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ON EXTENDING CONTINUOUS FUNCTIONS INTO A METRIZABLE AE

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We say that a subset S of a topological space X is M-embedded ($M^{\aleph_0}$-embedded) in X if every map from S to a (separable) metrizable AE can be extended over X. Characterizations of M- and $M^{\aleph_0}$-embedding are given and we prove that S is M-embedded ($M^{\aleph_0}$-embedded) in X iff (X, S) has the Homotopy Extension Property with respect to every (separable) ANR space.

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homotopy extension property

Let S be a subset of a topological space X. We say that S is P^{γ}-embedded in X (where γ is an infinite cardinal number), if every continuous γ-separable pseudometric on S extends to a continuous pseudometric on X. It is known that X is γ-collectionwise normal iff every closed subset is P^{γ}-embedded in X, that $P^{\aleph_0}$-embedding is equivalent to C-embedding and that S is P^{γ}-embedded in X iff every continuous function from S to a complete γ-separable metrizable AE extends to X. For information on these concepts see [3].

In this paper we define S to be M^{γ}-embedded (M-embedded) in X if every continuous function from S to a γ-separable metrizable (metrizable) AE extends to a continuous function on X. (Throughout this paper an AE, AR, or ANR means an AE, AR, or ANR for metric spaces.) We give necessary and sufficient conditions for S to be M^{γ}-embedded in X (one of which isolates what must be added to P^{γ}-embedding to produce M^{γ}-embedding). Several new classes of spaces with various subsets possessing this property are given. In particular it is shown that if X is paracompact (and completely regular T_1), then any closed, topologically complete subset of X is M-embedded in X. Finally, we give two applications. The first gives necessary and sufficient conditions for a metrizable subset of a topological space to possess a certain Dugundji extension property. The second involves a generalization of the Morita–Hoshina Homotopy Extension Theorem ([11], [12]). Namely, we prove: S is M^{γ}-embedded in X iff (X, S) has the HEP with respect to every ANR space of weight $\leq \gamma$.

In what follows, S will always denote a subspace of a topological space X and γ will denote an infinite cardinal number. No separation axioms are assumed unless stated.

Theorem 1. *The following are equivalent:*

(1) *S is M^γ-embedded in X.*

(2) *Every continuous function from S to a γ-separable metrizable convex subset K of a locally convex topological vector space extends to X with values in K.*

(3) *Given a continuous γ-separable pseudometric d on S, there is a continuous extension d^* of d to X such that $x \in d^*$-closure of S implies the existence of $x_0 \in S$ such that $d^*(x, x_0) = 0$.*

(4) *Given a continuous γ-separable pseudometric d on S, there exists a continuous extension d^* of d to X such that (S, d) is a retract (in the d^*-topology) of the d^*-closure of S.*

(5) *S is P^γ-embedded in X and given a continuous γ-separable pseudometric d on X, there exists a zero-set Z of X such that*

$$S \subset Z \subset \{x \in X : \exists x_0 \in S \text{ such that } d(x, x_0) = 0\}.$$

(6) *S is P^γ-embedded in X and given a continuous function f from X into a γ-separable metric space, there exists a zero-set Z of X such that $S \subset Z \subset f^{-1}(f(S))$.*

(7) *Every continuous function from S to a γ-separable normed linear space extends to X.*

Proof. Clearly (5) and (6) are the most interesting characterizations as they show what must be added to P^γ-embedding to produce M^γ-embedding. (1) implies (2) is clear as any such K is a metrizable AE. To show that (2) implies (3), let d be a γ-separable continuous pseudometric on S and define f from S into $C^*(S, d)$ by $f(x) = d_x - d_a$, where a is a fixed point of S and $d_x(y) = d(x, y)$. Let L denote the convex hull of $f(S)$. Then L is γ-separable, hence f extends to f^* on X with $f^*(X) \subset L$. Define $d^*(x, y) = \|f^*(x) - f^*(y)\|$. Then d^* is a continuous pseudometric on X that extends d. Let x_0 be in the d^*-closure of S and let (x_n) be a sequence in S converging to x_0 in the d^*-topology. If

$$f^*(x_0) = \sum_{i=1}^{n} r_i f(y_i),$$

where $r_i > 0$, $\sum_{i=1}^{n} r_i = 1$ and $y_i \in S$, then we have

$$\left| d(x_n, z) - \sum_{i=1}^{n} r_i d(y_i, z) \right|$$

converges to 0 as $n \to \infty$ for each $z \in S$. Let $z = x_m$, and let $n, m \to \infty$. Since $d(x_n, x_m)$ converges to 0, we must have $\sum_{i=1}^{n} r_i d(y_i, x_m)$ converging to 0 as $m \to \infty$. But this implies for each i that $d(y_i, x_m)$ converges to 0, hence $d^*(x_0, y_i) = 0$.

