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Weighted Subcoercive Operators on Lie Groups

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with their multi-commutators span $\mathfrak{g}$. Let $A_i = dU(a_i)$ denote the infinitesimal generator of the continuous one-parameter group $t \mapsto U(\exp(-ta_i))$ and set $A^\alpha = A_{i_1} \cdots A_{i_n}$ where $\alpha = (i_1, \ldots, i_n)$ with $i_j \in \{1, \ldots, d'\}$. We analyze properties of mth order differential operators

$$dU(C) = \sum_{\alpha;\, |\alpha| \leqslant m} c_\alpha A^\alpha$$

with coefficients $c_\alpha \in \mathbf{C}$.

If L denotes the left regular representation of G in $L_2(G)$ then $dL(C)$ satisfies a Gårding inequality on $L_2(G)$ if, and only if, the closure of each $dU(C)$ generates a holomorphic semigroup S on $\mathcal{X}$, the action of S_z is determined by a smooth, representation independent, kernel K_z which, together with its derivatives $A^\alpha K_z$, satisfies mth order Gaussian bounds and, if U is unitary, S is quasi-contractive in an open representation independent subsector of the sector of holomorphy. Alternatively, $dL(C)$ satisfies a Gårding inequality on $L_2(G)$ if, and only if, the closure of $dL(C)$ generates a holomorphic, quasi-contractive, semigroup satisfying bounds $\|A_i S_t\|_{2\to 2} \leqslant c t^{-1/m} e^{\omega t}$ for all $t > 0$ and $i \in \{1, \ldots, d'\}$.

These results extend to operators for which the directions $a_1, \ldots, a_{d'}$ are given different weights. The unweighted Gårding inequality is a stability condition on the principal part, i.e., the highest-order part, of $dL(C)$ but in the weighted case the condition is on the part of $dL(C)$ with the highest weighted order. © 1998 Academic Press

1. INTRODUCTION

The theory of partial differential operators extends naturally from the Euclidean space $\mathbf{R}^d$ to a general d-dimensional Lie group. The operators are defined in any continuous Banach space representation U of G as polynomials in the associated representatives of the Lie algebra $\mathfrak{g}$ of G.

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Operators formed from the representatives of a vector space basis of $\mathfrak{g}$ with polynomials satisfying the strong ellipticity conditions of the $\mathbf{R}^d$-theory are called strongly elliptic. Langlands, in an unpublished thesis [Lan1] (see also [Lan2]), proved that the closure of each strongly elliptic operator H generates a holomorphic semigroup S with a smooth, fast decreasing, representation-independent, integral kernel K. More recently Bratteli, Goodman, Jørgensen, and Robinson [BGJR] proved that in each unitary representation S is quasi-contractive and H satisfies a Gårding inequality, i.e., a coercivity condition. Conversely a limiting argument of Folland [Fol2] shows that the Gårding inequality for H in the left regular representation of G on $L_2(G)$ implies strong ellipticity. Thus strong ellipticity, or $\mathbf{R}^d$-coercivity, is equivalent to G-coercivity for a partial differential operator H expressed in terms of a vector space basis of $\mathfrak{g}$. These conditions then imply that H is the pregenerator of a semigroup with good boundedness and analyticity properties and a universal "Gaussian" kernel (see, for example, [Rob]). One of our results is a converse of the last conclusion: if H is the pregenerator of a quasi-contractive semigroup on $L_2(G)$ with a good "Gaussian" kernel then H must be a G-coercive operator. Hence one concludes that there is an equivalence between $\mathbf{R}^d$-coercivity, G-coercivity and good semi-group properties. Our main result establishes a similar equivalence for weighted subelliptic operators.

We consider operators H which are polynomials in the representatives of a (Lie)-algebraic basis of $\mathfrak{g}$ with different weights assigned to each of the directions in the basis. The order of H is defined as the weighted order of the polynomial and the weighting is taken into account in the definition of distance etc. Since there is no obvious direct definition of coercivity in terms of the coefficients of the polynomial we introduce a notion of (weighted) G-subcoercivity in terms of a weighted Gårding inequality. We then establish that H is G-subcoercive if, and only if, it generates a holomorphic, quasi-contractive, semigroup on $L_2(G)$ with a universal "Gaussian" kernel. This equivalence encompasses all earlier known results and gives a straightforward characterization of the "heat" semigroups on the Lie group G. The proofs rely on a combination of earlier arguments and two new techniques.

First, we introduce the notion of a reduced weighted algebraic basis. The reduced algebraic basis is an algebraic subbasis of the original algebraic basis in which certain "over-weight" directions have been eliminated. Our strategy is to establish the main structural features for operators defined with a reduced algebraic basis and then to lift the results to operators expressed in terms of the original unreduced basis. If all weights are equal to one, or if the weights satisfy the compatibility conditions used for weighted strongly elliptic operators in [ElR5], then the reduction process has no effect. It is, however, interesting to note that a weighted vector space basis of $\mathfrak{g}$ usually yields a weighted algebraic basis after reduction. Therefore subelliptic techniques automatically enter the analysis of weighted strongly elliptic operators.

Secondly, we associate with each weighted algebraic basis of $\mathfrak{g}$ a homogeneous (nilpotent) group G_0 which serves as a local approximation of G. The group G_0 is uniquely determined by G and the weighted algebraic basis by a canonical contraction of $\mathfrak{g}$. Since the work of Rothschild and Stein [RoS] local nilpotent approximates have become a standard tool. Given an algebraic basis of $\mathfrak{g}$ with d' elements and rank r the Rothschild–Stein approximate $\tilde{G}$ of G is the nilpotent group with d' generators which is free of step r. This group was used in our earlier work on subelliptic operators [ElR3, ElR6]. The disadvantage of the Rothschild–Stein approach is that $\tilde{G}$ is usually of larger dimension than G. But the approximate G_0 used in the current analysis has the same dimension as G and this is advantageous for the parametrix arguments used to lit results from G_0 to G.

In the sequel we adopt the notation of [Rob] as modified in [ElR3] and [AER]. Let G be a d-dimensional connected Lie group with Lie algebra $\mathfrak{g}$ and $(\mathcal{X}, G, U)$ a strongly, or weakly*, continuous representation of G on the Banach space $\mathcal{X}$ by bounded operators $g \mapsto U(g)$. If $a_i \in \mathfrak{g}$ then A_i $(= dU(a_i))$ will denote the generator of the one-parameter subgroup $t \mapsto U(\exp(-ta_i))$ of the representation. Let $a_1, ..., a_{d'}$ be an algebraic basis of $\mathfrak{g}$, i.e., a set of linearly independent elements which together with their multi-commutators span $\mathfrak{g}$, and $w_1, ..., w_{d'} \in [1, \infty\rangle$ a d'-tuple of numbers which we call *weights*. The group G can be equipped with a modulus $|\cdot|'$ which is naturally determined by the algebraic basis $a_1, ..., a_{d'}$ and the weights $w_1, ..., w_{d'}$. The detailed definition of this modulus will be given in Section 6. The modulus then determines a local "dimension" $D' > 0$ of the group such that $c^{-1}\delta^{D'} \leqslant |B'_\delta| \leqslant c\delta^{D'}$ for some $c > 0$ and all $\delta \in \langle 0, 1]$ where $|B'_\delta|$ denotes the volume of the ball $B'_\delta = \{g \in G : |g|' < \delta\}$ with respect to left invariant Haar measure dg.

Next for each $n \in \mathbf{N}_0$ set

$$J_n(d') = \bigoplus_{k=0}^{n} \{1, ..., d'\}^k, \qquad J_n^+(d') = \bigoplus_{k=1}^{n} \{1, ..., d'\}^k$$

and

$$J(d') = \bigcup_{n=0}^{\infty} J_n(d'), \qquad J^+(d') = \bigcup_{n=1}^{\infty} J_n^+(d').$$

Then $A^\alpha = A_{i_1} \cdots A_{i_n}$ for $\alpha = (i_1, ..., i_n)$, etc. Alternatively, we set $a^\alpha = a_{i_1} \cdots a_{i_n}$ in the universal enveloping algebra and write $A^\alpha = dU(a^\alpha)$.

The *weighted length* $\|\alpha\|$ of $\alpha=(i_1, ..., i_n)\in J(d')$ is defined by

$$\|\alpha\| = \sum_{k=1}^{n} w_{i_k}$$

and the *Euclidean length* n is denoted by $|\alpha|$.

If the algebraic basis is extended to a full vector space basis $a_1, ..., a_d$ and $n\in\mathbf{N}$ we define $\mathscr{X}_n=\mathscr{X}_n(U)=\bigcap_{\alpha\in J_n(d)} D(A^\alpha)$ and introduce norms and seminorms by

$$\|x\|_n=\|x\|_{U,n}=\max_{\substack{\alpha\in J(d)\\|\alpha|\leqslant n}}\|A^\alpha x\|, \qquad N_n(x)=N_{U,n}(x)=\max_{\substack{\alpha\in J(d)\\|\alpha|=n}}\|A^\alpha x\|.$$

These spaces are independent of the choice of the full basis up to equivalence of norms. Similarly, for $n\in\mathbf{R}$ with $n\geqslant 0$, we define the weighted spaces

$$\mathscr{X}'_n=\mathscr{X}'_n(U)=\bigcap_{\substack{\alpha\in J(d')\\\|\alpha\|\leqslant n}} D(A^\alpha)$$

corresponding to the weighted algebraic basis. The associated norms and seminorms are given by

$$\|x\|'_n=\|x\|'_{U,n}=\begin{cases}\max\limits_{\substack{\alpha\in J(d')\\\|\alpha\|\leqslant n}}\|A^\alpha x\| & \text{if there exists } \alpha\in J(d') \\ & \qquad \text{with } \|\alpha\|=n,\\ 0 & \text{otherwise,}\end{cases}$$

$$N'_n(x)=N'_{U,n}(x)=\begin{cases}\max\limits_{\substack{\alpha\in J(d')\\\|\alpha\|=0}}\|A^\alpha x\| & \text{if there exists } \alpha\in J(d') \\ & \qquad \text{with } \|\alpha\|=n,\\ 0 & \text{otherwise.}\end{cases}$$

The definition of $\|x\|'_n=0$ in case $n\notin\{\|\alpha\|: \alpha\in J(d')\}$ is to avoid complications in the proofs of some statements. In Section 11 we remove this part of the definition.

Let $\mathscr{X}_\infty=\mathscr{X}_\infty(U)=\bigcap_{n=1}^\infty \mathscr{X}_n$. Since $a_1, ..., a_{d'}$ is an algebraic basis one also has $\mathscr{X}_\infty=\bigcap_{n=1}^\infty \mathscr{X}'_n$. It then follows by the proof of Lemma 2.4 of [ElR1] that the space $\mathscr{X}_\infty$ is weakly, or weakly*, dense in $\mathscr{X}'_n$ for all $n\geqslant 0$. If U is the left regular representation on $L_p(G)=L_p(G; dg)$ we denote the corresponding spaces by $L_{p;n}$, $L'_{p;n}$, $L_{p;\infty}$ and the norms and seminorms by $\|\cdot\|_{p;n}$ etc. Further we let $L=L_G$ denote the left regular representation of G in $L_2(G; dg)$.

A function $C: J(d')\to\mathbf{C}$ such that $C(\alpha)=0$ if $\|\alpha\|>m$ but $C(\alpha)\neq 0$ for at least one $\alpha\in J(d')$ with $\|\alpha\|=m$ is defined to be an *m-th order form C*.

Here, and in the sequel, the order m is understood to be the weighted order. We write $c_\alpha = C(\alpha)$.

The *principal part* P of the mth order form C is the mth order form given by

$$P(\alpha) = \begin{cases} C(\alpha) & \text{if } \quad \|\alpha\| = m, \\ 0 & \text{if } \quad \|\alpha\| < m, \end{cases}$$

and C is called *homogeneous* if $C = P$.

The *formal adjoint* $C^\dagger$ of C is the function $C^\dagger \colon J_m(d') \to \mathbf{C}$ defined by

$$C^\dagger(\alpha) = (-1)^{|\alpha|} \overline{C(\alpha_*)},$$

where $\alpha_* = (i_n, ..., i_1)$ if $\alpha = (i_1, ..., i_n)$. The *real* and *imaginary* parts of C are $\Re C = 2^{-1}(C + C^\dagger)$ and $\Im C = (2i)^{-1} (C - C^\dagger)$.

We consider the mth order operators

$$dU(C) = \sum_{\alpha \in J(d')} c_\alpha A^\alpha$$

with domain $D(dU(C)) = \mathscr{X}'_m$ associated with the form. If $(\mathscr{F}, G, U_*)$ is the dual representation of $(\mathscr{X}, G, U)$ then $dU_*(C^\dagger)$ with the domain $D(dU_*(C^\dagger)) = \mathscr{F}'_m$ is called the *dual operator*.

The mth order form C is defined to be a *G-weighted subcoercive form* if $m/w_i \in 2\mathbf{N}$ for each $i \in \{1, ..., d'\}$ and the corresponding operator $dL_G(C)$ satisfies a local *Gårding inequality*;

$$\mathrm{Re}(\varphi, dL_G(C)\, \varphi) \geqslant \mu (N'_{2; m/2}(\varphi))^2 - \nu \, \|\varphi\|_2^2$$

for some $\mu > 0$ and $\nu \in \mathbf{R}$, uniformly for all $\varphi \in C_c^\infty(V)$ where V is some open neighbourhood of the identity $e \in G$. For example, let $c_{\alpha, \beta} \in \mathbf{C}$, with $\alpha, \beta \in J(d')$ and $\|\alpha\| = m/2 = \|\beta\|$, satisfy $\mathrm{Re} \sum_{\alpha, \beta} c_{\alpha, \beta} \overline{\xi_\alpha} \xi_\beta > 0$ for all non-zero complex (ξ_α). Then the operator $H = \sum_{\alpha, \beta} (-1)^{|\alpha|} c_{\alpha, \beta} A^{\alpha *} A^\beta$ satisfies the Gårding inequality. This follows because

$$\begin{aligned} \mathrm{Re}(\varphi, H\varphi) &= \mathrm{Re} \sum_{\alpha, \beta} c_{\alpha, \beta} (A^\alpha \varphi, A^\beta \varphi) \\ &\geqslant \mu \sum_{\|\alpha\| = m/2} \|A^\alpha \varphi\|_2^2 \geqslant \mu (N'_{2; m/2}(\varphi))^2 \end{aligned}$$

where μ is the strictly positive lowest eigenvalue of the real part of the matrix $(c_{\alpha, \beta})$.

Our main result establishes that subcoercivity gives an infinitesimal characterization of generators of semigroups with kernels satisfying Gaussian bounds.

THEOREM 1.1. *Let C be an m-th order form and assume that the weights w_i satisfy $m/w_i \in 2\mathbf{N}$. Then the following conditions are equivalent.*

I. *The form C is G-weighted subcoercive.*

II. *There are $c, \mu > 0$ and an open neighbourhood V of the identity of G such that*

$$\mu\varepsilon^{2w_i} \|A_i\varphi\|_2^2 \leqslant \varepsilon^m \operatorname{Re}(\varphi, dL_G(C)\,\varphi) + c\,\|\varphi\|_2^2$$

for all $\varphi \in C_c^\infty(V)$, all $\varepsilon \in \langle 0, 1]$ and all $i \in \{1, ..., d'\}$.

III. *The closure of $dL_G(C)$ generates a holomorphic semigroup S on $L_2(G)$ with the following properties.*

i. *The semigroup S is quasi-contractive in an open subsector of the sector of holomorphy, i.e., there exists $\varphi \in \langle 0, \pi/2]$ and $\omega \geqslant 0$ such that $\|S_z\| \leqslant e^{\omega |z|}$ for all $z \in \Lambda(\varphi) = \{z \in \mathbf{C}\backslash\{0\} : |\arg z| < \varphi\}$.*

ii. *$S_t L_2(G) \subseteq \bigcap_{i=1}^{d'} D(A_i)$ and there exist $c > 0$ and $\omega \geqslant 0$ such that*

$$\|A_i S_t\|_{2\to 2} \leqslant c t^{-w_i/m} e^{\omega t}$$

for all $t > 0$ and $i \in \{1, ..., d'\}$.

