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Some Koszul rings from geometry

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ABSTRACT

We give examples of Koszul rings that arise naturally in algebraic geometry. In the first part, we prove a general result on Koszul property associated to an ample line bundle on a projective variety. Specifically, we show how Koszul property of multiples of a base point free ample line bundle depends on its Castelnuovo–Mumford regularity. In the second part, we give examples of Koszul rings that come from adjoint line bundles on minimal irregular surfaces of general type.

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0. Introduction

Let k be a field. A standard graded k -algebra $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ is said to be *Koszul* if k has a linear minimal resolution as an R -algebra.

Let

$$\cdots \rightarrow E_p \rightarrow E_{p-1} \rightarrow \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow k \rightarrow 0$$

be a minimal resolution of k over R . Then R is Koszul if and only if $E_0 = R$ and $E_p = R(-p)^{\oplus r(p)}$ for $p \geq 1$. Equivalently, $\text{Tor}_i^R(k, k)$ has pure degree i for all i .

Koszul algebras were introduced by Stewart Priddy [20] and they have applications in many areas of mathematics, such as algebraic geometry, commutative algebra and representation theory to name a few. For a sample of these applications, see [1,3,4,11]. See [17] for a general introduction to Koszul property with historical notes. [4] also has a general treatment of Koszul property.

Part of the algebraic geometer's interest in Koszul rings stems from the following observation:

Let L be a very ample line bundle on a projective variety X over k . Let I_X be the ideal defining X under the embedding in a projective space defined by L . Define

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$$R(L) = \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n}).$$

If $R(L)$ is Koszul then X is projectively normal and I_X is generated by quadrics. In the notation of N_p property, this means that L satisfies the property N_1 . (See [13], 1.8.D for details on N_p property.) If $R(L)$ is Koszul we say that L has Koszul property.

There are several results establishing Koszul property for line bundles on curves. For instance, see [5,16,18,23]. Koszul property for line bundles on elliptic ruled surfaces is studied in [7]. Koszul property for adjoint line bundles on regular surfaces is studied in [8]. Some general results on Koszul property for adjunction bundles are discussed in [15].

Quite generally, high enough powers of ample line bundles have Koszul property [2]. The relation between the precise powers that achieve Koszul property and Castelnuovo–Mumford regularity of the bundle is of general interest. In the first part of this paper (Section 3), we prove a general result (Theorem 3.3) establishing such a relation. If B is a base point free ample line bundle on a projective variety X and if $\text{reg}(B)$ is r (cf. Definition 3.1), we show that $B^{\otimes n}$ has Koszul property for $n \geq r + 1$. This result is also proved in a preprint [10] invoking the notion of multigraded regularity. The proof in [10] and our proof are both motivated by Theorem 1.3 in [8] and essentially follow the methods developed there.

A similar result is proved in [2] and [6].¹ Let R be a polynomial ring and let $I \subset R$ be a homogeneous ideal. Set $A = R/I$. In these papers, authors develop useful criteria to determine if the d th Veronese subring $A_{(d)}$ of A is Koszul. One of their results says that if $d \geq \text{reg}(I)/2$, then $A_{(d)}$ is Koszul. In our situation, this means the following: Let B be a base point free ample line bundle that defines a map whose image is projectively normal in the projective space. Then $B^{\otimes d}$ has Koszul property for $d \geq \text{reg}(B)/2$.

In the second part of the paper (Section 4) we give examples of Koszul rings associated to certain adjoint line bundles on a minimal irregular surface of general type. This extends an analogous result for regular surfaces in [8]. As mentioned above, the Koszul property implies N_1 , but the converse is, in general, not true [21]. Our theorem establishes the converse in this case. Our method is similar to [19].

Establishing N_1 property involves proving that a certain multiplication map of global sections of vector bundles is surjective. Koszul property is equivalent to the surjectivity of infinitely many multiplication maps of global sections of certain vector bundles, first of which is the multiplication map that appears in the N_1 property. In most examples of Koszul rings arising in algebraic geometry, the surjectivity required for Koszul property is proved by methods very similar to those used in establishing the N_1 property, after an appropriate inductive framework is set up. However, in the case of adjoint line bundles on irregular surfaces that we study, the methods used in establishing the N_1 property [19] do not work for the subsequent surjections required for Koszul property. This suggests a potential example where N_1 property does not imply Koszul property. In this paper, we establish the Koszul property under a stronger assumption than was made in [19], namely the canonical bundle is base point free.

Many of the results cited here are directly connected to the cases we study. They represent only a sliver of the research on Koszul property in algebraic geometry. There are many results of a similar flavor that we do not mention here, but that are interesting nevertheless.

1. Preliminaries

Let k be a field and let X be a projective variety over k .

Notation: For a coherent sheaf F on X , we write $H^i(F)$ to denote the i th sheaf cohomology group $H^i(X, F)$.

¹ I sincerely thank Burt Totaro for bringing this result to my notice.

Let L be a line bundle on X . As before consider the ring:

$$R(L) = \bigoplus_{n=0}^{\infty} H^0(L^{\otimes n}).$$

The question of whether $R(L)$ is Koszul has a nice cohomological interpretation due to Lazarsfeld. Given any vector bundle F on X that is generated by its global sections, we have a canonical surjective map

$$H^0(F) \otimes \mathcal{O}_X \rightarrow F. \tag{1}$$

Let M_F be the kernel of this map. We have then the natural exact sequence

$$0 \rightarrow M_F \rightarrow H^0(F) \otimes \mathcal{O}_X \rightarrow F \rightarrow 0. \tag{2}$$

Now set $M^{(0),L} := L$. If L is globally generated, define

$$M^{(1),L} := M_L \otimes L = M_{M^{(0),L}} \otimes L.$$

If $M^{(1),L}$ is generated by its global sections, define

$$M^{(2),L} := M_{M^{(1),L}} \otimes L.$$

Inductively, define $M^{(h),L} := M_{M^{(h-1),L}} \otimes L$, provided that $M_{M^{(h-1),L}}$ is generated by its global sections.

Then we have the following proposition that characterizes the Koszul property of L in terms of certain cohomology groups.

