Fusion Rules for the Free Bosonic Orbifold Vertex Operator Algebra

Toshiyuki Abe

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail: sm3002at@ex.ecip.osaka-u.ac.jp

Communicated by Geoffrey Mason
Received July 27, 1999

Fusion rules among irreducible modules for the free bosonic orbifold vertex operator algebra are completely determined.

1. INTRODUCTION

We determine the fusion rules for the free bosonic orbifold vertex operator algebra \( M(1)^+ \) which is the fixed point set of the free bosonic vertex operator algebra \( M(1) \) under an automorphism \( \theta \) of order 2.

In [Z], Zhu introduced an associative algebra \( A(V) \) (called Zhu’s algebra) associated to a vertex operator algebra \( V \). Zhu’s algebra \( A(V) \) inherits a part of the vertex operator algebra structure of \( V \) which affords much information on \( V \)-modules. For example, there exists a one-to-one correspondence between the set of equivalence classes of irreducible \( \mathbb{N} \)-gradable \( V \)-modules and the set of equivalence classes of irreducible \( A(V) \)-modules. Later, in [FZ], the notion of Zhu’s algebra is generalized to an \( A(V) \)-bimodule \( A(M) \) for an \( \mathbb{N} \)-gradable \( V \)-module \( M \), and the fusion rules of rational vertex operator algebras are completely characterized in terms of these bimodules (see also [Li]). More precisely, for irreducible \( V \)-modules \( M^i \) (\( i = 1, 2, 3 \)), with nontrivial top level \( M^i_0 \), there exists a natural injection from

\[
I \left( \begin{array}{c}
M^1 \\
M^2
\end{array} \right)
\]

with

\[
0021-8693/00 $35.00
Copyright © 2000 by Academic Press
All rights of reproduction in any form reserved.
which is the space of intertwining operators of type
\[
\begin{pmatrix}
M^3 \\
M^1 & M^2
\end{pmatrix},
\]
into the dual space of the contraction \((M^3)^* \otimes_{\Delta(V)} A(M^1) \otimes_{\Delta(V)} M^3_0\). Moreover, if \(V\) is rational, then this map becomes an isomorphism. By using this isomorphism, fusion rules were calculated for vertex operator algebras associated with finite-dimensional simple Lie algebras \([FZ]\), for the minimal series \([W]\), etc.

Let \(\mathfrak{h}\) be a \(d\)-dimensional complex vector space with a nondegenerate symmetric bilinear form and let \(\mathfrak{h}\) be its affinization. Then the Fock space \(F_{\mathfrak{h}}\) is a simple vertex operator algebra with central charge \(d\) and has the automorphism \(\theta\) of order 2 lifted from the map \(\mathfrak{h} \rightarrow \mathfrak{h}\), \(h \mapsto -h\) \([FLM]\). The fixed point set \(M(1)^\perp\) of \(M(1)\) is also a simple vertex operator algebra and the \(-1\)-eigenspace \(M(1)^\perp\) is an irreducible \(M(1)^\perp\)-module. It is well known that for every \(\lambda \in \mathfrak{h}\) the Fock space \(M(1, \lambda) = S(\mathfrak{h} \otimes \mathfrak{h}^{-1} \mathbb{C}[\mathfrak{h}^{-1}]) \otimes \mathbb{C} e^\lambda\) is an irreducible \(M(1)^\perp\)-module and the set \(\{M(1, \lambda) \mid \lambda \in \mathfrak{h}\}\) gives all the inequivalent irreducible \(M(1)^\perp\)-modules. Moreover, if \(\lambda \neq 0\), \(M(1, \lambda)\) is an irreducible \(M(1)^\perp\)-module, and \(M(1, \lambda)\) and \(M(1, -\lambda)\) are isomorphic to each other as the \(M(1)^\perp\)-module (see \([DM]\)). In addition, the \(\theta\)-twisted \(M(1)^\perp\)-module \(M(1)(\theta)\) defined as an induced module of the twisted affine Lie algebra \(\mathfrak{h}[-1]\) is also an \(M(1)^\perp\)-module, and the \(\pm 1\)-eigenspaces \(M(1)(\theta)^\pm\) for \(\theta\) give inequivalent irreducible \(M(1)^\perp\)-modules. It is known that these irreducible modules \(M(1, \lambda)^\pm, M(1, \lambda) (= M(1, -\lambda)) (\lambda \neq 0),\) and \(M(1)(\theta)^\pm\) give all the inequivalent irreducible \(M(1)^\perp\)-modules (see \([DN1]\) and \([DN2]\)).

In this paper, we only consider the rank one, i.e., the \(d = 1\), case and determine the fusion rules among any triples of irreducible \(M(1)^\perp\)-modules. One of the main results in this paper is the following theorem.

**Theorem.** Let \(M, N,\) and \(L\) be irreducible \(M(1)^\perp\)-modules.

1. If \(M = M(1)^\perp\), then \(N_{M(1)^\perp}^M = \delta_{N, L}\).
2. If \(M = M(1)^\perp\), then \(N_{M(1)^\perp}^M = 0\) or 1, and \(N_{M(1)^\perp}^N = 1\) if and only if the pair \((N, L)\) is one of the following pairs:

\[
\begin{align*}
(M(1)^\pm, M(1)^\mp), \quad & (M(1)(\theta)^\pm, M(1)(\theta)^\mp), \\
(M(1, \lambda), M(1, \mu)) \quad & (\lambda^2 = \mu^2).
\end{align*}
\]
(3) If \( M = M(1, \lambda) \) (\( \lambda \neq 0 \)), then \( N_{M(1, \lambda)N}^L = 0 \) or 1, and \( N_{M(1, \lambda)N}^L = 1 \) if and only if the pair \((N, L)\) is one of the following pairs:

\[
(M(1)^\pm, M(1, \mu)) \ (\lambda^2 = \mu^2), \quad (M(1, \mu), M(1, \nu)) \ (\nu^2 = (\lambda \pm \mu)^2),
\]

\[
(M(1)(\theta)^\pm, M(1)(\theta)^\pm), \quad (M(1)(\theta)^\pm, M(1)(\theta)^\mp).
\]

(4) If \( M = M(1)(\theta)^+ \), then \( N_{M(1)(\theta)^+N}^L = 0 \) or 1, and \( N_{M(1)(\theta)^+N}^L = 1 \) if and only if the pair \((N, L)\) is one of the following pairs:

\[
(M(1)^-, M(1)(\theta)^\pm), \quad (M(1, \lambda), M(1)(\theta)^\pm).
\]

(5) If \( M = M(1)(\theta)^- \), then \( N_{M(1)(\theta)^-N}^L = 0 \) or 1, and \( N_{M(1)(\theta)^-N}^L = 1 \) if and only if the pair \((N, L)\) is one of the following pairs:

\[
(M(1)^-, M(1)(\theta)^\mp), \quad (M(1, \lambda), M(1)(\theta)^\mp).
\]

Let \( M^i \) (\( i = 1, 2, 3 \)) be irreducible \( M(1)^+\)-modules. Then the classification result of irreducible \( M(1)^+\)-modules in [DN1] and formal characters for irreducible \( M(1)^+\)-modules show that the fusion rule \( N_{M^iM^j}^L \) is invariant under any permutation of \( \{1, 2, 3\} \). In more detail, the explicit forms of the formal characters of irreducible \( M(1)^+\)-modules imply that two irreducible \( M(1)^+\)-modules with the same formal characters are isomorphic to each other, and in particular that every irreducible \( M(1)^+\)-module is isomorphic to its contragredient module. Then the above symmetry-of-fusion rules follow from the fact that \( N_{M^iN}^L = N_{M^iM}^L = N_{M^iL}^L \) holds for modules \( M, N, L \) of a vertex operator algebra, where \( N^i \) and \( L^i \) are contragredient modules of \( N \) and \( L \) respectively. Next, we prove that \( A(M^1) \) is generated as an \( A(M(1)^+)^\ast \)-bimodule by at most two elements which are images of singular vectors of \( M^1 \) viewed as a module for the Virasoro algebra. This is obtained by using the fact that the \( M(1)^+ \) is generated by the Virasoro element and a singular vector of weight 4 (see [DG]). Further, using the Frenkel–Zhu injection, we prove that the fusion rule \( N_{M^iM^j}^L \) is less than 2. A more detailed study of the contraction \( (M^1_0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2_0 \) of \( A(M(1)^+)^\ast \)-modules implies that the fusion rule \( N_{M^iM^j}^L \) is in fact less than 1. In [FLM] and [DN1], the nontrivial intertwining operator of type

\[
\begin{pmatrix}
M(1, \lambda + \mu) \\
M(1, \lambda) & M(1, \mu)
\end{pmatrix}
\]
was constructed for every $\lambda, \mu \in \mathbb{C}$. This gives us nontrivial intertwining operators of types

$$\begin{pmatrix} M(1)^\pm \\ M(1)^+ M(1)^\pm \end{pmatrix}, \begin{pmatrix} M(1, \lambda) \\ M(1)^\pm M(1, \lambda) \end{pmatrix},$$

and

$$\begin{pmatrix} M(1, \lambda + \mu) \\ M(1, \lambda) M(1, \mu) \end{pmatrix}.$$

The fusion rules of corresponding types are nonzero. In addition, in [FLM], a twisted vertex operator from $M(1, \lambda)$ to $\text{Hom}(M(1)(\theta), M(1)(\theta)(e))$ was obtained for every $\lambda \in \mathbb{C}$. This provides the nontrivial intertwining operators of types

$$\begin{pmatrix} M(1)^+ \\ M(1)^\pm M(1)(\theta)^\pm \end{pmatrix}$$

and

$$\begin{pmatrix} M(1)(\theta)^\beta \\ M(1, \lambda) M(1)(\theta)^\alpha \end{pmatrix}$$

for any $\alpha, \beta \in \{+, -\}$. Thus the fusion rules of corresponding types are also nonzero. The study of the contractions also shows that all nonzero fusion rules are derived from these fusion rules by means of the above symmetry-of-fusion rules and the equivalency between $M(1, \lambda)$ and $M(1, -\lambda)$ for $\lambda \in \mathbb{C}$.

The organization of this paper is as follows. We recall the definitions of vertex operator algebras, modules, and fusion rules in Section 2.1, and those of Zhu’s algebras and their bimodule in Section 2.2, where we also explain the relation between fusion rules and the bimodules. We review the vertex operator algebra $M(1)^+$ and its irreducible modules in Section 2.3. In Section 3.1, we describe the irreducible decompositions of the irreducible $M(1)^+$-modules as modules for Virasoro algebra and prove that $N_{\mu_i}^{\lambda_i}$ is invariant under any permutation of $\{1, 2, 3\}$ for irreducible $M(1)^+$-modules $M^i$ ($1 \leq i \leq 3$). In Section 3.2 we prove some lemmas and a proposition (Proposition 3.7) which gives a generalization of Zhu’s anti-isomorphism of $A(V)$, and in Section 3.3 we give a set of generators of $A(M)$ for an irreducible $M(1)^+$-module $M$ and show that all fusion rules among irreducible $M(1)^+$-modules are less than 2. In Section 4.1 we explain that the vertex operators and twisted vertex operators constructed in [FLM] give some nonzero intertwining operators among irreducible $M(1)^+$-modules, and we state the main theorem. In Section 4.2 we prove the main theorem by studying the structure of the contraction $(M_0^i)^* \cdot A(M^i) \cdot M_0^i$ for irreducible $M(1)^+$-modules $M^i$ ($1 \leq i \leq 3$).
2. PRELIMINARIES

We recall the definitions of vertex operator algebras, their modules from [FLM, DLM1, and DMZ], and fusion rules from [FHL] in Section 2.1. In Section 2.2, following [Z] and [FZ], we review the definition of Zhu’s algebra $A(V)$ associated to a vertex operator algebra $V$ and its bimodule $A(M)$ for an $\mathbb{N}$-gradable $V$-module $M$. In Section 2.3, following [FLM], we recall the vertex operator algebra $M(1)^+$ and its irreducible modules.

Throughout the paper, $\mathbb{N}$ is the set of nonnegative integers and $\mathbb{Z}_{>0}$ is the set of positive integers. For vector space $V$, the vector space of the formal power series in $z$ is denoted by

$$V[z] = \left\{ \sum_{n \in \mathbb{C}} v_n z^n \middle| v_n \in V \right\},$$

and we set the subspaces $V[[z, z^{-1}]]$ and $V((z))$ as

$$V[[z, z^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} v_n z^n \middle| v_n \in V \right\},$$

$$V((z)) = \left\{ \sum_{n=k} v_n z^n \middle| k \in \mathbb{Z}, v_n \in V \right\}.$$ 

For $f(z) = \sum_{n \in \mathbb{C}} v_n z^n \in V[z]$, $v_{-1}$ is called the formal residue denoted by $\text{Res}_z f(z) = v_{-1}$.

2.1. Vertex Operator Algebras, Modules, and Fusion Rules

**Definition 2.1.** A $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ such that $\text{dim} V_n$ is finite for all integers $n$ and $V_0 = 0$ for a sufficiently small integer $n$ is called a vertex operator algebra if $V$ is equipped with a linear map

$$Y : V \rightarrow (\text{End } V)[[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End } V)$$

and with two distinguished vectors $1 \in V_0$ and $\omega \in V_2$ such that the conditions

$$Y(a, z)b \in V((z))$$

and (Jacobi identity)

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(a, z_1) Y(b, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y(b, z_2) Y(a, z_1)$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(a, z_0)b, z_2)$$
hold for \( a, b \in V \) and \( m, n \in \mathbb{Z} \), where \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \) and all binomial expressions are to be expanded as formal power series in the second variable.

\[
Y(1, z) = \text{id}_V, \quad Y(a, z)1 \in V[[z]], \quad \text{and} \quad Y(a, z)1 \mid_{z=0} = a.
\]

We set \( Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2} \), then \( L(n) \ (n \in \mathbb{Z}) \) forms a Virasoro algebra,

\[
[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c_V, \quad (2.1)
\]

for any \( m, n \in \mathbb{Z} \), where \( c_V \), which is called the central charge of \( V \), \( \in \mathbb{C} \).

\[
L(0)a = na \quad \text{for} \ n \in \mathbb{Z}, \ a \in V, \quad Y(L(-1)a, z) = \frac{d}{dz}Y(a, z). \quad (2.2)
\]

The vertex operator algebra is denoted by \( (V, Y, 1, \omega) \) or simply by \( V \). An element \( a \in V \) is called a homogeneous element of weight \( n \) and we write \( n = \text{wt}(a) \).

An automorphism \( g \) of a vertex operator algebra \( V \) is a linear automorphism of \( V \) such that \( gY(a, z)g^{-1} = Y(g(a), z) \) for all \( a \in V \) and \( g(\omega) = \omega \). Set \( \text{Aut}V \) to be the set of all automorphisms of \( V \) and let \( G \) be a subgroup of \( \text{Aut}V \). Then the fixed point set for \( G \) naturally becomes a vertex operator algebra. This vertex operator algebra is called the orbifold of \( V \) (cf. [DVVV, DM]).

Let \( g \) be an automorphism of a vertex operator algebra \( V \) of order \( T \). Then \( V \) is decomposed into the eigenspaces for \( g \):

\[
V = \bigoplus_{r=0}^{T-1} V^r, \quad V^r = \{a \in V \mid g(a) = e^{-2\pi ir/T}a\}.
\]

**Definition 2.2.** Let \( V \) be a vertex operator algebra and let \( g \) be an automorphism of order \( T \). A weak \( g \)-twisted \( V \)-module \( M \) is a vector space equipped with a linear map

\[
Y_M: V \to (\text{End} M)[z], \quad a \to Y_M(a, z) = \sum_{n \in \mathbb{Q}} a_n^Mz^{-n-1} \quad (a_n^M \in \text{End} M),
\]
such that the conditions
\[
Y_M(a, z) = \sum_{n \in \mathbb{Z}} a_n^M z^{-n-1}, \quad Y_M(a, z) v = z^{-r/T} M((z)),
\]
\[
Y_M(1, z) = \text{id}_M,
\]
and (twisted Jacobi identity)
\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(a, z_1) Y_M(b, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(b, z_2) Y_M(a, z_1)
= z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(a, z_0) b, z_2)
\]
(2.3)
hold for 0 \leq r \leq T - 1, a \in V', b \in V, and u \in M.