IV. *In each continuous representation $(\mathcal{X}, G, U)$ the closure of $dU(C)$ generates a continuous semigroup S with the following properties.*

i. *The semigroup S is holomorphic in a sector which contains an open representation independent subsector $\Lambda(\theta_C)$.*

ii. *If U is unitary then the semigroup S is quasi-contractive in each subsector of $\Lambda(\theta_C)$, i.e., for each $\varphi \in \langle 0, \theta_C\rangle$ there is an $\omega \geqslant 0$ such that $\|S_z\| \leqslant e^{\omega |z|}$ for all $z \in \Lambda(\varphi)$.*

iii. *The semigroup S has a representation independent, fast decreasing, kernel $K \in L_{1;\infty}(G) \cap C_{0;\infty}(G)$ such that*

$$A^\alpha S_z x = \int_G dg\,(A^\alpha K_z)(g)\, U(g)\, x$$

for all $\alpha \in J(d')$, $z \in \Lambda(\theta_C)$ and $x \in \mathcal{X}$.

iv. *For each $\varphi \in \langle 0, \theta_C\rangle$ and all $\alpha \in J(d')$ there exist $b, c > 0$ and $\omega \geqslant 0$ such that*

$$|(A^\alpha K_z)(g)| \leqslant c\,|z|^{-(D' + \|\alpha\|/m)} e^{\omega |z|} e^{-b((|g|')^m |z|^{-1})^{1/(m-1)}}$$

for all $g \in G$ and $z \in \Lambda(\varphi)$.

A crucial element in the proof is the local approximation of G by the homogeneous (nilpotent) group G_0 alluded to above. The group G_0 is constructed, following an idea of Kashiwara and Vergne [KaV], through exponentiation of a contraction of the Lie algebra $\mathfrak{g}$ of G. This contraction procedure was applied earlier to weighted operators on nilpotent Lie groups by Nagel, Ricci, and Stein [NRS] to obtain asymptotic properties of their fundamental solutions. More recently Hebisch [Heb2] applied the procedure to general Lie groups, but for a special class of weighted operators, to prove kernel bounds similar to these of Theorem 1.1, by using quite different arguments.

The contraction mechanism provides a family of groups G_t, $t \in [0, 1]$, which interpolate between $G = G_1$ and G_0. One can use this interpolation to establish that each G-weighted subcoercive form is automatically a G_0-weighted subcoercive form. Then the implication I $\Rightarrow$ IV in Theorem 1.1 is proved by applying the results of [AER] to C on the homogeneous group G_0 to obtain the implication for G_0 and subsequently lifting the result to G by parametrix arguments. The latter reasoning makes essential use of the results of Helffer and Nourrigat [HeN] for homogeneous groups.

The proof of IV $\Rightarrow$ III is straightforward. The L_2-bound on $A_i S_t$ follows from the corresponding kernel bound by a quadrature argument; one deduces that $\|A_i K_t\|_1 \leqslant c' t^{-w_i/m} e^{\omega' t}$ for some $c' > 0$ and $\omega' \geqslant 0$.

The circle of arguments used to prove I $\Rightarrow$ IV allows one to establish the equivalence of G_0-weighted subcoercivity and G-weighted subcoercivity. This equivalence is one of the most important structural features of the theory. It provides the starting point for the proof of III $\Rightarrow$ I and II $\Rightarrow$ I in the theorem since it then suffices to prove that C is G_0-subcoercive. The latter property follows by exploitation of the contraction mechanism and the homogeneity of G_0.

The proof that I $\Rightarrow$ II is straightforward. Since the A_i are group generators one has the inequalities $\varepsilon^{2w_i} \|A_i \varphi\|_2^2 \leqslant \varepsilon^m \|A_i^{m/(2w_i)} \varphi\|_2^2 + c \|\varphi\|_2^2$ for all $\varphi \in \langle 0, 1]$ (see [Rob], Lemma II.2.5).

A simple illustration of our results is given by the group $SO(3)$ of rotations in $\mathbf{R}^3$. If a_1, a_2, a_3 is a basis of $so(3)$ satisfying $[a_1, a_2] = a_3$, $[a_2, a_3] = a_1$ and $[a_3, a_1] = a_2$ then a_1, a_2 is an algebraic basis. If $w_1 = 3$, $w_2 = 2$ then the operator

$$H = A_1^4 - A_2^6 - A_1^2 A_2^3$$

has (weighted) order 12 and satisfies the Gårding inequality because a straightforward calculation gives

$$\operatorname{Re}(\varphi, H\varphi) \geqslant 2^{-1}(\|A_1^2 \varphi\|_2^2 + \|A_2^3 \varphi\|_2^2) \geqslant 2^{-1}(N'_{2;6}(\varphi))^2.$$

Hence H generates a holomorphic semigroup with a smooth kernel satisfying Gaussian bounds in each continuous representation of the group.

The paper is organized as follows. In Section 2 we introduce reduced algebraic bases and all proofs in Sections 3–10 are carried out for such bases. In Section 11 we remove this restriction on the bases. In Section 3 we introduce the contraction mechanism and give several uniform properties for the right invariant vector fields on each of the interpolating groups G_t. In Section 4 we analyze various structural properties of G-weighted subcoercive forms and prove that they are automatically G_0-weighted subcoercive forms. In Section 5 we prove the implication I $\Rightarrow$ IV in Theorem 1.1 for G_0 and U the left regular representation of G_0 in $L_2(G_0)$. In Section 6 we define distances on G and G_t associated with the weighted algebraic basis and in Section 7 the kernel on G_0 of Section 5 is lifted to a "kernel" on G by a parametrix argument which uses G_0 as a local approximation of G. In Section 8 we prove the implication I $\Rightarrow$ IV in Theorem 1.1 for reduced bases, but under the (weaker) assumption that C is merely a G_0-weighted subcoercive form. Under the same conditions we prove regularity results in Section 9. In Section 10 we prove that a form is a G-weighted subcoercive form if, and only if, it is a G_0-weighted subcoercive form. Moreover, we prove Theorem 1.1 for reduced bases and derive other equivalent characterizations for G-weighted subcoercive forms. In the last section we extend the results for reduced bases to general bases.

2. REDUCED BASES

Let $\mathfrak{g}$ be a d-dimensional Lie algebra with Lie product $[\cdot, \cdot]$. We adopt the multi-index notation introduced in Section 1. If $\alpha = (i_1, ..., i_n)$ with $i_j \in \{1, ..., d'\}$ is a multi-index of length $|\alpha| = n \neq 0$ and $a_1, ..., a_{d'} \in \mathfrak{g}$ we denote the *multi-commutator* $a_{[\alpha]}$ of *order* n by $a_{[\alpha]} = [a_{i_1}, [\cdots [a_{i_{n-1}}, a_{i_n}] \cdots]] \in \mathfrak{g}$ where $a_{[(i)]} = a_i$. Our principal interest is in algebraic bases $a_1, ..., a_{d'}$ of $\mathfrak{g}$. The smallest integer r for which the $a_1, ..., a_{d'}$ together with all their multi-commutators of order less than or equal to r span $\mathfrak{g}$ is called the *rank* of the algebraic basis.

We also consider algebraic bases with weights $w_1, ..., w_{d'} \in [1, \infty\rangle$ assigned to the d' directions. We call $a_1, ..., a_{d'}$ a *weighted algebraic basis*. The unweighted algebraic basis $a_1, ..., a_{d'}$ can be considered as a weighted algebraic basis with all $w_i = 1$.

Next we introduce a special class of weighted algebraic bases for which the weights are minimal. One only has directions for which the weight is not too large compared with the other directions and their weights. These bases are used in the analysis of subcoercive operators but the key results are independent of the special weightings.

A *filtration* for $\mathfrak{g}$ is a family of vector subspace $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ of $\mathfrak{g}$ with the following four properties. First $\mathfrak{g}_\lambda \subseteq \mathfrak{g}_\mu$ if $\lambda \leqslant \mu$, secondly $\mathfrak{g}_\lambda = \{0\}$ if $\lambda < 1$, thirdly $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$ for all $\lambda, \mu [0, \infty\rangle$ and fourthly $\mathfrak{g}_\lambda = \mathfrak{g}$ for large λ. If $a_1, ..., a_{d'}$ is a weighted algebraic basis, $\lambda \geqslant 0$ and we set

$$\mathfrak{g}_\lambda = \operatorname{span}\{a_{[\alpha]} : \alpha \in J(d'), 0 < \|\alpha\| \leqslant \lambda\}$$

then $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ is a filtration, which we call the *filtration corresponding to the weighted algebraic basis*. Note that it is possible that $a_i \in \mathfrak{g}_\lambda$ for a $\lambda \geqslant 0$ with $\lambda < w_i$.

Next for a general filtration define

$$\mathfrak{g}_{\underline{\lambda}} = \bigcup_{\lambda' < \lambda} \mathfrak{g}_{\lambda'}$$

for each $\lambda > 0$. Further let $1 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_k$ be such that $\mathfrak{g}_{\lambda_k} = \mathfrak{g}$ and

$$\{\lambda_j : j \in \{1, ..., k\}\} = \{\lambda \geqslant 0 : \mathfrak{g}_\lambda \neq \mathfrak{g}_{\underline{\lambda}}\}.$$

We call $\lambda_1, ..., \lambda_k$ the *weights* of the filtration $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$.

Note that in the sequel we sometimes set $\mathfrak{g}_{(\lambda+\mu)_-} = \mathfrak{g}_{\underline{\lambda+\mu}}$ for clarity of notation.

Algebraic bases $a_1, ..., a_{d'}$ with weights $w_1, ..., w_{d'}$ such that $a_i \notin \mathfrak{g}_{\underline{w_i}}$ for all $i \in \{1, ..., d'\}$, where $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ is the filtration corresponding to the weighted algebraic basis, are called *reduced weighted algebraic bases*. The definition of the reduced basis is such that the corresponding weights are minimal.

PROPOSITION 2.1. *Let $a_1, ..., a_{d'}$ be an algebraic basis with weights $w_1, ..., w_{d'}$ and corresponding filtration $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$. Then there exists a reduced weighted algebraic basis $b_1, ..., b_{d''}$ with weights $v_1, ..., v_{d''}$ such that $\{b_1, ..., b_{d''}\} \subseteq \{a_1, ..., a_{d'}\}$ and $v_i = w_j$ if $b_i = a_j$. Moreover, $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ is the filtration corresponding to $b_1, ..., b_{d''}$. Explicitly, one can take the algebraic basis $b_1, ..., b_{d''}$ to consist of precisely those a_i with $a_i \notin \mathfrak{g}_{\underline{w_i}}$.*

Proof. After reordering one may assume $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_{d'}$. Now suppose that $a_j \in \mathfrak{g}_{\underline{w_j}}$. Let $i \in \{1, ..., d'-1\}$ be such that $w_i < w_j = w_{i+1}$. Then there exist $c_{j\alpha} \in \mathbf{R}$ such that

$$a_j = \sum_{\substack{\alpha \in J^+(i) \\ \|\alpha\| < w_i}} c_{j\alpha} a_{[\alpha]}. \tag{1}$$

Hence the subbasis obtained from $a_1, ..., a_{d'}$ be removal of a_j remains a weighted algebraic basis with the same filtration $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ as the original weighted algebraic basis. Finite iteration of this process yields the desired reduced weighted algebraic basis. ∎

Note that the a_i in the directions eliminated in this construction of a reduced basis can be expressed in terms of the remaining directions by (1). In Section 11 this is used to lift properties of the reduced algebraic basis to the general algebraic basis.

LEMMA 2.2. *If $a_1, ..., a_{d'}$ is a reduced weighted algebraic basis then there is an extension to a (vector space) basis $b_{11}, ..., b_{1d_1}, ..., b_{k1}, ..., b_{kd_k}$ for $\mathfrak{g}$ with the following properties. First $b_{11}, ..., b_{1d_1}, ..., b_{i1}, ..., b_{id_i}$ is a basis for $\mathfrak{g}_{\lambda_i}$ for all $i \in \{1, ..., k\}$, where $\lambda_1 < \cdots < \lambda_k$ are the weights for the filtration $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$. Secondly*

$$\{a_1, ..., a_{d'}\} \subseteq \{b_{ij} : i \in \{1, ..., k\}, j \in \{1, ..., d_i\}\},$$

with $w_l = \lambda_i$ if $a_l = b_{ij}$. Thirdly the other b_{ij} equal some commutator $a_{[\alpha]}$ with $\|\alpha\| = \lambda_i$.

If b_{ij} is given the weight $w_{ij} = \lambda_i$ one obtains an extension of the weighted algebraic basis to a weighted vector space basis $a_1, ..., a_{d'}, ..., a_d$ such that a_l has the weight $w_l = \lambda_i$ if $a_l = b_{ij}$.

EXAMPLE 2.3. Let $a_1, ..., a_{d'}$ be an algebraic basis for $\mathfrak{g}$ and set all weights equal to one. Then $a_1, ..., a_{d'}$ is a reduced (weighted) algebraic basis for $\mathfrak{g}$. The operators we construct with respect to such a basis correspond to the subcoercive and subelliptic operators studied in [ElR3] and [ElR6] and if the basis is a vector space basis they correspond to the strongly elliptic operators described in [Rob].

EXAMPLE 2.4. Let $\mathfrak{g}$ be the four-dimensional Lie algebra $\mathfrak{k}_3$ with basis $a_1, ..., a_4$ and commutation relations $[a_4, a_3] = a_2$ and $[a_4, a_2] = a_1$. Then a_1, a_3, a_4 is an algebraic basis. Assign weights $w_1 = 8$, $w_3 = 3$ and $w_4 = 2$. Then the corresponding filtration is given by $\mathfrak{g}_2 = \text{span}\, a_4$, $\mathfrak{g}_3 = \text{span}\{a_3, a_4\}$, $\mathfrak{g}_5 = \text{span}\{a_2, a_3, a_4\}$ and $\mathfrak{g}_7 = \mathfrak{g}$. Therefore a_1, a_2, a_4 is not a reduced weighted algebraic basis since $a_1 \in \mathfrak{g}_7 = \mathfrak{g}_{\underline{8}}$. If one deletes the direction a_1 then a_3, a_4 is a reduced weighted algebraic basis with the same filtration as the weighted algebraic basis a_1, a_3, a_4.

EXAMPLE 2.5. Let $a_1, ..., a_d$ be a basis for $\mathfrak{g}$ with weights $w_1, ..., w_d \in \mathbf{N}$ and suppose that the structure constants c_{ij}^k, defined by $[a_i, a_j] = \sum_{k=1}^d c_{ij}^k a_k$, are such that $c_{ij}^k \neq 0$ implies $w_i + w_j - 1 \geqslant w_k$, i.e., one has

$$[a_i, a_j] = \sum_{\substack{k \in \{1, ..., d\} \\ w_k \leqslant w_i + w_j - 1}} c_{ij}^k a_k.$$

Let $\lambda_1 < \cdots < \lambda_k$ be such that $\{w_i : i \in \{1, ..., d\}\} = \{\lambda_j : j \in \{1, ..., k\}\}$ and let $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ be the filtration corresponding to the weighted basis $a_1, ..., a_d$. Then $\mathfrak{g}_\lambda = \{0\}$ if $\lambda < \lambda_1$ and $\mathfrak{g}_{\lambda_1} = \operatorname{span}\{a_i : i \in \{1, ..., d\}, w_i = \lambda_1\}$. Suppose that $j \in \{1, ..., k-1\}$ and $\mathfrak{g}_{\lambda_j} = \operatorname{span}\{a_i : i \in \{1, ..., d\}, w_i \leqslant \lambda_j\}$. Further suppose $\mathfrak{g}_\lambda \neq \mathfrak{g}_{\lambda_j}$ for some $\lambda \in \langle \lambda_j, \lambda_{j+1} \rangle$. Let $\lambda = \min\{\mu \in \langle \lambda_j, \lambda_{j+1} \rangle : \mathfrak{g}_\mu \neq \mathfrak{g}_{\lambda_j}\}$. The minimum exists since $\mathfrak{g}$ is finite-dimensional. Then there are $n \in \mathbf{N}$, $n \geqslant 2$ and $\alpha = (i_1, ..., i_n) \in J(d)$ such that $\|\alpha\| = \lambda$ and $a_{[\alpha]} \in \mathfrak{g}_\lambda \backslash \mathfrak{g}_{\lambda_j}$. But by assumption

$$a_{[\alpha]} \in \operatorname{span}\{a_i : i \in \{1, ..., d\}, w_i \leqslant \lambda - (n-1)\} \subseteq \mathfrak{g}_{\lambda - (n-1)} \subseteq \mathfrak{g}_{\lambda_j}$$

since $\mathfrak{g}_\mu = \mathfrak{g}_{\lambda_j}$ for all $\mu \in \langle \lambda_j, \lambda \rangle$. So $\mathfrak{g}_\lambda = \mathfrak{g}_{\lambda_j}$ for all $\lambda \in \langle \lambda_j, \lambda_{j+1} \rangle$.

Therefore $\mathfrak{g}_{\lambda_{j+1}} = \operatorname{span}\{a_i : i \in \{1, ..., d\}, w_i \leqslant \lambda_{j+1}\}$. It follows from the above argument that $a_1, ..., a_d$ is a reduced weighted algebraic basis for $\mathfrak{g}$. The operator which we construct with respect to such a basis are the weighted strongly elliptic operators studied in detail in [ElR5].