Proposition 1.1. (See [15, Lemma 1].) *Let X be a projective variety over a field k . Assume that L is a base point free line bundle on X such that the vector bundles $M^{(h),L}$ are globally generated for every $h \geq 0$. If $H^1(M^{(h),L} \otimes L^s) = 0$ for every $h \geq 0$ and every $s \geq 0$, then $R(L)$ is Koszul.*

2. Preparatory lemmas

In this section we will list and prove some well-known results that will be used repeatedly in what follows. k is any field and X is a projective variety over k .

Lemma 2.1. *Let E and L_1, L_2, \dots, L_r be coherent sheaves on X . Consider the multiplication maps*

$$\begin{aligned} \psi &: H^0(E) \otimes H^0(L_1 \otimes \dots \otimes L_r) \rightarrow H^0(E \otimes L_1 \otimes \dots \otimes L_r), \\ \alpha_1 &: H^0(E) \otimes H^0(L_1) \rightarrow H^0(E \otimes L_1), \\ \alpha_2 &: H^0(E \otimes L_1) \otimes H^0(L_2) \rightarrow H^0(E \otimes L_1 \otimes L_2), \\ &\vdots \\ \alpha_r &: H^0(E \otimes L_1 \otimes \dots \otimes L_{r-1}) \otimes H^0(L_r) \rightarrow H^0(E \otimes L_1 \otimes \dots \otimes L_r). \end{aligned}$$

If $\alpha_1, \dots, \alpha_{r-1}$ are surjective then so is ψ .

Proof. We have the following commutative diagram where id denotes the identity morphism:

$$\begin{array}{ccc}
 H^0(E) \otimes H^0(L_1) \otimes \cdots \otimes H^0(L_r) & \xrightarrow{\alpha_1 \otimes id} & H^0(E \otimes L_1) \otimes H^0(L_2) \otimes \cdots \otimes H^0(L_r) \\
 \downarrow \phi & & \downarrow \alpha_2 \otimes id \\
 H^0(E) \otimes H^0(L_1 \otimes \cdots \otimes L_r) & & H^0(E \otimes L_1 \otimes L_2) \otimes H^0(L_3) \otimes \cdots \otimes H^0(L_r) \\
 \downarrow \psi & & \downarrow \alpha_3 \otimes id \\
 & & \vdots \\
 & & \downarrow \alpha_{r-1} \otimes id \\
 H^0(E \otimes L_1 \otimes \cdots \otimes L_r) & \xleftarrow{\alpha_r} & H^0(E \otimes L_1 \otimes \cdots \otimes L_{r-1}) \otimes H^0(L_r)
 \end{array}$$

Since $\alpha_1, \alpha_2, \dots, \alpha_r$ are surjective and this diagram is commutative, a simple diagram chase shows that ψ is surjective. \square

Lemma 2.2. Let F be a locally free sheaf and A an ample line bundle on X . If the multiplication map $H^0(F \otimes A^{\otimes n}) \otimes H^0(A) \rightarrow H^0(F \otimes A^{\otimes n+1})$ is surjective for every $n \geq 0$, then F is generated by its global sections.

Proof. Since A is ample, there exists $m \geq 0$ such that $F \otimes A^{\otimes m}$ is generated by global sections. In other words, the morphism of sheaves $\nu : H^0(F \otimes A^{\otimes m}) \otimes \mathcal{O}_X \rightarrow F \otimes A^{\otimes m}$ is surjective.

The hypothesis implies, by Lemma 2.1, that $\psi : H^0(F) \otimes H^0(A^{\otimes m}) \rightarrow H^0(F \otimes A^{\otimes m})$ is surjective. Consider now the commutative diagram:

$$\begin{array}{ccc}
 H^0(F \otimes A^{\otimes m}) \otimes \mathcal{O}_X & & \\
 \uparrow \psi \otimes id & \searrow \nu & \\
 H^0(F) \otimes H^0(A^{\otimes m}) \otimes \mathcal{O}_X & & F \otimes A^{\otimes m} \\
 \downarrow id \otimes \phi & \nearrow \mu & \\
 H^0(F) \otimes A^{\otimes m} & &
 \end{array}$$

Since $\psi \otimes id$ and ν are surjective, a diagram chase shows that $\mu : H^0(F) \otimes A^{\otimes m} \rightarrow F \otimes A^{\otimes m}$ is surjective.

As $A^{\otimes m}$ is an invertible sheaf, the surjectivity of μ shows that F is generated by global sections. \square

Lemma 2.3. (See [14, CM Lemma].) Let E be a base point free line bundle on X and let F be a coherent sheaf on X . If $H^i(F \otimes E^{-i}) = 0$ for $i \geq 1$, then the multiplication map

$$H^i(F \otimes E^{\otimes i}) \otimes H^0(E) \rightarrow H^i(F \otimes E^{\otimes i+1})$$

is surjective for all $i \geq 0$.

Let N be a globally generated vector bundle and let A be a line bundle on X .

We have a short exact sequence

$$0 \rightarrow M_N \rightarrow H^0(N) \otimes \mathcal{O}_X \rightarrow N \rightarrow 0. \tag{3}$$

Remark 2.4. $H^1(M_N \otimes A) = 0$ if the following two conditions hold.

- The multiplication map $H^0(N) \otimes H^0(A) \rightarrow H^0(N \otimes A)$ is surjective.
- $H^1(A) = 0$.

This is easy to see: Tensor the sequence (3) by A and take global sections:

$$\dots \rightarrow H^0(N) \otimes H^0(A) \rightarrow H^0(N \otimes A) \rightarrow H^1(M_N \otimes A) \rightarrow H^0(N) \otimes H^1(A) \rightarrow \dots$$

Remark 2.5. $H^2(M_N \otimes A) = 0$ if the following two conditions hold.

- $H^1(N \otimes A) = 0$.
- $H^2(A) = 0$.

This is easy to see: Tensor the sequence (3) by A and take global sections:

$$\dots \rightarrow H^1(N \otimes A) \rightarrow H^2(M_N \otimes A) \rightarrow H^0(N) \otimes H^2(A) \rightarrow \dots$$

3. Koszul ring associated to an ample line bundle on a projective variety

In this section X denotes an arbitrary projective variety over a field k . Let B be a base point free ample line bundle on X .

