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An improved stability criterion for a class of neutral differential equations

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ABSTRACT

This work gives an improved criterion for asymptotical stability of a class of neutral differential equations. By introducing a new Lyapunov functional, we avoid the use of the stability assumption on the main operators and derive a novel stability criterion given in terms of a LMI, which is less restricted than that given by Park [J.H. Park, Delay-dependent criterion for asymptotic stability of a class of neutral equations, Appl. Math. Lett. 17 (2004) 1203–1206] and Sun et al. [Y.G. Sun, L. Wang, Note on asymptotic stability of a class of neutral differential equations, Appl. Math. Lett. 19 (2006) 949–953].

1. Introduction

We are concerned with the asymptotic stability of the following neutral differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+px(t-\tau)] = -ax(t) + b\tanh x(t-\sigma), \quad t \geq 0, \tag{1.1}$$

where a, τ and σ are positive constants, $\sigma \geq \tau$, b, p are real numbers, $|p| < 1$. For each solution $x(t)$ of Eq. (1.1), we assume the initial condition

$$x(t) = \phi(t), \quad t \in [-\sigma, 0],\ \phi \in C([-\sigma, 0], R).$$

Recently, the asymptotic stability of neutral differential equation (1.1) has been discussed in [1,4,5]. Using stability assumption on the operator

$$\mathcal{D}(x_t) = x(t) + px(t-\tau) + b\int_{t-\sigma}^{t} \tanh x(s)\mathrm{d}s,$$

Park in [4] has proposed a delay-dependent criterion on σ for asymptotical stability of Eq. (1.1). Lately, constructing an improved Lyapunov function based on the operator

$$\mathcal{D}^*(x_t) = x(t) + px(t-\tau) + \alpha\int_{t-\tau}^{t} x(s)\mathrm{d}s + b\int_{t-\sigma}^{t} \tanh x(s)\mathrm{d}s,$$

Sun et al. in [4] have derived a less conservative stability criterion. However, this condition still depends on the stability of the operator $\mathcal{D}^*$. Our aim is to improve these criteria and to break away from some of the assumptions of previous papers. Compared with the results in [4,5], our result has the following advantages:

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- First, we introduce a new Lyapunov functional and avoid the use of the stability assumption on the operator $\mathcal{D}$ in [4] or $\mathcal{D}^*$ in [5]. Hence, our criterion will be less restricted than [4,5] (see Example 2).
- Second, using the operator

$$\mathcal{D}_1(x_t) = x(t) + px(t-\tau) + \alpha \int_{t-\tau}^{t} x(s)\mathrm{d}s + b\int_{t-\sigma}^{t-\tau} \tanh x(s)\mathrm{d}s,$$

the obtained condition depends not only on τ, but also on $\sigma - \tau$. Thus, our criterion is more effective than [5] (see Example 1).

2. Main result

Throughout this section, the symbol $\star$ represents the elements below the main diagonal of a symmetric matrix. Also, $X > Y$ means $X - Y$ is positive definite.

Before present the main result, we need the following technical lemmas.

Lemma 1 (*Completing the Square*). *Assume that $S \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Then for every $Q \in \mathbb{R}^{n\times n}$,*

$$2\langle Qy, x\rangle \le \langle QS^{-1}Q^T x, x\rangle + \langle Sy, y\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

Lemma 2 (*[2]*). *For any symmetric positive definite matrix $M \in \mathbb{R}^{n\times n}$, scalar $\sigma \ge 0$ and vector function $w : [0, \sigma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then*

$$\left(\int_0^\sigma w(s)\mathrm{d}s\right)^T M \left(\int_0^\sigma w(s)\mathrm{d}s\right) \le \sigma \int_0^\sigma w^T(s)Mw(s)\mathrm{d}s.$$

For a real number α, Eq. (1.1) can be rewritten as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) + px(t-\tau) + \alpha\int_{t-\tau}^{t} x(s)\mathrm{d}s + b\int_{t-\sigma}^{t-\tau}\tanh x(s)\mathrm{d}s\right] = (\alpha - a)x(t) - \alpha x(t-\tau) + b\tanh x(t-\tau), \quad t \ge 0. \tag{2.1}$$

Define the following operators:

$$\mathcal{D}_1(x_t) = x(t) + px(t-\tau) + \alpha\int_{t-\tau}^{t} x(s)\mathrm{d}s + b\int_{t-\sigma}^{t-\tau}\tanh x(s)\mathrm{d}s, \qquad \mathcal{D}_2(x_t) = x(t) + px(t-\tau).$$

Now, we have the following theorem.