The implication (3) implies (4) is clear. To show that (4) implies (5), note that S is

P^γ-embedded in X. Let d be a continuous γ-separable pseudometric on X and let e denote the restriction of d to S. By (4), there exists a continuous extension e^* of e such that the e^*-closure of S retracts onto (S, e). Let r denote a retraction map and let $f(x)$ be the $(e^* \vee d)$-distance from x to S. Clearly $S \subset Z(f)$. Let $f(x) = 0$. This implies the existence of a sequence (x_n) in S such that (x_n) converges to x in both the e^* and d topologies. Since r is an e^* continuous function, we have that (x_n) converges to $r(x)$ in the e topology. We claim that $d(x, r(x)) = 0$. This follows from the inequalities:

$$d(x, r(x)) \leqslant d(x, x_n) + d(x_n, r(x)) = d(x, x_n) + e(x_n, r(x)).$$

It is easy to see that (5) and (6) are equivalent. To show that (5) implies (1), let f be a continuous function from S into a metric space (M, m) of weight $\leqslant \gamma$ that is an AE. Define $d(x, y) = m(f(x), f(y))$. Then d is a continuous γ-separable pseudometric on S hence has an extension to a continuous γ-separable pseudometric d^* on X [3]. By (5), there is a continuous function g on X, such that

$$S \subset Z(g) \subset \{x \in X : \exists x_0 \in S, \text{ such that } d^*(x, x_0) = 0\}.$$

Let $d' = d^* \vee \Psi_g$ where Ψ_g is the pseudometric on X defined by $\Psi_g(x, y) = |g(x) - g(y)|$. Then d' is a continuous pseudometric on X that extends d. M may be considered a retraction of a convex subset of a Banach space ([7], p. 95, 84). By the Dugundji Extension Theorem ([3], p. 166) it is only necessary to extend f to the d'-closure of S. Let x be in the d'-closure of S. Then one readily checks that $g(x) = 0$, hence there exists $x_0 \in S$ such that $d^*(x, x_0) = 0$. This implies that $d'(x, x_0) = 0$, hence setting $\hat{f}(x) = f(x_0)$ gives the desired extension.

Clearly (1) implies (7). To show that (7) implies (1), let f be a continuous function from S into a metrizable AE M of weight $\leqslant \gamma$. By the Arens–Eells Embedding Theorem [5], M can be embedded as a closed subspace of a normed linear space L. Let L^* denote the span of M in L. Then L^* is γ-separable, and M is closed in L^* (and hence a retract of L^*). This completes the proof.

It can be shown that (5) with the added condition that d be bounded is equivalent to M^γ-embedding. M^γ-embedding is also equivalent to requiring every continuous function from S into a bounded convex subset K of a γ-separable metrizable topological vector space to extend to X with values in K. Removing all cardinality references in the theorem produces characterizations of M-embedding.

Corollary 1. *If S is a P^γ-embedded zero-set of X, then S is M^γ-embedded in X.*

Corollary 2. *Every γ-separable metrizable* AE *is an* AE *for the class of γ-collectionwise normal, perfectly normal spaces.*

Corollary 2 was proven for collectionwise normal, perfectly normal spaces by Pasynkov [13]; earlier Michael ([7], p. 63) proved that metrizable (separable metrizable) AE's are AE's for the class of paracompact, perfectly normal (perfectly normal) spaces.

Corollary 3. *Every Lindelöf zero-set S of a completely regular T_1 space X is M-embedded in X.*

Proof. By Corollary 7.2 [1], S is P-embedded in X. The result follows from Corollary 1.