Definition 3.1. (See [13, Definition 1.8.4].) Let $m \geq 0$. We say that B is m -regular (with respect to B) if

$$H^i(B^{\otimes m+1-i}) = 0 \quad \text{for } i > 0.$$

If B is m -regular, then it is $(m + 1)$ -regular [13, Theorem 1.8.5.(iii)]. We define the *regularity* of B to be m if B is m -regular, but *not* $(m - 1)$ -regular. This notion of regularity is related to the classical notion of Castelnuovo–Mumford regularity as follows:

Let $f : X \rightarrow \mathbb{P}_k^N$ be the morphism to a projective space defined by B . Note that such a morphism exists because B is base point free. Let $L = f_*(B)$.

In the classical setting, we say L is m -regular if $H^i(\mathbb{P}_k^N, L(m - i)) = 0$ for $i \geq 0$.

Since $f^*(\mathcal{O}_{\mathbb{P}_k^N}(1)) = B$, by the projection formula we get

$$f_*(B^{\otimes 2}) = f_*(B \otimes f^*(\mathcal{O}_{\mathbb{P}_k^N}(1))) = f_*(B) \otimes \mathcal{O}_{\mathbb{P}_k^N}(1) = L(1).$$

By induction, we obtain for any $r \geq 1$,

$$f_*(B^{\otimes r}) = L(r - 1). \tag{4}$$

Since the morphism f is finite, we have $H^i(X, A) \cong H^i(\mathbb{P}_k^N, f_*(A))$ for any sheaf A on X . Hence, by (4), B is m -regular in the sense of Definition 3.1 if and only if $f_*(B)$ is m -regular in the sense of Castelnuovo–Mumford.

Suppose now that B is $(r - 1)$ -regular (in the sense of Definition 3.1). Then since B is $(n - 1)$ -regular for all $n \geq r$, we have

$$H^i(B^{\otimes n-i}) = 0 \quad \text{for all } i \geq 1 \text{ and } n \geq r. \tag{5}$$

Set $L = B^{\otimes r}$. We prove that $R(L)$ is a Koszul ring. Our methods will closely mirror those of [8].

Proposition 3.2. *We have*

- (A) $M^{(h),L}$ is globally generated for each $h \geq 0$, and
- (B) $H^i(M^{(h),L} \otimes B^{\otimes s-i}) = 0$, for all $h \geq 0, s \geq 0$, and $i \geq 1$.

Proof. We prove both assertions simultaneously by induction on h .

First suppose that $h = 0$. $M^{(0),L} = L$ is globally generated because B is. Further, for any $s \geq 0$ and $i \geq 1$, $H^i(M^{(0),L} \otimes B^{\otimes s-i}) = H^i(L \otimes B^{\otimes s-i}) = H^i(B^{\otimes r+s-i}) = 0$, by (5). Now fix some $h_1 > 0$ and suppose that the statements (A) and (B) hold for all $h < h_1$. So $M^{(h_1-1),L}$ is globally generated and $M^{(h_1),L}$ is defined.

We claim that the multiplication map

$$H^0(M^{(h_1),L} \otimes B^{\otimes n}) \otimes H^0(B) \rightarrow H^0(M^{(h_1),L} \otimes B^{\otimes n+1}) \tag{6}$$

is surjective for all $n \geq 0$. By Lemma 2.3, this follows if

$$H^i(M^{(h_1),L} \otimes B^{-i}) = 0 \quad \text{for all } i \geq 1. \tag{7}$$

We will first prove (7) for $i = 1$.

Tensor the sequence (2) corresponding to $F = M^{(h_1-1),L}$ by $B^{\otimes r-1}$. We obtain

$$0 \rightarrow M_{M^{(h_1-1),L}} \otimes B^{\otimes r-1} \rightarrow H^0(M^{(h_1-1),L}) \otimes B^{\otimes r-1} \rightarrow M^{(h_1-1),L} \otimes B^{\otimes r-1} \rightarrow 0.$$

Taking global sections, we get

$$\begin{aligned} 0 \rightarrow H^0(M_{M^{(h_1-1),L}} \otimes B^{\otimes r-1}) &\rightarrow H^0(M^{(h_1-1),L}) \otimes H^0(B^{\otimes r-1}) \xrightarrow{\gamma} H^0(M^{(h_1-1),L} \otimes B^{\otimes r-1}) \\ &\rightarrow H^1(M_{M^{(h_1-1),L}} \otimes B^{\otimes r-1}) \rightarrow H^0(M^{(h_1-1),L}) \otimes H^1(B^{\otimes r-1}) \rightarrow \dots \end{aligned}$$

By (5), $H^1(B^{\otimes r-1}) = 0$. So $H^1(M_{M^{(h_1-1),L}} \otimes B^{\otimes r-1}) = 0$ if and only if γ is surjective. Now, by Lemmas 2.1 and 2.3, γ is surjective if $H^i(M^{(h_1-1),L} \otimes B^{-i}) = 0$ for all $i \geq 1$. But this follows from induction hypothesis applied to $h_1 - 1$ and $s = 0$. Thus $H^1(M_{M^{(h_1-1),L}} \otimes B^{\otimes r-1}) = H^1(M^{(h_1),L} \otimes B^{-1}) = 0$, which is the statement (7) for $i = 1$.

Now suppose that $i \geq 2$.

Tensor the sequence (2) corresponding to $F = M^{(h_1-1),L}$ by $B^{\otimes r-i}$. We obtain

$$0 \rightarrow M_{M^{(h_1-1),L}} \otimes B^{\otimes r-i} \rightarrow H^0(M^{(h_1-1),L}) \otimes B^{\otimes r-i} \rightarrow M^{(h_1-1),L} \otimes B^{\otimes r-i} \rightarrow 0.$$

Taking global sections, we get

$$\begin{aligned} \dots \rightarrow H^0(M^{(h_1-1),L}) \otimes H^{i-1}(B^{\otimes r-i}) &\rightarrow H^{i-1}(M^{(h_1-1),L} \otimes B^{\otimes r-i}) \\ &\rightarrow H^i(M_{M^{(h_1-1),L}} \otimes B^{\otimes r-i}) \rightarrow H^0(M^{(h_1-1),L}) \otimes H^i(B^{\otimes r-i}) \rightarrow \dots \end{aligned}$$

$H^i(B^{\otimes r-i}) = 0$ by (5). $H^{i-1}(M^{(h_1-1),L} \otimes B^{\otimes r-i}) = 0$, by induction hypothesis (more precisely, statement (B) for $h_1 - 1$). Hence $H^i(M_{M^{(h_1-1),L}} \otimes B^{\otimes r-i}) = H^i(M^{(h_1),L} \otimes B^{-i}) = 0$ for all $i \geq 2$. This proves (7) and hence (6) for all $i \geq 1$. So we have (A) by Lemma 2.2.