EXAMPLE 2.6. Let $\mathfrak{g}$ be a homogeneous Lie algebra with respect to a family of dilations $(\gamma_t)_{t>0}$ and $a_1, ..., a_{d'}$ an algebraic basis for $\mathfrak{g}$ such that $\gamma_t(a_i) = t^{w_i} a_i$ for some $w_i \in [1, \infty\rangle$ and all $t > 0$. Then $a_1, ..., a_{d'}$ is a weighted algebraic basis with weights $w_1, ..., w_{d'}$. We describe the corresponding filtration and show that $a_1, ..., a_{d'}$ is a reduced weighted algebraic basis.

Extend the algebraic basis to a vector space basis $a_1, ..., a_{d'}, ..., a_d$ such that for each $i \in \{d'+1, ..., d\}$ there exists a $w_i \in [1, \infty\rangle$ such that $\gamma_t(a_i) = t^{w_i} a_i$ for all $t > 0$. For $\lambda > 0$ set $\mathfrak{g}^{(\lambda)} = \{a \in \mathfrak{g} : \gamma_t(a) = t^\lambda a \text{ for all } t > 0\}$. Then $\mathfrak{g}^{(\lambda)} = \operatorname{span}\{a_i ; i \in \{1, ..., d\}, w_i = \lambda\}$ and $\mathfrak{g} = \bigoplus_{\lambda > 0} \mathfrak{g}^{(\lambda)}$. By definition of $\mathfrak{g}_\lambda$ one obtains the inclusions

$$\mathfrak{g}_\lambda \subseteq \operatorname{span}\{a \in \mathfrak{g} : \exists_{\mu \in \langle 0, \lambda]} \forall_{t>0} [\gamma_t(a) = t^\mu a]\} \subseteq \bigoplus_{\mu \leqslant \lambda} \mathfrak{g}^{(\mu)}.$$

Conversely, let $i \in \{1, ..., d\}$. For all $\alpha \in J_r(d')$, with r the rank of the algebraic basis, there exist $c_\alpha \in \mathbf{R}$ such that $a_i = \sum_{\alpha \in J_r(d')} c_\alpha a_{[\alpha]}$. Then by scaling

$$a_i = \sum_{\substack{\alpha \in J_r(d') \\ \|\alpha\| = w_i}} c_\alpha a_{[\alpha]} \in \mathfrak{g}_{w_i}.$$

Therefore $\mathfrak{g}^{(w_i)} \subseteq \mathfrak{g}_{w_i}$ and hence $\mathfrak{g}_\lambda = \bigoplus_{\mu \leqslant \lambda} \mathfrak{g}^{(\mu)}$ for all $\lambda > 0$. So $\mathfrak{g}_\lambda \neq \mathfrak{g}_{\underline{\lambda}}$ if, and only if, $\mathfrak{g}^{(\lambda)} \neq \{0\}$.

Now suppose $a_i \in \mathfrak{g}_{\underline{w_i}}$ for some $i \in \{1, ..., d'\}$. Then

$$a_i \in \bigcup_{\lambda < w_i} \mathfrak{g}_\lambda \subseteq \bigoplus_{\mu < w_i} \mathfrak{g}^{(\mu)}$$

which is a contradiction. Therefore $a_1, ..., a_{d'}$ is a reduced weighted algebraic basis for $\mathfrak{g}$.

EXAMPLE 2.7. In this example we describe a general type of weighted algebraic basis. Let $a_1, ..., a_{d'}$ be indeterminates, $w_1, ..., w_{d'} \geqslant 1$ weights and let $\lambda > \max(w_1, ..., w_{d'})$. Let $V = \operatorname{span}\{a_1, ..., a_{d'}\}$ and for $t > 0$ define the linear map $\gamma_t \colon V \to V$ such that $\gamma_t(a_i) = t^{w_i} a_i$. Let $\mathfrak{T} = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ be the associative tensor algebra over V. We identify V with the subspace of tensors of degree one of $\mathfrak{T}$. There exists a unique algebra homomorphism $\bar{\gamma}_t$ on $\mathfrak{T}$ such that $\bar{\gamma}_t(a) = \gamma_t(a)$ for all $a \in V$. We will not distinguish between $\bar{\gamma}_t$ and γ_t and write γ_t. The associate tensor algebra $\mathfrak{T}$ is a Lie algebra with the usual commutation relations. Let $\mathfrak{G}$ be the Lie subalgebra of $\mathfrak{T}$ generated by $a_1, ..., a_{d'}$. Then $\mathfrak{G}$ is the free Lie algebra generated by $a_1, ..., a_{d'}$. The restriction, again denoted by γ_t, of γ_t to $\mathfrak{G}$ is a Lie algebra homomorphism. Let I be the ideal in $\mathfrak{G}$ spanned by all commutators $a_{[\alpha]}$ with $\|\alpha\| > \lambda$. Note that $a_i \notin I$ since $\lambda > w_i$. Then the *nilpotent Lie algebra* $\mathfrak{g}$ *with generators* $a_1, ..., a_{d'}$ *which is free of step* λ is equal to $\mathfrak{G}/I$. Since $\gamma_t(I) \subseteq I$, there exists a unique Lie algebra homomorphism $\bar{\gamma}_t \colon \mathfrak{g} \to \mathfrak{g}$ such that $\bar{\gamma}_t(a + I) = \gamma_t(a) + I$ for all $a \in \mathfrak{G}$. Again we write γ_t for $\bar{\gamma}_t$. One easily verifies that $\gamma_{st} = \gamma_s \circ \gamma_t$ for all $s, t > 0$, so $\mathfrak{g}$ equipped with the dilations γ_t, $t > 0$, becomes a homogeneous Lie algebra. Now it follows from Example 2.6 that $a_1, ..., a_{d'}$ is a reduced weighted algebraic basis for $\mathfrak{g}$.

We call $\mathfrak{g}$ the *weighted nilpotent Lie algebra with generators* $a_1, ..., a_{d'}$ *and weights* $w_1, ..., w_{d'}$ *which is free of step* λ. The corresponding connected simply connected Lie group G with Lie algebra $\mathfrak{g}$ is called the *weighted nilpotent Lie group with generators* $a_1, ..., a_{d'}$ *and weights* $w_1, ..., w_{d'}$ *which is free of step* λ. We denote $\mathfrak{g}$ and G by $\mathfrak{g}(d', \lambda, w_1, ..., w_{d'})$ and $G(d', \lambda, w_1, ..., w_{d'})$. These groups play a fundamental role in [NRS] and [ElR6], but are not directly relevant to our considerations. We reconsider these groups in Section 11.

3. HOMOGENIZATION BY CONTRACTION

We construct from each reduced weighted algebraic basis $a_1, ..., a_{d'}$ of the Lie algebra $\mathfrak{g}$ a family of Lie products $[\cdot, \cdot]_t$, $t > 0$, on $\mathfrak{g}$ and then examine the contraction of the Lie algebras $(\mathfrak{g}, [\cdot, \cdot]_t)$ as $t \to 0$. This yields a homogeneous Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_0)$. The corresponding simply connected homogeneous Lie group G_0 subsequently plays a fundamental role in the analysis of elliptic operators on the connected Lie group G corresponding to $\mathfrak{g}$.

Let $b_{11}, ..., b_{1d_1}, ..., b_{k1}, ..., b_{kd_k}$ be a basis for the filtration $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ corresponding to an extension of the reduced weighted algebraic basis described in Lemma 2.2. So $\{a_1, ..., a_{d'}\} \subseteq \{b_{ij}: i \in \{1, ..., k\}, j \in \{1, ..., d_i\}\}$ and $w_l = w_{ij} = \lambda_i$ if $a_l = b_{ij}$, where $\lambda_1 < \cdots < \lambda_k$ are the weights for the filtration. Moreover, for all i and j there exists a multi-index α_{ij} such that $b_{ij} = a_{[\alpha_{ij}]}$. Following the ideas of Kashiwara and Vergne [KaV], Nagel, Ricci, and Stein [NRS, Section 2], and Hebisch [Heb2, Lemma 4.1], we define the linear bijection $\gamma_t: \mathfrak{g} \to \mathfrak{g}$, for $t > 0$, by $\gamma_t(b_{ij}) = t^{w_{ij}} b_{ij} = t^{\lambda_i} b_{ij}$. Further we define $[\cdot, \cdot]_t: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$[a, b]_t = \gamma_t^{-1}([\gamma_t(a), \gamma_t(b)]).$$

Then $(\mathfrak{g}, [\cdot, \cdot]_t)$ is a Lie algebra, which equals $(\mathfrak{g}, [\cdot, \cdot])$ if $t = 1$, and the limit $[\cdot, \cdot]_0$ of the Lie brackets $[\cdot, \cdot]_t$ as $t \to 0$ defines an algebraic structure on $\mathfrak{g}$. The Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_0)$ is the contraction of $(\mathfrak{g}, [\cdot, \cdot]_t)$ in the sense of Saletan [Sal]. This contraction is uniquely determined by the reduced weighted algebraic basis $a_1, ..., a_{d'}$.

Proposition 3.1. I. *The limit* $[a, b]_0 = \lim_{t \downarrow 0} [a, b]_t$ *exists for all* $a, b \in \mathfrak{g}$.

II. $(\mathfrak{g}[\cdot, \cdot]_0)$ *is a homogeneous Lie algebra with dilations* $(\gamma_t)_{t>0}$.

III. *The homogeneous Lie algebra* $(\mathfrak{g}, [\cdot, \cdot]_0)$ *is uniquely determined, up to isomorphism, by the filtration corresponding to the reduced weighted algebraic basis.*

IV. *The* $a_1, ..., a_{d'}$ *form an algebraic basis for the Lie algebra* $(\mathfrak{g}, [\cdot, \cdot]_t)$ *for all* $t \in [0, \infty\rangle$.

V. *The reduced weighted algebraic basis* $a_1, ..., a_{d'}$ *is a reduced weighted algebraic basis for the Lie algebra* $(\mathfrak{g}, [\cdot, \cdot]_t)$ *for all* $t \in [0, \infty\rangle$. *Moreover, the filtrations with respect to the Lie algebras* $(\mathfrak{g}, [\cdot, \cdot]_t)$ *are equal to the filtration* $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ *as vector spaces.*

VI. *For all* i_1, j_1, i_2, j_2 *one has* $[b_{i_1 j_1}, b_{i_2 j_2}]_t - [b_{i_1 j_1}, b_{i_2 j_2}]_0 \in \mathfrak{g}_{(\lambda_{i_1} + \lambda_{i_2})_-}$ *for all* $t > 0$.

This is an elaboration of the results of Nagel, Ricci, and Stein [NRS] and Hebisch [Heb2]. The proof is a relatively straightforward extension of these earlier arguments so we omit the details.

For $t \in [0, \infty\rangle$ with $t \neq 1$ let G_t be the connected, simply connected, Lie group with the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_t)$. The group G_0 is unique and is called the *homogeneous contraction* of G. It is used as a "local approximation" of G in the subsequence analysis. The standard local approximation $G_0 = \mathbf{R}^d$ has a simple characterization in terms of the conditions considered in [ElR5].

PROPOSITION 3.2. *The following conditions are equivalent.*

I. $G_0 = \mathbf{R}^d$,

II. $d' = d$ *and* $w_i + w_j - w_k > 0$ *whenever the structure constant* c_{ij}^k *of the basis* $a_1, ..., a_{d'}$ *is non-zero.*

Proof. I ⇒ II. Assume that $d' < d$ then there exists $l \in \{2, ..., k\}$, to be chosen minimal, such that $\mathfrak{g}_{w_l} \neq \operatorname{span}\{a_i : i \in \{1, ..., d'\}, \ w_i \leqslant w_l\}$. Then there are $i, j \in \{1, ..., d'\}$ such that $w_i + w_j = w_l$ and $[a_i, a_j] \in \mathfrak{g}_{w_l} \backslash \underline{\mathfrak{g}_{w_l}}$. Since $[a_i, a_j]_0 = [a_i, a_j] \bmod \underline{\mathfrak{g}_{w_l}}$ this implies that $[a_i, a_j]_0 \neq 0$. Therefore $d' = d$. But then $[a_i, a_j]_t = \sum_{k=1}^d t^{w_i + w_j - w_k} c_{ij}^k a_k$ and $[a_i, a_j]_0 = 0$ if, and only if, $w_i + w_j - w_k > 0$ for those k such that $c_{ij}^k \neq 0$.

II ⇒ I. Since

$$[a_i, a_j] = \sum_{\substack{k \in \{1, ..., d\} \\ w_k < w_i + w_j}} c_{ij}^k a_k$$

it follows that $[a_i, a_j]_0 = 0$. ∎

In the unweighted case, i.e., if $w_i = 1$, the proposition demonstrates that $G_0 = \mathbf{R}^d$ if, and only if, one is dealing with a full vector space basis. Thus the analysis of strict algebraic bases enforces the introduction of non-commutative approximations G_0 of G. One advantage of G_0 as a local approximant is that it has the same dimension as G.

In order to analyze G and G_0 one needs information about the intermediate groups G_t, $t \in \langle 0, 1 \rangle$. We identify quantities associated with G_t by indices and suffices t but in the case $t = 1$ we often omit these indices or suffices.

Let $\exp_t : (\mathfrak{g}, [\cdot, \cdot]_t) \to G_t$, $t \in [0, \infty\rangle$, denote the exponential map. We use the map $\exp_0$ to lift the dilations γ_t on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_0)$ to dilations on G_0, which we also denote by γ_t. Note that for $t > 0$ with $t \neq 1$ the map $\gamma_t : (\mathfrak{g}, [\cdot, \cdot]_t) \to (\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra isomorphism which lifts to a Lie group isomorphism from G_t onto the enveloping group of G. This isomorphism frequently provides the necessary uniform estimates for $t \in \langle 0, 1 \rangle$. Complete the weighted algebraic basis $a_1, ..., a_{d'}$ to a full vector space basis $a_1, ..., a_d$ as in Lemma 2.2. Let (c_{ij}^k) be the structure constants of $(\mathfrak{g}, [\cdot, \cdot])$ with respect to the basis $a_1, ..., a_d$. We may assume that $c_{ij}^k \leqslant d^{-3}$ for all $i, j, k \in \{1, ..., d\}$ and we let $\|\cdot\|$ be the Euclidean norm with respect to the basis $a_1, ..., a_d$. Then $\|[a, b]\| \leqslant \|a\| \, \|b\|$ for all $a, b \in \mathfrak{g}$.

LEMMA 3.3. I. *There exists a* $u_1 \in \langle 0, 1 \rangle$ *such that* $\exp_t$ *is a diffeomorphism from* $\{a \in \mathfrak{g} : \|a\| < u_1\}$ *onto an open neighbourhood of the identity in* G_t, *uniformly for all* $t \in [0, 1]$.

II. *There exists a* $u_2 \in \langle 0, u_1]$ *such that the Campbell–Baker–Hausdorff formula with respect to* $(\mathfrak{g}, [\cdot, \cdot]_t)$ *is absolutely convergent on* $\{a \in \mathfrak{g}\colon \|a\| < u_2\}^2$ *uniformly for* $t \in [0, 1]$.

III. *There exists a* $u_3 \in \langle 0, u_2]$ *such that* $\exp_t a \exp_t b \in \exp_t\{c \in \mathfrak{g}\colon \|c\| < u_1\}$ *uniformly for all* $a, b \in \{c \in \mathfrak{g}\colon \|c\| < u_3\}$ *and* $t \in [0, 1]$.

IV. *Setting* $a *_t b = \log_t(\exp_t a \exp_t b)$, *where* $\log_t$ *denotes the local inverse of* $\exp_t$ *onto* $\{a \in \mathfrak{g}\colon \|a\| < u_1\}$, *one has*

$$a *_t b = \gamma_t^{-1}(\gamma_t(a) *_1 \gamma_t(b)) = \gamma_t^{-1}(\gamma_t(a) * \gamma_t(b))$$

for all $t \in \langle 0, 1]$ *and* $a, b \in \{c \in \mathfrak{g}\colon \|c\| < u_3\}$.

V. *There exists a* $u_4 \in \langle 0, u_3]$ *such that* $\exp_t a \exp_t b \in \exp_t\{c \in \mathfrak{g}\colon \|c\| < u_3\}$ *uniformly for all* $a, b \in W = \{c \in \mathfrak{g}\colon \|c\| < u_4\}$ *and* $t \in [0, 1]$.

The diffeomorphic property in Statement I is well-known for each $\exp_t$ and it is not difficult to establish the uniformly of u_1 in t.

For the proof of the second statement we need the following version of a standard result.