The weak \( g \)-twisted \( V \)-module is denoted by \( (M, Y_M) \) or simply by \( M \). In the case \( g \) is an identity of \( V \), the weak \( g \)-twisted \( V \)-module is called a weak \textit{V-module}. Here and later we denote the component operator \( a_n^M \) (\( n \in \mathbb{Z} \), \( n \in \mathbb{Q} \)) by \( a_n \) for simplicity.

Let \( (M, Y_M) \) be a weak \( g \)-twisted \( V \)-module and set \( Y_M(a, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \). Then the operators \{\( L(n); (n \in \mathbb{Z}), \text{id}_M \} \) also form a Virasoro algebra with central charge \( c_v \). Moreover, we also have (2.2) (see [DLM2]).

**Definition 2.3.** Let \( V \) be a vertex operator algebra and let \( g \) be an automorphism of \( V \) of order \( T \). A \( \frac{1}{T} \mathbb{N} \)-gradable \( g \)-twisted \( V \)-module \( M \) is a weak \( g \)-twisted \( V \)-module which has a \( \frac{1}{T} \mathbb{N} \)-grading \( M = \bigoplus_{n \in \mathbb{Z}} M_n \) such that \( a_n M_n \subset M_{\text{wt}(a) + n - m - 1} \) holds for any homogeneous \( a \in V, n \in \frac{1}{T} \mathbb{N} \), and \( m \in \mathbb{Q} \).

In the case \( g \) is an identity on \( V \), the \( \frac{1}{T} \mathbb{N} \)-gradable \( g \)-twisted \( V \)-module is called an \textit{\( \mathbb{N} \)-gradable \textit{V-module}}. An element \( u \in M_n \) is called a homogeneo-

**Definition 2.4.** An ordinary \( g \)-twisted \( V \)-module \( M \) is a weak \( g \)-twisted \( V \)-module on which \( L(0) \) acts semisimply,
\[
M = \bigoplus_{\lambda \in \mathbb{C}} M(\lambda), \quad M(\lambda) = \{ u \in M : L(0) u = \lambda u \},
\]
such that each eigenspace is finite dimensional and for fixed \( \lambda \in \mathbb{C} \), \( M(\lambda + n/T) = 0 \) for a sufficiently small integer \( n \).

In the case in which \( g \) is an identity of \( V \), the ordinary \( g \)-twisted \( V \)-module is called an ordinary \textit{V-module}, or more simply a \textit{V-module}. An
element \( u \in M(\lambda) \) is said to be \textit{homogeneous of weight} \( \lambda \) and we write \( \lambda = \text{wt}(u) \).

The notions of submodules and irreducible modules are defined in the obvious way. Let \( M \) be a \( V \)-module, then the restricted dual \( M' = \bigoplus_{\lambda \in \mathbb{C}} M(\lambda)^* \) is a \( V \)-module and the vertex operator \( Y_M(a, z) \) for \( a \in V \) is defined by

\[
\langle Y_M(a, z)u', v \rangle = \langle u', Y_M\left(e^{zL(1)}(-z^{-2})L(0)a, z^{-1}\right)v \rangle
\]

for \( u' \in M', v \in M \). This \( V \)-module \((M', Y_M')\) is called the \textit{contragredient module} of \( M \). It is known that if \( M \) irreducible, then \( M' \) is also irreducible (cf. [FHL]).

**Definition 2.5.** Let \( V \) be a vertex operator algebra and \((M^i, Y_{M^i})\) \((i = 1, 2, 3)\) be weak \( V \)-modules. An \textit{intertwining operator of type}

\[
\begin{pmatrix}
M^3 \\
M^1 & M^2
\end{pmatrix}
\]

is a linear map

\[
I: M^1 \to \left(\text{Hom}(M^2, M^3)\right)(z),
\]

\[
v \mapsto I(v, z) = \sum_{n \in \mathbb{C}} v_n z^{-n-1} \quad (v_n \in \text{Hom}(M^2, M^3))
\]

such that for \( a \in V, v \in M^1 \), and \( u \in M^2 \) the following conditions hold.

For fixed \( n \in \mathbb{C}, v_{n+k} u = 0 \) for sufficiently large integer \( k \),

\[
\begin{aligned}
(\text{Jacobi identity}) & \quad z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0}\right) Y_{M^1}(a, z_1) I(v, z_2) \\
& - z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0}\right) I(v, z_2) Y_{M^1}(a, z_1) \\
& = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2}\right) I(Y_{M^1}(a, z_0)v, z_2),
\end{aligned}
\]

\[
\frac{d}{dz} I(v, z) = I(L(-1)v, z).
\]

We denote the vector space composed of the intertwining operators of type

\[
\begin{pmatrix}
M^3 \\
M^1 & M^2
\end{pmatrix}
\]
The dimension of this vector space is called a fusion rule of corresponding type and is denoted by $N^M_{M'}$. It is well known that fusion rules have the following symmetry (see [FHL] and [HL]).

**Proposition 2.6.** Let $M^i (i = 1, 2, 3)$ be $V$-modules. Then

\[ N^M_{M^1 M^2} = N^M_{M^2 M^1} \quad \text{and} \quad N^M_{M^1 M^2} = N^M_{(M^1 M^2)^{\vee}}. \]

**2.2. Zhu’s Algebra $A(V)$ and the $A(V)$-Bimodule $A(M)$**

We recall the definition of Zhu’s algebra. Two bilinear products $\ast$ and $\circ$ on $V$ are defined as follows: For homogeneous $a \in V$ and $b \in V$, we define

\[ a \ast b = \left( \text{Res}_z \frac{(1 + z)^{wt(a)}}{z} Y(a, z) \right) b, \]

\[ a \circ b = \left( \text{Res}_z \frac{(1 + z)^{wt(a)}}{z^2} Y(a, z) \right) b, \]

and extend these to $V$ by linearity. Let $O(V)$ be the linear span of $a \ast b (a, \ b \in V)$ and set $A(V) = V/O(V)$. Let $M$ be an $\mathbb{N}$-gradable $V$-module. For every homogeneous $a \in V$, define $o(a) = a^M_{\text{wt}(a) - 1}$ and extend to $V$ linearly. The following proposition is due to Zhu (see [Z]).

**Proposition 2.7.** (1) The bilinear product $\ast$ induces $A(V)$, an associative algebra structure. The vector $1 + O(V)$ is the identity and $\omega + O(V)$ is in the center of $A(V)$.

(2) The linear map $o: V \to \text{End}_{M_0}, v \mapsto o(v) |_{M_0}$ induces an associative algebra homomorphism $o: A(V) \to \text{End}_{M_0}$. Thus $M_0$ is a left $A(V)$-module.

Next we recall the $A(V)$-bimodule $A(M)$. Let $(M, Y_M)$ be an $\mathbb{N}$-gradable $V$-module. Define bilinear maps $\ast: M \to M_1$, $\circ: V \times M \to M$, $\ast: M \times V \to M$ by

\[ a \ast u = \left( \text{Res}_z \frac{(1 + z)^{wt(a)}}{z} Y_M(a, z) \right) u = \sum_{i = 0}^{\text{wt}(a)} \binom{\text{wt}(a)}{i} a_{i - 1} u, \quad (2.5) \]

\[ a \circ u = \left( \text{Res}_z \frac{(1 + z)^{wt(a)}}{z^2} Y_M(a, z) \right) u = \sum_{i = 0}^{\text{wt}(a)} \binom{\text{wt}(a)}{i} a_{i - 2} u, \]
and
\[
    a \ast u = \left( \text{Res}_z \left( \frac{1 + z}{z} \right)^{\text{wt}(a) - 1} Y_M(a, z) \right) u = \sum_{i=0}^{\text{wt}(a) - 1} \binom{\text{wt}(a) - 1}{i} a_{i-1} u,
\]

for homogeneous \( a \in V \) and \( u \in M \) respectively, and extend these to linear operations on \( V \). Let \( O(M) \) be the linear span of \( a \ast u \) for \( a \in V \), \( u \in M \), and set \( A(M) = M/O(M) \). We denote the image of \( u \in M \) in \( A(M) \) by \([u]\). Then \( A(M) \) is an \( A(V) \)-bimodule, and the left and right actions are given by \([a] \ast [u] = [a \ast u]\) and \([u \ast a] = [u \ast a]\) respectively for \( a \in V \), \( u \in M \).

**Definition 2.8.** Let \( A \) be an associative algebra and \( R, B, L \) be a right \( A \)-module, an \( A \)-bimodule, and a left \( A \)-module respectively. The tensor product of \( R, B, \) and \( L \) as an \( A \)-module \( RBL \) is called a contraction of \( R, B, \) and \( L \) and is denoted by \( R \otimes B \otimes L \).

Now let \( M^i = \bigoplus_{n=0}^{\infty} M^i_n \) (\( i = 1, 2, 3 \)) be \( \mathbb{N} \)-gradable \( V \)-modules. Suppose that for any \( i \) there exists a scalar \( h_i \in \mathbb{C} \) such that \( L(0) \) acts on \( M^i_n \) as \( h_i + n \). Let \( I \) be an intertwining operator of type
\[
    \begin{pmatrix}
        M^3 \\
        M^1 \\
        M^2
    \end{pmatrix}.
\]

Then for each \( v \in M^1 \), we have \( I(v, z) \in z^{-h_1 - h_2 + h_3} (\text{Hom}(M^2, M^3))[[z, z^{-1}]] \) (see Proposition 1.5.1 in [FZ]). We define \( o(v) = \text{Res}_z z^{h_1 + h_2 - h_3 + \text{deg}(v)} \cdot I(v, z) \) for homogeneous \( v \in M^1 \) and extend it to \( M^1 \) by linearity. Then we have following theorem (see Theorem 1.5.2 in [FZ]).

**Theorem 2.9.** Let \( M^i = \bigoplus_{n=0}^{\infty} M^i_n \) (\( i = 1, 2, 3 \)) be \( \mathbb{N} \)-gradable \( V \)-modules. Suppose that for each \( M^i \) there exists an \( h_i \in \mathbb{C} \) such that \( L(0) \) acts on \( M^i_n \) as a scalar \( h_i + n \). Let \( (M^3)^* = \bigoplus_{n=0}^{\infty} (M^3_n)^* \) be the contragredient module of \( M^3 \). Then the linear map
\[
    \pi: I \left( \begin{pmatrix}
        M^3 \\
        M^1 \\
        M^2
    \end{pmatrix} \right) \rightarrow \left( \begin{pmatrix} M^3 \\
        M^1 \\
        M^2 \end{pmatrix} \right)^*, \quad I \mapsto \pi(I) \quad \text{(2.7)}
\]
given by the property \( \pi(I)(v'_3 \otimes [v_1] \otimes v_2) = \langle v'_3, o(v_1)v_2 \rangle \) for
\[
    I \in I \left( \begin{pmatrix}
        M^3 \\
        M^1 \\
        M^2
    \end{pmatrix} \right), \quad v'_3 \in \left( M^3_0 \right)^*, \quad v_1 \in M^1, \quad \text{and} \quad v_2 \in M^2.
\]
is well-defined.
If $V$ is rational, that is, all $\mathbb{N}$-gradable $V$-modules are completely reducible, then the linear map $\pi$ is an isomorphism. In general, $\pi$ is not surjective (see [Li]), but we have the following proposition.

**Proposition 2.10.** Let $M^i = \bigoplus_{n=0}^{\infty} M^i_n$ ($i = 1, 2, 3$) be as in Theorem 2.9. Suppose that $M^2$ is irreducible and that $M^3$ is an irreducible ordinary. Then $\pi$ is injective. Thus we have

$$N^{M^i}_{M^j} = \dim((M^3_n)^* \cdot A(M^1) \cdot M^2_n)^*.$$

**Proof.** Let $I$ be an intertwining operator of type

$$\begin{pmatrix}
M^3 \\
M^1 \\
M^2
\end{pmatrix}.$$

Suppose that $\pi(I) = 0$. Then we have $\langle v_3', I(v_1, z)v_2 \rangle = 0$ for any $v_3' \in (M^3_n)^*, v_1 \in M^1$, and $v_2 \in M^2_n$. Set $W = \{ u \in M^2 | \langle v_3', I(v_1, z)u \rangle = 0 \text{ for any } v_3' \in (M^3_n)^*, v_1 \in M^1 \}$.

Note that for any nonzero $u \in M^2_n$, we have $a_n u \in W$ for any homogeneous $a \in V, n \in \mathbb{Z}$. In fact, it is obvious that $a_n u \in W$ for $n \geq \text{wt}(a) - 1$, and for $n < \text{wt}(a) - 1$ we have $\langle v_3, a_n'(I(v_1, z)u) \rangle = 0$ for any $v_3' \in (M^3_n)^*$ and $v_1 \in M^1$. Hence we see that

$$\langle v_3', I(v_1, z)a_n u \rangle = \langle v_3', a_n I(v_1, z)u \rangle - \sum_{i=0}^{\infty} \binom{n}{i} z^{n-i} \langle v_3', I(a_n v_1, z)u \rangle = 0,$$

from the Jacobi identity (2.4). Thus $a_n u \in W$ for all $n < \text{wt}(a) - 1$. Since $M^2$ is irreducible, $M^2_n$ is spanned by $a_n u$ for homogeneous $a \in V$ (see [DM]), and then $W = M^2_n$.

Let $w' \in (M^3_n)^*$ be a nonzero element. Then for every $a \in V$ and $n \in \mathbb{Z}$, we have

$$\langle w', a_n(I(v_1, z)v_2) \rangle = \langle w', I(v_1, z)a_n v_2 \rangle + \sum_{i=0}^{\infty} \binom{n}{i} z^{n-i} \langle w', I(a_n v_1, z)v_2 \rangle = 0.$$

Hence $\langle w', Y_{M^i}(a, z_0)I(v_1, z)v_2 \rangle = 0$. This implies that

$$\langle Y_{M^i}(a, z_0)w', I(v_1, z)v_2 \rangle = \langle w', Y_{M^i}(e^{z_0 L(-1)} (-z_0^{-2}) L_{a}^{00} z_0^{-1}) I(v_1, z)v_2 \rangle = 0.$$
Therefore $\langle a_n^aw', I(v_1, z)v_2 \rangle = 0$ for any $a \in V$ and $n \in \mathbb{Z}$, where 
$Y_{a}^{\mathfrak{h}}(a, z) = \sum_{n \in \mathbb{Z}} a_n^a z^{-n-1}$. Since $(M')'$ is irreducible, it is spanned by $a_n^aw'$ for $a \in V$ and $n \in \mathbb{Z}$. Then for every $v'_3 \in (M')'$, $v' \in M^1$, and $v_2 \in M^2$, $\langle v'_3, I(v_1, z)v_2 \rangle = 0$, which means $I = 0$.

2.3. Vertex Operator Algebra $M(1)$

Let $\mathfrak{h}$ be a $d$-dimensional vector space with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, and let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be a Lie algebra with the commutation relation given by 
$[h \otimes t^m, h' \otimes t^n] = m \delta_{m+n, 0} \langle h, h' \rangle K$, 
$[K, \hat{\mathfrak{h}}] = 0$. Set $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$. For $\lambda \in \mathfrak{h}$, let $\mathbb{C}e^\lambda$ be a one-dimensional $\hat{\mathfrak{h}}^*$-module on which $\mathfrak{h} \otimes t\mathbb{C}[t]$ acts trivially, $\hat{\mathfrak{h}}$ acts as $\langle h, \lambda \rangle$ for $h \in \mathfrak{h}$, and $K$ acts as $1$. Let $M(1, \lambda)$ be an $\hat{\mathfrak{h}}$-module induced by the $\hat{\mathfrak{h}}^*$-module $\mathbb{C}e^\lambda$:

$$M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}})} \mathbb{C}e^\lambda = S(\mathfrak{h} \otimes t\mathbb{C}[t^{-1}]) \quad \text{(linearly)}.$$ 

Denote the action of $h \otimes t^n (h \in \mathfrak{h}, n \in \mathbb{Z})$ on $M(1, \lambda)$ by $h(n)$ and set 
$h(z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n-1}$.