Next, we prove **(B)** for $h_1, i = 1$ and any $s \geq 0$.

Tensor the sequence (2) corresponding to $F = M^{(h_1-1),L}$ by $B^{\otimes r+s-1}$. We obtain

$$0 \rightarrow M_{M^{(h_1-1),L}} \otimes B^{\otimes r+s-1} \rightarrow H^0(M^{(h_1-1),L}) \otimes B^{\otimes r+s-1} \rightarrow M^{(h_1-1),L} \otimes B^{\otimes r+s-1} \rightarrow 0.$$

Taking global sections, we get

$$\begin{aligned} 0 \rightarrow H^0(M_{M^{(h_1-1),L}} \otimes B^{\otimes r+s-1}) &\rightarrow H^0(M^{(h_1-1),L}) \otimes H^0(B^{\otimes r+s-1}) \xrightarrow{\gamma} H^0(M^{(h_1-1),L} \otimes B^{\otimes r+s-1}) \\ &\rightarrow H^1(M_{M^{(h_1-1),L}} \otimes B^{\otimes r+s-1}) \rightarrow H^0(M^{(h_1-1),L}) \otimes H^1(B^{\otimes r+s-1}) \rightarrow \dots \end{aligned}$$

By (5), $H^1(B^{\otimes r+s-1}) = 0$. So $H^1(M_{M^{(h_1-1),L}} \otimes B^{\otimes r+s-1}) = 0$ if and only if γ is surjective. Now, again by Lemmas 2.1 and 2.3, γ is surjective if $H^i(M^{(h_1-1),L} \otimes B^{-i}) = 0$ for all $i \geq 1$. But this follows from induction hypothesis applied to $h_1 - 1$ and $s = 0$. Thus $H^1(M_{M^{(h_1-1),L}} \otimes B^{\otimes r+s-1}) = H^1(M^{(h_1),L} \otimes B^{\otimes s-1}) = 0$. This proves **(A)** for h_1 and $i = 1$.

Now suppose that $i \geq 2$ and $s \geq 0$.

Tensor the sequence (2) corresponding to $F = M^{(h-1),L}$ by $B^{\otimes r+s-i}$. We obtain

$$0 \rightarrow M_{M^{(h-1),L}} \otimes B^{\otimes r+s-i} \rightarrow H^0(M^{(h-1),L}) \otimes B^{\otimes r+s-i} \rightarrow M^{(h-1),L} \otimes B^{\otimes r+s-i} \rightarrow 0.$$

Taking global sections, we get

$$\begin{aligned} \dots \rightarrow H^0(M^{(h_1-1),L}) \otimes H^{i-1}(B^{\otimes r+s-i}) &\rightarrow H^{i-1}(M^{(h_1-1),L} \otimes B^{\otimes r+s-i}) \\ &\rightarrow H^i(M_{M^{(h_1-1),L}} \otimes B^{\otimes r+s-i}) \rightarrow H^0(M^{(h_1-1),L}) \otimes H^i(B^{\otimes r+s-i}) \rightarrow \dots \end{aligned}$$

$H^i(B^{\otimes r+s-i}) = 0$, by (5). $H^{i-1}(M^{(h_1-1),L} \otimes B^{\otimes r+s-i}) = H^{i-1}(M^{(h_1-1),L} \otimes B^{\otimes r+s-1-(i-1)}) = 0$, by induction hypothesis because $r + s - 1 \geq s \geq 0$. So $H^i(M_{M^{(h_1-1),L}} \otimes B^{\otimes r+s-i}) = H^i(M^{(h_1),L} \otimes B^{\otimes s-i}) = 0$, as required. \square

Theorem 3.3. *Let X be a projective variety over a field k . Let B a base point free ample bundle on X with $\text{reg}(B) = r - 1$. Let $L = B^{\otimes n}$ with $n \geq r$. Then $R(L)$ is a Koszul ring.*

Proof. Since B is $(n - 1)$ -regular, Proposition 3.2 implies that, $H^1(M^{(h),L} \otimes B^{\otimes s-1}) = 0$ for all $s \geq 0$ and $h \geq 0$. So for $s \geq 0$ and $h \geq 0$, $H^1(M^{(h),L} \otimes L^s) = H^1(M^{(h),L} \otimes B^{\otimes rs}) = 0$. By Proposition 1.1, it follows that $R(L)$ is a Koszul ring. \square

4. Minimal irregular surfaces of general type

Let X be a nonsingular projective minimal² irregular³ surface of general type⁴ over the complex number field \mathbb{C} . Let K_X be the canonical line bundle on X . Suppose that K_X is base point free.

Notation: We write $L \equiv L'$ if the line bundles L and L' are numerically equivalent. We write $L \cdot L'$ to denote the intersection number of L and L' .

Let B be a base point free and ample divisor on X such that B' is base point free for all $B' \equiv B$ and $H^1(B') = 0$. Assume that $B^2 > B \cdot K_X$. Let $L = K_X \otimes B^{\otimes n}$, where $n \geq 2$.

Set $R(L) = \bigoplus_{n=0}^{\infty} H^0(L^{\otimes n})$. Our goal is to prove that $R(L)$ is Koszul. Our proof is similar to proofs in [19]. Theorem 5.14 in [8] proves an analogous result for regular surfaces.

² A surface X is *minimal* if every birational morphism $X \rightarrow Y$ is an isomorphism.