Theorem 1. *The zero solution of Eq. (1.1) is uniformly asymptotically stable if there exist a constant α with $0 < |\alpha| < 1$ and the positive scalars $\beta, \gamma, \eta, \theta$ such that the following linear matrix inequality holds:*

$$\Omega = \begin{pmatrix} \Omega_{11} & p(\alpha - a) - \alpha & b & \alpha - a & b(\alpha - a) \\ \star & -\beta - 2p\alpha & pb & -\alpha & -b\alpha \\ \star & \star & \theta(\sigma-\tau)^2 - \eta & b & b^2 \\ \star & \star & \star & -\gamma & 0 \\ \star & \star & \star & \star & -\theta \end{pmatrix} < 0, \tag{2.2}$$

where $\Omega_{11} = 2(\alpha - a) + \beta + \gamma\tau^2 + \eta$.

Proof. Since $\Omega < 0$, there is a number $\delta > 0$ such that

$$\Omega_1 = \begin{pmatrix} \Omega_{11} & p(\alpha - a) - \alpha & b & \alpha - a & b(\alpha - a) \\ \star & -\beta - 2p\alpha & pb & -\alpha & -b\alpha \\ \star & \star & \theta(\sigma-\tau)^2 - \eta + \delta & b & b^2 \\ \star & \star & \star & -\gamma & 0 \\ \star & \star & \star & \star & -\theta \end{pmatrix} < 0. \tag{2.3}$$

Consider the following Lyapunov functional:

$$V = V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7,$$

where $V_1 = \mathcal{D}_1^T(x_t)\mathcal{D}_1(x_t)$, $V_2 = \beta\int_{t-\tau}^{t} x^2(s)\mathrm{d}s$, $V_3 = \gamma\tau\int_{t-\tau}^{t}(\tau - t + s)(\alpha x(s))^2\mathrm{d}s$, $V_4 = \theta(\sigma-\tau)\int_{t-\sigma}^{t-\tau}(s - t + \sigma)\tanh^2 x(s)\mathrm{d}s$, $V_5 = \eta\int_{t-\tau}^{t}\tanh^2 x(s)\mathrm{d}s$, $V_6 = \delta\int_{t-\sigma}^{t-\tau}\tanh^2 x(s)\mathrm{d}s$ $V_7 = \epsilon\mathcal{D}_2^2(x_t)$, ϵ is a positive number that will be chosen later.

We first take the derivative of $V^* = V_1 + V_2 + V_3 + V_4 + V_5 + V_6$ along the solution of Eq. (1.1):

$$\begin{aligned}\frac{dV^*}{dt} &= 2\left(x(t) + px(t-\tau) + \alpha\int_{t-\tau}^{t} x(s)ds + b\int_{t-\sigma}^{t}\tanh x(s)ds\right)^T((\alpha - a)x(t) - \alpha x(t-\tau) + b\tanh x(t-\tau))\\ &\quad + \beta x^2(t) - \beta x^2(t-\tau) + \gamma\tau^2\alpha^2x^2(t) - \gamma\tau\int_{t-\tau}^{t}(\alpha x(s))^2ds + \theta(\sigma-\tau)^2\tanh^2 x(t-\tau)\\ &\quad - \theta(\sigma-\tau)\int_{t-\sigma}^{t-\tau}\tanh^2 x(s)ds + \eta\tanh^2 x(t) - \eta\tanh^2 x(t-\tau) + \delta\tanh^2 x(t-\tau) - \delta\tanh^2 x(t-\sigma).\end{aligned}$$

Using Lemma 2 and $\tanh^2 x(t) \le x^2(t)$, we have

$$\begin{aligned}\frac{dV^*}{dt} &\le 2\left(x(t) + px(t-\tau) + \alpha\int_{t-\tau}^{t} x(s)ds + b\int_{t-\sigma}^{t}\tanh x(s)ds\right)^T((\alpha - a)x(t) - \alpha x(t-\tau) + b\tanh x(t-\tau))\\ &\quad + \beta x^2(t) - \beta x^2(t-\tau) + \gamma\tau^2\alpha^2x^2(t) - \gamma\left(\int_{t-\tau}^{t}\alpha x(s)ds\right)^2 + \theta(\sigma-\tau)^2\tanh^2 x(t-\tau)\\ &\quad - \theta\left(\int_{t-\sigma}^{t-\tau}\tanh x(s)ds\right)^2 + \eta x^2(t) - \eta\tanh^2 x(t-\tau) + \delta\tanh^2 x(t-\tau) - \delta\tanh^2 x(t-\sigma).\end{aligned}$$