For $\lambda, \mu \in \mathfrak{h}$, we define a linear map $P_{\lambda\mu} : M(1, \mu) \to M(1, \lambda + \mu)$ by $P_{\lambda\mu}(u \otimes e^\mu) = u \otimes e^{\lambda + \mu}$ for $u \in S(h) \otimes t\mathbb{C}[t^{-1}]$. Then a vertex operator associated with $e^\lambda$ is defined by

$$I_{\lambda\mu}(e^\lambda, z) = \exp \left( \sum_{n \in \mathbb{Z}_{>0}} \frac{\lambda(-n)}{n} z^n \right) \exp \left( - \sum_{n \in \mathbb{Z}_{>0}} \frac{\lambda(n)}{n} z^{-n} \right) P_{\lambda\mu} z^{(\lambda, \mu)}.$$ 

(2.8)

The vertex operator associated with $h_1(-n_1)h_2(-n_2) \cdots h_k(-n_k)e^\lambda \in M(1, \lambda)(n_i \in \mathbb{Z}_{>0}, h_i \in \mathfrak{h})$ is defined by

$$I_{\lambda\mu}(h_1(-n_1)h_2(-n_2) \cdots h_k(-n_k)e^\lambda, z)$$

$$= \hat{\vartheta}^{(n_n-1)}(z) h_1(z) \hat{\vartheta}^{(n_k-1)}(z) \cdots \hat{\vartheta}^{(n_n-1)}(z) I_{\lambda\mu}(e^\lambda, z) \hat{\vartheta}^{(n_k-1)}(z),$$

(2.9)

where $\hat{\vartheta}^{(n)} = \hat{\vartheta}(\hat{\vartheta}^{(n)})^{n}$ for $n \in \mathbb{N}$ and the normal ordering $\hat{\vartheta} \cdot \hat{\vartheta}$ is an operation which reordered so that $h(n)$ ($n < 0$), $P_{\lambda\mu}$ are placed to the left of $h(n)$ ($n \in \mathbb{N}$). We extend $I_{\lambda\mu}$ to $M(1, \lambda)^3$ by linearity. Now let $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ $(d = \dim \mathfrak{h})$ be an orthonormal basis of $\mathfrak{h}$ and set $\mathbf{1} = 1 \otimes e^\lambda$, $\omega = (1/2) \sum_{i=1}^{d} \alpha_i (-1)^{i} \mathbf{1} \in M(1, 0)$. Then $(M(1, 0), I_{00}, I, \omega)$ is a simple vertex operator algebra with central charge $d$ and $(M(1, \lambda), I_{0\lambda}, I, \lambda) \mid \lambda \in \mathfrak{h}$ gives all irreducible $M(1, 0)$-modules (see [FLM]). The vertex operator algebra $M(1, 0)$ is called the free bosonic vertex operator algebra and is denoted by $M(1)$. 

TOSHIYUKI ABE

344
Let $\theta$ be an automorphism of $M(1)$ defined by

$$\theta(h_1(-n_1)h_2(-n_2)\cdots h_k(-n_k)1) = (-1)^kh_1(-n_1)h_2(-n_2)\cdots h_k(-n_k)1$$

for $h_i \in \mathbb{B}$, $n_i \in \mathbb{Z}_{>0}$. We denote the orbifold of $M(1)$ for $\theta$ by $M(1)^+$ and the $-1$-eigenspace of $M(1)$ by $M(1)^-$. Then the following proposition is proved in [DN1].

**Proposition 2.11.** The vertex operator algebra $M(1)^+$ is simple, and $M(1)^-, M(1, \lambda)$ ($\lambda \neq 0$) are irreducible $M(1)^+$-modules. Moreover, $M(1, \lambda)$ is isomorphic to $M(1, -\lambda)$ as an $M(1)^+$-module.

Next we consider the $\theta$-twisted $M(1)$-module. Let $\hat{\mathfrak{g}}[-1] = \mathfrak{g} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}K$ be a Lie algebra with commutation relation $[h \otimes t^n, h' \otimes t^m] = m\delta_{m+n,0}\langle h, h' \rangle K$, $[K, \hat{\mathfrak{g}}[-1]] = 0$ for $h, h' \in \mathfrak{g}$, $m, n \in 1/2 + \mathbb{Z}$. Set $\hat{\mathfrak{g}}[-1]^+ = \mathfrak{g} \otimes t^{1/2}\mathbb{C}[t] \otimes \mathbb{C}$. Then $\mathbb{C}$ is viewed as an $\hat{\mathfrak{g}}[-1]^+$-module on which $\mathfrak{g} \otimes t^{1/2}\mathbb{C}[t]$ acts trivially and $K$ acts as 1. Set $M(1)(\theta)$ to be the induced $\hat{\mathfrak{g}}[-1]^+$-module:

$$M(1)(\theta) = U(\hat{\mathfrak{g}}[-1]) \otimes_{U(\hat{\mathfrak{g}}[-1]\otimes \mathbb{C})} S(\mathfrak{g} \otimes t^{-1/2}\mathbb{C}[t^{-1/2}]) \quad \text{(linearly)}.$$  

Denote the action of $h \otimes t^n$ ($h \in \mathfrak{g}$, $n \in 1/2 + \mathbb{Z}$) on $M(1)(\theta)$ by $h(n)$ and set $h(z) = \sum_{n \in 1/2 + \mathbb{Z}} h(n)z^{-n-1}$. For $\lambda \in \mathfrak{g}$, a twisted vertex operator associated with $e^\lambda \in M(1, \lambda)$ is defined as

$$I_\lambda^\theta(e^\lambda, z) = z^{-\langle \lambda, \lambda \rangle / 2} \exp\left( \sum_{n \in 1/2 + \mathbb{N}} \frac{\lambda(-n)}{n} z^n \right) \times \exp\left( - \sum_{n \in 1/2 + \mathbb{N}} \frac{\lambda(n)}{n} z^{-n} \right). \quad (2.10)$$

For $h_1(-n_1)h_2(-n_2)\cdots h_k(-n_k) \otimes e^\lambda \in M(1, \lambda) (h_i \in \mathfrak{g}, n \in \mathbb{Z}_{>0})$, set

$$W^\theta(h_1(-n_1)h_2(-n_2)\cdots h_k(-n_k)e^\lambda, z) = \hat{\mathfrak{g}} \hat{\mathfrak{g}}^{(n_1-1)}h_1(z) \hat{\mathfrak{g}}^{(n_2-1)}h_2(z) \cdots \hat{\mathfrak{g}}^{(n_k-1)}h_k(z) I_\lambda(e^\lambda, z)$$

and extend to a linear operator on $M(1, \lambda)$, where the normal ordering $\hat{\mathfrak{g}} \hat{\mathfrak{g}}$ is an operation which sifts $h(n)$ ($n \in \mathbb{Z}_{>0}$) to the right and $h(n)$ ($n \in \mathbb{Z}_{<0}$) to the left. Let $c_{mn} \in \mathbb{Q} (m, n \in \mathbb{N})$ be constants defined by
the formal power series expansion
\[ \sum_{m, n \geq 0} c_{mn} x^m y^n = -\log \left( \frac{(1 + x)^{1/2} + (1 + y)^{1/2}}{2} \right). \]

and define an operator \( \Delta_z \) on \( M(1, \lambda) \) by
\[ \Delta_z = \sum_{m, n \geq 0} \sum_{i=0}^{d} c_{mn} \alpha_i(m) \alpha_i(n) z^{-m-n}. \]

Then the twisted vertex operator associated with \( u \in M(1, \lambda) \) is defined by
\[ I_\theta^u(u, z) = W_{\theta}(e^{d_1 u}, z). \tag{2.11} \]

Now define the action of \( \theta \) on \( M(1)(\theta) \) by
\[ \theta(h_1(-n_1) h_2(-n_2) \cdots h_k(-n_k) 1) \]
\[ = (-1)^k h_1(-n_1) h_2(-n_2) \cdots h_k(-n_k) 1 \tag{2.12} \]
for \( h_i \in \mathfrak{h} \), \( n_i \in 1/2 + \mathbb{N} \), and denote the \( \pm 1 \)-eigenspace of \( M(1)(\theta) \) by \( M(1)(\theta)^\pm \) respectively. Then we have following proposition (see Theorem 2.5 of [DN1]).

**Proposition 2.12.**
(1) \( (M(1)(\theta), I_\theta^u) \) is an irreducible \( \theta \)-twisted \( M(1) \)-module.

(2) \( (M(1)(\theta)^\pm, I_\theta^u) \) are irreducible \( M(1)^\pm \)-modules.

From now on, we consider the case \( \dim \mathfrak{h} = 1 \) and set \( \alpha_1 = h \). For \( \lambda \in \mathbb{C} \), we denote the irreducible \( M(1) \)-module \( M(1, \lambda h) \) by \( M(1, \lambda) \) and \( e^{\lambda h} \) by \( e^\lambda \).

Set \( J = h(-1)^4 1 - 2h(-3) h(-1) 1 + (3/2)^h(-2)^2 1 \in M(1)^+ \) which is the lowest weight vector of weight 4 for a Virasoro algebra. Then the vertex operator algebra \( M(1)^+ \) is generated by \( \omega \) and \( J \) (see [DG]), and the Zhu algebra \( A(M(1)^+) \) is generated by \( [\omega] \) and \( [J] \). More precisely, there is an isomorphism \( \mathbb{C}[x, y]/I \to A(M(1)^+) \), \( x + I \to [\omega] \), \( y + I \to [J] \), where \( I \) is an ideal generated by two polynomials \( (y - 4x^2 + x)(y - 908x^2 - 515x + 27) \) and \( (y - 4x^2 + x)(x - 1)(x - 1/16)(x - 9/16) \) (see [DN1, Theorem 4.4]).

In [DN1, Theorem 4.5], all equivalence classes of irreducible \( \mathbb{N} \)-gradable \( M(1)^\pm \)-modules are classified as follows:

**Theorem 2.13.** The set \( (M(1)^\pm, M(1)(\theta)^\pm, M(1, \lambda) \ (\lambda = M(1, -\lambda); \lambda \neq 0)) \) gives all inequivalent irreducible \( M(1)^\pm \)-modules.

Let \( M \) be an irreducible \( M(1)^\pm \)-module. Then the homogeneous space of the lowest weight (written by \( d_M \in \mathbb{C} \)) is one-dimensional, and \( M \) has
naturally an \( \mathbb{N} \)-gradation defined by \( M_n = M(a_M + n) \) for any \( n \in \mathbb{N} \), where \( M(\lambda) \) is the homogeneous space of weight \( \lambda \) of \( M \). One of the basis (written by \( v_M \)) of \( M_0 \) is given in Table I, and we denote the dual basis of \( (M_0)^* \) corresponding to \( v_M \) by \( v_M^* \). Since \( M_0 \) is one-dimensional, \( [J] \) also acts on this space as a scalar denoted by \( b_M \in \mathbb{C} \).

### 3. A SPANNING SET OF \( A(M) \)

This section is divided into three subsections. In the first subsection, we describe the irreducible decompositions of irreducible \( M(1)^+ \)-modules as modules for the Virasoro algebra. In the second subsection we prove some lemmas. In last subsection we give a spanning set of \( A(M) \) for the irreducible \( M(1)^+ \)-module \( M \).

#### 3.1. Irreducible Decompositions of Irreducible \( M(1)^+ \)-Modules

Let \( W \) be a module for the Virasoro algebra with central charge \( c \in \mathbb{C} \) such that \( L(0) \) acts on \( W \) semisimply and each eigenspace is finite dimensional. Then the *formal character* of \( W \) is defined by

\[
\text{ch}_W = \text{tr}_{\mathcal{H}} q^{L(0) - c/24} = q^{-c/24} \sum_{\lambda \in \mathbb{C}} (\text{dim } W(\lambda)) q^\lambda \in \mathbb{Z}[q],
\]

where \( W(\lambda) \) is the eigenspace of weight \( \lambda \) for \( L(0) \). Let \( L(1, \lambda) \) be the irreducible lowest weight module for Virasoro algebra with central charge 1 and lowest weight \( \lambda \). Then the formal character of \( L(1, \lambda) \) is given by

\[
\text{ch}_{L(1, \lambda)} = \begin{cases} 
q^\lambda / \eta(q) & \text{if } \lambda \neq \frac{n^2}{4} \text{ for any } n \in \mathbb{Z} \\
1/ \eta(q) (q^{n^2/4} - q^{(n+2)^2/4}) & \text{if } \lambda = \frac{n^2}{4} \text{ for some } n \in \mathbb{N},
\end{cases}
\]

where \( \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) is the Dedekind \( \eta \)-function (see [KR]).

<table>
<thead>
<tr>
<th>( M )</th>
<th>( M(1)^+ )</th>
<th>( M(1)^- )</th>
<th>( M(1, \lambda) )</th>
<th>( M(1, \lambda) )</th>
<th>( M(1, \lambda') )</th>
<th>( M(1, \lambda') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_M )</td>
<td>1</td>
<td>( h(-1/2) )</td>
<td>( e^h )</td>
<td>1</td>
<td>( h(-1/2) )</td>
<td></td>
</tr>
<tr>
<td>( a_M )</td>
<td>0</td>
<td>1</td>
<td>( \lambda^2/2 )</td>
<td>1/16</td>
<td>9/16</td>
<td></td>
</tr>
<tr>
<td>( b_M )</td>
<td>0</td>
<td>-6</td>
<td>( \lambda^4 - \lambda^2/2 )</td>
<td>3/128</td>
<td>-45/128</td>
<td></td>
</tr>
</tbody>
</table>
It is well known that the irreducible \( M(1) \)-module \( M(1, \lambda) \) (\( \lambda \in \mathbb{C} \)) is a completely reducible module for the Virasoro algebra, and decomposes into a direct sum of irreducible modules for the Virasoro algebra as follows (see [WY], [KR]):

\[
M(1, \lambda) = \begin{cases} 
L \left( 1, \frac{\lambda^2}{2} \right) & \text{if } \frac{\lambda^2}{2} \neq \frac{n^2}{4} \text{ for any } n \in \mathbb{Z} \\
\bigoplus_{p=0}^{\infty} L \left( 1, \frac{(n + 2p)^2}{4} \right) & \text{if } \frac{\lambda^2}{2} = \frac{n^2}{4} \text{ for some } n \in \mathbb{N}.
\end{cases}
\] (3.2)

Furthermore, in [DG] it is shown that the irreducible \( M(1)^{\pm} \)-modules \( M(1)^{\pm} \) are also completely reducible as modules for the Virasoro algebra, and the irreducible decompositions are given by

\[
M(1)^{+} = \bigoplus_{p=0}^{\infty} L(1, 4p^2), \quad M(1)^{-} = \bigoplus_{p=0}^{\infty} L(1, (2p + 1)^2). \quad (3.3)
\]

We show that the irreducible \( M(1)^{\pm} \)-modules \( M(1)(\theta)^{\pm} \) are also completely reducible as modules for the Virasoro algebra. To do this, we consider the character of \( M(1)(\theta) \).

**Lemma 3.1.** We have

\[
\text{ch}_{M(1)(\theta)} = \prod_{k=1}^{\infty} \frac{q^{1/16 - 1/24}}{(1 - q^{k^{2}/4})} = \frac{1}{\eta(q)} \sum_{p=0}^{\infty} q^{(2p+1)^2/16}. \quad (3.4)
\]

**Proof.** The first equality is clear by the fact that the weight space of weight \( n (n \in 1/16 + (1/2)\mathbb{Z}) \) has a basis

\[
\{ h(-m_1) \cdots h(-m_k) 1 \mid k \in \mathbb{N}, m_i \in \frac{1}{2} + \mathbb{N}, \\
m_1 \geq m_2 \geq \cdots \geq m_k > 0, m_1 + \cdots + m_k = n \}.
\]

To prove the second equality, we have to show that

\[
\prod_{k=1}^{\infty} \frac{(1 - q^k)}{(1 - q^{k^{2}/4})} = \sum_{p=0}^{\infty} q^{(p+1)^2/4}. \quad (3.5)
\]

From Jacobi’s triple product \( \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}z)(1 + q^{2n-1}z^{-1}) = \sum_{n \in \mathbb{Z}} q^{n^2}z^n \), we see

\[
\prod_{n=1}^{\infty} (1 - q^{4n})(1 + q^{2n}) = \sum_{n=0}^{\infty} q^{n^2+n}
\]
by substituting $z = q$. Replacing $q$ with $q^{1/4}$, we have
\[
\prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n/2}) = \sum_{n=0}^{\infty} q^{(n^2+n)/4}.
\]

On the other hand, direct calculations show
\[
\prod_{k=1}^{\infty} \frac{(1 - q^k)}{(1 - q^{k-1/2})} = \prod_{k=1}^{\infty} (1 - q^k)(1 + q^{k/2}).
\]

Thus the equality (3.5) holds.