³ The *irregularity* q of a surface X over a field k is defined to be $q = \dim_k H^1(\mathcal{O}_X)$. We say that X is irregular if $q > 0$.

⁴ A surface X is of *general type* if its Kodaira dimension $\kappa(X) = 2$.

4.1. Required lemmas

In this subsection we will prove some lemmas that will be used later in the proof of the main theorem.

A divisor D on X is nef if $D \cdot C \geq 0$ for every irreducible curve C in X . D is said to be big if a multiple mD , $m \in \mathbb{N}$, defines a birational map of X to a projective space.

As X is a minimal surface of general type, K_X is nef and big. In fact, a surface is minimal of general type if and only if K_X is nef and big.

Lemma 4.1 (Kawamata–Viehweg vanishing). *Let X be a nonsingular projective variety over the complex number field \mathbb{C} . Let D be a nef and big divisor on X . Then*

$$H^i(K_X \otimes D) = 0 \quad \text{for } i > 0.$$

For a proof, see [12] or [22]. We will refer to this result simply as K-V vanishing.

Recall that $\text{Pic}^0(X)$ denotes the group of divisors on X which are algebraically equivalent to zero modulo linear equivalence.

Lemma 4.2. *There exists a divisor $E \in \text{Pic}^0(X)$ such that $E^{\otimes 2} \neq \mathcal{O}_X$.*

Proof. We have the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_{X_h} \rightarrow \mathcal{O}_{X_h}^* \rightarrow 0,$$

where X_h is the complex analytic space associated to X .

Consider the resulting long exact sequence in cohomologies. Applying Serre’s GAGA and identifying the $\text{Pic}(X)$ with $H^1(X, \mathcal{O}_X^*)$, we obtain an exact sequence

$$0 \rightarrow H^1(X_h, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow \text{Pic}(X) \rightarrow H^2(X_h, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X) \rightarrow \dots$$

This gives $\text{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X_h, \mathbb{Z})$. This is an abelian variety. For more details on this see the discussion in Appendix B.5 in [9].

Since X is irregular $H^1(\mathcal{O}_X) \neq 0$. So $\text{Pic}^0(X)$ is a nontrivial abelian variety and hence contains 2-torsion elements. \square

Lemma 4.3. $H^1(B_1 \otimes B_2 \otimes \dots \otimes B_n) = 0$ for line bundles $B_1 \equiv B_2 \equiv \dots \equiv B_n \equiv B$ and $n \geq 1$.

Proof. Let $C \in |B|$ be a smooth curve. We have for every i ,

$$\text{deg}(B_i \otimes \mathcal{O}_C) = B_i \cdot C = B^2.$$

If $n > 3$, $\text{deg}(B_1 \otimes \dots \otimes B_n \otimes \mathcal{O}_C) = nB^2 > 2B^2 = B^2 + B^2 \geq B^2 + B \cdot K_X = 2g(C) - 2$, where $g(C)$ is the genus of C . So $H^1(B_1 \otimes \dots \otimes B_n \otimes \mathcal{O}_C) = 0$.

We have the short exact sequence:

$$0 \rightarrow B^{-1} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0. \tag{8}$$

Tensoring with $B_1 \otimes B_2 \otimes B_3$, we get

$$0 \rightarrow B_1 \otimes B_2 \otimes B_3 \otimes B^{-1} \rightarrow B_1 \otimes B_2 \otimes B_3 \rightarrow B_1 \otimes B_2 \otimes B_3 \otimes \mathcal{O}_C \rightarrow 0.$$

Note that $B_3 \otimes B^{-1} \equiv \mathcal{O}_B$, so we can write $B_1 \otimes B_2 \otimes B_3 = B_1 \otimes B'_2$, where $B'_2 \equiv B_2 \equiv B$.

Taking the long exact sequence in cohomology of the above short exact sequence we obtain

$$H^1(B_1 \otimes B'_2) \rightarrow H^1(B_1 \otimes B_2 \otimes B_3) \rightarrow H^1(B_1 \otimes B_2 \otimes B_3 \otimes \mathcal{O}_C).$$

Since $H^1(B_1 \otimes B_2 \otimes B_3 \otimes \mathcal{O}_C) = 0$, it is enough to prove the theorem for $n = 2$.

Exactly as above, we have the following exact sequence

$$H^1(B_1) \rightarrow H^1(B_1 \otimes B_2) \rightarrow H^1(B_1 \otimes B_2 \otimes \mathcal{O}_C).$$

$H^1(B_1) = 0$ by hypothesis. It is enough to prove that $H^1(B_1 \otimes B_2 \otimes \mathcal{O}_C) = 0$.

As before, $\text{deg}(B_1 \otimes B_2 \otimes \mathcal{O}_C) = 2B^2 \geq B^2 + B \cdot K_X = 2g(C) - 2$. If $\text{deg}(B_1 \otimes B_2 \otimes \mathcal{O}_C) > 2g(C) - 2$, then we are done.

Suppose that $\text{deg}(B_1 \otimes B_2 \otimes \mathcal{O}_C) = 2g(C) - 2$. Note that this implies that $B^2 = B \cdot K_X$. If $B_1 \otimes B_2 \otimes \mathcal{O}_C \neq K_C$, then $H^1(B_1 \otimes B_2 \otimes \mathcal{O}_C) = 0$ and we are done. Here K_C denotes the canonical divisor of C .

Assume that $B_1 \otimes B_2 \otimes \mathcal{O}_C = K_C$. By adjunction, we have $K_C = K_X \otimes B \otimes \mathcal{O}_C$. So $B_1 \otimes B_2 \otimes \mathcal{O}_C = K_X \otimes B \otimes \mathcal{O}_C$. This gives $B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1} \otimes \mathcal{O}_C = \mathcal{O}_C$.

