set theory [9], vague set theory [10] and the interval mathematics [11,12] are well known and often useful mathematical approaches for modeling uncertainty. However, what these theories can handle is merely a proper part of uncertainty. Each of these theories has its inherent difficulties as mentioned by Molodtsov [13]. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of these theories. Molodtsov [13] introduced soft sets as a mathematical tool for dealing with uncertainties which is free from the above-mentioned difficulties. This theory has proven useful in many different fields such as decision making [14–19], data analysis [20], forecasting [21] and simulation [22].

The notions and basic properties of soft set theory were presented in [13,23]. In classical soft set theory, a situation may be complex in real world because of the fuzzy nature of parameters. With this point of view, the classical soft sets have been extended to fuzzy soft sets [24,25], intuitionistic fuzzy soft sets [26–28], vague soft sets [29], interval-valued fuzzy soft sets [30] and interval-valued intuitionistic fuzzy soft sets [31], respectively. Up to the present, soft set theory has been applied to several algebra structures: groups [32,33], semigroups [34], BCK/BCI-algebras [35–37], ordered semigroups and BL-algebras [38,39]. Shabir and Naz [40] applied soft set theory to topological structure and pointed out that a soft set topological space gives a parameterized family of topologies on universe but the converse is not true. Algebraic structures of soft sets have been studied by some researchers. Maji et al. [23] presented some definitions on soft sets such as soft subset, the complement of s soft set. Based on the analysis of several operations on soft sets in [23], Ali et al. [41] presented some new algebraic operations for soft sets and proved that certain De Morgan’s laws hold in soft set theory with respect to these new definitions. Algebraic structures of soft set associated with new operations were studied by Ali et al. [42]. The concept...
of soft equality and some related properties are derived by Qin and Hong [43]. Çağman and Enginoğlu [18] studied products of soft sets and uni-int decision functions. Sezgin and Atagün [44] proved that certain De Morgan’s laws hold in soft set theory with respect to different operations and extend the theoretical aspect of operations on soft sets. Recently, Babitha and Sunil [45] have proposed soft set relations and many related concepts are discussed. Yang and Guo [46] introduced the notions of kernels and closures of soft set relation and soft set relation mappings and obtained some related properties. In this paper, we attempt to conduct a further study along Babitha and Sunil’s work.

The purpose of present paper is a further attempt to broaden the theoretical aspects of soft sets. We refer to [47,48] in order to refresh the fundamental concepts of set theory. The rest of this paper is organized as follows. In Section 2, basic notions about soft sets are reviewed. Section 3 focuses on equivalence soft set relations and gives soft analogues of many results concerning ordinary equivalence relations and partitions. Section 4 presents the concept of transitive closure of a soft set relation with related results. We prove that the poset (ESSR((F, A)), ⊆) of the equivalence soft set relations on a given soft set (F, A) is a complete lattice with the least element and greatest element. The last section summarizes all the contributions and points out our future research work.

2. Preliminaries

In this section, we recall some basic notions in soft set theory. Let U be an initial universe of objects and E the set of parameters in relation to objects in U. Parameters are often attributes, characteristics or properties of objects. Let \( \mathcal{P}(U) \) denote the power set of \( U \) and \( A, B \subseteq E \).

**Definition 1** [13]. A pair (\( F, A \)) is called a soft set over \( U \), where \( F \) is a function given by

\[
F : A \rightarrow \mathcal{P}(U).
\]

In other words, a soft set over \( U \) is a parameterized family of subsets of the universe \( U \). For any parameter \( x \in A, F(x) \) may be considered as the set of \( x \)-approximate elements of the soft set \( (F, A) \).

**Definition 2.** Let \((F, A)\) and \((G, B)\) be two soft sets over \( U \). Then we have the following.

1. \((F, A)\) is called a soft subset [23] of \((G, B)\), denoted by \((F, A) \subseteq (G, B)\), if \( A \subseteq B \) and \( F(a) \subseteq G(x) \) for all \( x \in A \).
2. \((F, A)\) is called a soft superset [23] of \((G, B)\), denoted by \((F, A) \supseteq (G, B)\), if \( G(B) \) is a soft subset of \((F, A)\).
3. \((F, A)\) is called soft equal [23] to \((G, B)\) if \((F, A) \subseteq (G, B)\) and \((F, A) \supseteq (G, B)\), denoted by \((F, A) = (G, B)\).
4. \((F, A)\) is called a relative null soft set [41] (with respect to the parameter set \( A \)), denoted by \( F_{A} \), if \( F(x) = \emptyset \) for all \( x \in A \).
5. \((F, A)\) is called a relative whole soft set [41] (with respect to the parameter set \( A \)), denoted by \( U_{A} \), if \( F(x) = U \) for all \( x \in A \).
6. The complement [41] of \((F, A)\), denoted by \((F, A)^{c}\), is defined by \((F, A)^{c} = (F^{c}, A)\), where \( F^{c} : A \rightarrow \mathcal{P}(U) \) is a function given by \( F^{c}(x) = U \setminus F(x) \) for all \( x \in A \).

The relative whole soft set with respect to the set of parameters \( E \) is called the absolute soft set over \( U \) and is simply denoted by \( U_{E} \). In a similar way, the relative null soft set with respect to \( E \) is called the null soft set over \( U \) and is denoted by \( U_{E} \).

Clearly, \( U_{A}^{c} = \Phi_{A} \), \( \Phi_{A}^{c} = U_{A} \), and \( \Phi_{A} \subseteq (F, A) \subseteq U_{A} \subseteq U_{E} \) for all soft sets \((F, A)\) over \( U [42]\).

**Definition 3.** Let \((F, A)\) and \((G, B)\) be two soft sets over \( U \). Then we have the following.

1. The union [23] of \((F, A)\) and \((G, B)\) is the soft set \((H, C)\), where \( C = A \cup B \) and for all \( x \in C \),

\[
H(x) = \begin{cases} 
F(x), & \text{if } x \in A \setminus B, \\
G(x), & \text{if } x \in B \setminus A, \\
F(x) \cup G(x), & \text{if } x \in A \cap B,
\end{cases}
\]

and is written as \((F, A) \cup (G, B) = (H, C)\).

2. The intersection [41] of \((F, A)\) and \((G, B)\) is the soft set \((H, C)\), where \( C = A \cap B \) and for all \( x \in C \),

\[
H(x) = \begin{cases} 
F(x), & \text{if } x \in A \setminus B, \\
G(x), & \text{if } x \in B \setminus A, \\
F(x) \cap G(x), & \text{if } x \in A \cap B.
\end{cases}
\]

and is written as \((F, A) \cap (G, B) = (H, C)\).

The following shows that the basic properties of operations such as union, intersection and De Morgan’s laws for union, intersection and complement on soft sets.