PROPOSITION 3.4 (Campbell–Baker–Hausdorff). *Let* G *be a Lie group with Lie algebra* $(\mathfrak{g}, [\cdot, \cdot])$ *and* $\|\cdot\|$ *a Euclidean norm on* $\mathfrak{g}$ *such that* $\|[a, b]\| \leqslant \|a\|\, \|b\|$ *for all* $a, b \in \mathfrak{g}$. *Then there exist* M, $s > 0$ *and* $\delta \in \langle 0, (2s)^{-1}\rangle$ *and for each* $\alpha \in J(2)$ *with* $|\alpha| \neq 0$ *there is a* $c_\alpha \in \mathbf{R}$ *with* $|c_\alpha| \leqslant M s^{|\alpha|}$, *all independent of* G, $\mathfrak{g}$ *and* $\|\cdot\|$, *such that* $\exp b_1 \exp b_2 = \exp c(b_1, b_2)$ *for all* $b_1, b_2 \in \mathfrak{g}$ *with* $\|b_1\|$, $\|b_2\| < \delta$ *where*

$$c(b_1, b_1) = \sum_{\substack{\alpha \in J(2)\\ |\alpha| \neq 0}} c_\alpha b_{[\alpha]}.$$

In particular this series converges absolutely.

Proof. This follows from the discussion in [Hoc, pp. 111–112]. ∎

We continue with the proof of Lemma 3.3 The structure constants of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_t)$ with respect to the basis $a_1, ..., a_d$ are equal to $t^{w_i + w_j - w_k} c_{ij}^k$, where (c_{ij}^k) are the structure constants of $(\mathfrak{g}, [\cdot, \cdot])$ with respect to the basis $a_1, ..., a_d$. Since $w_k \leqslant w_i + w_j$ if $c_{ij}^k \neq 0$, they are also bounded by d^{-3} if $t \in [0, 1]$. So $\|[a, b]_t\| \leqslant \|a\|\, \|b\|$ for all $a, b \in \mathfrak{g}$. Now Statement II follows from Proposition 3.4.

If M, s, δ, c_α are as in Proposition 3.4 then

$$\|c(b_1, b_2)\| \leqslant \sum_{\substack{\alpha \in J(2)\\ |\alpha| \neq 0}} M s^{|\alpha|} (2^{-1}\delta)^{|\alpha|} \leqslant \delta M s (1 - \delta s)^{-1}$$

for all $b_1, b_2 \in \mathfrak{g}$ with $\|b_1\|, \|b_2\| < 2^{-1}\delta$. Therefore $\|c(b_1, b_2)\| < u_1$ if δ is taken small enough. Finally, if $b_1, b_2 \in \mathfrak{g}$ with $\|b_1\|, \|b_2\| < \delta$ and $t \in \langle 0, 1]$ then

$$b_1 *_t b_2 = \sum_{\substack{\alpha \in J(2) \\ |\alpha| \neq 0}} c_\alpha b_{[\alpha]_t} = \sum_{\substack{\alpha \in J(2) \\ |\alpha| \neq 0}} c_\alpha \gamma_t^{-1} b'_{[\alpha]}$$
$$= \gamma_t^{-1}(b'_1 * b'_2) = \gamma_t^{-1}(\gamma_t(b_1) * \gamma_t(b_2)),$$

where $b'_1 = \gamma_t(b_1)$ and $b'_2 = \gamma_t(b_2)$. This completes the proof of Lemma 3.3. ∎

Next we need information on the vector fields in the directions $a_1, ..., a_d$ with respect to the left regular representation of G_t on $C^\infty(G_t)$. Let $t \in [0, 1]$. For $i \in \{1, ..., d\}$ and $\varphi \in C^\infty(G_t)$ define $Y_i^{(t)} \varphi \colon G_t \to \mathbf{C}$ by

$$(Y_i^{(t)} \varphi)(g) = \frac{d}{ds} \varphi(\exp_t(-s a_i)\, g) \Big|_{s=0}.$$

Moreover, for $g \in G_t$ define $R^{(t)}(g) \colon G_t \to G_t$ by $R^{(t)}(g)\, h = hg$. Further introduce $\pi \colon \mathfrak{g} \to \mathbf{R}^d$ by $\pi(\sum_{j=1}^d \xi_j a_j) = (\xi_1, ..., \xi_d)$. Then

$$(Y_i^{(t)} \varphi)(g) = -R^{(t)}(g)_* \big|_e \exp_{t*}\big|_0 \left(\frac{\partial}{\partial \pi^i}\Big|_0 \right)(\varphi),$$

where $R^{(t)}(g)_*\big|_e$ denotes the differential of $R^{(t)}(g)$ at the identity, etc. Next, for all $\psi \in C_c^\infty(W)$, where W is the uniform open neighbourhood introduced in Lemma 3.3.V, define $X_i^{(t)} \psi \in C_c^\infty(W)$ by

$$(X_i^{(t)} \psi)(a) = (Y_i^{(t)}(\psi \circ \log_t))(\exp_t a). \tag{2}$$

Then $\exp_{t*}\big|_a (X_i^{(t)}\big|_a) = Y_i^{(t)}\big|_{\exp_t a}$ for all $a \in W$.

Fix $a \in W$ and set $g = \exp_t a$. Let γ be a C^∞-path from an open neighbourhood of $0 \in \mathbf{R}$ to $\mathfrak{g}$ such that $\gamma(0) = a$. Since $X_1^{(t)}\big|_a, ..., X_d^{(t)}\big|_a$ span the tangent space at a there exists $c_1, ..., c_d \in \mathbf{R}$ such that

$$\dot\gamma(0) = \sum_{i=1}^d c_i X_i^{(t)}\big|_a.$$

We calculate the constants $c_1, ..., c_d$. Since $\exp_{t*}\big|_a \dot\gamma(0) = \sum_{i=1}^d c_i Y_i^{(t)}\big|_g$ one obtains

$$-\log_{t*}\big|_e R^{(t)}(g^{-1})_*\big|_g \exp_{t*}\big|_a \dot\gamma(0) = \sum_{i=1}^d c_i \frac{\partial}{\partial \pi^i}\Big|_0.$$

Hence

$$c_i = (-\log_{t*}|_e\, R^{(t)}(g^{-1})_*|_g\, \exp_{t*}|_a\, \dot{\gamma}(0))(\pi^i)$$
$$= -\frac{d}{ds}\, \pi^i(\gamma(s) *_t (-a))\Big|_{s=0}.$$

In particular,

$$\dot{\gamma}(0) = -\sum_{i=1}^{d} \frac{d}{ds}\, \pi^i(\gamma(s) *_0 (-a))\Big|_{s=0} X_i^{(0)}\big|_a.$$

Now let $t \in [0, 1]$, $\psi \in C_c^\infty(W)$ and $a \in W$. Then

$$(X_i^{(t)}\psi)(a) = (Y_i^{(t)}(\psi \circ \log_t))(\exp_t a)$$
$$= \frac{d}{ds}\, \psi(-sa_i *_t a)\Big|_{s=0} = -\dot{\gamma}(0)\, \psi, \tag{3}$$

where $\gamma(s) = sa_i *_t a$. So

$$(X_i^{(t)}\psi)(a) = \sum_{j=1}^{d} \frac{d}{ds}\, \pi^j(\gamma(s) *_0 (-a))\Big|_{s=0} X_j^{(0)}\big|_a(\psi)$$
$$= \sum_{j=1}^{d} \frac{d}{ds}\, \pi^j((sa_i *_t a) *_0 (-a))\Big|_{s=0} X_j^{(0)}\big|_a(\psi)$$

and hence

$$X_i^{(t)}\big|_a = \sum_{j=1}^{d} \frac{d}{ds}\, \pi^j((sa_i *_t a) *_0 (-a))\Big|_{s=0} X_j^{(0)}\big|_a.$$

Since the Campbell–Baker–Hausdorff formula converges absolutely on the set $\{c \in \mathfrak{g}: \|c\| < u_2\}^2$ in Lemma 3.3.II it follows that there exists $M, \delta > 0$ and for all $n \in \mathbf{N}$ and $\varepsilon_1, ..., \varepsilon_n \in \{0, 1\}$ there exists $c_{\varepsilon_1, ..., \varepsilon_n} \in \mathbf{R}$ such that $|c_{\varepsilon_1, ..., \varepsilon_n}| \leqslant M\delta^n$ and

$$\frac{d}{ds}\, (sa_i *_t a) *_0 (-a)\Big|_{s=0}$$
$$= a_i + \sum_{n=1}^{\infty} \sum_{\varepsilon_1, ..., \varepsilon_n \in \{0, 1\}} c_{\varepsilon_1, ..., \varepsilon_n} (\mathrm{ad}_{\varepsilon_1 t} a) \cdots (\mathrm{ad}_{\varepsilon_n t} a)(a_i) \tag{4}$$

for all $a \in W$ and $t \in [0, 1]$ and such that this series converges absolutely, uniformly for all $a \in W$ and $t \in [0, 1]$.

These observations imply the following continuity property of the vector fields.

LEMMA 3.5. *For each* $m \in \mathbf{N}$, $i_1, ..., i_m \in \{1, ..., d\}$ *and* $\psi \in C_c^\infty(W)$ *one has*

$$\lim_{t \to 0} X_{i_1}^{(t)} \cdots X_{i_m}^{(t)} \psi = X_{i_1}^{(0)} \cdots X_{i_m}^{(0)} \psi$$

uniformly on W.

Next for all $\psi \in C_c^\infty(\mathfrak{g})$ and $t > 0$ define $\psi_t \in C_c^\infty(\mathfrak{g})$ by

$$\psi_t(a) = t^{D'/2} \psi(\gamma_t(a))$$

where $D' = \sum_{i=1}^d w_i$.

LEMMA 3.6. *If* $t \in \langle 0, 1]$, $\psi \in C_c^\infty(\gamma_t(W))$ *and* $i \in \{1, ..., d\}$ *then* $X_i^{(t)} \psi_t = t^{w_i}(X_i \psi)_t$, *where* $X_i = X_i^{(1)}$.

Proof. The proof follows from Lemma 3.3.IV and (3). ∎

COROLLARY 3.7. *For each* $m \in \mathbf{N}$, $i_1, ..., i_m \in \{1, ..., d\}$ *and* $\psi \in C_c^\infty(\mathfrak{g})$ *one has*

$$(X_{i_1}^{(0)} \cdots X_{i_m}^{(0)} \psi)(a) = \lim_{t \to 0} t^{D'/2} t^{w_{i_1} + \cdots + w_{i_m}} (X_{i_1} \cdots X_{i_m} \psi_{t^{-1}})(\gamma_t(a))$$

uniformly for $a \in \mathfrak{g}$.

Proof. This result is a consequence of Lemmas 3.5 and 3.6 if $\operatorname{supp} \psi \subseteq W$ and it follows for general ψ by scaling. ∎

Finally it is necessary to examine the left regular representation of the groups G_t on the L_2-spaces, $L_2(G_t)$, with respect to a suitably normalized Haar measure. It follows from [Var, Theorem 2.14.3 and Exercise 2.26(d)], that there exists a unique Haar measure ρ_t on G_t such that

$$\int_{G_t} d\rho_t(g)\, \psi(\log_t g) = \int_W da\, \sigma_t(a)\, \psi(a) = \int_{\mathfrak{g}} da\, \sigma_t(a)\, \psi(a) \tag{5}$$

for all $\psi \in C_c(W)$, where

$$\sigma_t(a) = \left| \det \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (\mathrm{ad}_t\, a)^n \right| \tag{6}$$

for all $a \in W$ and $t \in [0, 1]$. In particular this fixes a Haar measure on $G = G_1$. Then $\sigma_0(a) = 1$ since $(\mathfrak{g}, [\cdot, \cdot]_0)$ is nilpotent.

LEMMA 3.8. *If $t \in \langle 0, 1]$ and $\varphi, \psi \in C_c^\infty(\gamma_t(W))$ then*

$$\int_{\mathfrak{g}} da\, \sigma_t(a)\, \overline{\varphi_t(a)}\, \psi_t(a) = \int_{\mathfrak{g}} da\, \sigma(a)\, \overline{\varphi(a)}\, \psi(a)$$

where $\sigma = \sigma_1$.

Proof. It follows from the identity $\mathrm{ad}_t\, a = \gamma_t^{-1}(\mathrm{ad}\, \gamma_t(a))\, \gamma_t$ that $\sigma_t(a) = \sigma(\gamma_t(a))$ for all $a \in W$. The lemma follows by a change of variables. ∎

EXAMPLE 3.9. Let $\mathfrak{g}$ be a homogeneous Lie algebra with respect to a family of dilations $(\gamma_t)_{t>0}$ and $a_1, ..., a_{d'}$ an algebraic basis such that $\gamma_t(a_i) = t^{w_i} a_i$ for some $w_i \in [1, \infty\rangle$. Then $[a, b]_t = [a, b]$ for all $t > 0$ and hence also for $t = 0$. Thus all the Lie algebras coincide.

LEMMA 3.10. *Let $a_1, ..., a_{d'}$ be an algebraic basis of rank r of a Lie algebra $\mathfrak{g}$ and set all weights equal to one as in Example* 2.3. *Then* $(\mathfrak{g}, [\cdot, \cdot]_0)$ *is a homogeneous nilpotent Lie algebra of rank r.*

Proof. Since all weights are equal to one, it follows that $\mathfrak{g}_i$ is the span of all commutators of $a_1, ..., a_{d'}$ of (unweighted) order less than or equal to i. Because the rank of the algebraic basis $a_1, ..., a_{d'}$ in $(\mathfrak{g}, [\cdot; \cdot])$ equals r there is a multi-index α with $\|\alpha\| = |\alpha| = r$ such that $a_{[\alpha]} \notin \mathfrak{g}_{r-1} = \mathfrak{g}_{\underline{r}}$. Then $a_{[\alpha]_0} = a_{[\alpha]} \bmod \mathfrak{g}_{\underline{r}} \neq 0$ and hence $a_{[\alpha]_0} \neq 0$. So the rank of the nilpotent Lie algebra $(\mathfrak{g}, [\cdot; \cdot]_0)$ is at least r. Here $a_{[\alpha]_0}$ is the multi-commutator with respect to $[\cdot; \cdot]_0$.

Conversely, the rank of the algebraic basis $a_1, ..., a_{d'}$ in $(\mathfrak{g}, [\cdot; \cdot]_0)$ is at most r by Proposition 3.1.V and the equality $\mathfrak{g}_r = \mathfrak{g}$.

Next suppose that the rank of the Lie algebra is larger than r. Then there is an $n > r$ together with $b_1, ..., b_n \in \mathfrak{g}$ such that $[b_1, [\cdots [b_{n-1}, b_n]_0 \cdots]_0]_0 \neq 0$. Expressing the b_i as linear combinations of a vector space basis $a_1, ..., a_{d'}, ..., a_d$ of $\mathfrak{g}$ one deduces that there exists an $\alpha \in J(d)$ with $|\alpha| = n$ such that $a_{[\alpha]_0} \neq 0$. But we have chosen the additional elements $a_{d'+1}, ..., a_d$ such that the basis, up to reordering, is the basis of Lemma 2.2. This implies, by the Jacobi identity, that there exists an $\alpha \in J(d')$ with $|\alpha| > r$ such that $a_{[\alpha]_0} \neq 0$. Since the rank of $a_1, ..., a_{d'}$ is at most r one then has $a_{[\alpha]_0} = \sum_{\beta \in J_r^+(d')} c_\beta a_{[\beta]_0}$, with $c_\beta \in \mathbf{R}$. It follows by scaling that the left hand side equals zero and hence $n \leqslant r$. ∎

In general the rank of the nilpotent algebra $(\mathfrak{g}, [\cdot, \cdot]_0)$ is larger than the rank of the algebraic basis $a_1, ..., a_{d'}$. Moreover, the rank of the algebraic basis $a_1, ..., a_{d'}$ in $(\mathfrak{g}, [\cdot, \cdot]_0)$ is larger than the rank of the algebraic basis in $(\mathfrak{g}, [\cdot, \cdot])$.

4. WEIGHTED SUBCOERCIVE FORMS: PART I

Let G be connected Lie group with Lie algebra $\mathfrak{g}$ and $a_1, ..., a_{d'}$ a weighted algebraic basis with weights $w_1, ..., w_{d'} \in [1, \infty\rangle$ having a common multiple, i.e., $\bigcap_{i=1}^{d'} w_i\mathbf{N} \neq \emptyset$. Set

$$w = \min\{x \in [1, \infty\rangle : x \in w_i\mathbf{N} \text{ for all } i \in \{1, ..., d'\}\}.$$

We adopt the definitions and notation introduced in Section 1.