We define an hermitian form $(\cdot): M(1)(\theta) 	imes M(1)(\theta) \to \mathbb{C}$ by
\[
\langle h(-m_k)^{p_k} \cdots h(-m_1)^{p_1} | h(-n_l)^{q_l} \cdots h(-n_1)^{q_1} \rangle
\]
\[
= \begin{cases} 
\prod_{i=1}^{l} (m_i)^{p_i} & \text{if } k = l \text{ and } m_i = n_i, p_i = q_i \text{ for all } 1 \leq i \leq k \\
0 & \text{otherwise},
\end{cases}
\]

for $k, l \in \mathbb{N}$, $m_i, n_j \in 1/2 + \mathbb{N}$, $m_k \geq \cdots \geq m_1$, $n_l \geq \cdots \geq n_1$, and $p_i, q_j \in \mathbb{N}$ $(1 \leq i \leq k, 1 \leq j \leq l)$. Then we can check that the hermitian form $(\cdot)$ is a nondegenerate positive-definite contravariant form on $M(1)(\theta)$. Thus $M(1)(\theta)$ is a unitary representation of the Virasoro algebra and is completely reducible as a module for the Virasoro algebra. Let
\[
M(1)(\theta) = \bigoplus_{h \in \mathbb{C}} L(1, h)^{m_h}, \quad m_h \in \mathbb{N},
\]
be the irreducible decomposition of $M(1)(\theta)$ as a Virasoro algebra module, where
\[
L(1, h)^{m_h} = \overline{L(1, h) \oplus \cdots \oplus L(1, h)}.
\]

Since weights which appear in $M(1)(\theta)$ are in $1/16 + (1/2)\mathbb{Z}$, we may assume that
\[
M(1)(\theta) = \bigoplus_{h \in \mathbb{C} + \frac{1}{16}} L(1, h)^{m_h}, \quad m_h \in \mathbb{N}.
\]

In addition, we can show that $1/16 + n/2 \neq m^2/4$ for any $n, m \in \mathbb{Z}$. By the formal characters (3.1), we find that the formal character of $M(1)(\theta)$
has the form

$$\text{ch}_{M(1|\theta)} = \frac{1}{\eta(q)} \sum_{h \in \mathbb{M} + \frac{1}{16} \mathbb{Z}} m_h q^h, \quad m_h \in \mathbb{N}.$$ 

Comparing with (3.4), we have

$$m_h = \begin{cases} 
1 & \text{if } h = \frac{(2p + 1)^2}{16} \text{ for some } p \in \mathbb{N}, \\
0 & \text{otherwise}. 
\end{cases}$$

Consequently we have irreducible decomposition of $M(1|\theta)$,

$$M(1|\theta) = \bigoplus_{p=0}^{\infty} L\left(1, \frac{(2p + 1)^2}{16}\right).$$

**Proposition 3.2.** $M(1)^{\dagger}$-modules $M(1|\theta)$ and $M(1|\theta)^{\dagger}$ are completely reducible as modules for the Virasoro algebra, and the irreducible decompositions are given by

$$M(1|\theta) = \bigoplus_{p=0}^{\infty} L\left(1, \frac{(2p + 1)^2}{16}\right),$$

$$M(1|\theta)^{\dagger} = \bigoplus_{p=0}^{\infty} L\left(1, \frac{(8p + 1)^2}{16}\right) \oplus L\left(1, \frac{(8p + 7)^2}{16}\right),$$

$$M(1|\theta)^{\dagger} = \bigoplus_{p=0}^{\infty} L\left(1, \frac{(8p + 3)^2}{16}\right) \oplus L\left(1, \frac{(8p + 5)^2}{16}\right).$$

**Proof.** We have already proved the irreducible decomposition (3.6). By the definitions of $M(1|\theta)^{\dagger}$, $M(1|\theta)^{\dagger}$ $(M(1|\theta)^{\dagger})$ is spanned by all homogeneous vectors whose weights are in $1/16 + (1/2)\mathbb{Z}$ (resp. $9/16 + (1/2)\mathbb{Z}$). Since each weight which appears in $L(1, (2p + 1)^2/16)$ is in $(2p + 1)^2/16 + \mathbb{Z}$, we see that if $(2p + 1)^2/16$ is in $1/16 + (1/2)\mathbb{Z}$ (resp. $9/16 + (1/2)\mathbb{Z}$), then $L(1, (2p + 1)^2/16)$ is contained in $M(1|\theta)^{\dagger}$ (resp. $M(1|\theta)^{\dagger}$). On the one hand, we have

$$\frac{(2p + 1)^2}{16} = \begin{cases} 
1/16 \pmod{\mathbb{Z}} & \text{if } p = 0, 3 \pmod{4}, \\
9/16 \pmod{\mathbb{Z}} & \text{if } p = 1, 2 \pmod{4}. 
\end{cases}$$
Thus we see that
\[
\bigoplus_{p=0}^\infty \left( L(1, (8p + 1)^2/16) \oplus L(1, (8p + 7)^2/16) \right) \subseteq M(1)(\theta)^+ \quad (3.9)
\]
\[
\bigoplus_{p=0}^\infty \left( L(1, (8p + 3)^2/16) \oplus L(1, (8p + 5)^2/16) \right) \subseteq M(1)(\theta)^- \quad (3.10)
\]
Since \(M(1)(\theta) = M(1)(\theta)^+ \oplus M(1)(\theta)^-\), the irreducible decomposition (3.6) shows that the inclusions of (3.9) and (3.10) are equal. \[\]

From irreducible decompositions (3.2), (3.3), (3.7), and (3.8), we can see that formal characters of irreducible \(M(1)^+\)-modules \(M(1)^\pm\), \(M(1)(\theta)^\pm\), and \(M(1, \lambda) (\lambda \neq 0)\) are distinct. So together with Theorem 2.13, we have the following lemma.

**Lemma 3.3.** Let \(M, N\) be irreducible \(M(1)^+\)-modules such that the formal character of \(M\) coincides with that of \(N\). Then \(M\) is isomorphic to \(N\).

In particular, if \(M\) is an irreducible \(M(1)^+\)-module, then the character of its contragredient module \(M'\) is the same as that of \(M\). Thus Lemma 3.3 shows that \(M\) and its contragredient module \(M'\) are isomorphic as \(M(1)^+\)-modules. Together with Proposition 2.6, we thus have the following proposition.

**Proposition 3.4.** Let \(M^i (i = 1, 2, 3)\) be irreducible \(M(1)^+\)-modules. Then the fusion rule \(N_{M^i M^j}^{M^k}\) is invariant under the permutation of \(\{1, 2, 3\}\).

### 3.2. Some Useful Lemmas

**Lemma 3.5.** Let \((M, V_M)\) be an \(\mathbb{N}\)-gradable \(V\)-module. Then the following hold for any \(a \in V\) homogeneous, \(u \in M\), and homogeneous vector \(v \in M\) of weight \(\text{wt}(v)\):

1. For all \(m, n \in \mathbb{N}\) such that \(n \geq m\),

\[
\text{Res}_z \frac{(1 + z)^{\text{wt}(a) + m}}{z^{\text{wt}(a) + n}} Y_M(a, z) u \in O(M).
\]

2. For any \(n \in \mathbb{Z}_{>0}\), \([L(-n)v] = (-1)^{n-1}[\omega * v - n v * \omega - \text{wt}(v)v]\).

**Proof.** Part 1 is shown in Lemma 1.5.3 in [FZ]. For Part 2, we can show that

\[
[L(-n)v] = (-1)^{n-1}\left( (L(-1) - (n - 1)(L(-2) + L(-1)))v \right)
\]
for all $u \in M$ and $n \in \mathbb{Z}_{\geq 0}$ by induction on $n$ and Part 1. Hence

$$[L(-n)v] = (-1)^{n-1}[(L(-2) + 2L(-1) + L(0) - n(L(-2)) + L(-1)) - L(0)]v.$$ 

$$= (-1)^{n-1}[(\omega \ast v - nw \ast \omega - \text{wt}(v)v].$$

\[\square\]

**Lemma 3.6.** Let $M$ be a $V$-module and let $U$ be a subspace of $M$ such that $a \ast U \subset U$ and $U \ast a \subset U$ hold for any $a \in V$. If $a \in V$ homogeneous of positive weight and $u \in U$ satisfy that $a_{n}u \in U + O(M)$ for any $1 \leq n \leq \text{wt}(a) - 1$, then $a_{n}u \in U + O(M)$ for all $n \leq \text{wt}(a) - 1$. In particular, if $u \in M$ is homogeneous, then $L(-n)u \in U + O(M)$ for any $n \in \mathbb{N}$.

**Proof.** We show that

$$a_{-n}u \in U + O(M), \quad (3.11)$$

for any $n \in \mathbb{N}$ by induction on $n$. From the assumption of this lemma, we have

$$a_{n}u = a \ast u - u \ast a - \sum_{i=1}^{\text{wt}(a) - 1} \binom{\text{wt}(a) - 1}{i} a_{i}u \in U + O(M).$$

So (3.11) holds for $n = 0$. In the case $n = 1$, we have

$$a_{-1}u = u \ast a - \sum_{i=1}^{\text{wt}(a) - 1} \binom{\text{wt}(a) - 1}{i} a_{i-1}u.$$ 

Since $a_{n}u \in U + O(M)$ for every $0 \leq n \leq \text{wt}(a) - 1$, we see that (3.11) holds for $n - 1$. Assume that $l \in \mathbb{Z}_{\geq 0}$ and (3.11) holds for any $0 \leq n \leq l$. Then by Lemma 3.5 (1),

$$a_{-l-1}u + \sum_{i=1}^{\text{wt}(a) - 1} \binom{\text{wt}(a) - 1}{i} a_{i-1}u \in O(M).$$

By induction hypothesis, we have $a_{-n-1}u \in U + O(M)$. \[\square\]

Let $M$ be a $V$-module. We define a linear endomorphism $\phi_{M}: M \rightarrow M$ by

$$\phi_{M}(u) = e^{L(1)_d} \text{id}_{L(0)}u$$

for $u \in M$. We remark that the operator $\phi_{V}$ induces an anti-automor-
Proposition 3.7. The linear map $\phi_M: M \to M$ satisfies the properties

$$
\phi_M(a * u) = \phi_M(u) * \phi_M(a), \phi_M(u * a) = \phi_M(a) * \phi_M(u),
$$

for any $a \in V$ and $u \in M$.

Proof. One can find in [FHL] the conjugation formulas

$$
z^{L(0)}_1 Y_M(a, z_0) z^{-L(0)}_1 = Y_M(z^{L(0)}_1 a, z_1 z_0),
$$

$$
e^{-z_i L(1)} Y_M(a, z_0) e^{-z_i L(1)} = Y_M \left( e^{-z_i (1 - z z_0)} L(1) (1 - z z_0)^{-2L(0)} a, \frac{z_0}{1 - z z_0} \right)
$$

for every $a \in V$. So we have

$$
\text{Res}_{z_0} \frac{(1 + z_0)^{wt(a) + n}}{z_0^m} e^{L(1) e \pi i L(0)} Y_M(a, z_0) u
$$

$$
= \text{Res}_{z_0} \left( \frac{(1 + z_0)^{wt(a) + n}}{z_0^m} Y_M \right)
$$

$$
\times \left( e^{(1 + z_0) L(1)} (1 + z_0)^{-2wt(a)} e \pi i L(0) a, \frac{-z_0}{1 + z_0} \right) \phi_M(u)
$$

for every $m, n \in \mathbb{Z}$, $a \in V$, and $u \in M$. Here we replace $-z_0/(1 + z_0)$ with $w$ and apply the formula for a change of variables (see [Z]),

$$
\text{Res}_w g(w) = \text{Res}_z \left( g(f(z)) \frac{d}{dz} f(z) \right),
$$

where $g(w) \in M((w))$ and $f(z) \in \mathbb{C}[[z]]$. Then we have

$$
\text{Res}_{z_0} \frac{(1 + z_0)^{wt(a) + n}}{z_0^m} e^{L(1) e \pi i L(0)} Y_M(a, z_0) u
$$

$$
= (-1)^{m+1} \text{Res}_w \frac{(1 + w)^{wt(a) - n + m - 2}}{w^m} Y_M
$$

$$
\times \left( \sum_{k=0}^{\infty} \frac{(1 + w)^{-k}}{k!} L(1)^k e \pi i L(0) a, w \right) \phi_M(u)
$$
\[
= (-1)^{m+1} \sum_{k=0}^{\infty} \operatorname{Res}_w \frac{(1 + w)^{w(a) - n + m - 2-k}}{w^m} Y_M
\times \left( \frac{L(1)^k}{k!} e^{\pi i L(0) a} w \right) \phi_M(u).
\]

Hence if we take \( m = 1 \) and \( n = 0 \), then
\[
\phi_M(a \ast u) = \sum_{k=0}^{\infty} \phi_M(u) \ast \left( \frac{L(1)^k}{k!} e^{\pi i L(0) a} \right) \phi_M(u)
= \phi_M(u) \ast \phi_M(a).
\]

Similarly, if we take \( m = 1 \) and \( n = -1 \), then \( \phi_M(u \ast a) = \phi_M(a) \ast \phi_M(u) \), and if \( m = 2 \) and \( k = 0 \), then \( \phi_M(a \ast u) = -\phi_M(a) \ast \phi_M(u) \).

Thus \( \phi_M \) induces a linear map on \( \mathcal{A}(M) \) (also denoted by \( \phi_M \)) such that \( \phi_M([a \ast u]) = \phi_M([u]) \ast \phi_M([a]), \phi_M([u \ast a]) = \phi_M([a]) \ast \phi_M([u]) \) hold for all \( a \in V \) and \( u \in M \).

Let \( M \) be a weak \( V \)-module. An element \( u \in M \) is called a \textit{lowest weight vector of weight} \( wt(u) \in \mathbb{C} \) if \( L(n)u = wt(u) \delta_n u \) for any \( n \in \mathbb{N} \). If \( a \in V \) is a lowest weight vector, we have the commutation relation
\[
\left[ L(m), a_n \right] = ((wt(a) - 1)(m + 1) - n) a_{m+n}, \quad \text{for } m, n \in \mathbb{Z}.
\]

\textbf{Lemma 3.8.} Let \( M \) be a \( V \)-module, and let \( a \in V \) and \( v \in M \) be lowest weight vectors. Then the subset of \( M \) spanned by vectors \( a_{n_1} a_{n_2} \cdots a_{n_k} v \), \( k \in \mathbb{Z}_{\geq 0} \) and \( n_i \in \mathbb{Z} \) \( (i = 1, 2, \ldots, k) \), is invariant under the action of \( L(n) \) for any \( n \in \mathbb{N} \).

\textbf{Proof.} Set \( U = \mathrm{Span}(a_{n_1} a_{n_2} \cdots a_{n_k} v \mid k \in \mathbb{Z}_{\geq 0}, n_i \in \mathbb{Z}) \). To show \( L(n)U \subset U \) for all \( n \in \mathbb{N} \), we prove that \( L(n)a_{n_1} a_{n_2} \cdots a_{n_k} v \in U \) for any \( n_i \in \mathbb{Z} \). By the commutation relation (3.12), we have
\[
L(n)a_{n_1} a_{n_2} \cdots a_{n_k} v = \sum_{i=1}^{k} a_{n_1} \cdots \left[ L(n), a_{n_i} \right] \cdots a_{n_k} v + a_{n_1} a_{n_2} \cdots a_{n_{i-1}} L(n) v
= \sum_{i=1}^{k} ((wt(a) - 1)(n + 1) - n_i) a_{n_1} \cdots a_{n_{i-1}} a_{n_i} a_{n_{i+1}} \cdots a_{n_k} v
+ a_{n_1} a_{n_2} \cdots a_{n_k} L(n) v.
\]
Since $L(n)v = \text{wt}(v)\delta_{n,0}v$ for $n \in \mathbb{N}$, the right-hand side of (3.13) is in $U$.