Tensoring (8) with $B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1}$, we obtain

$$0 \rightarrow B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-2} \rightarrow B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1} \rightarrow B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1} \otimes \mathcal{O}_C \rightarrow 0.$$

Taking cohomology long exact sequence, we have

$$\begin{aligned} H^0(B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{\otimes -2}) &\rightarrow H^0(B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1}) \\ &\rightarrow H^0(B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1} \otimes \mathcal{O}_C) \rightarrow H^1(B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{\otimes -2}). \end{aligned}$$

Now $H^0(B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{\otimes -2}) = H^2(K_X^{\otimes 2} \otimes B_1^{-1} \otimes B_2^{-1} \otimes B^{\otimes 2})$, by Serre duality. Since $B_1^{-1} \otimes B_2^{-1} \otimes B^{\otimes 2} \equiv \mathcal{O}_X$ and K_X is nef and big (X is minimal of general type), it follows that $H^2(K_X^{\otimes 2} \otimes B_1^{-1} \otimes B_2^{-1} \otimes B^{\otimes 2}) = 0$ by K-V vanishing.

Similarly, $H^1(B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{\otimes -2}) = H^1(K_X^{\otimes 2} \otimes B_1^{-1} \otimes B_2^{-1} \otimes B^{\otimes 2}) = 0$. Thus we obtain

$$H^0(B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1}) \cong H^0(B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1} \otimes \mathcal{O}_C) \cong H^0(\mathcal{O}_C) \cong k.$$

So $H^0(B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1}) \neq 0$ and $B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1}$ is effective. But

$$B \cdot (B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1}) = B \cdot B_1 + B \cdot B_2 - B \cdot K_X - B^2 = B^2 - B \cdot K_X = 0.$$

So $B_1 \otimes B_2 \otimes K_X^{-1} \otimes B^{-1} = \mathcal{O}_X \Rightarrow B_1 \otimes B_2 = K_X \otimes B$. Finally, $H^1(B_1 \otimes B_2) = H^1(K_X \otimes B) = 0$, by K-V vanishing, thus concluding the proof. \square

Lemma 4.4. $H^2(B^{\otimes n} \otimes \delta) = 0$ for $n \geq 1$ and any numerically trivial line bundle δ .

Proof. $H^2(B^{\otimes n} \otimes \delta) = H^0(K_X \otimes B^{-n} \otimes \delta^{-1})$, by Serre duality.

If $H^0(K_X \otimes B^{-n} \otimes \delta^{-1}) \neq 0$, then there is an effective divisor D that is linearly equivalent to $K_X - nB - \delta$. (By abuse of notation, we denote the divisor associated to a line bundle by the same letter.) So we have $B \cdot D = B \cdot (K_X - nB - \delta) \geq 0$, because B is ample. So $B \cdot K_X \geq nB^2$. But this contradicts the hypothesis that $B^2 > B \cdot K_X$. \square

Lemma 4.5. $H^2(K_X^{-1} \otimes B^{\otimes n} \otimes \delta) = 0$ for $n \geq 2$.

Proof. By Serre duality, $H^2(K_X^{-1} \otimes B^{\otimes n} \otimes \delta) = H^0(K_X^{\otimes 2} \otimes B^{-n} \otimes \delta^{-1})$.

If $H^0(K_X^{\otimes 2} \otimes B^{-n} \otimes \delta^{-1}) \neq 0$, then there is an effective divisor D that is linearly equivalent to $2K_X - nB - \delta$. So we have $B \cdot D = B \cdot (2K_X - nB - \delta) \geq 0$, because B is ample. So $2B \cdot K_X \geq nB^2$. But this contradicts the hypothesis that $B^2 > B \cdot K_X$. \square

4.2. Main theorem

In this subsection we will prove our main theorem: $R(L)$ is a Koszul ring.

By Lemma 4.2, there exists $E \in \text{Pic}^0(X)$ such that $E^{\otimes 2} \neq \mathcal{O}_X$. Note that E is numerically trivial. Set $B_1 = B \otimes E^{-1}$ and $B_2 = B \otimes E$. Then $L = B_1 \otimes B_2 \otimes K_X$. Let δ be a numerically trivial line bundle such that

$$\delta^{\otimes 2} \neq E^{\otimes 2}.$$

Let m, r, t be nonnegative integers such that $m + r + t > 0$.

Recall the definition of $H^1(M^{(h),L})$ from Section 1. $H^1(M^{(h-1),L})$ has to be globally generated to define $H^1(M^{(h),L})$. The discussion below will establish that $H^1(M^{(h),L})$ is globally generated for all $h \geq 0$.

Consider the following statements for a nonnegative integer h :

$$(V_h) \quad H^1(M^{(h),L} \otimes B_1^{\otimes m} B_2^{\otimes r} \otimes K^{\otimes t} \otimes \delta) = 0.$$

$$(S_h) \quad \begin{cases} \text{The multiplication map} \\ H^0(M^{(h),L}) \otimes H^0(B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t} \otimes \delta) \rightarrow H^0(M^{(h),L} \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t} \otimes \delta) \\ \text{is surjective.} \end{cases}$$

Our goal is to prove that (S_h) and (V_h) hold for all $h \geq 0$.

Lemma 4.6. *The statements (S_0) and (V_0) hold.*

Proof. (V_0) : $H^1(L \otimes B_1^{\otimes m} B_2^{\otimes r} \otimes K^{\otimes t} \otimes \delta) = 0$ holds by K-V vanishing.

To prove (S_0) , we will use Lemma 2.1 iteratively. First, let us observe that the following map is surjective for $m \geq 0$:

$$H^0(L \otimes B_1^{\otimes m}) \otimes H^0(B_1 \otimes \delta) \rightarrow H^0(L \otimes B_1^{\otimes m+1} \otimes \delta). \tag{9}$$

By Lemma 2.3, we need $H^1(L \otimes B_1^{\otimes m-1} \otimes \delta^{-1}) = 0$ and $H^2(L \otimes B_1^{\otimes m-2} \otimes \delta^{-2}) = 0$.

$$H^1(L \otimes B_1^{\otimes m-1} \otimes \delta^{-1}) = H^1(K_X \otimes B_1^{\otimes m} \otimes B_2 \otimes \delta^{-1}) = 0, \quad \text{by K-V vanishing,}$$

$$H^2(L \otimes B_1^{\otimes m-2} \otimes \delta^{-2}) = H^2(K_X \otimes B_1^{\otimes m-1} \otimes B_2 \otimes \delta^{-2}).$$

If $m = 0$, then $H^2(K_X \otimes B_1^{-1} \otimes B_2 \otimes \delta^{-2}) = H^2(K_X \otimes E^{\otimes 2} \otimes \delta^{-2}) = H^0(E^{-2} \otimes \delta^{\otimes 2}) = 0$ because $E^{-2} \otimes \delta^{\otimes 2} \neq \mathcal{O}_X$ is numerically trivial.