Corollary 4. *Let X be a completely regular T_1 space such that υX has non-measurable cardinality. Then X is M-embedded in υX.*

Proof. By ([3], p. 187), X is P-embedded in υX. Let d be a continuous pseudometric on υX. We claim that

$$\{x \in \upsilon X : \exists x_0 \in X \text{ such that } d(x, x_0) = 0\} = \upsilon X.$$

For $x \in \upsilon X$, let $f(y) = d(x, y)$. Since every zero-set in υX meets X, there exists $x_0 \in X$ such that $f(x_0) = 0$, i.e. $d(x, x_0) = 0$.

The next corollary will be important later in constructing P-embedded, non-M-embedded subspaces.

Corollary 5. *If X contains a metric topology, then S is M-embedded in X iff S is a P-embedded zero-set of X.*

Proposition 1. *Let S be a P^γ-embedded subset of X with the property that every continuous γ-separable pseudometric on S is majorized by a complete continuous γ-separable pseudometric on S. Then S is M^γ-embedded in X.*

Proof. We will verify (5) of Theorem 1. Let d be a continuous γ-separable pseudometric on X and let e be a complete continuous γ-separable pseudometric on S such that $e \geq d \mid S$. Let e^* be an extension of e to X. Claim that $f(x) = (d \vee e^*)(x, S)$ satisfies the required condition. Suppose the $d \vee e^*$ distance of x to S is 0. There exists a sequence (x_n) in S that converges with respect to $d \vee e^*$ to x. Since (x_n) is e-cauchy, there exists $x_0 \in S$ such that (x_n) converges to x_0 in the e topology. Since e majorizes $d \mid S$, the sequence (x_n) also converges to x_0 in the d topology. Then $d(x, x_0) \leq d(x, x_n) + d(x_n, x_0)$, which implies that $d(x, x_0) = 0$.

One obvious way to attempt to weaken the hypothesis of this result would be to require only that every complete continuous γ-separable pseudometric on S extend to a continuous pseudometric on X. But in [15] we show that this is actually equivalent to P^γ-embedding.

Corollary 1. *If S is a pseudocompact C^*-embedded subset of X, then S is M-embedded in X.*

Proof. It is easy to see that every continuous pseudometric on a pseudocompact space is complete and totally bounded. It is known ([3], p. 208) that C^*-embedding is equivalent to the extendability of every totally bounded continuous pseudometric.

Corollary 2. *Any compact space is absolutely M-embedded, i.e. every embedding in a completely regular T_1 space is an M-embedding.*

Proof. Since any compact space is absolutely C^*-embedded, this follows from Corollary 1.

Corollary 3. *If S is a complete γ-separable metric space that is P^γ-embedded in X, then S is M^γ-embedded in X.*

Proof. If (S, m) is a complete γ-separable metric space and if d is a continuous pseudometric on S, then $d \vee m$ is a complete γ-separable pseudometric majorizing d.

Corollary 4. *If S is a discrete P-embedded subspace, it is M-embedded.*

Corollary 5. *If S is a paracompact, P-embedded, topologically complete subset of a completely regular T_1 space X, then S is M-embedded in X.*

Proof. Z. Frolík has shown that a completely regular T_1 space S is paracompact and topologically complete iff there is a perfect map f from S onto a complete metric space (M, m). (For a proof, see for example [6], p. 73.) Defining $e(x, y) = m(f(x), f(y))$, we see that e is a complete continuous pseudometric on S. If d is any continuous pseudometric on S, then $e \vee d$ is complete. To see this, let (x_n) be an $e \vee d$ cauchy sequence. Then (x_n) is e cauchy and hence converges in the e topology to $x \in S$. We claim that there exists $y \in f^{-1}f(x)$ such that (x_n) converges to y in the d topology. This will complete the proof, since (x_n) will also converge to this y in the e topology. Suppose such a y does not exist. Then for each y such that $f(y) = f(x)$, there exists $\varepsilon(y) > 0$ and $N(y)$ such that $(x_n)_{n \geq N(y)} \subset S - S_d(y, \varepsilon(y))$. Since $f^{-1}f(x)$ is compact, there exists $\{y_i; i, \ldots, n\}$ such that

$$f^{-1}f(x) \subset \bigcup_{i=1}^{n} S_d(y_i, \varepsilon(y_i)) = G.$$

Let $N = \max\{N(y_i): i = 1, \ldots, n\}$ and let $n \geq N$. Then $x_n \notin G$, hence $f(x_n) \notin A = M - f(S - G)$. But A is open in M and contains $f(x)$. Since $(f(x_n))$ converges to $f(x)$, this is a contradiction.