Let $C: J(d') \to \mathbf{C}$ be an mth order form with $m \in 2w\mathbf{N}$. We called C a *G-weighted subcoercive form*, with respect to the weighted algebraic basis $a_1, ..., a_{d'}$ of $\mathfrak{g}$, if

$$\mathrm{Re}(\varphi, dL_G(C)\,\varphi) \geqslant \mu(N'_{2;\,m/2}(\varphi))^2 - \nu\,\|\varphi\|_2^2 \tag{7}$$

for some $\mu > 0$ and $\nu \in \mathbf{R}$, uniformly for all $\varphi \in C_c^\infty(V)$ where V is some open neighbourhood of the identity $e \in G$. (The condition $m \in 2w\mathbf{N}$ ensures there exist $\alpha \in J(d')$ with $\|\alpha\| = m/2$.) The least upper bound $\mu_{C,G}$ of the μ for which (7) is satisfied is called the *elliptic constant*.

The mth order operators $dU(C)$ associated with the G-weighted subcoercive forms C, a general representation $(\mathcal{X}, G, U)$ of G and a weighted algebraic basis $a_1, ..., a_{d'}$ of the corresponding Lie algebra $\mathfrak{g}$ are called *G-weighted subcoercive operators*.

We assign an angle to each subcoercive form, which subsequently provides an estimate for the lower bound of the holomorphy sector. Set

$$\theta_{C,G} = \theta_C = \sup\{\theta \in [0, \pi/2] : \forall_{\psi \in [-\theta, \theta]}$$
$$[e^{i\psi}C \text{ is a } G\text{-weighted subcoercive form}]\}.$$

Then $\theta_{C,G} \in [0, \pi/2]$. In Section 5 and Theorem 10.1 we prove that $\theta_{C,G} > 0$.

The foregoing notation explicitly identifies the relevant group G. But if this is clear from the context we omit the G.

EXAMPLE 4.1. If $G = \mathbf{R}^d$ with the usual basis $a_1, ..., a_d$ and with $w_1, ..., w_d = 1$ then an mth order form C is a (weighted) subcoercive form if, and only if, $\sum_{\alpha;\,|\alpha| = m} \mathrm{Re}\, c_\alpha (i\xi)^\alpha > 0$ for all $\xi \in \mathbf{R}^d \backslash \{0\}$. Alternatively, for general weights C is a weighted subcoercive form if, and only if, $\sum_{\alpha;\,\|\alpha\| = m} \mathrm{Re}\, c_\alpha (i\xi)^\alpha > 0$ for all $\xi \in \mathbf{R}^d \backslash \{0\}$. The conditions are equivalent to the corresponding Gårding inequalities as a consequence of Fourier theory; the differential operator is a multiplication operator on the Fourier transform.

EXAMPLE 4.2. Let $\mathfrak{g}$ be a general Lie algebra, $m \in 2w\mathbf{N}$ and for all $\alpha, \beta \in J(d')$ with $\|\alpha\| = m/2 = \|\beta\|$ let $c_{\alpha, \beta} \in \mathbf{C}$ satisfy $\operatorname{Re} \sum_{\alpha, \beta} c_{\alpha, \beta} \overline{\xi_\alpha} \xi_\beta > 0$ for all non-zero complex (ξ_α). Then the argument given in Section 1 establishes that the operator $H = \sum_{\alpha, \beta} (-1)^{|\alpha|} c_{\alpha, \beta} A^{\alpha *} A^\beta$ is a weighted subcoercive operator with respect to any representation.

EXAMPLE 4.3. Let $w_i = 1$ and $d', s \in \mathbf{N}$. Consider the free group $G = G(d', s, 1, ..., 1)$. Then a form $C \colon J(d') \to \mathbf{C}$ of order $m \in \mathbf{N}$ is a G-weighted subcoercive form of order m if, and only if, C is a subcoercive form of order m and step s (see [ElR3]).

EXAMPLE 4.4. Let $\mathfrak{g}$ be a homogeneous Lie algebra with respect to a family of dilations $(\gamma_t)_{t>0}$ and fix an algebraic basis $a_1, ..., a_{d'}$ such that $\gamma_t(a_i) = t^{w_i} a_i$ for some $w_i \in [1, \infty\rangle$. Let G be the corresponding connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Then G is homogeneous group with the dilations $(\gamma_t)_{t>0}$. A form P is called a *Rockland form* if P is homogeneous and the operator $dU(P)$ is injective on the space $\mathcal{X}_\infty(U)$ for every nontrivial irreducible unitary representation U of G. The Helffer–Nourrigat theorem, [HeN], states that a homogeneous form P is a positive Rockland form if, and only if, the operator $dL(P)|_{C_c^\infty(G)}$ is hypoelliptic. A Rockland form P is called a *positive Rockland form* if $dL(P)$ is symmetric and positive (see [AER]). In that case the operator $dL(P)$ is referred to as a *positive Rockland operator*.

Let P be a positive Rockland form of order m. By [ElR7, Lemma 2.2], there exist a basis $b_1, ..., b_d$ of $\mathfrak{g}$, $d'' \in \{1, ..., d\}$ and $v_1, ..., v_d \in [1, \infty\rangle$ such that $[\mathfrak{g}, \mathfrak{g}] \subseteq \operatorname{span}\{b_{d''+1}, ..., b_d\}$ and $\gamma_t(b_i) = t^{v_i} b_i$ for all $i \in \{1, ..., d\}$ and $t > 0$. Moreover, $b_1, ..., b_{d''}$ is an algebraic basis of $\mathfrak{g}$. We give $b_1, ..., b_{d''}$ the weights $v_1, ..., v_{d''}$. It follows from Example 2.6 that the filtration corresponding to the algebraic basis $a_1, ..., a_{d'}$ equals the filtration corresponding to the weighted algebraic basis $b_1, ..., b_{d''}$. It then follows from Lemma 2.4 in [ElR7] that $m \in 2v_i\mathbf{N}$ for all $i \in \{1, ..., d''\}$. Set $v = \min\{x \in [1, \infty\rangle : x \in v_i\mathbf{N}$ for all $i \in \{1, ..., d'\}\}$. Then by definition of v one deduces that $m \in 2v\mathbf{N}$. (We do not assume that the v_i are integers, in which case $v = \operatorname{lcm}(v_1, ..., v_{d'})$, and in which case it is well known that $m \in 2v\mathbf{N}$. In the present situation one writes $m = 2qv + x$, with $q \in \mathbf{N}$ and $x \in [0, 2v\rangle$ and easily establishes that $x = 0$.) Moreover, it follows from [ElR7, Theorem 2.5], that $dL(P)$ satisfies a Gårding inequality. So every positive Rockland operator is a weighted subcoercive operator associated with a weighted subcoercive form with respect to a suitable weighted algebraic basis of $\mathfrak{g}$.

On the other hand, if $m \in 2w\mathbf{N}$ then it follows from [ElR7, Theorem 2.5], that $dL(P)$ satisfies a Gårding inequality and hence P is a G-weighted subcoercive form.

If $m \in 2w\mathbf{N}$ then there are many positive Rockland operators of order m. For example, if P is the form such that for any representation $(\mathcal{X}, G, U)$

$$dU(P) = \sum_{i=1}^{d'} (-1)^{m/(2w_i)} A_i^{m/w_i} \tag{8}$$

(see [FoS] (4.20)) then $dL(P)$ is a positive Rockland operator. These operators have been studied in [FoS, Heb1, DzH, Dzi, DHZ, AER, ElR7].

The definition of G-subcoercivity is local insofar the Gårding inequality (7) is only required for φ supported in some arbitrarily small neighbourhood V of the identity. We show, however, that this is equivalent to a global condition, i.e., we conclude that the local Gårding inequality implies that (7) is valid for all $\varphi \in L_{2;\infty}(G)$. If the group is homogeneous this equivalence is a direct consequence of the dilation structure.

In the sequel we establish that the local Gårding inequality for C is in fact equivalent to a global inequality for the principal part P. But the proof is very indirect. It follows by the passage to a reduced weighted algebraic basis and the introduction of the corresponding homogeneous contraction G_0 of G. It is remarkable fact that the Gårding inequalities on G and G_0 are equivalent. The next proposition compares various versions of the Gårding inequalities for G and G_0. It should be emphasized that in the following proposition *all the conditions are equivalent* and, in addition, *all the ellipticity constants are equal.* But at this stage we are only able to establish some of these connections. (We prove the equivalence of the remaining implications in Section 10.)

PROPOSITION 4.5. *Let G be a connected Lie group, $a_1, ..., a_{d'}$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of G, G_0 the corresponding homogeneous contraction of G and V, V_0 open neighbourhoods of the identity in G and G_0, respectively. Further let $m \in 2w\mathbf{N}$ and C be an m-th order form with principal part P. Consider the following conditions.*

1(1′). *There is a $\mu > 0$ and $\nu \in \mathbf{R}$ such that*

$$\operatorname{Re}(\varphi, dL_G(C)\,\varphi) \geqslant \mu(N'_{2;\,m/2}(\varphi))^2 - \nu\,\|\varphi\|_2^2$$

for all $\varphi \in L_{2;\infty}(G)$ (for all $\varphi \in C_c^\infty(V)$).

2(2′). *There is a $\mu > 0$ and $\nu \in \mathbf{R}$ such that*

$$\operatorname{Re}(\varphi, dL_G(P)\,\varphi) \geqslant \mu(N'_{2;\,m/2}(\varphi))^2 - \nu\,\|\varphi\|_2^2$$

for all $\varphi \in L_{2;\infty}(G)$ (for all $\varphi \in C_c^\infty(V)$).

3(3′). *There is a* $\mu > 0$ *and* $\nu \in \mathbf{R}$ *such that*

$$\mathrm{Re}(\varphi, dL_{G_0}(C)\,\varphi) \geqslant \mu(N'_{2;\,m/2}(\varphi))^2 - \nu\,\|\varphi\|_2^2$$

for all $\varphi \in L_{2;\,\infty}(G_0)$ (*for all* $\varphi \in C_c^\infty(V_0)$).

4(4′). *There is a* $\mu > 0$ *and* $\nu \in \mathbf{R}$ *such that*

$$\mathrm{Re}(\varphi, dL_{G_0}(P)\,\varphi) \geqslant \mu(N'_{2;\,m/2}(\varphi))^2 - \nu\,\|\varphi\|_2^2$$

for all $\varphi \in L_{2;\,\infty}(G_0)$ (*for all* $\varphi \in C_c^\infty(V_0)$).

5(5′). *There is a* $\mu > 0$ *such that*

$$\mathrm{Re}(\varphi, dL_{G_0}(P)\,\varphi) \geqslant \mu(N'_{2;\,m/2}(\varphi))^2$$

for all $\varphi \in L_{2;\,\infty}(G_0)$ (*for all* $\varphi \in C_c^\infty(V_0)$).

Then $1 \Rightarrow 1' \Rightarrow 3 \Leftrightarrow 3' \Leftrightarrow 4 \Leftrightarrow 4' \Leftrightarrow 5 \Leftrightarrow 5' \Leftarrow 2' \Leftarrow 2$. *Moreover, if* $1'$ *is valid then* $\mu_{C,\,G} \leqslant \mu_{P,\,G_0} = \mu_{C,\,G_0}$ *and if* $2'$ *is valid* $\mu_{P,\,G} \leqslant \mu_{P,\,G_0} = \mu_{C,\,G_0}$.

Proof. Clearly each of the five unprimed conditions implies its primed version and $5 \Rightarrow 4$ and $5' \Rightarrow 4'$. But we have already argued that $3' \Rightarrow 5$ and hence $4' \Rightarrow 5$. Therefore we have $3' \Rightarrow 5 \Leftrightarrow 5' \Leftrightarrow 4 \Leftrightarrow 4'$. Hence it suffices to prove that $1' \Rightarrow 5'$ and $5 \Rightarrow 3$, because $2' \Rightarrow 4'$ follows from $1' \Rightarrow 4'$.

First we prove $5 \Rightarrow 3$. Let P_0 be the form defined in (8). It was established in [AER, Proposition 2.1], that $dL(P_0)$ is a positive self-adjoint operator and there exists a $c > 0$ such that $\|\varphi\|'_{2;\,m} \leqslant c(\|dL(P_0)\,\varphi\|_2 + \|\varphi\|_2)$ for all $\varphi \in D(dL(P_0)) = L'_{2;\,m}$. Now let C_1 be a form of order less than or equal to m. Then the operator $dL(C_1 + C_1^\dagger)$ is symmetric and $dL(P_0)$-bounded, by the foregoing bounds. Hence by Theorem VI.1.38 of [Kat] one deduces that there exist $c_1 > 0$ and $c_2 \in \mathbf{R}$ such that $|(\varphi, dL(C_1 + C_1^\dagger)\,\varphi)| \leqslant c_1(\varphi, dL(P_0)\,\varphi) + c_2\,\|\varphi\|_2^2$ for all $\varphi \in L'_{2;\,m}$. The same argument applies to the operator $i(dL(C_1 - C_1^\dagger))$ and by linear combination it follows that there exist $c'_1 > 0$ and $c'_2 \in \mathbf{R}$ such that $|(\varphi, dL(C_1)\,\varphi)| \leqslant c'_1(\varphi, dL(P_0)\,\varphi) + c'_2\,\|\varphi\|_2^2$. Next there exists a $c_3 > 0$, such that $(\varphi, dL(P_0)\,\varphi) = \|dL(P_0)^{1/2}\,\varphi\|_2^2 \leqslant c_3(N'_{2;\,m/2}(\varphi))^2$ for all $\varphi \in L'_{2;\,m}$ (see [ElR7, Corollary 2.6.II]). Combining these estimates it follows that for all $\alpha \in J(d')$ with $\|\alpha\| \leqslant m$ there exist $c_1, c_2 > 0$ such that $|(\varphi, A^\alpha \varphi)| \leqslant c_1(N'_{2;\,m/2}(\varphi))^2 + c_2\,\|\varphi\|_2^2$ for all $\varphi \in L'_{2;\,m}$. Therefore, by a scaling argument, one concludes that

$$|(\varphi, A^\alpha \varphi)| \leqslant c_1 \varepsilon^{m - \|\alpha\|}(N'_{2;\,m/2}(\varphi))^2 + c_2 \varepsilon^{-\|\alpha\|}\,\|\varphi\|_2^2 \tag{9}$$

for all $\varepsilon > 0$ if $\|\alpha\| < m$ and

$$|(\varphi, A^\alpha \varphi)| \leqslant c_1(N'_{2;\,m/2}(\varphi))^2 \tag{10}$$

if $\|\alpha\| = m$. Then $5 \Rightarrow 3$ follows from (9). Moreover, it follows from the proof that $\mu_{C, G_0} = \mu_{P, G_0}$.

Now we prove $1' \Rightarrow 5'$. We temporally denote by $(\cdot, \cdot)_0$ the inner product on $L_2(G_0)$ and $(\cdot, \cdot)$ the inner product on $L_2(G)$. Let W be the open neighbourhood of 0 in $\mathfrak{g}$ as introduced in Lemma 3.3.V. We may assume $\exp(W) \subseteq V$ and $\exp_0(W) = V_0$ and use the notation of Section 3. Let $\varphi \in C_c^\infty(V_0)$ and set $\psi = \varphi \circ \exp_0 \in C_c^\infty(W)$. For $t \in [0, 1]$ set

$$P_{X^{(t)}} = \sum_{\substack{\alpha \in J(d') \\ \|\alpha\| = m}} c_\alpha X^{(t)\,\alpha},$$

where we have used multi-index notation. Then

$$\mathrm{Re}(\varphi, dL_{G_0}(P)\,\varphi)_0 = \mathrm{Re} \int da\, \overline{\psi(a)}\, (P_{X^{(0)}}\psi)(a)$$

and we approximate $\mathrm{Re} \int da\, \overline{\psi(a)}\, (P_{X^{(0)}}\psi)(a)$ by $\mathrm{Re} \int da\, \sigma_t(a)\, \overline{\psi(a)}\, (P_{X^{(t)}}\psi)(a)$.