If $m > 0$, then $H^2(K_X \otimes B_1^{\otimes m-1} \otimes B_2 \otimes \delta^{-2}) = H^2(K_X \otimes B^{\otimes m} \otimes \delta_1) = 0$ by K-V vanishing (δ_1 is a numerically trivial line bundle).

Second, let us show that the following map is surjective for $m, r \geq 0$:

$$H^0(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r}) \otimes H^0(B_2 \otimes \delta) \rightarrow H^0(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r+1} \otimes \delta). \tag{10}$$

By Lemma 2.3, we need $H^1(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r-1} \otimes \delta^{-1}) = 0$ and $H^2(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r-2} \otimes \delta^{-2}) = 0$.

$$H^1(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r-1} \otimes \delta^{-1}) = H^1(K_X \otimes B_1^{\otimes m+1} \otimes B_2^{\otimes r} \otimes \delta^{-1}) = 0, \text{ by K-V vanishing,}$$

$$H^2(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r-2} \otimes \delta^{-2}) = H^2(K_X \otimes B_1^{\otimes m+1} \otimes B_2^{\otimes r-1} \otimes \delta^{-2}).$$

If $m = r = 0$, then $H^2(K_X \otimes B_1^{\otimes m+1} \otimes B_2^{-1} \otimes \delta^{-2}) = H^2(K_X \otimes E^{\otimes 2} \otimes \delta^{-2}) = H^0(E^{-2} \otimes \delta^{\otimes 2}) = 0$ because as above $E^{-2} \otimes \delta^{\otimes 2} \neq \mathcal{O}_X$ is numerically trivial.

If $m + r > 0$, then $H^2(K_X \otimes B_1^{\otimes m+1} \otimes B_2^{\otimes r-1} \otimes \delta^{-2}) = H^2(K_X \otimes B^{\otimes m+r} \otimes \delta_1) = 0$ by K-V vanishing (δ_1 is a numerically trivial line bundle).

Finally, we will prove the following map is surjective:

$$H^0(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K_X^{\otimes t}) \otimes H^0(K_X \otimes \delta) \rightarrow H^0(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K_X^{\otimes t+1} \otimes \delta). \quad (11)$$

By Lemma 2.3, we need $H^1(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K_X^{\otimes t-1} \otimes \delta^{-1}) = 0$ and $H^2(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K_X^{\otimes t-2} \otimes \delta^{-2}) = 0$.

$H^1(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K_X^{\otimes t-1} \otimes \delta^{-1}) = H^1(K_X^{\otimes t} \otimes B_1^{\otimes m+1} \otimes B_2^{\otimes r+1} \otimes \delta^{-1})$. If $t = 0$, it is zero by Lemma 4.3. If $t > 0$, it is zero by K-V vanishing.

$$H^2(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K_X^{\otimes t-2} \otimes \delta^{-2}) = H^2(K_X^{\otimes t-1} \otimes B_1^{\otimes m+1} \otimes B_2^{\otimes r+1} \otimes \delta^{-2}).$$

If $t = 0$, then H^2 is zero by Lemma 4.5. If $t = 1$ it is zero by Lemma 4.3. If $t > 0$, it is zero by K-V vanishing.

The proof is now complete by the surjectivity of (9), (10) and (11), and Lemma 2.1. \square

Note that (9) shows that L is globally generated (taking $\delta = \mathcal{O}_X$ and applying Lemma 2.2). So we can define $M^{(1),L}$.

Theorem 4.7. *The statements (V_h) and (S_h) hold for all $h \geq 0$.*

Proof. The proof is by induction on h . Both statements hold when $h = 0$ by Lemma 4.6. Suppose that the statements hold for all nonnegative integers $\leq h - 1$ for some $h \geq 1$.

Proving that (V_h) holds is easy: By Remark 2.4, (V_h) follows if

- (i) $H^0(M^{(h-1),L}) \otimes H^0(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t} \otimes \delta) \rightarrow H^0(M^{(h-1),L} \otimes L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t} \otimes \delta)$ is surjective, and
- (ii) $H^1(B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t} \otimes \delta) = 0$.

(ii) follows by K-V vanishing. (i) is simply the statement (S_{h-1}) .

To prove (S_h) , we will need to do some work. We are going to use Lemma 2.1 iteratively. Lemma 2.1 allows us to prove the surjectivity separately for B_1 , B_2 and K_X , as in Lemma 4.6. We will deal with these three cases in the three lemmas that follow.

First, we prove the following.

Lemma 4.8. *The multiplication map*

$$H^0(M^{(h),L} \otimes B_1^{\otimes m}) \otimes H^0(B_1 \otimes \delta) \rightarrow H^0(M^{(h),L} \otimes B_1^{\otimes m+1} \otimes \delta)$$

is surjective for $m \geq 0$.

Proof. We use Lemma 2.3. We need the following two statements for $m \geq 0$:

$$H^1(M^{(h),L} \otimes B_1^{\otimes m-1} \otimes \delta^{-1}) = H^1(M_{M^{(h-1),L}} \otimes L \otimes B_1^{\otimes m-1} \otimes \delta^{-1}) = 0, \quad (12)$$

$$H^2(M^{(h),L} \otimes B_1^{\otimes m-2} \otimes \delta^{-2}) = H^2(M_{M^{(h-1),L}} \otimes L \otimes B_1^{\otimes m-2} \otimes \delta^{-2}) = 0. \quad (13)$$

By Remark 2.4, (12) follows if $H^1(L \otimes B_1^{\otimes m-1}) = 0$ and if the following map is surjective:

$$H^0(M^{(h-1),L}) \otimes H^0(L \otimes B_1^{\otimes m-1}) \rightarrow H^0(M^{(h-1),L} \otimes L \otimes B_1^{\otimes m-1}).$$

The H^1 is zero by K-V vanishing and the surjectivity is simply (S_{h-1}) .