**Lemma 3.9.** Let $M$, $a$, and $v$ be as in Lemma 3.8. Then the subspace of $M$ spanned by the vectors

$$L(-m_1)L(-m_2) \cdots L(-m_k)a_{n_1}a_{n_2} \cdots a_nv,$$

$$k, l \in \mathbb{N}, m_i \in \mathbb{Z}_{>0}, n_j \in \mathbb{Z},$$

is invariant under the actions of $L(n)$ ($n \in \mathbb{Z}$) and $a_m$ ($m \in \mathbb{Z}$).

**Proof.** Set $U$ to be the subspace of $M$ spanned by vectors (3.14). By the Poincaré–Birkhoff–Witt (PBW) Theorem for the Virasoro algebra and Lemma 3.8, we see that $U$ is invariant under the action of $L(n)$ for $n \in \mathbb{Z}$.

Using induction on $k$, we prove that

$$a_mL(-m_1)L(-m_2) \cdots L(-m_k)a_{n_1}a_{n_2} \cdots a_nv \in U, \quad (3.15)$$

for any $k, l \in \mathbb{N}$, $m, m_1 \in \mathbb{Z}_{>0}$, and $n_j \in \mathbb{Z}$. The case $k = 0$ is clear. Assume that (3.15) holds for $k = p \in \mathbb{N}$. Then if we put $v' = L(-m_2) \cdots L(-m_{p+1})a_{n_1}a_{n_2} \cdots a_nv$, we have

$$a_mL(-m_1)v' = -[L(-m_1), a_m]v' + L(-m_1)a_mv'$$

$$= ((\text{wt}(a) - 1)(m_1 + 1) + m)a_{n-m}v' + L(-m_1)a_nv'.$$

By the induction hypothesis and the previous paragraph, $a_{n-m}v'$, $a_nv'$, and $L(-m_1)a_nv'$ belong to $U$. Hence (3.15) holds for $k = p + 1$.

3.3. A Spanning Set of $A(M)$

Let $M$ be an irreducible $M(1)^+$-module. Then from Lemma 3.9, the subspace of $M$ spanned by vectors

$$L(-m_1)L(-m_2) \cdots L(-m_k)J_{n_1}J_{n_2} \cdots J_{n/M}$$

$$(3.16)$$

for $k, l \in \mathbb{N}$, $m_i \in \mathbb{Z}_{>0}$, and $n_i \in \mathbb{Z}$ is invariant under the actions of $L(m)$ and $J_n$ for all $m, n \in \mathbb{Z}$, where $v_M$ is the lowest weight vector of $M$ given in Table I. Since $M(1)^+$ is generated by $\omega$ and $J_n$, this subspace is invariant under the action of $M(1)^+$. Thus this is an $M(1)^+$-submodule of $M$. By the irreducibility of $M$, we have the following lemma.

**Lemma 3.10.** An irreducible $M(1)^+$-module $M$ is spanned by vectors (3.16).

One of main results of this subsection is the following proposition.
PROPOSITION 3.11. Let $M$ be an irreducible $M(1)^+$-module. Suppose that there exists a lowest weight vector $u \in M$ such that for any $n \in \mathbb{Z}_{>0}$, $J_n u$ and $J_n u$ belong to the submodule for the Virasoro algebra of $M$ generated by $v_M$ and $u$. Then $M$ is spanned by the set
\[ \left\{ a \ast v_M \ast b, a \ast u \ast b \mid a, b \in M(1)^+ \right\} + O(M). \] (3.17)

To prove the proposition, we need some lemmas. The proof of the proposition is given after Lemma 3.14.

LEMMA 3.12. Let $M$ be an $M(1)$-module. Then for any $m, n \in \mathbb{Z}$, the commutator $[J_m, J_n]$ is a linear combination of
\[ L(m_1) L(m_2) \cdots L(m_p) J_{m_{p+1}} \] and $L(n_1) L(n_2) \cdots L(n_q)$,
for $m, n \in \mathbb{Z}$.

Proof. This is proved by an argument similar to that for Lemma 3.1 of [DN]. $lacksquare$

LEMMA 3.13. Let $u \in M$ be a lowest weight vector and set $U$ to be the submodule of $M$ spanned by the set
\[ \left\{ J_{n_1} \cdots J_{n_k} L(p_1) \cdots L(p_s) J_{n_{s+1}} \cdots J_{n_{k+s}} u \mid n_1, \ldots, n_k, p_1, \ldots, p_s \in \mathbb{Z} \right\}. \] (3.18)

Proof. Let $v$ be either $v_M$ or $u$. We have to show that
\[ J_{n_1} \cdots J_{n_{k+s}} L(p_1) \cdots L(p_s) J_{n_{s+1}} \cdots J_{n_{k+s}} v \in U_{M,u}, \]
for any $s \in \mathbb{Z}_{>0}$ and $n_1, \ldots, n_k, p_1, \ldots, p_s \in \mathbb{Z}$. By the PBW Theorem for the Virasoro algebra, we may assume that $p_1 \leq p_2 \leq \cdots \leq p_s$. If $p_1 \geq 0$, then $p_1 \geq 0$. Then we view (3.18) as follows:
\[ J_{n_1} \cdots J_{n_{k+s}} L(p_1) \cdots L(p_s) J_{n_{s+1}} \cdots J_{n_{k+s}} v = 3 \sum_{i=j}^k \delta(p_i + 1) - n_i J_{n_1} \cdots J_{n_{i-1}} L(p_i) \cdots L(p_{i-1}) J_{n_i} \cdots J_{n_{k+i}} v + \text{wt}(v) \delta_{p_1,0} J_{n_1} \cdots J_{n_{i-1}} L(p_1) \cdots L(p_{i-1}) \cdots J_{n_i} v. \]
If \( p_1 < 0 \), then we view it as follows:

\[
\begin{align*}
J_{n_1} \cdots J_{n_{j-1}} L(p_1) \cdots L(p_k) J_{n_j} \cdots J_{n_k}^v \\
&= - \sum_{i=1}^{j-1} J_{n_i} \cdots [L(p_1), J_{n_i}] \cdots J_{n_{j-1}} L(p_2) \cdots L(p_k) J_{n_j} \cdots J_{n_k}^v \\
&\quad + L(p_1) J_{n_1} \cdots J_{n_{j-1}} L(p_2) \cdots L(p_k) J_{n_j} \cdots J_{n_k}^v.
\end{align*}
\]

\[
\begin{align*}
&= - \sum_{i=1}^{j-1} (3(p_1 + 1) - n_i) J_{n_i} \cdots J_{n_{p_1+n_i}} \cdots J_{n_{j-1}} L(p_2) \\
&\quad \cdots L(p_1) J_{n_1} \cdots J_{n_{p_1-1}} L(p_2) \cdots L(p_k) J_{n_j} \cdots J_{n_k}^v \\
&\quad + L(p_1) J_{n_1} \cdots J_{n_{p_1}} L(p_2) \cdots L(p_k) J_{n_j} \cdots J_{n_k}^v.
\end{align*}
\]

Hence by using induction on \( s \), we see that (3.18) follows from Lemma 3.6.

**Lemma 3.14.** Let \( M \) and \( u \) be as in Proposition 3.11. Then for any positive integer \( l \) and \( n_i \in \mathbb{Z} \) \((i = 1, \ldots, l)\), we have \( J_{n_1} \cdots J_{n_{l-1}} L(p) \cdots L(p_k) J_{n_l} \cdots J_{n_k}^v \in U_{M, u} \), where \( U_{M, u} \) is the subset of \( M \) defined in Lemma 3.13.

**Proof.** Let \( v \) be either \( v_M \) or \( u \). By using induction on \( l \), we prove

\[
J_{n_1} \cdots J_{n_l}^v \in U_{M, u}, \tag{3.19}
\]

for any \( n_i \in \mathbb{Z} \). In the case \( l = 1 \), the assumptions of this lemma imply that for any positive integer \( n \), \( J_{n}^v \) is a linear combination of \( L(-m_1) \cdots L(-m_p) v_M \) and \( L(-n_1') \cdots L(-n_q') u \) for \( m_i, n_j' \in \mathbb{Z}_{>0} \). Hence by Lemma 3.5(2), \( J_{n}^v \in U_{M, u} \) holds for any \( n \in \mathbb{Z}_{>0} \). Thus (3.19) for \( l = 1 \) follows from Lemma 3.6. Assume that \( p \in \mathbb{Z}_{>0} \) and that (3.19) holds for any \( l \leq p \). Then we have to show (3.19) for \( l = p + 1 \). By the induction hypothesis, \( J_{n_1} \cdots J_{n_{p+1}}^v \in U_{M, u} \). Hence by Lemma 3.6, it is sufficient to prove that (3.19) for \( l = p + 1 \) holds for any \( n_i \in \mathbb{Z}_{>0} \). Then we have

\[
J_{n_1} \cdots J_{n_{p+1}}^v = \sum_{i=2}^{p+1} J_{n_1} \cdots [J_{n_i}, J_{n_i}] \cdots J_{n_{p+1}}^v + J_{n_2} \cdots J_{n_{p+1}}^v J_{n_1}^v. \tag{3.20}
\]

By Lemma 3.12, \([J_{n_i}, J_{n_i}]\) is expressed by a linear combination of \( L(m_1') \cdots L(m_i) J_{m_{i+1}} \) and \( L(n_1') \cdots L(n_i') \) for \( m_i, n_j' \in \mathbb{Z} \), and \( J_{n_i}^v \) can be expressed by a linear combination of \( L(-m_1') \cdots L(-m_i) v_M \) and \( L(-n_1') \cdots L(-n_i') u \) for \( m_i, n_j' \in \mathbb{Z}_{>0} \) by assumption of this lemma. Hence by the induction hypothesis and Lemma 3.13, the right-hand side of the equality (3.20) lies in \( U_{M, u} \).
Now we prove Proposition 3.11.

Proof of Proposition 3.11. Let \( U_{M, u} \) be the same set as in Lemma 3.13. Then we show \( M = U_{M, u} \). By Lemma 3.10, it is enough to show that

\[
L(-m_1)L(-m_2)\cdots L(-m_k)J_{n_1}J_{n_2}\cdots J_{n_l}v_M \in U_{M, u} \tag{3.21}
\]

for any \( m_i \in \mathbb{Z}_{>0} \) and \( n_i \in \mathbb{Z} \). But Lemma 3.14 tells us that \( J_{n_1}\cdots J_{n_l}v_M \in U_{M, u} \), and since the vectors of this form are homogeneous, (3.21) follows from Lemma 3.6.

We consider the case \( M = M(1)^\pm \), \( M(1, \lambda) (\lambda \neq 0, 1/2) \), and \( M(1)(\theta)^\pm \). Since \( \deg(J_nv_M) = 3 - n \leq 2 \) if \( n \geq 1 \), by the irreducible decomposition (3.2), (3.3), and (3.7), \( J_nv_M (n \geq 1) \) lies in the submodule for the Virasoro algebra of \( M \) generated by \( v_M \). Thus if we take \( u = v_M \) in Proposition 3.11, we have \( M = \text{Span}(a*v_M \mid a, b \in M(1)^\pm + O(M)) \), then \( A(M) \) is generated by \([v_M]\) as an \( A(M(1)^\pm) \)-bimodule.

In the case \( M = M(1, \lambda) (\lambda^2 = 1/2) \), by (3.2) we have the irreducible decomposition \( M = L(1,1/4) \oplus L(1,9/4) \oplus L(1,25/4) \oplus \cdots \). Now let \( u \) be the lowest weight vector of \( L(1,9/4) \), then \( \deg(J_nv_M) = 3 - n \leq 2 \) and \( \deg(J_nu) = 5 - n \leq 4 \) for any positive integer \( n \). Hence \( J_nv_M \) and \( J_nu (n \geq 1) \) lie in \( L(1,1/4) \oplus L(1,9/4) \). Thus Proposition 3.11 implies that \( M(1, \lambda) = \text{Span}(a*v_M \mid a, b \in M(1)^\pm + O(M)) \). In this case, \( A(M(1, \lambda)) \) is generated by \([v_M]\) and \([u]\) as an \( A(M(1)^\pm) \)-bimodule.

In the remaining case \( M = M(1)(\theta)^+ \), by (3.8), we have the irreducible decomposition \( M = L(1,9/16) \oplus L(1,25/16) \oplus L(1,121/16) \oplus \cdots \). Let \( u' \) be the lowest weight vector of \( L(1,25/16) \), then \( \deg(J_nv_M) = 3 - n \leq 2 \) and \( \deg(J_nu') = 4 - n \leq 3 \) for any positive integer \( n \). Hence \( J_nv_M \) and \( J_nu' (n \geq 1) \) are in \( L(1,9/16) \oplus L(1,25/16) \). Thus by Proposition 3.11, \( M(1)(\theta)^+ = \text{Span}(a*v_M \mid a, b \in M(1)^\pm + O(M)) \), and \( A(M(1)(\theta)^+) \) is generated by \([v_M]\) and \([u']\) as an \( A(M(1)^\pm) \)-bimodule. Summarizing, we have the following proposition.

**Proposition 3.15.** Let \( M \) be an irreducible \( M(1)^\pm \)-module. Then the following hold.

1. If \( M = M(1)^\pm \), \( M(1, \lambda) (\lambda^2 \neq 0, 1/2) \), or \( M(1)(\theta)^+ \), then \( A(M) \) is generated by \([v_M]\) as an \( A(M(1)^\pm) \)-bimodule.
2. If \( M = M(1, \lambda) (\lambda^2 = 1/2) \), then \( A(M) \) is generated by \([v_M]\) and \([u]\) as an \( A(M(1)^\pm) \)-bimodule, where \( u \) is the lowest weight vector of weight 9/4.
3. If \( M = M(1)(\theta)^+ \), then \( A(M) \) is generated by \([v_M]\) and \([u']\) as an \( A(M(1)^\pm) \)-bimodule, where \( u' \) is the lowest weight vector of weight 25/16.

**Remark 3.16.** If \( M \) is the vertex operator algebra \( M(1)^+ \) itself, this result is clear because one has \( a*1 = a = 1*a \) for all \( a \in M(1)^+ \).
As a corollary of Proposition 3.15, we have:

**Corollary 3.17.** Let $M$, $N$, $L$ be irreducible $M(1)^+$-modules.

1. If $M = M(1)^{\pm}$, $M(1, \lambda) (\lambda^2 \neq 0, 1/2)$, or $M(1)(\theta)^+$, then $N_{MN}^L \leq 1$.
2. If $M = M(1, \lambda) (\lambda^2 = 1/2)$ or $M(1)(\theta)^-$, then $N_{MN}^L \leq 2$.

**Proof.** If $A(M)$ is generated by $n$ vectors $[v_1, \ldots, v_n]$ as an $A(M(1)^+)$-bimodule, then the contraction $(L_0)^* \cdot A(M) \cdot N_0$ is spanned by $n$ vectors $v'_L \otimes [v_1] \otimes v_N, \ldots, v'_L \otimes [v_n] \otimes v_N$. Hence we have $\dim_c(L_0)^* \cdot A(M) \cdot N_0 \leq n$. On the other hand, by Proposition 2.10,

$$N_{MN}^L \leq \dim_c((L_0)^* \cdot A(M) \cdot N_0)^* = \dim_c(L_0)^* \cdot A(M) \cdot N_0 \leq n.$$ 

Thus the corollary follows from Proposition 3.15.

4. FUSION RULES

This section consists of two subsections. In the first subsection we prove some propositions which give triples of irreducible $M(1)^+$-modules whose fusion rules are nonzero and state the main theorem. In the second subsection we prove the main theorem.