Let $t \in \langle 0, 1]$. Now Lemma 3.6, applied to $\psi_{t^{-1}}$ gives $P_{X^{(t)}}\psi = P_{X^{(t)}}(\psi_{t^{-1}})_t = t^m (P_X \psi_{t^{-1}})_t$ where $P_X = P_{X^{(1)}}$. Since $\psi_{t^{-1}}, P_{X^{(t)}}\psi_{t^{-1}} \in C_c^\infty(\gamma_t(W))$ one can then use Lemma 3.8 to deduce that

$$\begin{aligned}
\mathrm{Re} \int da\, \sigma_t(a)\, \overline{\psi(a)}\, (P_{X^{(t)}}\psi)(a) \\
&= t^m\, \mathrm{Re}((\psi_{t^{-1}} \circ \log), dL_G(C)(\psi_{t^{-1}} \circ \log)) \\
&\quad - \sum_{\|\alpha\| < m} t^m\, \mathrm{Re}((\psi_{t^{-1}} \circ \log), c_\alpha\, dL_G(A^\alpha)(\psi_{t^{-1}} \circ \log)).
\end{aligned}$$

Since $dL_G(C)$ satisfies the Gårding inequality on $C_c^\infty(V)$ one then estimates that

$$\begin{aligned}
\mathrm{Re} \int da\, \sigma_t(a)\, \overline{\psi(a)}\, (P_{X^{(t)}}\psi)(a) \\
&\geqslant \mu t^m \max_{\|\alpha\| = m/2} \int da\, \sigma(a)\, |(X^{(1)\,\alpha}\psi_{t^{-1}})(a)|^2 - \nu t^m \int da\, \sigma(a)\, |\psi_{t^{-1}}(a)|^2 \\
&\quad - \sum_{\|\alpha\| < m} t^m\, \mathrm{Re}\, c_\alpha \int da\, \sigma(a)\, \overline{\psi_{t^{-1}}(a)}\, (X^{(1)\,\alpha}\psi_{t^{-1}})(a).
\end{aligned}$$

Since $\psi_{t^{-1}}, X^{(1)\alpha}\psi_{t^{-1}} \in C_c^\infty(\gamma_t(W))$ one can again use Lemmas 3.8 and 3.6 to deduce that

$$\begin{aligned}
&\operatorname{Re} \int da\, \sigma_t(a)\, \overline{\psi(a)}\, (P_{X^{(t)}}\psi)(a) \\
&\quad \geqslant \mu \max_{\|\alpha\| = m/2} \int da\, \sigma_t(a)\, |(X^{(t)\alpha}\psi)(a)|^2 - \nu t^m \int da\, \sigma_t(a)\, |\psi(a)|^2 \\
&\qquad - \sum_{\|\alpha\| < m} t^{m - \|\alpha\|} \operatorname{Re} c_\alpha \int da\, \sigma_t(a)\, \overline{\psi(a)}\, (X^{(t)\alpha}\psi)(a).
\end{aligned}$$

Now it follows from (6) that $\lim_{t\to 0} \sigma_t(a) = \lim_{t\to 0} \sigma(\gamma_t(a)) = \sigma(0) = 1 = \sigma_0(a)$ uniformly for all $a \in W$. Therefore, as an application of Lemma 3.5, one estimates that

$$\begin{aligned}
&\operatorname{Re}(\varphi, dL_{G_0}(P)\, \varphi)_0 \\
&\quad = \lim_{t\to 0} \operatorname{Re} \int da\, \sigma_t(a)\, \overline{\psi(a)}\, (P_{X^{(t)}}\psi)(a) \\
&\quad \geqslant \lim_{t\to 0} \Big(\mu \max_{\|\alpha\| = m/2} \int da\, \sigma_t(a)\, |(X^{(t)\alpha}\psi)(a)|^2 - \nu t^m \int da\, \sigma_t(a)\, |\psi(a)|^2 \\
&\qquad - \sum_{\|\alpha\| < m} t^{m - \|\alpha\|} \operatorname{Re} c_\alpha \int da\, \sigma_t(a)\, \overline{\psi(a)}\, (X^{(t)\alpha}\psi)(a) \Big) \\
&\quad = \mu \max_{\|\alpha\| = m/2} \int da\, \sigma_0(a)\, |(X^{(0)\alpha}\psi)(a)|^2 = \mu (N'_{(0);\,2;\,m/2}(\varphi))^2,
\end{aligned}$$

where $N'_{(0);\,2;\,m/2}$ denotes the seminorm on G_0. This completes the proof. ∎

5. HOMOGENEOUS GROUPS

Our analysis of subcoercive operators on the Lie group G proceeds by studying the comparable problems on the homogeneous contraction G_0 of G and then extending the results to G by a parametrix argument.

Let $\mathfrak{g}$ be a homogeneous Lie algebra with respect to a family of dilations $(\gamma_t)_{t>0}$ and $a_1, ..., a_{d'}$ an algebraic basic such that $\gamma_t(a_i) = t^{w_i} a_i$ for some $w_i \in [1, \infty\rangle$. Then $a_1, ..., a_{d'}$ is a reduced weighted algebraic basis by Example 2.6. Further let G be the corresponding connected, simply connected, homogeneous Lie group and $C\colon J(d') \to \mathbf{C}$ a G-weighted subcoercive form of order m where $m \in 2w\mathbf{N}$. In this section we assume the form C is homogeneous, so $C = P$, the principal part of C. Let $H = dL(C)$ be the corresponding homogeneous weighted subcoercive operator on $L_2(G)$.

First we prove that $H|_{C_c^\infty}$ is hypoelliptic and then deduce that H generates a holomorphic semigroup S on the sector $\Lambda(\theta_C)$ with a smooth kernel K satisfying Gaussian type bounds. The homogeneous structure is exploited through scaling arguments. The arguments are an amalgamation of methods used in [HeN, ElR3, AER].

As a preliminary we observe that $\theta_C > 0$. Set

$$\tau = \sup\{|\mathrm{Im}(dL_G(P)\,\varphi, \varphi)|/(N'_{2;\,m/2}(\varphi))^2 : \varphi \in L_{2;\,\infty}(G),\ \varphi \neq 0\}.$$

Then $\tau < \infty$ by (10) and $\theta_C \geqslant \arctan(\mu\tau^{-1}) > 0$.

Lemma 5.1. *The operator* $H|_{C_c^\infty} = dL(C)|_{C_c^\infty}$ *is hypoelliptic.*

Proof. We prove that $dL(C)$ is a Rockland operator, i.e., the operator $dU(C)$ is injective on $\mathscr{H}_\infty(U)$ for each non-trivial irreducible representation $(\mathscr{H}, G, U)$. But this is equivalent to hypoellipticity by the Helffer–Nourrigat theorem [HeN].

It follows from Proposition 4.5 that $\mathrm{Re}(\varphi, H\varphi) \geqslant \mu(N'_{2;\,m/2}(\varphi))^2$ for all $\varphi \in L_{2;\,\infty}(G)$, where $\mu = \mu_{C,\,G}$ is the ellipticity constant. So

$$\mathrm{Re}(\varphi, dL(C)\,\varphi) \geqslant \mu \max_{\substack{\alpha \in J(d')\\ \|\alpha\| = m/2}} \mathrm{Re}(-1)^{|\alpha|}\,(\varphi, A^{\langle \alpha_*,\, \alpha\rangle}\varphi) \tag{11}$$

for all $\varphi \in L_{2;\,\infty}(G)$. Now we argue as in Section 2 and the proof of Proposition 4.6.2 in Helffer–Nourrigat [HeN] that

$$\mathrm{Re}(x, dU(C)\,x) \geqslant \mu \max_{\substack{\alpha \in J(d')\\ \|\alpha\| = m/2}} \mathrm{Re}(-1)^{|\alpha|}\,(x, A^{\langle \alpha_*,\, \alpha\rangle}x) \tag{12}$$

for every irreducible unitary representation $(\mathscr{H}, G, U)$ and every $x \in \mathscr{H}_\infty(U)$.

At this point we merely sketch the proof since Proposition 10.2 requires a refinement of the result and we then give full details. First note that Proposition 2.1 of [HeN], which gives a norm comparison for two operators, has a version expressed in terms of the forms of the operators. This is evident from the proof of the proposition which relies on Fourier transformation. Replacing the Fourier transforms of the operator norms by similar transforms of the operator forms does not affect the argument.

Secondly, one deduces (12) from (11) by the same argument that establishes Proposition 4.6.2 as a consequence of Proposition 4.6.1 in [HeN]. This relies upon the form version of Proposition 2.1.

One immediately deduces from (12) that

$$\mathrm{Re}(x, dU(C)\,x) \geqslant \mu(N'_{U;\,m/2}(x))^2 \tag{13}$$

for every irreducible unitary representation $(\mathscr{H}, G, U)$ and every $x \in \mathscr{H}_\infty(U)$.

Now let $(\mathscr{H}, G, U)$ be a non-trivial irreducible unitary representation and for $x \in \mathscr{H}_\infty(U)$ suppose $dU(C)\, x = 0$. Then $(x, dU(C)x) = 0$ and $N'_{U;\, m/2}(x) = 0$ by (13). Hence $dU(a_i)^{m/(2w_i)}\, x = 0$ and, by spectral theory, one also has $dU(a_i)\, x = 0$ for all $i \in \{1, ..., d'\}$. Since $a_1, ..., a_{d'}$ is an algebraic basis for $\mathfrak{g}$ it follows that $dU(a)\, x = 0$ for all $a \in \mathfrak{g}$ and since U is non-trivial this implies that $x = 0$. Thus $dL(C)$ is a Rockland operator and therefore hypoelliptic. ▮

The Helffer–Nourrigat theorem even states that the operator $H + \lambda I$ is hypoelliptic for all $\lambda \in \mathbf{C}$. We need a variant of hypoellipcity involving an extension of H.

Lemma 5.2. *Let $\varphi \in D((H^\dagger)^*)$ where $H^\dagger = dL(C^\dagger)$. If $((H^\dagger)^* + \lambda I)\, \varphi \in L_{2;\, \infty}$ for some $\lambda \in \mathbf{R}$ then $\varphi \in L_{2;\, \infty}$.*

Proof. This follows as in the proof of Lemma 4.6 in [ElR3]. ▮

The hypoelliptic of H has many consequences. In particular by special choice of H one can deduce regularity properties.

Corollary 5.3. I. *The operator H is closed on $L'_{2;\, m}$.*

II. *For all $n \in \mathbf{N}$ one has $D(H^n) = L'_{2;\, nm}$, with equivalent norms. There exists a $c > 0$ such that $cN'_{2;\, nm}(\varphi) \leqslant \|H^n \varphi\|_2$ for all $\varphi \in D(H^n)$.*

III. *The spaces $L_{2;\, \infty}$ and $C_c^\infty(G)$ are cores for H^n, for all $n \in \mathbf{N}$.*

IV. *If $n \in \mathbf{N}$ and $k \in \langle 0, nm \rangle$ then there exists a $c > 0$ such that*

$$N'_{2;\, k}(\varphi) \leqslant \varepsilon^{nm-k} N'_{2;\, nm}(\varphi) + c\varepsilon^{-k}\, \|\varphi\|_2$$

for all $\varepsilon > 0$ and $\varphi \in L'_{2;\, nm}$.

V. *If $n \in \mathbf{N}$ and $k \in \langle 0, nm \rangle$ then there exists a $c > 0$ such that*

$$\|\varphi\|'_{2;\, k} \leqslant \varepsilon^{nm-k}\, \|\varphi\|'_{2;\, nm} + c\varepsilon^{-k}\, \|\varphi\|_2$$

for all $\varepsilon > 0$ and $\varphi \in L'_{2;\, nm}$.

Proof. This follows from the Helffer–Nourrigat theory and scaling. See also [AER, Proposition 2.1]. The last two statements are a consequence of the specific example (8). ▮

Proposition 5.4. *The operator $H = dL(C)$ generates a holomorphic semigroup S on $L_2(G)$, with a holomorphy sector containing the sector $\Lambda(\theta_C)$, which satisfies $\|S_z\|_{2 \to 2} \leqslant 1$ for all $z \in \Lambda(\theta_C)$.*

Proof. The Gårding inequality establishes that

$$\|(H + \lambda I)\, \varphi\|_2\, \|\varphi\|_2 \geqslant \mathrm{Re}(\varphi, (H + \lambda I)\, \varphi) \geqslant \lambda\, \|\varphi\|_2^2$$

for all $\lambda > 0$ and $\varphi \in L_{2;\,\infty}$. Hence

$$\|(H+\lambda I)\,\varphi\|_2 \geqslant \lambda\,\|\varphi\|_2 \tag{14}$$

for all $\varphi \in D(H)$. So $H+\lambda I$ is injective for all $\lambda > 0$. If we show that the range $R(H+\lambda I)$ of $H+\lambda I$ is equal to L_2 for some $\lambda > 0$ then it follows from the Hille–Yosida theorem that H generates a contraction semigroup. It suffices to prove that $R(H+\lambda I)$ is dense since the space $R(H+\lambda I)$ is closed by (14).

The formal adjoint $C^\dagger$ is also a G-weighted subcoercive form so we can apply the above reasoning to $H^\dagger = dL(C^\dagger)$ and deduce that

$$\|(H^\dagger+\lambda I)\,\varphi\|_2 \geqslant \lambda\,\|\varphi\|_2 \tag{15}$$

for all $\varphi \in D(H^\dagger) = L'_{2;\,m}$ and $\lambda > 0$. Fix $\lambda > 0$. Let P be the projection of L_2 onto $R(H^\dagger+\lambda I)$. By (15) the map $T\colon (H^\dagger+\lambda I)\,\varphi \mapsto \varphi$ from $R(H^\dagger+\lambda I)$ into L_2 is continuous. Let $E = TP$. Then E is continuous and $E(H^\dagger+\lambda I)\,\varphi = \varphi$ for all $\varphi \in L'_{2;\,m}$. So for all $\psi \in L_2$ one has

$$(E^*\psi, (H^\dagger+\lambda I)\,\varphi) = (\psi, E(H^\dagger+\lambda I)\,\varphi) = (\psi, \varphi)$$

for all $\varphi \in L'_{2;\,m}$. Therefore $E^*\psi \in D((H^\dagger+\lambda I)^*)$ and $(H^\dagger+\lambda I)^*\,E^*\psi = \psi$. Now it follows from Lemma 5.2 that $E^*\psi \in L_{2;\,\infty} \subseteq D(H+\lambda I)$ for all $\psi \in L_{2;\,\infty}$. Hence $(H+\lambda I)\,E^*\psi = \psi$ for all $\psi \in L_{2;\,\infty}$. Therefore $L_{2;\,\infty} \subseteq R(H+\lambda I)$ and $R(H+\lambda I)$ is dense. Thus H generates a contraction semigroup.

Finally, for any $\theta \in \langle -\theta_C, \theta_C\rangle$ the form $e^{i\theta}C$ is also a G-weighted subcoercive form of order m, so the operator $e^{i\theta}H = dU(e^{i\theta}C)$ is the generator of a contradiction semigroup. Then by [Kat, Theorem IX.1.23], it follows that H is the generator of a holomorphic semigroup which is holomorphic in a sector with angle at least θ_C. This completes the proof of the proposition. ∎

Let $|\cdot|$ be a homogeneous modulus on G (see [HeS]). Extend the algebraic basis to a vector space basis $a_1, ..., a_{d'}, ..., a_d$ such that for each $i \in \{d'+1, ..., d\}$ there exists a $w_i \in [1, \infty\rangle$ such that $\gamma_t(a_i) = t^{w_i} a_i$ for all $t > 0$. Set $D' = \sum_{i=1}^d w_i$.

Proposition 5.5. *The holomorphic semigroup S generated by H has a smooth kernel $K \in L_{1;\,\infty}(G) \cap C_{0;\,\infty}(G)$ such that*

$$(A^\alpha S_z \varphi)(g) = \int_G dh\,(A^\alpha K_z)(h)\,\varphi(h^{-1}g)$$

for all $\alpha \in J(d')$, $z \in \Lambda(\theta_C)$, $\varphi \in L_2(G)$ *and* $g \in G$. *Moreover, the function* $z \mapsto K_z(g)$ *is analytic on* $\Lambda(\theta_C)$, *uniformly for* $g \in G$, *and for each* $\alpha \in J(d')$ *and* $\varepsilon \in \langle 0, \theta_C \rangle$ *there exists* $b, c > 0$ *such that*

$$|(A^\alpha K_z)(g)| \leqslant c\, |z|^{-(D' + \|\alpha\|)/m}\, e^{-b(|g|^m\, |z|^{-1})^{1/(m-1)}}$$

for all $z \in \Lambda(\theta_C - \varepsilon)$ *and all* $g \in G$.

Proof. The proof is almost the same as the proofs of Proposition 2.2 and Corollary 2.3 in [AER], with one significant difference. This occurs at the beginning of the proof of Proposition 2.2, where the spectral theorem was used to establish that

$$(\|(\lambda I + H)^{-n}\|_{L_2 \to L'_{2;\, nm}})^{-1} \geqslant c_1^{-1} \tag{16}$$

for a $c_1 > 0$ and all $n \in \mathbf{N}$ uniformly for $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$ with $|\lambda| = 1$ for the self-adjoint operator H in [AER]. Since the present operator H is not necessarily symmetric we have to give a new proof of (16), uniformly for $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$ with $|\lambda| = 1$.

Let $n \in \mathbf{N}$. By Corollary 5.3 there exist $c_1, c_2 > 0$ such that $N'_{2;\, nm}(\varphi) \leqslant c_1\, \|H^n \varphi\|_2$ and $\|H^{n-j}\varphi\|_2 \leqslant \delta^j\, \|H^n\varphi\|_2 + c_2\, \delta^{-n+j}\, \|\varphi\|_2$ for all $\varphi \in L'_{2;\, nm}$, $j \in \{1, ..., n-1\}$ and $\delta > 0$. Let $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$. Then

$$\begin{aligned} \|H^n\varphi\| &\leqslant \|(\lambda I + H)^n\, \varphi\|_2 + \sum_{j=1}^{n} \binom{n}{j} |\lambda|^j\, \|H^{n-j}\varphi\|_2 \\ &\leqslant \|(\lambda I + H)^n\, \varphi\|_2 + ((1 + |\lambda|\, \delta)^n - 1)\, \|H^n\varphi\|_2 \\ &\quad + \delta^{-n} c_2((1 + |\lambda|\, \delta)^n - 1)\, \|\varphi\|_2 \end{aligned}$$

for all $\varphi \in L'_{2;\, nm}$.