By Remark 2.5, (13) follows if $H^1(M^{(h-1),L} \otimes L \otimes B_1^{\otimes m-2} \otimes \delta^{-2}) = 0$ and $H^2(L \otimes B_1^{\otimes m-2} \otimes \delta^{-2}) = 0$.

The H^1 vanishes by (V_{h-1}) . If $m = 0$, $H^2(K_X \otimes E^{\otimes 2} \otimes \delta^{-2}) = 0$, as in the proof of Lemma 4.6. H^2 is zero by K-V vanishing. If $m > 0$, then H^2 is zero by K-V vanishing. \square

Lemma 4.8 implies that $M^{(h),L}$ is globally generated (by Lemma 2.2) for all $h \geq 0$.

Now to the next step:

Lemma 4.9. *The multiplication map*

$$H^0(M^{(h),L} \otimes B_1^{\otimes m} \otimes B_2^{\otimes r}) \otimes H^0(B_2 \otimes \delta) \rightarrow H^0(M^{(h),L} \otimes B_1^{\otimes m} \otimes B_2^{\otimes r+1} \otimes \delta)$$

is surjective for $m, r \geq 0$.

Proof. According to Lemma 2.3, we need the following two statements for $m, r \geq 0$:

$$H^1(M_{M^{(h-1),L}} \otimes L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r-1} \otimes \delta^{-1}) = 0, \tag{14}$$

$$H^2(M_{M^{(h-1),L}} \otimes L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r-2} \otimes \delta^{-2}) = 0. \tag{15}$$

By Remark 2.4, (14) follows if $H^1(L \otimes B_1^{\otimes m} \otimes B_2^{r-1} \otimes \delta^{-1}) = 0$ and if the following map is surjective:

$$H^0(M^{(h-1),L}) \otimes H^0(L \otimes B_1^{\otimes m} \otimes B_2^{r-1} \otimes \delta^{-1}) \rightarrow H^0(M^{(h-1),L} \otimes L \otimes B_1^{\otimes m} \otimes B_2^{r-1} \otimes \delta^{-1}).$$

The H^1 is zero by K-V vanishing and the surjectivity is simply (S_{h-1}) .

By Remark 2.5, (15) follows if $H^1(M^{(h-1),L} \otimes L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r-2} \otimes \delta^{-2}) = 0$ and $H^2(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r-2} \otimes \delta^{-2}) = 0$.

The H^1 vanishes by (V_{h-1}) . If $m = r = 0$, then H^2 is zero as in Lemma 4.9. If $m + r > 0$, then H^2 is zero by K-V vanishing. \square

Finally we have the following.

Lemma 4.10. *The multiplication map*

$$H^0(M^{(h),L} \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t}) \otimes H^0(K \otimes \delta) \rightarrow H^0(M^{(h),L} \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t+1} \otimes \delta)$$

is surjective for $m, r, t \geq 0$.

Proof. According to Lemma 2.3, we need the following two statements for $m, r, t \geq 0$:

$$H^1(M_{M^{(h-1),L}} \otimes L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t-1} \otimes \delta^{-1}) = 0, \tag{16}$$

$$H^2(M_{M^{(h-1),L}} \otimes L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t-2} \otimes \delta^{-2}) = 0. \tag{17}$$

By Remark 2.4, (16) follows if $H^1(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t-1} \otimes \delta^{-1}) = 0$ and if the following map is surjective:

$$\begin{aligned}
 &H^0(M^{(h-1),L}) \otimes H^0(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t-1} \otimes \delta^{-1}) \\
 &\rightarrow H^0(M^{(h-1),L} \otimes L \otimes B_1^{\otimes m} \otimes B_2^{r-1} \otimes \delta^{-1}).
 \end{aligned}$$

The surjectivity is simply (S_{h-1}) . For $t \geq 1$, the H^1 is zero by K-V vanishing. For $t = 0$, the vanishing follows from Lemma 4.4.

By Remark 2.5, (17) follows if $H^1(M^{(h-1),L} \otimes L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t-1} \otimes \delta^{-2}) = 0$ and $H^2(L \otimes B_1^{\otimes m} \otimes B_2^{\otimes r} \otimes K^{\otimes t-1} \otimes \delta^{-2}) = 0$.

The H^1 vanishes by (V_{h-1}) . For $t \geq 2$, H^2 is zero by K-V vanishing; for $t = 1$, H^2 vanishes by Lemma 4.4; for $t = 0$, it vanishes by Lemma 4.5. \square

(S_h) follows now from Lemma 2.1 and Lemmas 4.8, 4.9, 4.10. \square

Theorem 4.11. *Let X be a nonsingular projective minimal irregular surface of general type over \mathbb{C} . Suppose that the canonical divisor K_X of X is base point free. Let B be a base point free ample divisor on X such that B' is base point free for all $B' \equiv B$ and $H^1(B') = 0$. Assume that $B^2 > B \cdot K_X$. Let $L = K_X \otimes B^{\otimes n}$, where $n \geq 2$. Then $R(L)$ is a Koszul ring.*

Proof. By Theorem 4.7, (V_h) and (S_h) hold for all $h \geq 0$.

If $s > 0$, then taking $s = m = r = t$ and $\delta = \mathcal{O}_X$, (V_h) gives us $H^1(M^{(h),L} \otimes L^{\otimes s}) = 0$.

If $s = 0$, we need to prove that $H^1(M^{(h),L}) = 0$ for $h \geq 0$. If $h = 0$ this follows by K-V vanishing. Suppose that $h > 0$. We need to prove that $H^1(M^{(h),L}) = H^1(M_{M^{(h-1),L}} \otimes L) = 0$. By Remark 2.4, this follows if the multiplication map

$$H^0(M^{(h-1),L}) \otimes H^0(L) \rightarrow H^0(M^{(h-1),L} \otimes L)$$

is surjective and if $H^1(L) = 0$. Surjectivity is simply the statement (S_{h-1}) and $H^1(L) = 0$ by K-V vanishing.

This implies $H^1(M^{(h),L} \otimes L^{\otimes s}) = 0$ for $h \geq 0$ and $s \geq 0$, thereby proving that $R(L)$ is Koszul, by Proposition 1.1. \square

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