4.1. Main Theorem

Recall the linear map $I_{\lambda \mu}: M(1, \lambda) \to z^{\lambda \mu} \text{Hom}(M(1, \mu), M(1, \lambda + \mu)[[z, z^{-1}]]$ defined in (2.8) and (2.9) for $\mu \in \mathbb{C}$. By the arguments of Section 8.6 of [FLM], we see that $I_{\lambda \mu}$ is an intertwining operator of type

$$\begin{pmatrix}
M(1, \lambda + \mu) \\
M(1, \lambda) \\
M(1, \mu)
\end{pmatrix}.$$

Since $e^\mu \in M(1, \mu)_0$, we have

$$I_{\lambda \mu}(e^\lambda, z)e^\mu = z^{\lambda \mu} \exp \left( \sum_{n \geq 0} \frac{\lambda h(-n)}{n} z^n \right) e^{\lambda + \mu},$$

and its coefficient of $z^{\lambda \nu}$ is $e^{\lambda + \mu}$ which is nonzero. Thus this intertwining operator $I_{\lambda \mu}$ is nonzero. This implies that the fusion rule $N_{M(1, \lambda)(M(1, \mu))}^{M(1, \lambda + \mu)(M(1, \mu))}$ is nonzero for any $\lambda, \mu \in \mathbb{C}$. By Proposition 2.11, there exists an isomorphism $f$ from $M(1, \mu)$ onto $M(1, -\mu)$ of $M(1)^+$-modules. We define a linear map $Y_\lambda \circ f: M(1, \lambda) \to z^{-\lambda \mu}(\text{Hom}(M(1, \mu), M(1, \lambda - \mu))[\langle z, z^{-1} \rangle])$,

$$u \mapsto Y_\lambda \circ f(u, z) = Y_\lambda(u, z) \circ f.$$
Then we can easily see that $Y \circ f$ gives a nonzero intertwining operator of type

$$\begin{pmatrix}
M(1, \lambda - \mu) \\
M(1, \lambda) & M(1, \mu)
\end{pmatrix}.$$ 

Thus the following proposition holds.

**Proposition 4.1.** For $\lambda, \mu, \nu \in \mathbb{C}$, the fusion rule $N^{M(1, \nu)}_{M(1, \lambda)M(1, \mu)}$ is nonzero if $\nu^2 = (\lambda \pm \mu)^2$.

In particular, if $\lambda = 0$ we have that $N^{M(1, \nu)}_{M(1)M(1, \mu)}$ is nonzero if $\mu^2 = \nu^2$. In fact, we have the following proposition.

**Proposition 4.2.** For $\mu, \nu \in \mathbb{C}$ such that $\mu^2 = \nu^2$, the fusion rules

$$N^{M(1, \nu)}_{M(1)^\pm M(1, \mu)}$$

are nonzero. Furthermore, the fusion rules

$$N^{M(1, \nu)}_{M(1)^\pm M(1)}$$

are nonzero.

**Proof.** Since $(M(1, \mu), I_{\theta_0})$ is an irreducible $M(1)$-module, $I_{\theta_0}$ gives nonzero intertwining operators of types

$$\begin{pmatrix}
M(1, \mu) \\
M(1)^\mp & M(1, \mu)
\end{pmatrix}$$

by Proposition 11.9 of [DL]. Because $M(1, -\mu)$ is isomorphic to $M(1, \mu)$ as an $M(1)^\pm$-module, the first statement holds. If $\mu = 0$, then $Y = I_{\theta_0}$ gives nonzero intertwining operators of types

$$\begin{pmatrix}
M(1) \\
M(1)^\pm & M(1)
\end{pmatrix}.$$ 

Since $\theta$ is an automorphism of $M(1)$, we have $\theta Y(a, z)\theta = Y(\theta(a), z)$ for every $a \in M(1)$. This implies that $Y(a, z)M(1)^\pm \subset M(1)^\pm ((z))$ for every $a \in M(1)^\pm$ and $Y(a, z)M(1)^\mp \subset M(1)^\mp ((z))$ for every $a \in M(1)^\mp$. Thus $Y$ gives nonzero intertwining operators of types

$$\begin{pmatrix}
M(1)^\pm \\
M(1)^\mp & M(1)^\pm
\end{pmatrix}$$

and

$$\begin{pmatrix}
M(1)^\mp \\
M(1)^\pm & M(1)^\pm
\end{pmatrix}.$$
FUSION RULES FOR BOSONIC ORBIFOLD 361

Next we recall the linear map $I_\alpha^\beta: M(1, \lambda) \rightarrow (\text{End } M(1)(\theta))[z]$ defined in (2.10) and (2.11). The arguments of Chapter 9 of [FLM] show that $I_\alpha^\beta$ is an intertwining operator of type

\[
\begin{pmatrix}
M(1)(\theta) \\
M(1, \lambda) & M(1)(\theta)
\end{pmatrix}.
\]

Let $p_\pm$ be projections from $M(1)(\theta)$ onto $M(1)(\theta)^\pm$ and let $\iota_\pm$ be inclusions from $M(1)(\theta)^\pm$ into $M(1)(\theta)$ respectively. Define a linear map $p_\beta \circ I_\alpha^\beta \circ \iota_\alpha$ by

\[
p_\beta \circ I_\alpha^\beta \circ \iota_\alpha: M(1, \lambda) \rightarrow \left\{ \text{Hom} \left( M(1)(\theta)^\alpha, M(1)(\theta)^\beta \right) \right\}[z],
\]

\[u \mapsto p_\beta \circ I_\alpha^\beta \circ \iota_\alpha(u, z) = p_\beta \left( I_\alpha^\beta(u, z) \iota_\alpha \right),
\]

where $\alpha, \beta$ are $+ \leftrightarrow -$. Then we have the following lemma.

**Lemma 4.3.** For any $\alpha, \beta \in \{+, -, \}, p_\beta \circ I_\alpha^\beta \circ \iota_\alpha$ is a nonzero intertwining operator of type

\[
\begin{pmatrix}
M(1)(\theta)^\beta \\
M(1, \lambda) & M(1)(\theta)^\alpha
\end{pmatrix}
\]

if $\lambda \neq 0$.

**Proof.** It is clear that $p_\beta \circ I_\alpha^\beta \circ \iota_\alpha$ is an intertwining operator of type

\[
\begin{pmatrix}
M(1)(\theta)^\beta \\
M(1, \lambda) & M(1)(\theta)^\alpha
\end{pmatrix}
\]

We next show that $p_\beta \circ I_\alpha^\beta \circ \iota_\alpha$ is nonzero if $\lambda \neq 0$. By direct calculation, we can see that the coefficient of $z^{-\lambda^2/2}$ in $I_\alpha^\beta(e^\lambda, z)1$ is 1 and that of $z^{-\lambda^2/2+1/2}$ is $\lambda h(-1/2)1$, and the coefficient of $z^{-\lambda^2/2-1/2}$ in $I_\alpha^\beta(e^\lambda, z)h(-1/2)1$ is $-\lambda$ and that of $z^{-\lambda^2/2+1}$ is $-2\lambda^2(h(3/2)1 + 2\lambda^2(1 - 3/2\lambda^2)h(-1/2)1$. If $\lambda \neq 0$, all of these coefficients are nonzero. This proves that $p_\beta \circ I_\alpha^\beta \circ \iota_\alpha(e^\lambda, z)$ is nonzero for all $\alpha, \beta \in \{+, -, \}$ if $\lambda$ is nonzero. 

**Proposition 4.4.** (1) For any nonzero $\lambda \in \mathbb{C}$ and $\alpha, \beta \in \{+, -, \}$,

\[
N_{M(1)(\theta)^\beta}^{M(1)(\theta)^\alpha} \quad \text{is nonzero.}
\]

(2) The fusion rules

\[
N_{M(1)}^{M(1)(\theta)^\pm} \quad \text{and} \quad N_{M(1)}^{M(1)(\theta)^\pm}
\]

are nonzero.
Proof. Part 1 follows from Lemma 4.3. By the definition of the action of $\theta$ on $M(1)(\theta)$ (2.12), we can show that $\theta I_0^s(a, z) = I_0^s(\theta(a), z)$ holds for every $a \in M(1)$. This implies that
\[
I_0^s(a, z) M(1)(\theta)_{\pm} \subset M(1)(\theta)_{\mp} ((z)) \quad \text{if } a \in M(1)^+, \\
I_0^s(a, z) M(1)(\theta)_{\pm} \subset M(1)(\theta)_{\mp} ((z)) \quad \text{if } a \in M(1)^-.
\]
Because $I_0^s(1, z) = \text{id}_{M(1)(\theta)}$ and $I_0^s(h(-1)1, z) = h(z)$, we see that $p_+ \circ I_0^s \circ t_\pm (p_- \circ I_0^s \circ t_\pm)$ give nonzero intertwining operators of type
\[
\left( M(1)(\theta)_{\mp}^\pm, M(1)(\theta)_{\pm}^\mp \right) \quad \text{resp.} \quad \left( M(1)(\theta)_{\mp}^\mp, M(1)(\theta)_{\pm}^\pm \right).
\]
This shows Part 2.

The series of Propositions 4.1, 4.2, and 4.4 together with Proposition 3.4 give us triples $(M, N, L)$ of irreducible $M(1)$-modules for which the fusion rule $N_{M(1)N}^L$ is nonzero. In fact, we have following theorem.

**Theorem 4.5 (Main Theorem).** Let $M, N,$ and $L$ be irreducible $M(1)$-modules.

1. If $M = M(1)^+$, then $N_{M(1)^+N}^L = \delta_{N,L}$.
2. If $M = M(1)^-$, then $N_{M(1)^-N}^L$ is 0 or 1, and $N_{M(1)^-N}^L = 1$ if and only if the pair $(N, L)$ is one of the following pairs:
   \[
   \left( M(1)^{\pm}, M(1)^{\mp} \right), \quad \left( M(1)(\theta)^{\pm}, M(1)(\theta)^{\mp} \right), \\
   \left( M(1,\lambda), M(1,\mu) \right) \left( \lambda^2 = \mu^2 \right).
   \]
3. If $M = M(1,\lambda) \ (\lambda \neq 0)$, then $N_{M(1,\lambda)N}^L$ is 0 or 1, and $N_{M(1,\lambda)N}^L = 1$ if and only if the pair $(N, L)$ is one of the following pairs:
   \[
   \left( M(1)^{\pm}, M(1,\mu) \right) \left( \lambda^2 = \mu^2 \right), \quad \left( M(1,\mu), M(1,\nu) \right) \left( \nu^2 = (\lambda \pm \mu)^2 \right), \\
   \left( M(1)(\theta)^{\pm}, M(1)(\theta)^{\mp} \right), \quad \left( M(1)(\theta)^{\mp}, M(1)(\theta)^{\pm} \right).
   \]
4. If $M = M(1)(\theta)^+$, then $N_{M(1)(\theta)^+N}^L$ is 0 or 1, and $N_{M(1)(\theta)^+N}^L = 1$ if and only if the pair $(N, L)$ is one of the following pairs:
   \[
   \left( M(1)^{\pm}, M(1)(\theta)^{\mp} \right), \quad \left( M(1,\lambda), M(1)(\theta)^{\pm} \right).
   \]
5. If $M = M(1)(\theta)^-$, then $N_{M(1)(\theta)^-N}^L$ is 0 or 1, and $N_{M(1)(\theta)^-N}^L = 1$ if and only if the pair $(N, L)$ is one of the following pairs:
   \[
   \left( M(1)^{\pm}, M(1)(\theta)^{\mp} \right), \quad \left( M(1,\lambda), M(1)(\theta)^{\pm} \right).
   \]
4.2. Proof of the Main Theorem

Recall the direct sum decomposition $M(1)^+ = L(1,0) \oplus L(1,4) \oplus \cdots$ (see (3.3)). The lowest weight vector of $L(1,4)$ is $J$. Hence $h(-3)h(-1)1$ is a linear combination of $L(-4)1$, $L(-2)^21$ and $J$ (note that $L(-1)1 = 0$). In fact, $h(-3)h(-1)1 = (-1/9)(J - 3L(-4)1 - 4L(-2)^21)$. By Lemma 3.5(2), we have the equality in $A(M(1)^+)$,

$$[J - 4\omega^2 - 17\omega + 9h(-3)h(-1)1] = 0,$$  \hspace{1cm} (4.1)

where we use the notation $[a]^{*n} = [a^n] = [a \times a \times \cdots \times a]$ for $a \in M(1)^+$ and $n \in \mathbb{N}$.

Now we prove the main theorem. We divide the proof into five steps, Step 1—Step 5, where (1)–(5) are proved respectively.

Proof of Theorem 4.5. By Theorem 2.13 and Table I, we see that two irreducible $M(1)^+$-modules $N$ and $L$ are isomorphic to each other if and only if $a_N = a_L$ and $b_N = b_L$.

Step 1. If $M = M(1)^+$, then $v_M = 1$ (see Table I), and $(L_0)^* \cdot A(M) \cdot N_0 = \mathbb{C}v_L' \otimes [1] \otimes v_N$ by Proposition 3.15(1). If the fusion rule $N_{MN}^L$ is nonzero, then $v_L' \otimes [1] \otimes v_M$ is also nonzero. On the one hand, we have the equalities

$$a_L v_L' \otimes [1] \otimes v_N = v_L' \otimes [a] \otimes v_N = a_N v_L' \otimes [1] \otimes v_N,$$

$$b_L v_L' \otimes [1] \otimes v_N = v_L' \otimes [J] \otimes v_N = b_N v_L' \otimes [1] \otimes v_N.$$

This implies that if the fusion rule $N_{MN}^L$ is nonzero, then $a_N = a_L$ and $b_N = b_L$, hence $M$ and $N$ are equivalent. Then (1) follows from Proposition 4.2 and Proposition 4.4.

Step 2. Let $M = M(1)^-$. Then we have the irreducible decomposition (3.3). Hence $h(-3)h(-1)1 \ast v_M$ is in $L(1,1)$ and it is a linear combination of $L(-m_1)\cdots L(-m_k)v_M$ ($m_i \in \mathbb{Z}_{>0}, m_1 + \cdots + m_k \leq 4$). In fact,

$$h(-3)h(-1)1 \ast v_M$$

$$= 3v_M + 12L(-1)v_M + 12L(-1)^2v_M - 8L(-3)v_M$$

$$+ 16L(-2)L(-1)v_M - \frac{1}{2}L(-4)v_M$$

$$+ \frac{1}{4}L(-3)L(-1)v_M + \frac{3}{2}L(-2)L(-1)^2v_M.$$  \hspace{1cm} (4.2)
On the other hand, we have
\[ v'_L \otimes \left[ L(-m_1) \cdots L(-m_\lambda) v_M \right] \otimes v_N = F(a_L, a_N) v'_L \otimes [v_M] \otimes v_N \]
\[ (4.3) \]
for any \( m_i \in \mathbb{Z}_{>0} \) by Lemma 2.6, where \( F \in \mathbb{C}[x, y] \) is given by
\[ F = \prod_{i=1}^{k} (-1)^{m_{i-1}} \left( x - m_i y - \sum_{j=i+1}^{k} m_j - \text{wt}(v_M) \right). \]
\[ (4.4) \]
Hence by (4.1) and (4.2)-(4.4), we have
\[ v'_L \otimes \left[ (J - 4\omega^* - 17\omega + 9h(-3)h(-1)1) v_M \right] \otimes v_N \]
\[ = f(a_L, a_N, b_L) v'_L \otimes [v_M] \otimes v_N = 0, \]
where \( f \in \mathbb{C}[x, y, z] \) is given by
\[ f = z - 4x^2 + x + \frac{9}{4}(x - y)(6x^2 - 18xy - 12y^2 - 21x - 23y + 11). \]

Proposition 3.7 and (4.1) show that
\[ v'_L \otimes \left[ \phi_M(v_M) * \phi_M(1)(J - 4\omega^* - 17\omega + 9h(-3)h(-1)1) v_M \right] \otimes v_N \]
\[ = v'_L \otimes \phi_M(\left[ (J - 4\omega^* - 17\omega + 9h(-3)h(-1)1) v_M \right]) \otimes v_N \]
\[ = -f(a_N, a_L, b_N) v'_L \otimes [v_M] \otimes v_N = 0, \]
since \( \phi_M(\omega) = [\omega], \phi_M(1)(J) = [J], \) and \( \phi_M([v_M]) = -[v_M]. \) Moreover, the Verma module of the Virasoro algebra with central charge 1 and lowest weight 1 has a singular vector \( 2L(-3)v - 4L(-2)L(-1)v + L(-1)^3v \) of weight 4, where \( v \) is the cyclic vector of the Verma module. The image of this singular vector in \( M(1)^- \) is zero, then by (4.3) and (4.4) we have \( g(a_N, a_L, b_N) v'_L \otimes [v_M] \otimes v_N = 0, \) where \( g(x, y) = (x - y)(x^2 - 2xy + y^2 - 2x - 2y + 1)/2. \)

Now suppose that the fusion rule \( N_{LM}^N \) is nonzero. Then \( v'_L \otimes [v_M] \otimes v_N \) is nonzero by Proposition 3.15(1). Hence \( f(a_L, a_N, b_L), f(a_N, a_L, b_N), \) and \( g(a_N, a_L) \) are necessarily zero. By (1) and Proposition 3.4, we assume that \( N, L \in \{ M(1)^-, M(1, \lambda) (\lambda \neq 0), M(1)(1) \}. \)

In the cases \( N = L = M(1)^-, M(1)(1)^+ \) and \( M(1)(1)^- \), and the cases \( L = M(1)^- \) and \( N = M(1)(\lambda)^- \), we see that \( f(a_L, a_N, b_L) \) are nonzero. Therefore the fusion rules of corresponding types are zero.