**Proposition 1.** For two soft sets \((F, A)\) and \((G, B)\) over \( U \), the following are true.

1. \((F, A) \cap (F, A) = (F, A) [23]\), \((F, A) \cup (F, A) = (F, A) [41]\).
3. Equivalence soft set relations

**Definition 4** ([45]). Let \((F, A)\) and \((G, B)\) be two soft sets over a universe \(U\). Then the Cartesian product of \((F, A)\) and \((G, B)\) is defined as \((F, A) \times (G, B) = (H, A \times B)\), where \(H : A \times B \rightarrow P(U \times U)\) and \(H(a, b) = F(a) \times G(b)\) for all \((a, b) \in A \times B\), i.e., \(H(a, b) = \{(h_i, h_j) : h_i \in F(a)\} \cap \{(h_i, h_j) : h_j \in G(b)\}\).

**Example 1** ([45]). Consider the soft set \((F, A)\) which describes the “peoples having different jobs” and the soft set \((G, B)\) which describes the “peoples qualified in various courses” in a social gathering. Suppose that \(U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}\}\) denotes the set of peoples in a social gathering, \(A = \{\text{chartered account, doctor, engineer, teacher}\}\) and \(B = \{\text{B.Sc., B.Tech., MBBS, M.Sc.}\}\). Let \(F\) (chartered account) = \(h_1, h_2\), \(F\) (doctor) = \(h_3, h_4\), \(F\) (engineer) = \(h_5, h_6\) and \(F\) (teacher) = \(h_7, h_8\), \(G\) (B.Sc.) = \(h_1, h_2, h_3, h_{10}\), \(G\) (B.Tech.) = \(h_4, h_5, h_7, h_9\) and \(G\) (MBBS) = \(h_6, h_8, h_9\) and \(G\) (M.Sc.) = \(h_3, h_5\).

Now \((F, A) \times (G, B) = (H, A \times B)\) where a typical element will look like

\[
H(\text{doctor, MBBS}) = (h_4, h_5) \times (h_3, h_4, h_5, h_8) = \{(h_4, h_3), (h_4, h_4), (h_4, h_5), (h_4, h_8), (h_5, h_3), (h_5, h_4), (h_5, h_5), (h_5, h_8)\}.
\]

**Definition 5** ([45]). Let \((F, A)\) and \((G, B)\) be two soft sets over a universe \(U\). Then a soft set relation from \((F, A)\) to \((G, B)\) is a soft subset of \((F, A) \times (G, B)\).

In other words, a soft set relation from \((F, A)\) to \((G, B)\) is of the form \((H_1, S)\), where \(S \subseteq A \times B\) and \(H_1(a, b) = H(a, b)\) for all \((a, b) \in S\), where \((H, A \times B) = (F, A) \times (G, B)\) as defined in **Definition 4**. Any soft subset of \((F, A) \times (F, A)\) is called a soft set relation on \((F, A)\).

In an equivalent way, we can define the soft set relation \(R\) on \((F, A)\) in the parameterized form as follows: if \((F, A) = \{F(a), F(b), \ldots\}\), then

\[
F(a) \circ R \circ F(b) \iff F(a) \times F(b) \in R.
\]

**Definition 6.** Let \(R\) and \(\delta\) be soft set relations on \((F, A)\). Then

1. The inverse of the relation \(R\) is the soft set relation on \((F, A)\), denoted by \(R^{-1}\), is defined by \(R^{-1} = \{F(a) \times F(b) : F(a) \times F(b) \in R\}\) [45];

2. The union of two soft set relations \(R\) and \(\delta\) on \((F, A)\), denoted by \(R \cup \delta\), is defined by \(R \cup \delta = \{F(a) \times F(b) : F(a) \times F(b) \in R\} \cup \{F(a) \times F(b) : F(a) \times F(b) \in \delta\}\) [46];

3. The intersection of two soft set relations \(R\) and \(\delta\) on \((F, A)\), denoted by \(R \cap \delta\), is defined by \(R \cap \delta = \{F(a) \times F(b) : F(a) \times F(b) \in R\} \cap \{F(a) \times F(b) : F(a) \times F(b) \in \delta\}\) [46];

4. \(R \subseteq \delta\) if for any \(a, b \in A\), \(F(a) \times F(b) \in R \Rightarrow F(a) \times F(b) \in \delta\) [46].

**Example 2.** Consider a soft set \((F, A)\) over \(U\) where \(U = \{h_1, h_2, h_3, h_4\}\), \(A = \{m_1, m_2\}\), \(F(m_1) = \{h_1, h_2\}\) and \(F(m_2) = \{h_2, h_3, h_4\}\). Two soft set relations \(R\) and \(\delta\) on \((F, A)\) are given by

\[
R = \{F(m_1) \times F(m_1), F(m_2) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_2)\}, \]

\[
\delta = \{F(m_1) \times F(m_1), F(m_2) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_2)\}.
\]

Then the union and intersection of \(R\) and \(\delta\) are

\[
R \cup \delta = \{F(m_1) \times F(m_1), F(m_2) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_2)\},
\]

\[
R \cap \delta = \{F(m_1) \times F(m_1)\}.
\]

Consider another soft set relation \(Q\) on \((F, A)\) given by

\[
Q = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_2)\}.
\]

Then \(R \subseteq Q\) but \(\delta \not\subseteq Q\).
Definition 7 ([45]). Let $\mathcal{R}$ be a soft set relation from $(F, A)$ to $(G, B)$ and $\mathcal{S}$ be a soft set relation from $(G, B)$ to $(H, C)$. Then the composition of $\mathcal{R}$ and $\mathcal{S}$, denoted by $\mathcal{S} \circ \mathcal{R}$, is a soft set relation from $(F, A)$ to $(H, C)$ defined as follows: if $F(a) \in (F, A)$ and $H(c) \in (H, C)$, then

$$F(a) \times H(c) \in \mathcal{S} \circ \mathcal{R} \iff F(a) \times G(b) \in \mathcal{R} \quad \text{and} \quad G(b) \times H(c) \in \mathcal{S} \quad \text{for some } G(b) \in (G, B).$$

Definition 8 ([45]). Let $(F, A)$ be a soft set over $U$. The identity soft set relation $I_{FA}$ on $(F, A)$ is defined as follows: $F(a) \times F(b) \in I_{FA}$ if and only if $a = b$. That is, $I_{FA} = \{F(a) \times F(a) : F(a) \in (F, A)\}$.

Clearly, $I_{FA}^{-1} = I_{FA}$ and $I_{FA} \circ I_{FA} = I_{FA}$.