Setting $\alpha\int_{t-\tau}^{t} x(s)ds = u(t)$, $\int_{t-\sigma}^{t-\tau}\tanh x(s)ds = v(t)$ and using $0 < |\alpha| < 1$, we have

$$\begin{aligned}\frac{dV^*}{dt} &\le [2(\alpha - a) + \beta + \gamma\tau^2 + \eta]x^2(t) + [2\alpha + 2p(\alpha - a)]x(t)x(t-\tau)\\ &\quad + 2bx(t)\tanh x(t-\tau) + 2(\alpha - a)x(t)u(t) + [2(\alpha - a)b]x(t)v(t) - (2p\alpha + \beta^2)x^2(t-\tau)\\ &\quad + 2pbx(t-\tau)\tanh x(t-\tau) - 2\alpha x(t-\tau)u(t) - 2\alpha bx(t-\tau)v(t) + [\theta(\sigma-\tau)^2 - \eta + \delta]\tanh^2 x(t-\tau)\\ &\quad - \delta\tanh^2 x(t-\sigma) + 2b\tanh x(t-\tau)u(t) + 2b^2\tanh x(t-\tau)v(t) - \gamma u^2(t) - \theta v^2(t)\\ &\le \zeta^T(t)\Sigma\zeta(t),\end{aligned}$$

where

$$\zeta(t) = \begin{pmatrix} x(t)\\ x(t-\tau)\\ \tanh x(t-\tau)\\ u(t)\\ v(t)\\ \tanh x(t-\sigma)\end{pmatrix},$$

$$\Sigma = \begin{pmatrix} \Omega_{11} & p(\alpha-a)ap - \alpha & b & \alpha - a & b(\alpha - a) & 0\\ \star & -\beta - 2p\alpha & pb & -\alpha & -b\alpha & 0\\ \star & \star & \theta(\sigma-\tau)^2 - \eta + \delta & b & b^2 & 0\\ \star & \star & \star & -\gamma & 0 & 0\\ \star & \star & \star & \star & -\theta & 0\\ \star & \star & \star & \star & \star & -\delta\end{pmatrix}.$$

Using the condition (2.3), we have $\Sigma < 0$ and

$$\frac{dV^*}{dt} \le \zeta^T(t)\Sigma\zeta(t) < 0.$$

Thus, there is a positive number λ such that

$$\frac{dV^*}{dt} \le -\lambda\left(\|x(t)\|^2 + \|x(t-\tau)\|^2 + \|\tanh x(t-\tau)\|^2 + \|u(t)\|^2 + \|v(t)\|^2 + \|\tanh x(t-\sigma)\|^2\right).$$

Now taking the derivative of V_7 and using Lemma 1, we have

$$\begin{aligned}\frac{dV_7}{dt} &= 2\epsilon[x(t) + px(t-\tau)][-ax(t) + b\tanh x(t-\sigma)]\\ &= -2\epsilon ax^2(t) + 2\epsilon bx(t)\tanh x(t-\sigma) - 2\epsilon apx(t)x(t-\tau) + 2\epsilon bpx(t-\tau)\tanh x(t-\sigma)\\ &\le \epsilon(-2a + |b|^2 + |ap|^2)x^2(t) + \epsilon(1 + |bp|^2)x^2(t-\tau) + 2\epsilon\tanh^2 x(t-\sigma).\end{aligned}$$

Choosing

$$\epsilon = \begin{cases} \dfrac{\lambda}{2}\min\left\{\dfrac{1}{1+|bp|^2}, \dfrac{1}{2}\right\}, & \text{if } -2a + |b|^2 + |ap|^2 \le 0\\ \dfrac{\lambda}{2}\min\left\{\dfrac{1}{-2a+|b|^2+|ap|^2}, \dfrac{1}{1+|bp|^2}, \dfrac{1}{2}\right\} & \text{if } -2a + |b|^2 + |ap|^2 > 0,\end{cases}$$

we obtain that $\frac{dV}{dt} < -\frac{\lambda}{2}\|x(t)\|^2$. On the other hand, we have $V \geq \epsilon \mathcal{D}_2^2 x(t)$ and $\mathcal{D}_2 x(t)$ is stable due to $|p| < 1$. Therefore, according to Theorem 8.1 in [3] (pp. 292–293), the zero solution of Eq. (1.1) is uniformly asymptotically stable. The proof of the theorem is completed. □