In the case \( L = M(1)^- \) and \( N = M(1, \lambda) \), we see that either \( f(1, \lambda^2/2, -6) \) or \( g(1, \lambda^2/2) \) is nonzero. Then the fusion rule of corresponding type is zero.
In the case \( L = M(1, \lambda) \) and \( N = M(1, \mu) \), assume that the fusion rule is nonzero. Then if we put \( s = \lambda^2 \) and \( \mu^2 = t \), we have

\[
\begin{align*}
f\left( \frac{s}{2}, \frac{t}{2}, s^2 - \frac{s}{2} \right) + f\left( \frac{t}{2}, \frac{s}{2}, \frac{t}{2} - \frac{t}{2} \right) &= \frac{9}{16} (s - t)^2 (3s + 3t - 2) = 0, \\
\frac{9}{2} (s - t) (s + t - 1) &= 0.
\end{align*}
\]

Hence we have \( s = t \), that is, \( \lambda^2 = \mu^2 \). Thus \( N^{M(1, \lambda)}_{M(1, \mu)} = \delta_{\lambda \mu} \lambda^2 \) holds.

In the case \( L = M(1, \lambda) \) and \( N = M(1) \theta^\ast \), if we put \( s = \lambda^2 \), then we have

\[
\begin{align*}
f\left( \frac{s}{2}, \frac{1}{16}, s^2 - \frac{s}{2} \right) &= \frac{27}{4096} (8s - 1)(32s^2 - 236s + 205), \\
\frac{27}{4096} (8s - 9)(384s^2 - 1160s + 131).
\end{align*}
\]

But there is no common zero of these two polynomials, and the corresponding fusion rule is zero.

If \( L = M(1, \lambda) \) and \( N = M(1) \theta^\ast \), then we have

\[
\begin{align*}
f\left( \frac{s}{2}, \frac{9}{16}, s^2 - \frac{s}{2} \right) &= \frac{9}{4096} (8s - 9)(96s^2 - 996s + 119), \\
\frac{9}{8192} (8s - 1)(384s^2 - 2504s + 2211).
\end{align*}
\]

Thus there is no common zero of these two polynomials, and the corresponding fusion rule is zero in this case too. Now (2) follows from Proposition 4.2 and 4.4.

**Step 3.** Let \( M = M(1, \lambda) \ (\lambda \neq 0) \). We prove (3) by dividing the problem into the following four cases: (i) \( \lambda^2 \neq 1/2, 2, 9/2 \), (ii) \( \lambda^2 = 2 \), (iii) \( \lambda^2 = 9/2 \), and (iv) \( \lambda^2 = 1/2 \).

(i) First we assume that \( \lambda^2 \neq 1/2, 2, 9/2 \). Then we can see that \( h(-3)h(-111) * v_M \) belongs to \( L(1, \lambda^2 / 2) \) by the direct sum decomposition (3.2). Thus \( h(-3)h(-111) * v_M \) can be expressed by a linear combination of \( L(-m_1) \cdots L(-m_k) v_M \ (m_i \in \mathbb{Z}_{>0}, m_1 + \cdots + m_k \leq 4) \). Then by using (4.3) and (4.4), we have the following equalities in the contraction \( L_0^a \cdot A(M) \cdot N_0 \),

\[

\begin{align*}
v_L' \otimes \left[ (J - 4 \omega \omega^2 - 17 \omega + 9h(-3)h(-111) * v_M) \right] &\otimes v_N \\
&= f(a_L, a_N, b_L) v_L' \otimes [v_M] \otimes v_N = 0, \quad (4.5)
\end{align*}
\]

\[

\begin{align*}
v_L' \otimes \left[ (J - 4 \omega \omega^2 - 17 \omega + 9h(-3)h(-111) * v_M) \right] &\otimes v_N \\
&= f(a_L, a_N, b_L) v_L' \otimes [v_M] \otimes v_N = 0, \quad (4.5)
\end{align*}
\]
\[ v_L' \otimes \left[ \phi_M(v_M) \ast \phi_{M(1)} \left( J - 4\omega^2 - 17\omega + 9h(-3)h(-1)1 \right) \right] \otimes v_N \\
= v_L' \otimes \phi_M \left( \left( J - 4\omega^2 - 17\omega + 9h(-3)h(-1)1 \right) \ast v_M \right) \otimes v_N \\
= e^{x^2 \pi i / 2} f(a_N, a_L, b_N) v_L' \otimes [v_M] \otimes v_N = 0, \quad (4.6) \]

where \( f = f(x, y, z) \in \mathbb{C}[x, y, z] \) is given by

\[
f = z - 4x^2 + x + \frac{9(\lambda^4 - 4\lambda^2(x + y) + 4(x - y)^2)}{32\lambda^2(\lambda^2 - 2)(2\lambda^2 - 9)(2\lambda^2 - 1)} \\
\times \left( -3(\lambda^2 - 2)^3 + 4(8\lambda^4 - 29\lambda^2 + 6)x \\
+ 4(7\lambda^2 + 6)y - 4(16\lambda^2 + 3)(x - y)^2 \right).
\]

We consider the case \( N = M(1)(\theta)^+ \) and \( L = M(1, \mu) \) (\( \mu \neq 0 \)). If we assume that the fusion rule \( N_{LM}^N \) is nonzero, then \( v_L' \otimes [v_M] \otimes v_N \) is nonzero by Proposition 3.15(1). Hence we have \( f(a_L, a_N, b_M) = f(a_N, a_L, b_N) = f(a_N, a_L, b_M) = 0. \) This shows the equations

\[
p(t, u) = (8u - 1)(8u - 9)((1024t + 192)u^2 - (2048t^2 + 512t + 624)u + 1024t^3 - 256t^2 + 816t + 75) = 0, \\
q(t, u) = ((8t + 8u - 1)^2 - 256tu)((1024t + 192)u^2 - (1024t^2 - 3456t + 816)u + 1192t^2 + 864t + 675) = 0,
\]

respectively. If \( t \in \{1/2, 2, 9/2\} \), then there is no common solution of \( p(t, u) = q(t, u) = 0. \) Hence we may assume that \( t \notin \{1/2, 2, 9/2\} \) and discuss it by interchanging \( \lambda \) and \( \mu. \) Thus we have \( p(u, t) = q(u, t) = 0. \) If \( t = 1/8, 9/8 \), then \( p(t, u) \) and \( q(t, u) \) have no common solution. Therefore \( t \neq 1/8, 9/8. \) Similarly, \( u \neq 1/8, 9/8. \) If we put \( r(t, u) = (u - 1/8)(u - 9/8)r(t, u), \) then \( r(t, u) = r(u, t) = 0 \) hold. Since \( r(t, u) \) and \( q(t, u) \) have no common solution, we may also assume that \( t \neq u. \) From \( r(t, u) - r(u, t) = q(t, u) - q(u, t) = q(t, u) + q(u, t) = 0, \) we have following three equations:

\[
32\alpha^2 - 14\alpha + 45 - 128\beta = 0, \\
(128\beta + 3)(64\alpha^2 - 16\alpha + 1 - 256\beta) = 0, \quad (4.7) \\
(64\alpha^2 - 16\alpha + 1 - 256\beta)(64\alpha^2 - 280\alpha + 225 + 1024\beta) = 0,
\]

where we put \( \alpha = t + u \) and \( \beta = tu. \) If \( 64\alpha^2 - 16\alpha + 1 - 256\beta \neq 0, \) then \( \beta = -3/128. \) But in this case, (4.7) and \( 64\alpha^2 - 280\alpha + 225 + 1024\beta = 0. \)
1024\beta = 0 don’t have a common solution. So we have 64\alpha^2 - 16\alpha + 1 - 256\beta = 0 and 32\alpha^2 - 14\alpha + 45 - 128\beta = 0. This implies \alpha = 89/12 and \beta = 30625/2304. But the solutions (t, u) \in \mathbb{C}^2 of the equation \(x^2 - \alpha x + \beta = 0\) don’t satisfy \(p(t, u) = 0\). Therefore we see that the corresponding fusion rule is zero.

In the case \(N = M(1, \theta)^{-1}\) and \(L = M(1, \mu) (\mu \neq 0)\) we can show that the fusion rule \(N_{LM}^N\) is zero by the same method as that in the preceding case.

We next prove that \(N_{LM(1, \lambda)M(1, \mu)}^{(1, \nu)} = \delta_\nu^{-1} \cdot (\lambda \pm \mu)^2\) if \(\lambda^2 \neq 1/2, 2, 9/2\). By Proposition 4.1 and Corollary 3.17, it is enough to prove that if \(N_{LM(1, \lambda)M(1, \mu)}^{(1, \nu)} \neq 0\) then \(v^2 = (\lambda \pm \mu)^2\). Let \(N = M(1, \mu)\) and \(L = M(1, \nu)\), and assume that \(N_{LM}^N\) is nonzero. By Proposition 3.15, we see that \(v_L^* \otimes [v_M^*] \otimes v_N \neq 0\). Then the equalities (4.5) and (4.6) follow two equations

\[
(s^2 + t^2 + u^2 - 2st - 2su - 2tu)p(s, t, u) = (s^2 + t^2 + u^2 - 2st - 2su - 2tu)p(s, u, t) = 0,
\]

where \(s = \lambda^2, t = \nu^2, u = \mu^2\), and \(p \in \mathbb{C}[x, y, z]\) is given by

\[
p = (-3x + 16xy + z + 32yz - z^2)
\times (-2 - 12x + 58xy + 16xy^2 - 12y + 3y^2 - 6yz - 12z + 3z^2).
\]

Suppose that \(s^2 + t^2 + u^2 - 2st - 2su - 2tu \neq 0\), then we have \(p(s, t, u) = p(s, u, t) = 0\). In addition, we assume that \(\lambda^2 \neq 8\). Then the circle relations \(h(-3)h(-1)1 \ast v_M\) and \(h(-2)^21 \ast v_M\) belong to \(L(1, \lambda^2/2)\) and can be expressed by linear combinations of \(L(-m_i) \cdots L(-m_k)v_M\) \((m_i \in \mathbb{Z}_{>0}, m_i + \cdots + m_k \leq 5)\). By (4.3) and (4.4), we have the equalities

\[
v_L^* \otimes [h(-3)h(-1)1 \ast v_M] \otimes v_N = g_1(a_L, a_N)v_L^* \otimes [v_M] \otimes v_N = 0,
\]

\[
v_L^* \otimes [h(-2)^21 \ast v_M] \otimes v_N = g_2(a_L, a_N)v_L^* \otimes [v_M] \otimes v_N = 0,
\]

where \(g_1(a_L, a_N)\) and \(g_2(a_L, a_N)\) become

\[
g_1(a_L, a_N) = (s^2 + t^2 + u^2 - 2st - 2su - 2tu)(t - u)q(s, t, u),
\]

\[
g_2(a_L, a_N) = (s^2 + t^2 + u^2 - 2st - 2su - 2tu)r(s, t, u),
\]
and \( q, r \in \mathbb{C}[x, y, z] \) are given by

\[
q = -12 + 24x - 5x^2 + 12y - 16xy + 4x^2y + 3y^2 - 4xy^2 + 12z \\
- 16xz + 4x^2z + 6yz + 8xyz - 3z^2 - 4yz^2, \\
r = 192 - 245x + 108x^2 - 18x^3 + x^4 - 240y - 28xy + 8x^2y + x^3y \\
+ 96y^2 + 86x^2y^2 - 21x^2y^2 - 12y^3 + 19xy^3 - 144z + 460xz \\
- 152x^2z + 11x^3z - 96yz - 100xyz + 14x^2yz + 36y^2z \\
- 152xy^2z + 14xz^2 + 7x^2z^2 + 57xyz^2 - 36y^2z^2 + 12z^3 - 19xz^3.
\]

Since \( s^2 + t^2 + u^2 - 2st - 2su - 2tu \) is nonzero, we have \((t - u)q(s, t, u) = r(s, t, u) = 0\). Similarly, we have \((u - t)q(s, u, t) = r(s, u, t) = 0\) by interchanging \( \mu^2 \) and \( \nu^2 \). Assume that \( t = u \). Then \( r(s, t, t) = 0 \) follows if \( s = 6(1 - 2t) \), and

\[
p(6(1 - 2t), t, t) = 0 \text{ implies that } t \notin \{1/2, 2, 9/2, 8\}.
\]

Hence we can interchange \( s \) and \( t \), and we have \( r(t, 6(1 - 2t), t) = 0 \) and \( r(t, t, 6(1 - 2t)) = 0 \). But the common solutions of these equations don't satisfy \( p(6(1 - 2t), t, t) = 0 \). Hence we see that \( t \neq u \), which shows \( q(s, t, u) = q(s, u, t) = 0 \). Next assume that \( t = n^2/2 \) \((n = 1, 2, 3, 4)\). Then \( p(s, n^2/2, u) = 0 \) follows

\[
s = \frac{16u^2 - (16n^2 + 1)u}{8n^2 - 3} \quad \text{or} \quad \frac{-12u^2 - 12(n^2 + 4)u + 3n^4 - 24n^2 - 8}{4(4n^2 + 29n^2 - 12)}.
\]

But in any case, there is no common solution of equations \( p(s, u, n^2/2) = 0 \) and \( q(s, u, n^2/2) = 0 \). Thus \( t \notin \{1/2, 2, 9/2, 8\} \). Similarly, we have \( u \notin \{1/2, 2, 9/2, 8\} \). Therefore if we put \( x_1 = s, x_2 = t \) and \( x_3 = u \),

\[
p(x_{i_1}, x_{i_2}, x_{i_3}) = q(x_{i_1}, x_{i_2}, x_{i_3}) = r(x_{i_1}, x_{i_2}, x_{i_3}) = 0 \quad \text{and} \quad x_{i_1} \neq x_{i_3}
\]

if \( i \neq j \) \hspace{1cm} (4.11)

hold for every permutation \( \{i_1, i_2, i_3\} \) of \( \{1, 2, 3\} \) and \( i, j = 1, 2, 3 \). In particular, we have \( (t - u)q(q(s, t, u) - q(t, s, u)) - (t - s)q(q(u, t, s) - q(t, u, s)) = 0 \) and this follows \( s + t + u + 5 = 0 \) since \( s, t, u \) are distinct from each other. On the other hand, if we put \( r(s, t, u) - r(t, s, u) = (s - t)\alpha(s, t, u) \), then we have \( \alpha(s, t, u) = \alpha(u, t, s) = 0 \). If we next put \( \alpha(s, t, u) - \alpha(t, u, s) = (s - u)\beta(s, t, u) \), we have \( \beta(s, t, u) = \beta(s, u, t) = 0 \). Now we see that \( \beta(s, t, u) - \beta(s, u, t) = -16(t - u)(3s + 3t + 3u - 10) = 0 \) and this follows \( 3s + 3t + 3u - 10 = 0 \). It is inconsistent with \( s + t + u + 5 = 0 \). Thus we see that there is no solution that satisfies (4.11). Hence \( s^2 + t^2 + u^2 - 2st - 2su - 2tu = 0 \). By substituting \( \lambda^2 \) for \( s \), \( \nu^2 \) for \( t \), and \( \mu^2 \) for \( u \), we have \( \nu^2 = (\lambda \pm \mu)^2 \) if \( \lambda \neq 1/2, 2, 9/2, 8 \).