Proposition 2. Let $\mathcal{R}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{S}, \mathcal{S}_1$ and $\mathcal{S}_2$ be soft set relations on $(F, A)$. Then

1. $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$.
2. If $\mathcal{R}_1 \subseteq \mathcal{S}_1$ and $\mathcal{R}_2 \subseteq \mathcal{S}_2$, then $\mathcal{R}_1 \circ \mathcal{R}_2 \subseteq \mathcal{S}_1 \circ \mathcal{S}_2$.
3. $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$.
4. $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) \subseteq (\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3)$.
5. $(\mathcal{R}_1 \circ \mathcal{R}_2)^{-1} = \mathcal{R}_2^{-1} \circ \mathcal{R}_1^{-1}$ [46].
6. $(\mathcal{R}_1 \cup \mathcal{R}_2)^{-1} = \mathcal{R}_1^{-1} \cup \mathcal{R}_2^{-1}$, $(\mathcal{R}_1 \cap \mathcal{R}_2)^{-1} = \mathcal{R}_1^{-1} \cap \mathcal{R}_2^{-1}$ [46].
7. $\mathcal{R} \subseteq \mathcal{S}, \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{S}$.
8. $\mathcal{R} \subseteq \mathcal{S}, \mathcal{R}_1 \subseteq \mathcal{S}_2, \mathcal{R}_2 \subseteq \mathcal{S}_3, \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_3$.

Proof. Obviously, (1) and (2) hold. We only show (3).

(3) By Definitions 6(2) and 7.

$$F(a) \times F(b) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) \iff F(a) \times F(c) \in \mathcal{R}_1 \quad \text{and} \quad F(c) \times F(b) \in \mathcal{R}_2 \cup \mathcal{R}_3 \quad \text{for some } F(c) \in (F, A).$$

Hence $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$.

Example 3. Consider a soft set $(F, A)$ over $U$ where $U = \{h_1, h_2, h_3, h_4\}, A = \{m_1, m_2, m_3\}, F(m_1) = \{h_1, h_2\}, F(m_2) = \{h_2, h_4\}$ and $F(m_3) = \{h_1, h_3, h_4\}$. Three soft set relations $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ on $(F, A)$ are given by

$$\mathcal{R}_1 = \{(F(m_1) \times F(m_1), F(m_2) \times F(m_1))\},$$
$$\mathcal{R}_2 = \{(F(m_1) \times F(m_2), F(m_2) \times F(m_2))\},$$
$$\mathcal{R}_3 = \{(F(m_1) \times F(m_3), F(m_3) \times F(m_1))\}.$$

Then $\mathcal{R}_2 \cup \mathcal{R}_3, \mathcal{R}_1 \circ \mathcal{R}_2$ and $\mathcal{R}_1 \circ \mathcal{R}_3$ are given by

$$\mathcal{R}_2 \cup \mathcal{R}_3 = \{(F(m_1) \times F(m_2), F(m_1) \times F(m_3), F(m_2) \times F(m_2), F(m_3) \times F(m_1))\},$$
$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{(F(m_1) \times F(m_1), F(m_2) \times F(m_1))\}, \quad \mathcal{R}_1 \circ \mathcal{R}_3 = \{(F(m_3) \times F(m_1))\},$$
and thus

$$\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = \{(F(m_1) \times F(m_1), F(m_2) \times F(m_1), F(m_3) \times F(m_1))\}$$
$$= (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3).$$

Now, we redefine the notions of reflexivity, symmetry and transitivity of soft set relations and effectively used to prove their properties.

Definition 9. Let $\mathcal{R}$ be a soft set relation on $(F, A)$, then $\mathcal{R}$ is said to be

1. reflexive if $I_{FA} \subseteq \mathcal{R}$;
2. symmetric if $\mathcal{R}^{-1} = \mathcal{R}$;
3. transitive if $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$;
4. equivalence soft set relation if it is reflexive, symmetric and transitive.

Theorem 1. Let $\mathcal{R}$ and $\mathcal{S}$ be two soft set relations on $(F, A)$.

1. $\mathcal{R}$ is equivalence if and only if $\mathcal{R}^{-1}$ is equivalence.
2. If $\mathcal{R}$ and $\mathcal{S}$ are equivalence, then so are $\mathcal{R} \circ \mathcal{R}$ and $\mathcal{R} \cup \mathcal{S}$.
3. If $\mathcal{R}$ is equivalence, $\mathcal{R} \circ \mathcal{R} = \mathcal{R}$.
4. If $\mathcal{R}$ and $\mathcal{S}$ are equivalence, then $\mathcal{R} \cup \mathcal{S}$ is equivalence if and only if $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{R} \cup \mathcal{R}$ and $\mathcal{S} \circ \mathcal{R} \subseteq \mathcal{R} \cup \mathcal{R}$. 
Proof. (1) Since $I_{FA} \subseteq R \iff I_{FA} \subseteq R^{-1}$, $R$ is reflexive $\iff R^{-1}$ is reflexive.

By Proposition 2(5), $R$ is symmetric $\iff R^{-1} = R = (R^{-1})^{-1} \iff R^{-1}$ is symmetric.

By Proposition 2(4), $R$ is transitive $\iff R \circ R \subseteq R \iff R \circ R \subseteq R^{-1} \iff R^{-1}$ is transitive.

(2) First, we show that $R \circ R$ is an equivalence soft set relation. Since $R$ is reflexive, $I_{FA} \subseteq R$ and hence by Proposition 2(2), $I_{FA} = I_{FA} \circ I_{FA} \subseteq R \circ R$, i.e., $R \circ R$ is reflexive. Since $R$ is symmetric, $R^{-1} = R$ and hence by Proposition 2(6), $(R \circ R)^{-1} = R^{-1} \circ R^{-1} = R \circ R$, i.e., $R \circ R$ is symmetric. Since $R$ is transitive, $R \circ R \subseteq R$ and hence by Proposition 2(2), $(R \circ R) \circ (R \circ R) \subseteq R \circ R$, i.e., $R \circ R$ is transitive.

Next, we show that $R \cap \delta$ is an equivalence soft set relation. Since $R$ and $\delta$ are reflexive, $I_{FA} \subseteq R$ and $I_{FA} \subseteq \delta$, and hence $I_{FA} \subseteq R \cap \delta$, i.e., $R \cap \delta$ is reflexive. Since $R$ and $\delta$ are symmetric, $R^{-1} = R$ and $\delta^{-1} = \delta$ and hence by Proposition 2(6), $(R \cap \delta)^{-1} = R^{-1} \cap \delta^{-1} = R \cap \delta$, i.e., $R \cap \delta$ is symmetric. Since $R$ and $\delta$ are transitive, $R \circ R \subseteq R$ and $\delta \circ \delta \subseteq \delta$ and hence by Proposition 2(2), $(R \cap \delta) \circ (R \cap \delta) = (R \cap \delta \circ R) \cap (R \cap \delta \circ \delta) \subseteq (R \circ R) \cap (\delta \circ \delta) \subseteq R \cap \delta$, i.e., $R \cap \delta$ is transitive.

(3) Since $R$ is transitive, $R \circ R \subseteq R$. So we show that $R \subseteq R \circ \delta$. Let $F(a) \times F(b) \in R$. Since $R$ is reflexive, $F(b) \times F(b) \in R$ and thus $F(a) \times F(b) \in R \circ R$, i.e., $R \subseteq R \circ \delta$. Hence $R = R \circ \delta$.