R.A. Alò and H.L. Shapiro [4] have introduced the notion of a subspace S being strongly paracompact in X if every cover of S by sets open in X has a refinement by sets open in X that is locally finite in X. They show that if X is a normal T_1 space and S a subset of X, then S is strongly paracompact in X iff S is paracompact and P-embedded in X. Thus Corollary 5 can be restated.

Corollary 6. *Let X be a normal T_2 space and S a strongly paracompact topologically complete subspace. Then S is M-embedded in X.*

Corollary 7. *If X is a completely regular T_1 paracompact space, then any closed topologically complete subset is M-embedded in X.*

Corollary 8. *If S is a Lindelöf, topologically complete, C-embedded subset of a completely regular T_1 space X, then S is M-embedded in X.*

Proof. In [6], p. 74 it is shown that a completely regular T_1 space S is Lindelöf and topologically complete iff there exists a perfect map from X onto a complete separable metric space. Making the obvious modifications in the proof of Corollary 5, we see that every continuous separable pseudometric on S is majorized by a complete continuous separable pseudometric on S. By Corollary 7.1 of [1] S is P-embedded in X. It is easy to see that every continuous pseudometric on a Lindelöf space is separable. The result now follows from Proposition 1.

J.T. Lisica has shown that any AR is an AE for paracompact p-spaces (those spaces which admit a perfect map to a metric space). (For a proof, see [14].) We give another proof here based on our theorem.

Proposition 2. *Let X be a completely regular T_1 paracompact p-space. Then every closed subset of X is M-embedded in X.*

Proof. Let S be a closed subset of a paracompact p-space X and let φ be a perfect map from X onto the metric space (M, m). Since X is collectionwise normal S is P-embedded in X. Now let d be a continuous pseudometric on X. Define $e(x, y) = m(\varphi(x), \varphi(y))$ and let $f(x)$ be the $e \vee d$-distance from x to S. Clearly $S \subset Z(f)$. Now let $f(x) = 0$. There exists a sequence (x_n) in S converging to x in the e and d topologies. This implies $(\varphi(x_n))$ converges to $\varphi(x)$ in M. But $\varphi(S)$ is closed in M, hence $\varphi(x) \in \varphi(S)$. We claim there exists $y \in S \cap \varphi^{-1}(\varphi(x))$ such that (x_n) converges to y in the d topology. If we can show this, we will have $d(x, y) \leq d(x, x_n) + d(x_n, y)$ which will imply that $d(x, y) = 0$. The proof that such a y exists is similar to the proof in Corollary 5 so we omit it here.

We will generalize an example due to Michael [10] to produce several P-embedded, non-M-embedded subspaces. Let (S, m) be a non-topologically complete metric space with completion (S^*, m^*). Let X denote its completion with the following topology: The basic open sets are of the form $U \cup P$, where U is open in (S^*, m^*) and P is any subset of $X - S$. Let us call X the *Michael modification* of (S, m). One can show that X is hereditarily paracompact, and that S is a closed, non-zero-set of X. Since X is collectionwise normal, S is P-embedded in X. Since X contains a metric topology (that generated by m^*), by Corollary 5 of Theorem 1, S is not M-embedded in X.

The original example given by Michael involved taking S to be the rationals. One can also let S be any normed linear space that is not a Banach space or any metrizable non-complete Abelian topological group [8].

The remainder of the paper deals with two applications of M-embedding. The first concerns a simultaneous extension property. There are many of these properties; the one we will be considering is defined as follows. Let S be a subspace of X and let $C(S)$ denote the vector space of real-valued continuous functions on S, similarly with $C(X)$. We say that there is a *simultaneous linear extender* from $C(S)$ to $C(X)$ if there is a linear function e from $C(S)$ to $C(X)$ assigning to each $f \in C(S)$ an extension of f to X in such a manner that $e(f)(X)$ is contained in the convex hull of $f(S)$. We say that (X, S) has the *Dugundji Extension Property* if every continuous function from S into a convex subset Z of a locally convex topological vector space has an extension to X with values in Z.