Remark 1. Note that the stability criterion of (1.1) obtained in [5] is given as

$$\begin{pmatrix} -2a+2\alpha+\beta+\gamma\tau^2+\eta & p(\alpha-a)-\alpha & b & \alpha-a & b(\alpha-a) \\ \star & -\beta-2p\alpha & pb & -\alpha & -b\alpha \\ \star & \star & \theta\sigma^2-\eta & b & b^2 \\ \star & \star & \star & -\gamma & 0 \\ \star & \star & \star & \star & -\theta \end{pmatrix} < 0. \tag{2.4}$$

To compare with the stability criterion obtained in [4], we consider the case $\alpha = 0$. In this case, we have

$$\mathcal{D}_1(x_t) = x(t) + px(t-\tau) + b\int_{t-\sigma}^{t-\tau} \tanh x(s)\mathrm{d}s, \qquad V_3 = 0.$$

By the same arguments used in the proof of Theorem 1 with the Lyapunov functional chosen as

$$V = V_1 + V_2 + V_4 + V_5 + V_6 + V_7$$

we can derive the following stability criterion.

Corollary 1. *The zero solution of Eq. (1.1) is uniformly asymptotically stable if there exist the positive scalars $\beta, \gamma, \theta, \eta$ such that the following linear matrix inequality holds:*

$$\overline{\Omega} = \begin{pmatrix} -2a+\beta+\eta & -ap & b & -ab \\ \star & -\beta & pb & 0 \\ \star & \star & \theta(\sigma-\tau)^2-\eta & b^2 \\ \star & \star & \star & -\theta \end{pmatrix} < 0. \tag{2.5}$$

Remark 2. In [1], the delay-independent sufficient condition for the stability of the Eq. (1.1) is

$$1 + \frac{pa^2+b^2}{2(1-pb)} - a < 0 \quad \text{and} \quad pb < 1. \tag{2.6}$$

To show the advantages of our criterion, we consider the following examples:

Example 1. Consider the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + 0.6x(t-0.15)] = -0.6x(t) + 0.493\tanh x(t-0.3). \tag{2.7}$$

By solving the LMI (2.2) with respect to $\alpha, \beta, \gamma, \theta, \eta$, we obtain its solution:

$$\beta = 1.3947, \qquad \gamma = 9.0979, \qquad \theta = 4.4533, \qquad \eta = 0.7499, \qquad \alpha = -0.8718,$$

which implies the asymptotical stability of Eq. (2.7). However, the LMI (2.4) is infeasible, that is the criterion of [5] cannot be applied. Moreover, the criterion obtained in [1] cannot be applied since the inequality (2.6) does not hold. This shows that our criterion is more effective than that obtained in [1,5].

Example 2. Consider the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - 0.7x(t-9.9)] = -0.6x(t) - 0.3\tanh x(t-10). \tag{2.8}$$

By solving the LMI (2.5) with respect to $\beta, \gamma, \theta, \eta$, we obtain its solution:

$$\beta = 0.3923, \qquad \gamma = 6.1108, \qquad \theta = 4.9347, \qquad \eta = 0.2840,$$

which implies the asymptotical stability of Eq. (2.8). However, the criterion obtained in [4] cannot be applied since the operator $\mathcal{D}$ used in [4] is unstable. Indeed, we consider the equation $\mathcal{D}x_t = 0$ with initial condition

$$x(s) = c = \text{constant} > 0, \quad s \in [-10, 0].$$

If $c \geqslant 1$ then $\tanh(x(s)) > \frac{1}{2}, \quad \forall s \in [-10, 0]$ and hence $x(t) \geqslant 0.7c + 1.5 > 1, \forall t \in (0, 10]$. Similarly, we also have $x(t) \geqslant 1, \forall t \in (10, 20]$. Continuing the progress, we will have $x(t) \geqslant 1, \forall t \in (0, +\infty)$. If $0 < c < 1$ then $\tanh(x(s)) > \frac{x(s)}{2} = \frac{c}{2}$ and hence $x(t) \geqslant 0.7c + 1.5c, \forall t \in (0, 10]$. Similarly, we also have $x(t) \geqslant 0.7c + 2 \times 1.5c, \forall t \in (10, 20]$. Since $c > 0$, there is a natural number $n_0 \in N$ such that $0.7c + (n_0 - 1) \times 1.5c > 1$. Then, we will have $x(t) \geqslant 0.7c + n_0 \times 1.5c > 1, \forall t \in (10n_0, 10(n_0+1)]$. Similarly to above, we have $x(t) \geqslant 1, \forall t \in [10n_0, +\infty)$. Thus, the operator $\mathcal{D}$ is unstable.

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