(4) Suppose that $R \cup \delta$ is equivalence. Then

$$F(a) \times F(b) \in R \cup \delta \iff F(a) \times F(c) \in \delta \quad \text{and} \quad F(c) \times F(b) \in R \quad \text{for some } c \in A$$

$$\Rightarrow (F(a) \times F(c)) \in \delta \cup R \quad \text{and} \quad (F(c) \times F(b)) \in \delta \cup R$$

$$\Rightarrow (F(a) \times F(b)) \in \delta \cup R, \quad \text{by equivalence of } \delta \cup R.$$

Hence $R \circ \delta \subseteq R \cup \delta$. Similarly, we have $\delta \circ R \subseteq R \cup \delta$.

Conversely, suppose that $R \circ \delta \subseteq R \cup \delta$ and $\delta \circ R \subseteq R \cup \delta$. Since $R$ and $\delta$ are reflexive, by the hypothesis, $I_{FA} = I_{FA} \circ I_{FA} \subseteq R \circ \delta \subseteq \delta \circ R \subseteq R \cup \delta$, i.e., $R \cup \delta$ is reflexive. Since $R$ and $\delta$ are symmetric, by Proposition 2(6), $(R \cup \delta)^{-1} = R^{-1} \cup \delta^{-1} = R \cup \delta$, i.e., $R \cup \delta$ is symmetric. Since $R$ and $\delta$ are transitive, by Proposition 2(3), (7) and the hypothesis, $(R \cup \delta) \circ (R \cup \delta) = [(R \cap \delta) \cup (\delta \cap R)] \cup [(\delta \cap \delta) \cap (\delta \cap \delta)] \subseteq [R \cap (R \cup \delta) \cup (\delta \cap \delta)] = R \cup \delta$, i.e., $R \cup \delta$ is transitive. Hence $R \cup \delta$ is equivalence.

Proposition 3. Let $R$ and $\delta$ be soft set relations on $(F, A)$.

(1) If $R$ is reflexive and $\delta$ is reflexive and transitive, then $\delta \subseteq R$ if and only if $R \circ \delta = \delta$.

(2) If $R$ and $\delta$ are reflexive, then so is $R \circ \delta$.

Proof. (1) Suppose that $R \subseteq \delta$. Since $R$ is reflexive, $I_{FA} \subseteq R$ and then $\delta = I_{FA} \circ \delta \subseteq R \circ \delta$. On the other hand, since $\delta$ is transitive, by Proposition 2(5), $R \circ \delta \subseteq \delta \circ \delta \subseteq \delta$. Hence $R \circ \delta = \delta$.

Conversely, suppose that $R \circ \delta = \delta$. Since $\delta$ is reflexive, by Proposition 2(2), $R = R \circ I_{FA} \subseteq R \circ \delta = \delta$. Hence $R \subseteq \delta$.

(2) Since $R$ and $\delta$ are reflexive, by Proposition 2(2), $I_{FA} = I_{FA} \circ I_{FA} \subseteq R \circ \delta$. Hence $R \circ \delta$ is reflexive.

Theorem 2. Let $R$ and $\delta$ be equivalence soft set relations on $(F, A)$. Then $R \circ \delta$ is equivalence if and only if $R \circ \delta = \delta \circ R$.

Proof. Suppose that $R \circ \delta = \delta \circ R$. Since $R$ and $\delta$ are symmetric, $R^{-1} = R$ and $\delta^{-1} = \delta$. Since $R \circ \delta$ is symmetric, by Proposition 2(5), $R \circ \delta = (R \circ \delta)^{-1} = \delta^{-1} \circ R^{-1} = \delta \circ R$.

Conversely, suppose that $R \circ \delta = \delta \circ R$. Then, by Proposition 3(2), $R \circ \delta$ is reflexive. Since $R$ and $\delta$ are symmetric, by the hypothesis, $(R \circ \delta)^{-1} = \delta^{-1} \circ R^{-1} = \delta \circ R = R \circ \delta$, i.e., $R \circ \delta$ is symmetric. On the other hand, since $R$ and $\delta$ are transitive, by Proposition 2(3) and the hypothesis, $(R \circ \delta) \circ (R \circ \delta) = (R \circ R) \circ (\delta \circ \delta) \subseteq R \circ \delta$, i.e., $R \circ \delta$ is transitive. Hence $R \circ \delta$ is an equivalence soft set relation.

Remark 1. Let $\{R_{\gamma} : \gamma \in \Gamma\}$ be a family of equivalence soft set relations on $(F, A)$. Then, clearly, $\bigcap_{\gamma \in \Gamma} R_{\gamma}$ is a soft set relation on $(F, A)$. But, in general, $\bigcup_{\gamma \in \Gamma} R_{\gamma}$ need not be equivalence soft set relation on $(F, A)$.

Example 4. Let $(F, A)$ be a soft set over $U$ where $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$, $A = \{m_1, m_2, m_3\}$ and $F(m_1) = \{h_1, h_2, h_5, h_6\}$, $F(m_2) = \{h_3, h_4, h_7, h_8\}$, $F(m_3) = \{h_2, h_4, h_9\}$. Consider equivalence soft set relations $R$ and $\delta$ on $(F, A)$ given by

$$R = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_2) \times F(m_2), F(m_2) \times F(m_2), F(m_2) \times F(m_3), F(m_3) \times F(m_2), F(m_3) \times F(m_3), F(m_3) \times F(m_3)\}.$$

$$\delta = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_3), F(m_2) \times F(m_2), F(m_2) \times F(m_3), F(m_3) \times F(m_1), F(m_3) \times F(m_2), F(m_3) \times F(m_3)\}.$$

Then $R \cup \delta$ is a soft set relation on $(F, A)$ given by

$$R \cup \delta = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_1) \times F(m_3), F(m_2) \times F(m_1), F(m_2) \times F(m_2), F(m_2) \times F(m_3), F(m_3) \times F(m_1), F(m_3) \times F(m_2), F(m_3) \times F(m_3)\}.$$

Since $F(m_2) \times F(m_2), F(m_1) \times F(m_3) \in R \cup \delta$, $F(m_2) \times F(m_3) \in (R \cup \delta) \circ (R \cup \delta)$ but $F(m_2) \times F(m_3) \notin R \cup \delta$, i.e., $(R \cup \delta) \circ (R \cup \delta) \not\subseteq (R \cup \delta)$. This shows that $R \cup \delta$ is not transitive. Hence $R \cup \delta$ is not an equivalence soft set relation.
Hence, let $F$ be an equivalence soft set relation on $(F, A)$ and $a \in A$. Then equivalence class of $F(a)$, denoted by $F(a)/\mathcal{R}$, is defined as $F(a)/\mathcal{R} = \{F(b) : F(a) \times F(b) \in \mathcal{R}\}$. The set of $\{F(a)/\mathcal{R} : a \in A\}$ is called the quotient soft set of $(F, A)$ and is denoted by $(F, A)/\mathcal{R}$.