Proposition 3. *If (X,S) has the Dugundji Extension Property, then there exists a simultaneous linear extender from $C(S)$ to $C(X)$.*

Proof. Let L be the locally convex topological vector space formed by taking a product of real lines, one for each $f \in C(S)$, and let φ denote the cannonical mapping of S into L. Let Z denote the convex hull of $\varphi(S)$ in L, and let $\hat{\varphi}$ denote an extension of φ to X with values in Z. Defining $e(f)(x) = \hat{\varphi}(x)_f$, one checks that all conditions are satisfied.

It is clear that one could replace L with a product of copies of any locally convex topological vector space E and thus get a simultaneous linear extender from $C(S, E)$ to $C(X, E)$. For further development of this idea see [16].

Proposition 4. *Let S be a metrizable subspace of a topological space X. Then (X, S) has the Dugundji Extension Property iff S is M-embedded in X.*

Proof. Necessity follows from (2) of Theorem 1. To show sufficiency, let f be a continuous function from (S, m) into a convex subset Z of a locally convex topological vector space. Since S is M-embedded in X, there exists an extension m^* of m to X such that $x \in m^*$-closure of S implies there exists $x_0 \in S$ such that $m^*(x, x_0) = 0$. Defining $f^*(x) = f(x_0)$ extends f to the m^*-closure of S in X. The Dugundji Extension Theorem allows us to extend f^* to all of X.

This improves a result of D. Lutzer and H. Martin [9] in which they showed that if S is a closed, metrizable zero-set of a collectionwise normal space X, then there is a simultaneous linear extender from $C(S)$ to $C(X)$. As an example of our result, there exists a simultaneous linear extender from $C(\{\Omega\} \times [0, \omega])$ to $C([0, \Omega] \times [0, \omega])$.

Our second application is a generalization of the Morita–Hoshina generalization of the Borsuk Homotopy Extension Theorem. Namely, they proved (Theorem 3.7 of [12]) that S is P^γ-embedded in X iff (X, S) has the HEP with respect to every complete ANR space with weight $\leq \gamma$. We will show that S is M^γ-embedded in X iff (X, S) has the HEP with respect to every ANR space of weight $\leq \gamma$. The following preliminary results are needed.

Proposition 5. *Let Y be a compact T_2 space of weight $\leq \gamma$. Then S is M^γ-embedded in X iff $S \times Y$ is M^γ-embedded in $X \times Y$.*

Proof. If S is M^γ-embedded in X, then $S \times Y$ is P^γ-embedded in $X \times Y$ by Corollary 3.4 of [2]. Let d be a bounded γ-separable continuous pseudometric on $X \times Y$ and define d^* on X by

$$d^*(x_1, x_2) = \sup\{d((x_1, y), (x_2, y)) : y \in Y\}.$$

It can be shown that d^* is γ-separable by a method similar to that used in the proof of Theorem 3.3 of [2]. Then there exists a continuous function f on X, such that

$$S \subset Z(f) \subset \{x \in X : \exists x_0 \in S, \text{ such that } d^*(x, x_0) = 0\}.$$

Defining g on $X \times Y$ by $g(x, y) = f(x)$, one checks that the necessary conditions in (5) of Theorem 1 are satisfied.

To prove necessity, observe that by Corollary 3.4 of [2], S is P^γ-embedded in X. Let d be a bounded γ-separable continuous pseudometric on X and define d^* on $X \times Y$ by $d^*((x_1, y_1), (x_2, y_2)) = d(x_1, x_2)$. There is a continuous f on $X \times Y$ with $0 \leq f \leq 1$ such that

$$S \times Y \subset Z(f) \subset \{(x, y) \in X \times Y : \exists (x_0, y_0) \in S \times Y,$$
$$\text{such that } d^*((x, y), (x_0, y_0)) = 0\}.$$

Defining g on X by $g(x) = \inf\{f(x, y) : y \in Y\}$ one checks that g has the required properties.