In the case \( \lambda^2 = 8 \), we may assume that \( N \) and \( L \) are any of \( M(1, \mu) \) \((\mu^2 = 1/2, 2, 9/2, 8)\), that is to say, that \( \mu^2 \) and \( \nu^2 \) are any of \( 1/2, 2, 9/2, 8 \).
and 8. But then $p(8, \mu^2, \nu^2)$ is nonzero. Hence we have $\nu^2 = (\mu \pm 2\sqrt{2})^2$ by (4.8). Consequently we see that (3) holds if $\lambda^2 \neq 1/2, 2, 9/2$.

(ii) We next consider the case $\lambda^2 = 2$. Then we have the irreducible decomposition for the Virasoro algebra $M(1, \lambda) = L(1, 1) \oplus L(1, 4) \oplus L(1, 9) \oplus \cdots$ (see (3.2)). Put $u = \sqrt{2} h(-3)v_M - 3h(-2)h(-1)v_M + \sqrt{2}(-1)^3v_M$ which is the lowest weight vector of $L(1, 4)$. Since $h(-3)h(-1)1* v_M \in L(1, 1) \oplus L(1, 4)$, it is a linear combination of $L(-1)u, u$ and $L(-m_1)\cdots L(-m_k)v_M$ ($m_i \in \mathbb{Z}_{>0}, m_i + \cdots + m_k \leq 4$).

Then by formulas (4.3) and (4.4), we have the equation

$$L \phi^2 [\frac{(J - 4\omega \phi^2 - 17\omega + 9h(-3)h(-1)1) \phi^2}{v_M} \otimes v_N] = f_1(a_L, a_N)v_L \otimes [u] \otimes v_N + f_2(a_L, a_N, b_L)v_L' \otimes [v_M] \otimes v_N = 0,$$

where $f_1 = f(x, y) \in \mathbb{C}[x, y]$ and $f_2 = f(x, y, z) \in \mathbb{C}[x, y, z]$ are given by

$$f_1 = \frac{3}{2} + \frac{21}{8}(x - y),$$

$$f_2 = z + \frac{95}{8}x - \frac{99}{8}y - \frac{173}{8}x^2 - \frac{9}{4}xy + \frac{207}{8}y^2 + \frac{47}{4}x^3 - 27x^2y + \frac{135}{4}xy^2 - \frac{27}{2}y^3.$$

Since $\phi_M([u]) = [u]$ and $\phi_M([v_M]) = -[v_M]$, we have

$$-f_1(a_N, a_M)v_M \otimes [u] \otimes v_N + f_2(a_N, a_M, b_N)v_M' \otimes [v_M] \otimes v_N = 0.$$

We may assume that $N$ and $L$ are any of $M(1, \mu) (\mu^2 = 1/2, 2, 9/2), M(1, \theta)^\pm$. But then we can see that the determinant of the matrix

$$\left( \begin{array}{cc} f_1(a_L, a_N) & f_2(a_L, a_N, a_L) \\ -f_1(a_N, a_L) & f_2(a_N, a_L, a_N) \end{array} \right)$$

is nonzero except for the cases

$$(N, L) = (M(1, \mu), M(1, \nu));$$

$$(\mu^2, \nu^2) = (1/2, 9/2), (9/2, 1/2), (1/2, 1/2), (M(1)(\theta)^\alpha, M(1)(\theta)^\beta);$$

$\alpha, \beta \in \{+, -\}.$

Note that $9/2 = (\sqrt{2} + 1/\sqrt{2})^2$ and $1/2 = (\sqrt{2} - 1/\sqrt{2})^2 = (\sqrt{2} - \sqrt{2}).$
3/ \sqrt{2})^2. Then we have

\[ N_{M(1,\lambda)M(1,\mu)}^{M(1,\nu)} = \delta_{\nu, (\lambda + \mu)}^2, \quad N_{M(1,\lambda)M(1,\mu)}^{M(1,\nu)} = 0 \]

for every \( \mu^2, \nu^2 = 1/2, 2, 9/2 \) by Corollary 3.17 and Proposition 4.1. Together with the results of (i) and Proposition 4.4, we see that (3) holds for \( \lambda^2 = 2 \).

(iii) We consider the case \( \lambda^2 = 9/2 \). In this case, the submodule for the Virasoro algebra of \( M = M(1, \lambda) \) generated by \( v_M \) is isomorphic to the irreducible module \( L(1,9/4) \). The Verma module for the Virasoro algebra with central charge 1 and lowest weight 9/4 has a singular vector \( 18L(-4)w - 14L(-3)L(-1)w - 9L(-2)^2w + 10L(-2)L(-1)^2w - L(-1)^4w \), where \( w \) is the cyclic vector of the Verma module. Since the image of the singular vector in \( M(1, \lambda) \) is zero, by using (4.3) and (4.4) we have the equality

\[ f(a_L, a_N)v_L' \otimes [v_M] \otimes v_N = 0, \]

where \( f = f(x, y) \in \mathbb{C}[x, y] \) is given by

\[ f = (81 - 72(x + y) + 16(x - y)^2)(1 - 8(x + y) + 16(x - y)^2). \]

By the results of Step 1, Step 2, and (i), (ii) of Step 3, we may assume that \( N \) and \( L \) are any of \( M(1, \mu) \) (\( \mu^2 = 1/2, 9/2 \), \( M(1) \theta \)). If \( a_L, a_N \in \{ 1/4, 9/4, 1/16, 9/16 \} \), then \( f(a_L, a_N) \) is nonzero except for the pairs \( (a_L, a_N) \in \{ 1/16, 1/16 \}, 1/16, 9/16 \}, 9/16, 1/16 \}, 9/16, 9/16 \}. By Corollary 3.17 and Proposition 4.1, we see that for \( \lambda^2 = 9/2, \mu^2, \nu^2 = 1/2, 9/2 \), \( N_{M(1,\lambda)M(1,\mu)}^{M(1,\nu)} = N_{M(1,\lambda)M(1,\mu)}^{M(1,\nu)} = 0 \). Then Proposition 4.4 implies that (3) holds for \( \lambda^2 = 9/2 \).

(iv) We prove that (3) holds for \( \lambda^2 = 1/2 \). In this case we have the irreducible decomposition \( M(1, \lambda) = L(1,1/4) \oplus L(1,9/4) \oplus L(1,25/4) \oplus \cdots \) (see (3.2)). Let \( u = \sqrt{2}h(-2)v_M - 2h(-1)^2v_M \) which is the lowest weight vector of \( L(1,9/4) \). Then \( h(-3)h(-1)^1 * v_M \) belongs to \( L(1,1/4) \oplus L(1,9/4) \) and is a linear combination of \( L(-m_1) \cdots L(-m_k)v_M \) and \( L(-n_1) \cdots L(-n_l)u(m_1, n_1 + \cdots + m_k \leq 4, n_1 + \cdots + n_l \leq 3) \). By (4.3) and (4.4), we have the equality

\[ v_L' \otimes [(J - 4\omega^2 - 17\omega + 9h(-3)h(-1)^1 * v_M)] \otimes v_N \]

\[ = f_1(a_L, a_N)v_L' \otimes [u] \otimes v_N + f_2(a_L, a_N, b_L)v_L' \otimes [v_M] \otimes v_N = 0, \]

(4.12)
where \( f_1 = f(x, y) \in \mathbb{C}[x, y] \) and \( f_2 = f(x, y, z) \in \mathbb{C}[x, y, z] \) are given by

\[
f_1 = \frac{27}{128} + \frac{13}{16}y - \frac{19}{16}(x - y)^2,
\]

\[
f_2 = z - \frac{3}{2} + 12(x + y) - 24(x - y)^2.
\]

Since \( \phi_M([u]) = e^{\pi i/4}[u] \) and \( \phi_M([v_M]) = e^{\pi i/4}[v_M] \), we have also

\[
f_1(a_N, a_L) v_L^i \otimes [u] \otimes v_N + f_2(a_N, a_L, b_N) v_L^i \otimes [v_M] \otimes v_N = 0.
\]

In addition, we have \( L(-1)^3 v_M + L(-2)v_M = 0 \), since the Verma module of central charge 1 and highest weight 1/4 has nontrivial singular vector of weight 9/4. Therefore we have the equality

\[
u_L^i \otimes \left[ (L(-1)^3 v_M + L(-2)v_M) \right] \otimes v_N = g(a_L, a_N) v_L^i \otimes [v_M] \otimes v_N = 0,
\]

where \( g = g(x, y) = -1/16 + (x + y)/2 - (x - y)^2 \). By Step 1, Step 2 and (i)-(iii) of Step 3, we may assume that \( N \) and \( L \) are any of \( M(1, \lambda) \) \((\lambda^2 = 1/2, M(1)\theta)^\pm\).

In the case \( N = L = M(1, \lambda) \) \((\lambda^2 = 1/2)\), since both \( f_1(a_L, a_N) \) and \( g(a_L, a_N) \) are nonzero, \( v_L^i \otimes [u] \otimes v_N = v_L^i \otimes [v_M] \otimes v_N = 0 \) by (4.12) and (4.13). Hence we have \( \dim L_0^M \cdot A(M) \cdot N_0 = 0 \) by Proposition 3.15. Thus Proposition 2.10 shows that the fusion rule \( N_{L/N}^L \) is zero.

In the case \( L = M(1, \mu) \) \((\mu^2 = 1/2)\) and \( N = M(1)(\theta)^\pm \), the determinant of the matrix

\[
\begin{pmatrix}
f_1(a_N, a_L) & f_2(a_N, a_L, b_N) \\
f_1(a_N, a_L) & f_2(a_N, a_L, b_N)
\end{pmatrix}
\]

is nonzero. Hence by Proposition 2.10, the fusion rule \( N_{L/N}^L \) is zero in this case too.

In the case that \( N \) and \( L \) are either \( M(1)(\theta)^+ \) or \( M(1)(\theta)^- \), \( v_L^i \otimes [u] \otimes v_N \) and \( v_L^i \otimes [v_M] \otimes v_N \) are linearly dependent, so the dimension of \( L_0^M \otimes A(M) \otimes N_0 \) is less than one by Proposition 3.15(3). Now Proposition 4.4 (1) shows that the fusion rule \( N_{L/N}^L \) is 1 in this case. Consequently, (3) holds for \( \lambda^2 = 1/2 \). Thus we see that (3) holds for all \( \lambda \in \mathbb{C} - \{0\} \).

**Step 4.** In the case \( M = M(1)(\theta)^\pm \), we have \( M(1)(\theta)^+ = L(1, 1/16) \oplus L(1, 49/16) \oplus L(1, 81/16) \oplus \cdots \) (see (3.7)). We put \( u = 9h(-5/2)h(-1/2)I - 5h(-3/2)I v_M - 10h(-3/2)h(-1/2)I + 4h(-1/2)I \) which is the lowest weight vector of \( L(1, 49/16) \). Since \( h(-3)h(-1)I \cdot v_M \) is in \( L(1, 1/16) \oplus L(1, 49/16) \), it can be expressed as a linear combination of \( L(-1)u, u \) and \( L(-m_1) \cdots L(-m_k)v_M \) \((m_i \in \mathbb{Z}_{\geq 0}, m_1 + \cdots + m_k \leq 4)\).
So we have equality

\[ v'_L \otimes \left[(J - 4\omega^2 - 17\omega + 9h(-3)h(-1)\mathbf{1})\otimes v_M\right] \otimes v_N = f(a_L, a_N)v'_L \otimes [u] \otimes v_N + g(a_L, a_N, b_L)v'_L \otimes [v_M] \otimes v_N = 0, \]

where \( f = f(x, y) \in \mathbb{C}[x, y] \) and \( g = g(x, y) \in \mathbb{C}[x, y, z] \) are given by

\[
\begin{align*}
    f &= \frac{1}{2} + \frac{8}{7}(x - y), \\
    g &= \frac{135}{1792} - \frac{1}{56}x + \frac{73}{28}y - \frac{82}{7}x^2 + \frac{212}{7}xy - \frac{180}{7}y^2 \\
    &\quad + \frac{32}{7}(x - y)^2(5x + 12y) - \frac{256}{7}(x - y)^4.
\end{align*}
\]

Since \( \phi_M([u]) = -e^{\alpha/16}[u] \) and \( \phi_M([v_M]) = e^{\alpha/16}[v_M] \), we have equality

\[ -f(a_L, a_N)v'_L \otimes [u] \otimes v_N + g(a_L, a_N, b_L)v'_L \otimes [v_M] \otimes v_N = 0. \]

By the results of Step 1--Step 3 and Proposition 3.4, we assume that \( N \) and \( L \) are either \( M(1(\theta)^+ \) or \( M(1(\theta)^- \). But then the determinant of the matrix

\[
\begin{pmatrix}
    f(a_L, a_N) & g(a_L, a_N, a_L) \\
    -f(a_N, a_L) & g(a_N, a_L, a_N)
\end{pmatrix}
\]

is nonzero. Hence we have \( v'_L \otimes [u] \otimes v_N = v'_L \otimes [v_M] \otimes v_N = 0 \). This proves that

\[
N_{M(1(\tilde{\theta}))}^{M(1(\theta))} = 0
\]

for any \( \alpha, \beta \in \{+, -\} \). Thus we see that (4) holds.

**Step 5.** By the results of Step 1--Step 4; it is enough to show that \( N_{\tilde{N}} = 0 \) for \( M = M = L = M(1(\theta)^+ \). Since we have the direct product decomposition \( M(1(\theta)^- = L(1, 9/16) \oplus L(1, 25/16) \oplus L(1, 121/16) \oplus \cdots \) (see 3.8), if we put \( u = -(1/2)h(-3/2)1 + h(-3/2)^1 \) which is the lowest weight vector of \( L(1, 25/16) \), then \( h(-3)h(-1)\mathbf{1} \otimes v_M \) can be expressed as a linear combination of \( L(-m_1) \cdots L(-m_k) v_M \) and \( L(-n_1) \cdots L(-n_l) u \) \((m_i, n_j \in \mathbb{Z}_{\geq 0}, m_1 + \cdots + m_k \leq 4, n_1 + \cdots + n_l \leq 3)\). Calculating the vector \( v'_L \otimes [(J - 4\omega^2 - 17\omega + 9h(-3)h(-1)\mathbf{1})\otimes v_M] \otimes v_N \) by means of (4.3) and (4.4) we have the linear equalities

\[
\frac{75}{224}v'_L \otimes [u] \otimes v_N - \frac{135}{256}v'_L \otimes [v_M] \otimes v_N = 0.
\]
Since \( \phi_M([u]) = -e^{\frac{3\pi i}{16}}[u] \) and \( \phi_M([v_M]) = e^{\frac{9\pi i}{16}}[v_M] \), by Proposition 3.7, we have

\[
-\frac{75}{224} v'_L \otimes [u] \otimes v_N - \frac{135}{256} v'_L \otimes [v_M] \otimes v_N = 0.
\]

This follows that \( v'_L \otimes [u] \otimes v_N = v'_L \otimes [v_M] \otimes v_N = 0 \). So we see that the fusion rule \( N^L_M = 0 \) if \( M = N = L = M(1 \theta)^+ \) by Proposition 3.15 (3). The proof of the theorem is complete.

**ACKNOWLEDGMENTS**

I thank Professor Kiyokazu Nagatomo for useful discussions and suggestions. I also thank Doctor Yoshiyuki Koga and Akihiko Ogawa for their many helpful opinions.

**REFERENCES**