**Theorem 3.** Let $\mathcal{R}$ be an equivalence soft set relation on $(F, A)$ and $a, b \in A$. Then

1. Every $F(a)/\mathcal{R}$ is a non null soft subset of $(F, A)$.
2. $F(a)/\mathcal{R} = F(b)/\mathcal{R}$ if and only if $F(a) \times F(b) \in \mathcal{R}$ if and only if $F(a)/\mathcal{R} \sim F(b)/\mathcal{R} \neq \emptyset$.

**Proof.** (1) Since $\mathcal{R}$ is reflexive, $F(a) \times F(a) \in \mathcal{R}$ for any $a \in A$ and hence by **Definition 10**, $F(a) \in F(a)/\mathcal{R}$. Hence $F(a)/\mathcal{R}$ is non null soft subset of $(F, A)$.

(2) By Lemma 4.5 of [45], $F(a)/\mathcal{R} = F(b)/\mathcal{R}$ if and only if $F(a) \times F(b) \in \mathcal{R}$ if and only if $F(a)/\mathcal{R} \sim F(b)/\mathcal{R} \neq \emptyset$. Since $\mathcal{R}$ is an equivalence soft set relation on $(F, A)$,

$F(a)/\mathcal{R} \sim F(b)/\mathcal{R} \neq \emptyset$ ⇔ $F(c) \in F(a)/\mathcal{R} \sim F(b)/\mathcal{R}$ for some $F(c) \in F(a)/\mathcal{R}$

⇒ $F(a) \times F(c) \in \mathcal{R}$ and $F(c) \times F(b) \in \mathcal{R}$

⇒ $F(a) \times F(b) \in \mathcal{R}$.

Hence $F(a) \times F(b) \in \mathcal{R}$ if and only if $F(a)/\mathcal{R} \sim F(b)/\mathcal{R} \neq \emptyset$. □

**Definition 11 ([45]).** A collection $\mathcal{P} = \{\langle F_i, A_i \rangle : i \in I\}$ of nonempty soft subsets of soft set $(F, A)$ is called a partition of $(F, A)$ if

1. $\bigcup_{i \in I} \langle F_i, A_i \rangle = \langle F, A \rangle$;
2. $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

**Definition 12 ([45]).** Let $\mathcal{P} = \{\langle F_i, A_i \rangle \}$ be a partition of $(F, A)$. We define a soft set relation $(F, A)/\mathcal{P}$ on $(F, A)$ by $F(a) \times F(b) \in (F, A)/\mathcal{P}$ if and only if there exists $\langle F_i, A_i \rangle \in \mathcal{P}$ such that $F(a), F(b) \in \langle F_i, A_i \rangle$.

Babitha and Sunil [45] proved that an equivalence soft set relation on soft set gives rise to a partition of soft set, and each partition of soft set gives rise to an equivalence soft set relation as follows.

**Theorem 4 ([45]).** Let $\mathcal{R}$ be an equivalence soft set relation on $(F, A)$ and $\mathcal{P}$ be a partition of $(F, A)$. Then we have the following.

1. $(F, A)/\mathcal{R}$ is a partition of $(F, A)$.
2. $(F, A)/\mathcal{P}$ is an equivalence soft set relation on $(F, A)$.

The following gives the intimate connection between equivalence soft set relations and partitions.

**Theorem 5.** Let $\mathcal{R}$ be an equivalence soft set relation on $(F, A)$ and $\mathcal{P} = \{\langle F_i, A_i \rangle \}$ be a partition of $(F, A)$. Then

1. $\langle F, A \rangle/\mathcal{P} = \bigcup_{i \in I} \langle F_i, A_i \rangle$.
2. $(F, A)/((F, A)/\mathcal{P}) = \mathcal{P}$.
3. $(F, A)/((F, A)/\mathcal{R}) = \mathcal{R}$.

**Proof.** (1) By **Definition 12**, we have

$F(a) \times F(b) \in (F, A)/\mathcal{P} \iff F(a) \in \langle F_i, A_i \rangle$ and $F(b) \in \langle F_i, A_i \rangle$ for some $\langle F_i, A_i \rangle \in \mathcal{P}$

$\iff F(a) \times F(b) \in \langle F_i, A_i \rangle \times \langle F_i, A_i \rangle$ for some $\langle F_i, A_i \rangle \in \mathcal{P}$

$\iff F(a) \times F(b) \in \bigcup_{i \in I} \langle F_i, A_i \rangle \times \langle F_i, A_i \rangle$.

Hence $\langle F, A \rangle/\mathcal{P} = \bigcup_{i \in I} \langle F_i, A_i \rangle$.

(2) Let $F(a)/((F, A)/\mathcal{P}) \in (F, A)/((F, A)/\mathcal{P})$. Since $F(a) \in F(a)/((F, A)/\mathcal{P})$ and $\mathcal{P}$ is a partition, there exists unique $\langle F_i, A_i \rangle \in \mathcal{P}$ such that $F(a) \in \langle F_i, A_i \rangle$. By **Definition 12**, we have $\langle F_i, A_i \rangle = F(a)/((F, A)/\mathcal{P})$. Hence $F(a)/((F, A)/\mathcal{P}) \in \mathcal{P}$. On the other hand, let $\langle F_i, A_i \rangle \in \mathcal{P}$. Since $\langle F_i, A_i \rangle$ is a non null soft set, there exists an $F(a) \in \langle F, A \rangle$ such that $F(a) \in \langle F_i, A_i \rangle$. By our previous argument, $F(a)/((F, A)/\mathcal{P}) = \langle F_i, A_i \rangle$. Hence $\langle F_i, A_i \rangle \in (F, A)/((F, A)/\mathcal{P})$. Therefore, we have $(F, A)/((F, A)/\mathcal{P}) = \mathcal{P}$.

(3) By (1), **Definition 12** and **Theorem 4**, we have

$F(a) \times F(b) \in (F, A)/((F, A)/\mathcal{R}) \iff F(a) \times F(b) \in \langle F_i, A_i \rangle \times \langle F_i, A_i \rangle$ for some $\langle F_i, A_i \rangle \in (F, A)/\mathcal{R}$

$\iff F(a) \times F(b) \in F(c)/\mathcal{R} \times F(c)/\mathcal{R}$ for some $F(c) \in (F, A)$

$\iff F(a) \times F(b) \in \mathcal{R}$.

Hence $(F, A)/((F, A)/\mathcal{R}) = \mathcal{R}$. □
Babitha and Sunil [45] introduced the induced soft set relation from the relation on a set of parameters as follows.

**Definition 13** ([45]). Let \((F, A)\) be a soft set defined on the universal set \(U\) and \(\mathcal{R}\) be a relation defined on \(A\), i.e., \(\mathcal{R} \subseteq A \times A\). Then the induced soft set relation \(\mathcal{R}_A\) on \((F, A)\) is defined as follows:

\[
F(a) \times F(b) \in \mathcal{R}_A \iff (a, b) \in \mathcal{R}.
\]

**Theorem 6.** Let \((F, A)\) be a soft set defined on \(U\) and \(\mathcal{R}\) be a relation defined on \(A\). Then \(\mathcal{R}\) is an equivalence relation if and only if the induced relation \(\mathcal{R}_A\) is an equivalence soft set relation.