Let $\theta \in \langle -\theta_C + \varepsilon/2, \theta_C - \varepsilon/2 \rangle$ be such that $\rho = e^{i\theta}\lambda \in \Lambda(\pi/2 - \varepsilon/2)$. Then the Gårding inequality, applied to the form $e^{i\theta}C$, gives

$$\begin{aligned} \|(\lambda I + H)\, \varphi\|_2\, \|\varphi\|_2 = &\geqslant \mathrm{Re}(\varphi, (\rho I + e^{i\theta}H)\, \varphi) \geqslant (\mathrm{Re}\, \rho)\, \|\varphi\|_2^2 \\ &\geqslant |\lambda|\, \sin(\varepsilon/2)\, \|\varphi\|_2^2 \end{aligned}$$

for all $\varphi \in L'_{2;\, m}$. Hence by induction $\|\varphi\|_2 \leqslant |\lambda|^{-n}\, (\sin(\varepsilon/2))^{-n}\, \|(\lambda I + H)^n\, \varphi\|_2$ uniformly for all $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$ and $\varphi \in L'_{2;\, nm}$.

Taking $\delta > 0$ such that $(1 + |\lambda|\, \delta)^n - 1 = 1/2$ one establishes that

$$\begin{aligned} N'_{2;\, nm}(\varphi) &\leqslant c_1\, \|H^n\varphi\|_2 \\ &\leqslant (2c_1 + 2c_1 c_2(((3/2)^{1/n} - 1)\, \sin(\varepsilon/2))^{-n})\, \|(\lambda I + H)^n\, \varphi\|_2 \end{aligned}$$

uniformly for all $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$ and $\varphi \in L'_{2;\, nm}$. Next, using Proposition 5.3.IV one deduces that there exists a $c_3 > 0$ such that

$$\|\varphi\|'_{2;\, nm} \leqslant 2^{-1} c_3 (1 + |\lambda|^{-n}) \, \|(\lambda I + H)^n \varphi\|_2.$$

So the operators $(\lambda I + H)^{-n}$ map the Hilbert space L_2 continuously into the Banach space $L'_{2;\, nm}$ and (16) is valid with c_1 replaced by c_3, uniformly for all $\lambda \in \Lambda(\pi/2 + \theta_C - \varepsilon)$ with $|\lambda| = 1$. Now one can proceed as in [AER]. We omit the details. ▮

Next define $K_t = 0$ for $t \leqslant 0$. Then $(t, g) \mapsto K_t(g)$ is a C^∞-function on $(\mathbf{R} \times G) \backslash \{(0, e)\}$ and the Gaussian bounds imply that this function is a distribution on $\mathbf{R} \times G$.

Proposition 5.6. *One has* $((\partial_t + H) K_t)(g) = \delta(t)\, \delta(g)$ *as distributions.*

Proof. It follows from [AER] that $((\partial_t + H) K_t)(g) = 0$ pointwise if $(t, g) \neq (0, e)$. The fact that $(t, g) \mapsto K_t(g)$ is a fundamental solution of the operator $\partial_t + H$ then follows as in Folland [Fol1, Proposition 3.3]. ▮

Now we introduce a class of differential operators which is useful in Sections 7 and 8 to prove that weighted subcoercive operators generate holomorphic semigroups. One has

$$|g|^n \, |(A^\alpha K_t)(g)| \leqslant c \, |t|^{-(D' + \|\alpha\| - n)/m} \, e^{-b(|g|^m \, |t|^{-1})^{1/(m-1)}}$$

for all $\alpha \in J(d')$ and $n \in [0, \infty\rangle$. Thus differentiation introduces an additional singularity $t^{-\|\alpha\|/m}$ but multiplication with $|g|^n$ introduces a factor $t^{n/m}$, which effectively removes the singularity. This motivates the following definition. Let M_f denote the operator of multiplication with the function f. An nth order differential operator

$$L = \sum_{\substack{\alpha \in J(d') \\ \|\alpha\| \leqslant n}} M_{f_\alpha} A^\alpha,$$

with variable C^∞-coefficients f_α, on an open set V containing the identity element e, is defined to be an *operator of actual order* N if there exists a $c > 0$ and an open neighbourhood B of the identity e such that $|f_\alpha(g)| \leqslant c \, |g|^{n_\alpha(N)}$ for all $\alpha \in J(d')$ with $\|\alpha\| \leqslant n$ and $g \in B \cap V$ where $n_\alpha(N) = (\|\alpha\| - N) \vee 0$.

One can restate the foregoing inequalities.

COROLLARY 5.7. *Let L be a differential operator on V of actual order N with $N \in [0, \infty\rangle$. Then for each compact subset B of V there exist $b, c > 0$ and $\omega \geqslant 0$ such that*

$$|(LK_t)(g)| \leqslant c\,|t|^{-(D'+N)/m}\, e^{\omega t} e^{-b(|g|^m\,|t|^{-1})^{1/(m-1)}}$$

uniformly for all $g \in B$ and $t > 0$.

The next lemma gives another description of the actual order of a differential operator.

LEMMA 5.8. *Let V be an open neighbourhood of the identity element e in G and $\varphi\colon V \to \mathbf{C}$ a C^∞-function. Then for each $n \in [1, \infty\rangle$ the following are equivalent.*

I. *$(A^\alpha \varphi)(e) = 0$ for all $\alpha \in J(d')$ with $\|\alpha\| < n$.*

II. *For every compact neighbourhood K of e such that $K \subset V$ there exists $c > 0$ such that $|\varphi(g)| \leqslant c\,|g|^n$ for all $g \in K$.*

III. *There exist a compact neighbourhood K of e such that $K \subset V$ and a $c > 0$ such that $|\varphi(g)| \leqslant c\,|g|^n$ for all $g \in K$.*

Proof. Extend the algebraic basis $a_1, ..., a_{d'}$ to a vector space basis $a_1, ..., a_d$ such that for all $i > d'$ there exists a $w_i \in [1, \infty\rangle$ with $\gamma_t(a_i) = t^{w_i} a_i$ for all $t > 0$. Define a modulus $|\cdot|$ on $\mathfrak{g}$ by

$$\left|\sum_{i=1}^{d} \xi_i a_i\right|^{2\bar{w}} = \sum_{i=1}^{d} |\xi_i|^{2\bar{w}/w_i},$$

where $\bar{w} = \min\{x \in [1, \infty\rangle : x \in w_i \mathbf{N} \text{ for all } i \in \{1, ..., d\}\}$. Then by scaling there exists a $c \geqslant 1$ such that $c^{-1}\,|a| \leqslant |\exp a| \leqslant c\,|a|$ for all $a \in \mathfrak{g}$. Moreover, $\|a\| \leqslant d^{1/2}\,|a|$ if $\|a\| \leqslant 1$. Next, let $X_i = X_i^{(1)}$ be the vector fields defined by (2), but now for the full basis of the Lie algebra $\mathfrak{g}$. Let $\psi = \varphi \circ \exp$ and $N \in \mathbf{N}$ with $N \geqslant n$. Then for all $a = \sum_{i=1}^{d} \xi_i a_i \in \mathfrak{g}$ one has by the usual Taylor formula

$$\begin{aligned}
\varphi(\exp a) = \psi(a) &= \sum_{\substack{\alpha \in J(d)\\ |\alpha| \leqslant N}} \alpha!^{-1} (X^\alpha \psi)(0)\, \xi^\alpha + O(\|a\|^N) \\
&= \sum_{\substack{\alpha \in J(d)\\ |\alpha| \leqslant N}} \alpha!^{-1} (A^\alpha \varphi)(e)\, \xi^\alpha + O(|a|^N) \\
&= \sum_{\substack{\alpha \in J(d)\\ |\alpha| \leqslant N}} \alpha!^{-1} (A^\alpha \varphi)(e)\, \xi^\alpha + O(|a|^n)
\end{aligned}$$

as $a \to 0$. Here $\alpha! = k_1! \cdots k_d!$ if $\alpha = (i_1, ..., i_m)$ and $k_l = \#\{j \in \{1, ..., m\} : i_j = l\}$. But if $\alpha \in J(d)$ with $|\alpha| \leqslant N$ and $\|\alpha\| \geqslant n$ then

$$(A^\alpha \varphi)(e)\, \xi^\alpha = O(|a|^{\|\alpha\|}) = O(|a|^n), \tag{17}$$

where we have used the inequality $|\xi_i| \leqslant |a|^{w_i}$. Hence

$$\varphi(\exp a) = \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| < n}} \alpha!^{-1} (A^\alpha \varphi)(e)\, \xi^\alpha + O(|a|^n). \tag{18}$$

Now the implication I $\Rightarrow$ II is obvious.

The implication II $\Rightarrow$ III is trivial.

III $\Rightarrow$ I. Suppose that $|\varphi(g)| \leqslant c\,|g|^n$ for all $g \in K$. Then $\varphi(e) = 0$. We shall prove by induction on k that $(A^\alpha \varphi)(e) = 0$ for all $\varphi \in J(d)$ with $\|\alpha\| = k$. Let $k \in [0, n\rangle$ and suppose that $(A^\alpha \varphi)(e) = 0$ for all α with $\|\alpha\| < k$. Then for all $a = \sum_{i=1}^d \xi_i a_i \in \mathfrak{g}$ one has

$$\sum_{\substack{\alpha \in J(d) \\ \|\alpha\| = k}} \alpha!^{-1} (A^\alpha \varphi)(e)\, \xi^\alpha$$

$$= \varphi(\exp a) - \sum_{\substack{\alpha \in J(d) \\ n > \|\alpha\| > k}} \alpha!^{-1} (A^\alpha \varphi)(e)\, \xi^\alpha + O(|a|^n) = o(|a|^k)$$

by (17) and (18). Next fix $a = \sum_{i=1}^d \xi_i a_i \in \mathfrak{g}$. Then the scaling $\eta_i = u^{w_i} \xi_i$ gives

$$u^k \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| = k}} (A^\alpha \varphi)(e)\, \xi^\alpha = \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| = k}} (A^\alpha \varphi)(e)\, \eta^\alpha = o(|\gamma_u(a)|^k) = o(u^k)$$

for all small $u > 0$. Therefore

$$\sum_{\substack{\alpha \in J(d) \\ \|\alpha\| = k}} \alpha!^{-1} (A^\alpha \varphi)(e)\, \xi^\alpha = 0. \tag{19}$$

We next prove that $(A^\alpha \varphi)(e) = 0$ for all $M \in \mathbf{N}$ and all $\alpha \in J(d)$ with $\|\alpha\| = k$ and $|\alpha| = M$. The proof is by induction on M. If $M = 1$ then $\alpha = (i)$ for some $i \in \{1, ..., d\}$ and one substitutes $\xi = (0, ..., 1, ..., 0)$ with the non-zero entry in the ith place. Therefore $(A^\alpha \varphi)(e) = 0$ if $|\alpha| = 1$. Next let $M \in \mathbf{N}$, $M \geqslant 2$ and suppose that $(A^\alpha \varphi)(e) = 0$ for all $\alpha \in J(d)$ with $\|\alpha\| = k$ and $|\alpha| < M$.

Let $\alpha=(i_1, ..., i_M)\in J(d)$, $j\in\{1, ..., M-1\}$ with $\|\alpha\|=k$. If $\beta=(i_1, ..., i_{j-1}, i_{j+1}, i_j, i_{j+2}, ..., i_M)$ then

$$(A^\alpha\varphi)(e)=(A^\beta\varphi)(e)+\sum_{l=1}^{d} c^l_{i_j i_{j+1}}(A^{\delta_l}\varphi)(e)$$

where the c^l_{ij} are the structure constants of $\mathfrak{g}$ with respect to the basis $a_1, ..., a_d$ and $\delta_l=(i_1, ..., i_{j-1}, l, i_{j+2}, ..., i_M)$. Since $\mathfrak{g}$ is homogeneous one has $c^l_{i_j i_{j+1}}=0$ if $w_l\neq w_{i_j}+w_{i_{j+1}}$. Therefore $\|\delta_l\|=k$ for all l such that the term in the sum does not vanish. But $|\delta_l|=M-1$. Hence by the induction hypothesis $(A^{\delta_l}\varphi)(e)=0$. It follows that $(A^\alpha\varphi)(e)=(A^\beta\varphi)(e)$. Consequently, $(A^{\alpha_\sigma}\varphi)(e)=(A^\alpha\varphi)(e)$ for all $\alpha=(i_1, ..., i_M)\in J(d)$ with $\|\alpha\|=k$ and all $\sigma\in S_M$, the permutation group, where $\alpha_\sigma=(i_{\sigma(1)}, ..., i_{\sigma(M)})$. Now it follows from (19) by differentiation that $(A^\alpha\varphi)(e)=0$. ∎

One important implication of the Lemma 5.8 is that if L_1 and L_2 are differential operators with variable coefficients and actual orders N_1 and N_2, respectively, then $L_1\circ L_2$ is a similar operator but with actual order N_1+N_2.

Finally we need an estimate for the kernel of the resolvent

$$(\lambda I+H)^{-1}=\int_0^\infty dt\, e^{-\lambda t}S_t,$$

where $\lambda\in\mathbf{C}$ with $\operatorname{Re}\lambda>0$, of the closed operator $H=dL(C)$. The estimates of Proposition 5.5 establish that $(\lambda I+H)^{-1}$ has a kernel R_λ given by

$$R_\lambda(g)=\int_0^\infty dt\, e^{-\lambda t}K_t(g)$$

and $R_\lambda\in L_1(G)$ with $\|R_\lambda\|_1\leqslant c(\operatorname{Re}\lambda)^{-1}$ for a suitable $c>0$ and all $\lambda\in\mathbf{C}$ such that $\operatorname{Re}\lambda>0$. Since $K_t\in C^\infty(G)$ for $t>0$ it follows that $R_\lambda\in C^\infty(G\backslash\{e\})$. Moreover, $A^\alpha R_\lambda\in L_1(G)$ for all $\alpha\in J(d)$ with $\|\alpha\|<m$ and $\|A^\alpha R_\lambda\|_1\leqslant c'(\operatorname{Re}\lambda)^{-(m-\|\alpha\|)/m}$ for $\operatorname{Re}\lambda>0$. (For a related discussion of strongly elliptic operators see [Rob] Section III.6b and Appendix A of [ElR4].) Higher derivatives of R_λ are, however, not in $L_1(G)$ because of singularities at the identity e. Nevertheless differential operators of order larger or equal to m but with actual order less than m do map R_λ into $L_1(G)$.

LEMMA 5.9. *Let L be a differential operator on V of actual order N with $N\in[0, m\rangle$ and B a compact subset of V. If 1_B denotes the characteristic function of B then $1_B L R_\lambda\in L_1$ and there are $c>0$ and $\rho\geqslant 0$ such that*

$$\|1_B L R_\lambda\|_1\leqslant c(\operatorname{Re}\lambda)^{-(m-N)/m}$$

for all $\lambda\in\mathbf{C}$ with $\operatorname{Re}\lambda>\rho$.

Proof. It follows from Corollary 5.7 that

$$\begin{aligned}\|1_B L R_\lambda\|_1 &\leqslant c' \int_B dg \int_0^\infty dt\; e^{-(\lambda-\omega)t} t^{-(D'+N)/m} e^{-b(|g|^m\,|t|^{-1})^{1/(m-1)}}\\ &\leqslant c' \int_0^\infty dt \int_G dh\; e^{-(\lambda-\omega)t} t^{-N/m} e^{-b\,|h|^{m/(m-1)}}\\ &\leqslant c''(\lambda-\omega)^{-1+N/m}.\end{aligned}$$

Therefore one can choose $\rho = 2\omega$ and $c = 2c''$. ∎

6. DISTANCES

In this section we define a distance on the (general) connected Lie group C, together with a new distance on the homogeneous contraction G_0. Let $a_1, ..., a_{d'}$ be a reduced weighted algebraic basis for the (general) Lie algebra $\mathfrak{g}$ and extend this algebraic basis to a full basis $a_1, ..., a_{d'}, ..., a_d$ with weights $w_1, ..., w_d$ as in Lemma 2.2. So each a_i with $i > d'$ is a commutator of elements of the algebraic basis. Set

$$D' = \sum_{i=1}^{d} w_i = \sum_{\lambda > 0} \lambda \dim(\mathfrak{g}_\lambda / \mathfrak{g}_{\underline{\lambda}}).$$

Then D' is independent of the extension of the algebraic basis to a full basis and we refer to it as the *local dimension* of G with respect to the (reduced) weighted algebraic basis. This name is justified by the estimates of Proposition 6.1.II given below.