Proposition 6. *Let A and B be M^γ-embedded subsets of X. Then $A \cup B$ is M^γ-embedded iff $A \cup B$ is P^γ-embedded in X iff $A \cup B$ is C^*-embedded in X.*

Proof. It is only necessary to show that if $A \cup B$ is C^*-embedded in X, then it is M^γ-embedded in X. By Theorem 2.9 of [12] we have that $A \cup B$ is P^γ-embedded in X. Let d be a continuous γ-separable pseudometric on X. There exist continuous functions f_1 and f_2 on X, with $0 \leq f_1, f_2 \leq 1$, such that

$$A \subset Z(f_1) \subset \{x \in X : \exists x_0 \in A, \text{ such that } d(x, x_0) = 0\}$$

and

$$B \subset Z(f_2) \subset \{x \in X : \exists x_0 \in B, \text{ such that } d(x, x_0) = 0\}.$$

Then the function $f_1 \wedge f_2$ has the desired properties.

Corollary. *Let A be M^γ-embedded in X and B a P^γ-embedded zero set of X. Then $A \cup B$ is M^γ-embedded in X.*

Proof. This follows from Corollary 1 of Theorem 1, Theorem 2.10 of [12], and Proposition 6.

Theorem 2. *The following are equivalent:*

(1) *S is M^{γ}-embedded in X.*

(2) *$(X \times \{0\}) \cup (S \times I)$ is M^{γ}-embedded in $X \times I$.*

(3) *(X, S) has the* HEP *with respect to every* ANR *space of weight $\leq \gamma$.*

Proof. To prove that (1) implies (2) observe that $S \times I$ is M^{γ}-embedded in $X \times I$, by Proposition 5. Clearly, $X \times \{0\}$ is M^{γ}-embedded in $X \times I$. Hence (2) will follow if we can show that $(X \times \{0\}) \cup (S \times I)$ is P^{γ}-embedded in $X \times I$. But this follows from Theorem 3.7 [12], (1) implies (3). To show that (2) implies (3) we will simply observe that the proof of (3) implies (4) of Theorem 3.7 [12] outlined in Remark 3.8 goes through, where the only change necessary is to assume that Y is an ANR of weight $\leq \gamma$ and Z is the convex hull of Y in L. To show that (3) implies (1), observe that by Theorem 3.7 [12] again we have that S is P^{γ}-embedded in X. Let d be a continuous γ-separable pseudometric on X. Define d^* on $X \times I$ by

$$d^*((x_1, t_1), (x_2, t_2)) = d(x_1, x_2) \vee |t_1 - t_2|.$$

Then d^* is a continuous γ-separable pseudometric on $X \times I$. Clearly, (3) implies that (X, S) has the HEP with respect to every AR space of weight $\leq \gamma$ which implies (X, S) has the HEP with respect to every AE space of weight $\leq \gamma$. This in turn implies that $(X \times \{0\}) \cup (S \times I)$ is M^{γ}-embedded in $X \times I$. Hence there exists a continuous function f on $X \times I$ with $0 \leq f \leq 1$ such that

$$(X \times \{0\}) \cup (S \times I) \subset Z(f) \subset \{(x, t) \in X \times I : \exists (x_0, t_0) \in (X \times \{0\}) \cup (S \times I) \text{ such that } d^*((x, t), (x_0, t_0)) = 0\}.$$

Define g on X by $g(x) = \sup\{f(x, t) : t \in I\}$. It is easy to check that g has the properties required in (5) of Theorem 1.

Observe that Theorem 2 is not the complete analogue of Morita and Hoshina's Theorem 3.7 [12]. The following question remains: If X is a topological space, Y a compact T_2 space and B a closed subset of Y, is $X \times B$ necessarily M-embedded in $X \times Y$? We conjecture that the answer is no.[1]

Setting $\gamma = \aleph_0$ in Theorem 2, we obtain:

Corollary 1. *S is $M^{\aleph_0}$-embedded in X iff (X, S) has the* HEP *with respect to every separable* ANR *space.*

Corollary 2. *S is M-embedded in X iff (X, S) has the* HEP *with respect to every* ANR *space.*

[1] R. Levy, M.D. Rice and the author have shown: If X is a $T_{3\frac{1}{2}}$ space, then X is compact iff whenever X is embedded in Z and Y is any space, then $X \times Y$ is M-embedded in $Z \times Y$.

Let us make the final observation that Theorem 2 is a special case of a theorem of Morita's (Theorem 6 [11]): If S is a P^{γ}-embedded zero-set of a topological space X, then (X, S) has the HEP with respect to every ANR space of weight $\leq \gamma$.

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