**Proof.** By Definitions 9 and 13, we have the following.

(a) \(\mathcal{R}\) is reflexive \(\iff \Delta_A \subseteq \mathcal{R}\) \(\iff I_{A_a} \subseteq \mathcal{R}_A \iff \mathcal{R}_A\) is reflexive. Here \(\Delta_A = \{(a, a) : a \in A\}\) is the diagonal relation on \(A\).

(b) \(\mathcal{R}\) is symmetric \(\iff \mathcal{R} = \mathcal{R}^{-1}\) \(\iff \mathcal{R}_A = \mathcal{R}_A^{-1}\) \(\iff \mathcal{R}_A\) is symmetric.

(c) \(\mathcal{R}\) is transitive \(\iff \mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R} \iff \mathcal{R}_A \circ \mathcal{R}_A \subseteq \mathcal{R}_A \iff \mathcal{R}_A\) transitive.

From (a), (b) and (c), \(\mathcal{R}\) is an equivalence relation if and only if \(\mathcal{R}_A\) is an equivalence soft set relation. \(\square\)

Let \((F, A)\) be a soft set on universal set \(U\), \(\mathcal{R}\) be an equivalence relation on \(A\) and \(f : A \to A\) be a function. Then we say that \(f\) is compatible with \(\mathcal{R}\) if and only if for all \(a, b \in A\),

\[ (a, b) \in \mathcal{R} \implies (f(a), f(b)) \in \mathcal{R}. \]

**Theorem 7.** Let \((F, A)\) be a soft set on \(U\), \(f : A \to A\) be a function, and \(\mathcal{R}\) be an equivalence relation on \(A\). If \(f\) is compatible with \(\mathcal{R}\), then there exists a unique function \(g : \langle F, A \rangle/\mathcal{R}_A \to \langle F, A \rangle/\mathcal{R}_A\) such that

\[
g(F(a)/\mathcal{R}_A) = F(f(a)/\mathcal{R}_A) \quad \text{for all } F(a) \in \langle F, A \rangle. \quad (*)
\]

**Proof.** Suppose that \(f\) is compatible with \(\mathcal{R}\). Let \(g = \{(F(a)/\mathcal{R}_A, F(f(a))/\mathcal{R}_A) : (a, b) \in \mathcal{R}\}\). We shall prove that \(g\) is a function. Clearly, the domain of \(g\) is \(\langle F, A \rangle/\mathcal{R}_A\). Let \((F(a)/\mathcal{R}_A, F(f(a))/\mathcal{R}_A)\) and \((F(b)/\mathcal{R}_A, F(f(b))/\mathcal{R}_A)\) be elements in \(g\). Then we have

\[
F(a)/\mathcal{R}_A = F(b)/\mathcal{R}_A \Rightarrow F(a) \times F(b) \in \mathcal{R}_A, \quad \text{by Theorem 3}
\]

\[
\Rightarrow (a, b) \in \mathcal{R}, \quad \text{by Definition 13}
\]

\[
\Rightarrow (f(a), f(b)) \in \mathcal{R}, \quad \text{by compatibility}
\]

\[
\Rightarrow F(f(a)/\mathcal{R}_A) = F(f(b)/\mathcal{R}_A), \quad \text{by Theorem 3}
\]

and hence \(g\) is a function. Finally, \((*)\) holds because \((F(a)/\mathcal{R}_A, F(f(a))/\mathcal{R}_A) \in g\). The uniqueness can be easily checked. \(\square\)

**Example 5.** Let \((F, A)\) be a soft set on the universal set \(U\), \(A\) and \(B\) be two sets of parameters and \(f : A \to B\) be a function. Define the relation \(\mathcal{R}\) on \(A\) by, for points in \(A\),

\[ (a, b) \in \mathcal{R} \iff f(a) = f(b). \]

Then, clearly, \(\mathcal{R}\) is an equivalence relation on \(A\) and thus there is a unique one-to-one function \(\tilde{f} : A/\mathcal{R} \to B\) such that \(f = \tilde{f} \circ \varphi\) (where \(\varphi : A \to A/\mathcal{R}\) is the natural map).

By **Definition 13** and **Theorem 6**, the induced soft set relation \(\mathcal{R}_A\) is an equivalence soft set relation on \((F, A)\). Then there is a function \(\tilde{F} : A/\mathcal{R} \to \langle F, A \rangle/\mathcal{R}_A\) such that \(\tilde{F}(a/\mathcal{R}) = F(a)/\mathcal{R}_A\) for all \(a/\mathcal{R} \in A/\mathcal{R}\). In fact, let \(\tilde{F} = \{(a/\mathcal{R}, F(a)/\mathcal{R}_A) : a/\mathcal{R} \in A/\mathcal{R}\}\). Consider pairs \((a/\mathcal{R}, F(a)/\mathcal{R}_A)\) and \((b/\mathcal{R}, F(b)/\mathcal{R}_A)\) in \(\tilde{F}\). Then the calculation

\[
a/\mathcal{R} = b/\mathcal{R} \Rightarrow (a, b) \in \mathcal{R}, \quad \text{by equivalence}
\]

\[
\Rightarrow F(a) \times F(b) \in \mathcal{R}_A, \quad \text{by Definition 13}
\]

\[
\Rightarrow F(a)/\mathcal{R}_A = F(b)/\mathcal{R}_A, \quad \text{by Theorem 3}
\]

shows that \(\tilde{F}\) is a function. Hence there is a unique one-to-one function \(f^* : (A/\mathcal{R})/\mathcal{R}_A \to B\) such that \(\tilde{F} = f^* \circ \tilde{F}\). In fact, let \(f^* = \{(F(a)/\mathcal{R}_A, F(f(a))/\mathcal{R}_A) : (a, b) \in \mathcal{R}\}\). Consider the pairs \((F(a)/\mathcal{R}_A, F(f(a))/\mathcal{R}_A)\) and \((F(a)/\mathcal{R}_A, F(f(a))/\mathcal{R}_A)\) in \(f^*\). The calculation

\[
F(a)/\mathcal{R}_A = F(b)/\mathcal{R}_A \Rightarrow F(a) \times F(b) \in \mathcal{R}_A, \quad \text{by Theorem 3}
\]

\[
\Rightarrow a/\mathcal{R} = b/\mathcal{R}, \quad \text{by definition of } \tilde{F}
\]

\[
\Rightarrow f(a) = f(b), \quad \text{by definition of } f^*
\]

shows that \(f^*\) is a function. The uniqueness and one-to-one of \(f^*\) can be easily checked. Finally, by the definitions of \(\tilde{F}\) and \(\tilde{F}\), for any \(a/\mathcal{R} \in A/\mathcal{R}\),

\[
(f^* \circ \tilde{F})(a/\mathcal{R}) = f^*(\tilde{F}(a/\mathcal{R})) = f^*(F(a)/\mathcal{R}_A) = f(a) = \tilde{f}(a/\mathcal{R})
\]

and hence \(\tilde{F} = f^* \circ \tilde{F}\).
4. Transitive closure of a soft set relation

For a soft set relation \( R \) on \( (F, A) \), there exists at least one transitive soft set relation containing \( R \), namely the trivial one \( (F, A) \times (F, A) \). Furthermore, the intersection of any family of transitive soft set relations is again transitive. Thus we need the smallest transitive soft set relation containing the soft set relation \( R \). Now we define the transitive closure of soft set relation as follows.

**Definition 14.** Let \( R \) be a soft set relation on \( (F, A) \). Then the transitive closure of \( R \), denoted by \( \hat{R} \), is the soft set relation on \( (F, A) \) defined as follows:

\[
\hat{R} = R \cup R^2 \cup R^3 \cup \ldots \cup R^n \cup \ldots
\]

where \( R^1 = R \) and \( R^n = R \circ R \circ \cdots \circ R \) \( n \) times, \( n \geq 2 \).

**Remark 2.** By Definition 14, \( R \subseteq \hat{R} \). Since every element of \( \hat{R} \) is in one of \( R^i \), \( \hat{R} \) must be transitive by the following reasoning: if \( F(a) \times F(b) \in R^i \) and \( F(b) \times F(c) \in R^k \), then from composition's associativity, \( F(a) \times F(c) \in R^{i+k} \) (and thus in \( \hat{R} \)) due to the definition of \( R^i \). Let \( \delta \) be any transitive soft set relation on \( (F, A) \) containing \( R \). Since \( \delta \) is transitive, whenever \( R^i \subseteq \delta \), \( R^{i+1} \subseteq \delta \) according to the construction of \( R^i \). Then, by induction, \( \delta \) contains every \( R^i \) and thus \( \hat{R} \). Therefore, the transitive closure \( \hat{R} \) is the smallest transitive soft set relation containing \( R \).

**Proposition 4.** Let \( R \) be a soft set relation on \( (F, A) \). Then we have the following.

1. \( \hat{R} \) is transitive.
2. If there exists \( n \in \mathbb{N} \) such that \( R^{n+1} = R^n \), then \( \hat{R} = R \cup R^2 \cup \ldots \cup R^n \).

**Proof.** It follows from Definition 14 and Remark 2.

**Example 6.** Let \( (F, A) \) be a soft set over \( U \) where \( U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\} \), \( A = \{m_1, m_2, m_3\} \) and \( F(m_1) = \{h_1, h_2, h_5, h_6\}, F(m_2) = \{h_3, h_4, h_7, h_8\}, F(m_3) = \{h_2, h_4, h_3\} \). Consider soft set relations \( R \) on \( (F, A) \) given by

\[
R = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_3) \times F(m_2)\}.
\]

Then

\[
R^2 = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_3) \times F(m_2), F(m_3) \times F(m_2)\},
\]

\[
R^3 = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_2) \times F(m_2), F(m_3) \times F(m_3), F(m_3) \times F(m_2), F(m_3) \times F(m_2), F(m_3) \times F(m_3)\},
\]

\[
R^4 = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_2) \times F(m_2), F(m_3) \times F(m_3), F(m_3) \times F(m_2), F(m_3) \times F(m_2), F(m_3) \times F(m_3)\},
\]

\[
R^5 = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_1) \times F(m_2), F(m_2) \times F(m_2), F(m_2) \times F(m_2), F(m_2) \times F(m_2), F(m_2) \times F(m_2), F(m_2) \times F(m_2)\}.
\]

Thus the transitive closure of \( R \) is \( \hat{R} = R \cup R^2 \cup R^3 \cup R^4 \).

**Proposition 5.** Let \( R \) and \( \delta \) be two soft set relations on \( (F, A) \). Then we have the following.

1. \( \hat{R} \) is symmetric, then so is \( \hat{R} \).
2. If \( R \subseteq \delta \), then \( \hat{R} \subseteq \hat{\delta} \).
3. If \( R \) and \( \delta \) are equivalence soft set relations and \( R \circ \delta = \delta \circ R \), then \( (R \circ \delta)^n = R \circ \delta \).

**Proof.** (1) and (2) are proved from Proposition 2 and Definitions 9 and 14.

(3) By Theorem 2 and hypothesis, since \( R \circ \delta \) is an equivalence soft set relation, \( (R \circ \delta)^n \subseteq R \circ \delta \) for any \( n \geq 1 \). Hence \( (R \circ \delta) = R \circ \delta \).

Let \( R \) be a soft set relation on \( (F, A) \) and \( \{R_\gamma : \gamma \in \Gamma\} \) be a family of equivalence soft set relations on \( (F, A) \) such that \( R \subseteq R_\gamma \) for each \( \gamma \in \Gamma \). Then clearly \( \bigcap_{\gamma \in \Gamma} R_\gamma \) is the smallest equivalence soft set relation such that \( R \subseteq \bigcap_{\gamma \in \Gamma} R_\gamma \) and denoted by \( R^e \).

**Theorem 8.** If \( R \) is a soft set relation on \( (F, A) \), then \( R^e = (R \cup R^{-1} \cup I_{FA})^\perp \).

**Proof.** Let \( \delta = (R \cup R^{-1} \cup I_{FA})^\perp \). Then clearly \( R \subseteq \delta \). By Proposition 4(1), \( \delta \) is transitive. Since \( I_{FA} \subseteq (R \cup R^{-1} \cup I_{FA}), \delta \) is reflexive. By Proposition 2(6), \( (R \cup R^{-1} \cup I_{FA})^{-1} = R \cup R^{-1} \cup I_{FA} \), i.e., \( R \cup R^{-1} \cup I_{FA} \) is symmetric. By Proposition 5(1), \( \delta \) is symmetric. Thus \( \delta \) is an equivalence soft set relation such that \( R \subseteq \delta \). Now let \( K \) be an equivalence soft set relation on \( (F, A) \) such that \( R \subseteq K \). Since \( K \) is equivalence, by Proposition 2(4) and Definition 9, \( I_{FA} \subseteq K \) and \( R^{-1} \subseteq K^{-1} \subseteq K^{-1} = \hat{K} \).

Then, by Proposition 2(3), \( (R \cup R^{-1} \cup I_{FA})^n \subseteq K^\perp = K \) for any \( n \geq 1 \). Thus \( \delta \subseteq K \). This shows that \( R^e = (R \cup R^{-1} \cup I_{FA})^\perp \).
Theorem 9. Let \( R \) and \( \delta \) be two equivalence soft set relations on \((F, A)\). Then \((R \cup \delta)^\ast\) is an equivalence soft set relation.

Proof. By Proposition 4(1), \((R \cup \delta)^\ast\) is transitive. Since \( R \) and \( \delta \) are symmetric, \((R \cup \delta)^\ast = R^{-1} \cup \delta^{-1} = R \cup \delta\), i.e., \( R \cup \delta \) is symmetric. Then by Proposition 5(1), \((R \cup \delta)^\ast\) is symmetric. Since \( R \) and \( \delta \) are reflexive, \( I_{FA} = I_{FA} \cup I_{FA} \subseteq R \cup \delta \).

Thus \( I_{FA} \subseteq (R \cup \delta)^\ast\), i.e., \((R \cup \delta)^\ast\) is reflexive. Hence \((R \cup \delta)^\ast\) is an equivalence soft set relation. ■

Theorem 10. Let \( R \) and \( \delta \) be two equivalence soft set relations on \((F, A)\). If \( R \circ \delta \) is an equivalence soft set relation on \((F, A)\), then \( R \circ \delta \) is the least upper bound for \([R, \delta]\) with respect to \( \subseteq \).

Proof. Since \( \delta \) is reflexive, by Proposition 2(2), \( R = R \circ I_{FA} \subseteq R \circ \delta \). By the similar argument, \( \delta \subseteq R \circ \delta \). So, \( R \circ \delta \) is an upper bound for \([R, \delta]\) with respect to \( \subseteq \). Now let \( K \) be any equivalence soft set relation on \((F, A)\) such that \( R \subseteq K \) and \( \delta \subseteq K \). Since \( K \) is transitive, by Proposition 2(2), \( R \circ \delta \subseteq K \circ K \subseteq K \). Hence \( R \circ \delta \) is a least upper bound for \([R, \delta]\) with respect to \( \subseteq \). ■

Theorem 11. Let \( R \) and \( \delta \) be two equivalence soft set relations on \((F, A)\) such that \( R \circ \delta = \delta \circ R \). Then \((R \cup \delta)^\ast = (R \cup \delta)^\ast = (R \circ \delta)^\ast = R \circ \delta \).

Proof. Clearly, \( R \cup \delta \) is a soft set relation. Since \( R \) and \( \delta \) are equivalence soft set relations, by Theorem 8, Proposition 2(6) and Definition 9,

\[
(R \cup \delta)^\ast = ((R \cup \delta) \cup (R \cup \delta)^{-1}) \cup I_{FA} = (R \cup \delta)^\ast.
\]

By Proposition 2(2) and the hypothesis, we have \( R \circ \delta \subseteq (R \cup \delta) \circ (R \cup \delta) = R \cup \delta \) and thus, by Proposition 5(2), \( (R \circ \delta) \subseteq (R \cup \delta) \). On the other hand, by Theorems 2 and 10, \( R \circ \delta \) is an equivalence soft set relation and \( R \circ \delta \) is the least upper bound for \([R, \delta]\) with respect to \( \subseteq \). Since \( R \subseteq R \circ \delta \) and \( \delta \subseteq R \circ \delta \), by Proposition 5(2), \( (R \cup \delta) \subseteq (R \circ \delta) \). Hence \( (R \circ \delta) = (R \cup \delta) \).

Therefore, by Proposition 5(3), \( (R \cup \delta)^\ast = (R \cup \delta)^\ast = (R \circ \delta)^\ast = R \circ \delta \). ■

Example 7. Let \((F, A)\) be a soft set over \( U \) where \( U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\} \), \( A = \{m_1, m_2, m_3, m_4, m_5\} \) and \( F(m_1) = \{h_1, h_2, h_3, h_6\}, F(m_2) = \{h_3, h_4, h_7, h_8\}, F(m_3) = \{h_2, h_4, h_9\}, F(m_4) = \{h_1, h_6, h_8\}, F(m_5) = \{h_2, h_4, h_5\} \). If \( R \) and \( \delta \) are soft set relations on \((F, A)\) defined by

\[
R = \{(F(m_1) \times F(m_1), F(m_2) \times F(m_2), F(m_1) \times F(m_2), F(m_3) \times F(m_3), F(m_3) \times F(m_3), F(m_3) \times F(m_3), F(m_3) \times F(m_3))\},
\]

\[
\delta = \{(F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_2), F(m_3) \times F(m_3), F(m_3) \times F(m_3), F(m_3) \times F(m_3), F(m_3) \times F(m_3), F(m_3) \times F(m_3))\},
\]

then

\[
R \circ \delta = \delta \circ R = \{(F(m_1) \times F(m_1), F(m_2) \times F(m_2), F(m_2) \times F(m_2), F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_1) \times F(m_2), F(m_1) \times F(m_2), F(m_1) \times F(m_2))\},
\]

\[
R \cup \delta = \{(F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_2) \times F(m_2), F(m_2) \times F(m_2), F(m_2) \times F(m_2), F(m_2) \times F(m_2), F(m_2) \times F(m_2))\},
\]

and so \((R \cup \delta)^\ast = R \cup \delta \) and \((R \circ \delta)^\ast = R \circ \delta \) but \( R \cup \delta \neq R \circ \delta \) because \( R \) and \( \delta \) are not equivalence soft set relations on \((F, A)\).

If \( R \) is an equivalence soft set relation on \((F, A)\) given in Example 4 and \( \delta = I_{FA} \) is an identity soft set relation on \((F, A)\), then \( R \cup \delta = R \circ \delta = \delta \circ R \) and thus \((R \cup \delta)^\ast = (R \circ \delta)^\ast = (R \circ \delta)^\ast = R \circ \delta \).

Let \( \text{ESSR}(F, A) \) be a set of all equivalence soft set relations on \((F, A)\). Then \((\text{ESSR}(F, A), \subseteq)\) is a poset. Moreover, for any \( R, \delta \in \text{ESSR}(F, A) \), \( R \cap \delta \) is the greatest lower bound for \([R, \delta]\) with respect to \( \subseteq \).

Now we define two binary operation \( \land_\varepsilon \) and \( \lor_\varepsilon \) on \( \text{ESSR}(F, A) \) as follows: for any \( R, \delta \in \text{ESSR}(F, A) \),

\[
R \land_\varepsilon \delta = R \cap \delta \quad \text{and} \quad R \lor_\varepsilon \delta = (R \cup \delta)^\ast.
\]

Then we obtain the following result from Remark 1 and Theorems 8 and 11.

Theorem 12. \((\text{ESSR}(F, A), \land_\varepsilon, \lor_\varepsilon)\) is a complete lattice with the least element \( I_{FA} \) and the greatest element \((F, A) \times (F, A)\).

5. Conclusions

Soft set theory is an effective method for solving problems of uncertainty. Babitha and Sunil [45] extended the concepts of relation and functions in soft set theory. In this paper, we further study the equivalence soft set relations and obtain soft analogues of many results concerning ordinary equivalence relations and partitions. The transitive closure of a soft set relation is discussed and some basic properties are proved. There exists compact connections between soft sets and information systems and so one can apply the results deduced from the studies on soft set relations to solve these connections. Thus, one can get more affirmative solution in decision making problems in real life situations.
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References