Let G_t and $\exp_t$, for $t \in [0, 1]$, be as in Section 3. We introduce a distance $|\cdot|'_t$ on G_t. Although we are mainly interested in the cases $t = 0$ and $t = 1$ the construction is identical for all $t \in [0, 1]$. Let $B_i^{(t)}$, $i \in \{1, ..., d\}$, be the left invariant vector fields on G_t corresponding to a_i, i.e.,

$$(B_i^{(t)} \psi)(g) = \frac{d}{ds} \psi(g \exp_t(s a_i)) \Big|_{s=0}$$

for $\psi \in C^\infty(G_t)$. Then for $\delta > 0$ let $C_t(\delta)$ be the set of all absolutely continuous functions $\varphi\colon [0, 1] \to G_t$ which satisfy the differential equation

$$\dot{\varphi}(s) = \sum_{i=1}^{d'} \varphi_i(s)\, B_i^{(t)} \big|_{\varphi(s)}$$

almost everywhere with $|\varphi_i(s)| < \delta^{w_i}$ for all $i \in \{1, ..., d'\}$ and $s \in [0, 1]$. Now define the *distance* $d'_t(g; h)$ between two elements $g, h \in G_t$ by

$$d'_t(g; h) = \inf \{\delta > 0 \colon \exists_{\varphi \in C_t(\delta)} [\varphi(0) = g \text{ and } \varphi(1) = h]\}$$

and the *modulus* $|\cdot|'_t$ on G_t by $|g|'_t = d'_t(g; e)$. Since $a_1, ..., a_{d'}$ is an algebraic basis for $(\mathfrak{g}, [\cdot, \cdot]_t)$ it follows from a theorem of Carathéodory that $d'_t(g; h)$ is finite for all $g, h \in G_t$ (see also [NSW]). Moreover $d'_t(kg; kh) = d'_t(g; h)$ for all $g, h, k \in G_t$.

If $t = 0$ the modulus $|\cdot|'_0$ has the scaling property $|\gamma_s(g)|'_0 = s\,|g|'_0$ for all $g \in G_0$. Therefore, if $|\cdot|$ is the homogeneous modulus on G_0 introduced in Section 5 then there exists a $c > 0$ such that $c^{-1}\,|g| \leqslant |g|'_0 \leqslant c\,|g|$ for all $g \in G_0$.

Next, for $\delta > 0$ let $B^{(t)\prime}_\delta = \{g \in G_t \colon |g|'_t < \delta\}$ be the ball in G_t with radius δ. We denote by $|B^{(t)\prime}_\delta|_t$ the measure of $B^{(t)\prime}_\delta$ with respect to the fixed (left-)Haar measure ρ_t on G_t (see (5)). If $t = 1$ we drop the subscript t as before. Moreover, if confusion is possible, we write $|g|'_{(a)} = |g|'$ to indicate the dependence of the modulus on the reduced weighted algebraic basis.

Since $w_1, ..., w_{d'}$ have a common multiple $w_1, ..., w_d$ also have a common multiple. Set

$$\bar{w} = \min\{x \in [1, \infty\rangle \colon x \in w_i \mathbf{N} \text{ for all } i \in \{1, ..., d\}\}.$$

Then define a *modulus* $|\cdot|$ on $\mathfrak{g}$ by

$$\left| \sum_{i=1}^{d} \xi_i a_i \right|^{2\bar{w}} = \sum_{i=1}^{d} |\xi_i|^{2\bar{w}/w_i}.$$

The moduli $|\cdot|'_t$ are comparable locally.

Proposition 6.1. *Let* $t \in \{0, 1\}$.

I. *There exist* $c \geqslant 1$ *and* $\varepsilon \in \langle 0, 1]$ *such that* $c^{-1}\,|a| \leqslant |\exp_t(a)|'_t \leqslant c\,|a|$ *for all* $a \in \mathfrak{g}$ *with* $\|a\| \leqslant \varepsilon$, *where* $\|\cdot\|$ *is a Euclidean norm on* $\mathfrak{g}$.

II. *There exists a* $c \geqslant 1$ *such that* $c^{-1}\delta^{D'} \leqslant |B^{(t)\prime}_\delta|_t \leqslant c\delta^{D'}$ *for all* $\delta \in \langle 0, 1]$.

Proof. The proposition is trivial if $t = 0$ by scaling, so we only need to consider $t = 1$. We may assume that all weights are integers. Indeed, if one multiplies all weights with a positive constant then the distance and modulus are replaced by the appropriate root of the old distance and modulus and the constants c with the appropriate powers.

For all $n \in \mathbf{N}$ define $C_n \colon \mathfrak{g}^n \to G$ by setting $C_1(b_1) = \exp(b_1)$ and

$$C_n(b_1, ..., b_n) = \exp(b_1)\, C_{n-1}(b_2, ..., b_n) \exp(-b_1)\, C_{n-1}(b_2, ..., b_n)^{-1}$$

for $n > 1$.

LEMMA 6.2. *For all* $N \in \mathbf{N}$ *there exist* $\varepsilon_N > 0$ *and* $c_{N\beta} \in \mathbf{R}$ *with* $\beta \in J(N)$, $|\beta| \geqslant N+1$ *such that*

$$C_N(b_1, ..., b_N) = \exp([b_1, [\cdots [b_{N-1}, b_N] \cdots]] + R_N(b_1, ..., b_N))$$

for all $b_1, ..., b_N \in \mathfrak{g}$ *with* $\|b_i\| \leqslant \varepsilon_N$, *where*

$$R_N(b_1, ..., b_N) = \sum_{\substack{\beta \in J(N) \\ |\beta| \geqslant N+1}} c_{N\beta} b_{[\beta]}.$$

Moreover, the sum in R_N *is absolutely convergent and*

$$\sum_{\substack{\beta \in J(N) \\ |\beta| \geqslant N+1}} \|c_{N\beta} b_{[\beta]}\| \leqslant 1.$$

Proof. This follows from the Campbell–Baker–Hausdorff formula and induction on N (see [NSW, Lemma 2.21], and [VSC, Section III.3]). ▮

In the next lemma we replace one low-order term in an element of the Lie algebra by several a_i with $i \in \{1, ..., d'\}$ and another element of the Lie algebra which is not much larger than the original element. Since it is not possible to control all terms individually, we control all high-order terms together.

LEMMA 6.3. *For all* $M \in \mathbf{N}$ *and* $\alpha_0 \in J^+_M(d')$ *there exist* $n \in \mathbf{N}$ *and* $i_1, ..., i_n \in \{1, ..., d'\}$ *with the property that for each* $C \geqslant 1$ *there exist* $C' \geqslant 1$ *and* $\varepsilon \in \langle 0, 1]$ *such that for* $\delta \in \langle 0, \varepsilon \rangle$ *and* $c_\alpha \in \mathbf{R}$ *with* $\alpha \in J^+(d')$ *the properties*

1. $a = \sum_{\alpha \in J^+(d')} c_\alpha a_{[\alpha]}$ *converges absolutely*,
2. $|c_\alpha| \leqslant C\delta^{\|\alpha\|}$ *for* $\alpha \in J^+_M(d')$,
3. $\|\sum_{\alpha \in J(d');\, |\alpha| \geqslant M+1} c_\alpha a_{[\alpha]}\| \leqslant C\delta^{M+1}$ *and*,
4. $c_\alpha = 0$ *for all* $\alpha \in J^+_{|\alpha_0|-1}(d')$

imply the existence of $c'_\alpha \in \mathbf{R}$, $\alpha \in J^+(d')$ *and* $s_1, ..., s_n \in \mathbf{R}$ *such that*

5. $b = \sum_{\alpha \in J^+(d')} c'_\alpha a_{[\alpha]}$ *converges absolutely*,
6. $|c'_\alpha| \leqslant C'\delta^{\|\alpha\|}$ *for* $\alpha \in J^+_M(d')$,
7. $\|\sum_{\alpha \in J^+(d');\, |\alpha| \geqslant M+1} c'_\alpha a_{[\alpha]}\| \leqslant C'\delta^{M+1}$,
8. $|s_j| \leqslant C'\delta^{w_{i_j}}$ *for all* $j \in \{1, ..., n\}$,
9. $$\begin{cases} c'_\alpha = 0 & \text{for all} \quad \alpha \in J^+_{|\alpha_0|-1}(d') \\ c'_{\alpha_0} = 0 & \\ c'_\alpha = c_\alpha & \text{if} \quad |\alpha| = |\alpha_0| \text{ and } \alpha \neq \alpha_0, \text{ and,} \end{cases}$$
10. $\exp(a) = \exp(b) \exp(s_1 a_{i_1}) \cdots \exp(s_n a_{i_n})$.

Proof. We may assume that the Campbell–Baker–Hausdorff series converges absolutely with respect to $[\cdot, \cdot]$ on $\{a \in \mathfrak{g} : \|a\| \leqslant 2\}$. Let $M \in \mathbf{N}$ and $\alpha_0 \in J_M^+(d')$. Write $\alpha_0 = (j_1, ..., j_N)$ where $N = |\alpha_0|$. Let $c_{N\beta} \in \mathbf{R}$ and $\varepsilon_N > 0$ be as in Lemma 6.2. It is clear that there are $n \in \mathbf{N}$ and $i_1, ..., i_n \in \{1, ..., d'\}$ such that one has the identity

$$C_N(\tau_1 a_{j_1}, ..., \tau_N a_{j_N}) = \exp(\omega_1 a_{i_1}) \cdots \exp(\omega_n a_{i_n}) \tag{20}$$

for all $\tau_1, ..., \tau_N \in \mathbf{R}$, where for each l there exists a k such that $\omega_l a_{i_l} = \tau_k a_{j_k}$.

Let $C \geqslant 1$. Let $\varepsilon \in \langle 0, 1]$ be such that $\|a\| \leqslant 1$ for all $a = \sum c_\alpha a_{[\alpha]}$ where the c_α satisfy 1, 2, 3 and 4 in the statement of the lemma for some $\delta \in \langle 0, \varepsilon \rangle$, and, moreover, $\varepsilon < \varepsilon_N C^{-1} d^{-1}$.

Now for $\delta \in \langle 0, \varepsilon \rangle$ let $a = \sum c_\alpha a_{[\alpha]} \in \mathfrak{g}$ and suppose that 1, 2, 3 and 4 are valid. Let $\tau_1 = \operatorname{sgn}(c_{\alpha_0}) \, |c_{\alpha_0}|^{w_{j_1}/\|\alpha_0\|}$ and $\tau_l = |c_{\alpha_0}|^{w_{j_l}/\|\alpha_0\|}$ for all $l \in \{2, ..., N\}$. Then $|\tau_l| \leqslant C\delta^{w_{j_l}}$ for all $l \in \{1, ..., N\}$ and $\tau_1 \cdot \cdots \cdot \tau_N = c_{\alpha_0}$. Let $a'' = a - c_{\alpha_0} a_{[\alpha_0]}$. Then

$$a = (c_{\alpha_0} a_{[\alpha_0]} + R_N(\tau_1 a_{j_1}, ..., \tau_N a_{j_N})) + (a'' - R_N(\tau_1 a_{j_1}, ..., \tau_N a_{j_N})).$$

We estimate $R_N(\tau_1 a_{j_1}, ..., \tau_N a_{j_N})$. One has

$$R_N(\tau_1 a_{j_1}, ..., \tau_N a_{j_N}) = \sum_{\substack{\beta \in J(N) \\ |\beta| \geqslant N+1}} c_{N\beta} \tau^\beta b_{[\beta]},$$

where $b_k = a_{j_k}$ for all $k \in \{1, ..., N\}$. But $b_{[\beta]} = a_{[\alpha]}$ with $\alpha = (j_{k_1}, ..., j_{k_m})$ if $\beta = (k_1, ..., k_m) \in J^+(N)$. Then $|\tau^\beta| = |\prod_{l=1}^m \tau_{k_l}| \leqslant \prod_{l=1}^m C\delta^{w_{j_{k_l}}} = C^m \delta^{\|\alpha\|} \leqslant C^M \delta^{\|\alpha\|}$ if $m = |\beta| = |\alpha| \leqslant M$. Similarly one deduces that

$$\begin{aligned} \Big\| \sum_{\substack{\beta \in J(N) \\ |\beta| \geqslant M+1}} c_{N\beta} \tau^\beta b_{[\beta]} \Big\| &= \Big\| \sum_{\substack{\beta \in J(N) \\ |\beta| \geqslant M+1}} c_{N\beta} (d\varepsilon_N^{-1} \tau)^\beta \, (d^{-1} \varepsilon_N)^{|\beta|} \, b_{[\beta]} \Big\| \\ &\leqslant (C \, d\varepsilon_N^{-1} \delta)^{M+1} \sum_{\substack{\beta \in J(N) \\ |\beta| \geqslant M+1}} \| c_{N\beta} (d^{-1} \varepsilon_N)^{|\beta|} \, b_{[\beta]} \| \\ &\leqslant (C \, d\varepsilon_N^{-1})^{M+1} \, \delta^{M+1}, \end{aligned}$$

where we have used $C\delta < 1$ and Lemma 6.2. So one can write

$$a = (c_{\alpha_0} a_{[\alpha_0]} + R_N(\tau_1 a_{j_1}, ..., \tau_N a_{j_N})) + a'''$$

with $a''' = \sum c'''_\alpha a_{[\alpha]}$ absolutely convergent. The coefficients c'''_α satisfy the estimates $|c'''_\alpha| \leqslant C_1 \delta^{\|\alpha\|}$ for $\alpha \in J^+_M(d')$ with a $C_1 \geqslant 1$ which depends only on M, α_0 and C,

$$\Bigg\| \sum_{\substack{\alpha \in J(d') \\ |\alpha| \geqslant M+1}} c'''_\alpha a_{[\alpha]} \Bigg\| \leqslant C_1 \delta^{M+1}$$

and, in addition, $c'''_\alpha = 0$ for all $\alpha \in J^+_{|\alpha_0|-1}(d')$, $c'''_{\alpha_0} = 0$ and $c'''_\alpha = c_\alpha$ if $|\alpha| = |\alpha_0|$ and $\alpha \neq \alpha_0$. Then by the Campbell–Baker–Hausdorff formula one obtains as before that

$$\exp(-c_{\alpha_0} a_{[\alpha_0]} - R_N(\tau_1 a_{j_1}, ..., \tau_N a_{j_N})) \exp a = \exp b$$

where $b = \sum_{\alpha \in J(d')} c'_\alpha a_{[\alpha]}$ for some $c'_\alpha \in \mathbf{R}$ such that 5, 6, 7 and 9 in the statement of the lemma are valid for some $C' \geqslant 1$ which depends only on M, α_0 and C. Inverting the first element of the left hand side and using (20) gives 8, and 10. ▮

Now we complete the proof of Proposition 6.1. We need two more distances and a quasi-distance. For $j \in \{2, 3, 4\}$ and $\delta > 0$ let $C^{(j)}(\delta)$ be the set of all absolutely continuous functions $\varphi\colon [0, 1] \to G$ which satisfy the differential equation

$$\dot{\varphi}(s) = \sum_{i=1}^{d} \varphi_i(s)\, B_i\big|_{\varphi(s)}$$

almost everywhere with

$$\begin{aligned} &\varphi_{d'+1}(s) = \cdots = \varphi_d(s) = 0 \text{ and } |\varphi_i(s)| < \delta, && \text{if } j = 2, \\ &|\varphi_i(s)| < \delta^{w_i}, && \text{if } j = 3, \\ &\varphi_i(s) = \varphi_i(0) \text{ and } |\varphi_i(0)| < \delta^{w_i}, && \text{if } j = 4, \end{aligned}$$

for all $i \in \{1, ..., d'\}$ and $s \in [0, 1]$. Then the (quasi-)distance $d^{(j)}(g; h)$ between two elements $g, h \in G$ is defined by $d^{(j)}(g; h) = \inf\{\delta > 0\colon \exists_{\varphi \in C^{(j)}(\delta)} [\varphi(0) = g \text{ and } \varphi(1) = h]\}$. Next let $M = r \max(w_1, ..., w_{d'})$, where r is the rank of the algebraic basis $a_1, ..., a_{d'}$. Then

$$C^{(2)}(\delta^{M/r}) \subseteq C(\delta) \subseteq C^{(2)}(\delta)$$

for all $\delta \in \langle 0, 1]$. Hence $d^{(2)}(g; h) \leqslant d'(g; h) \leqslant (d^{(2)}(g; h))^{r/M}$ for all $g, h \in G$ with $d'(g; h) \leqslant 1$. It follows, however, from [NSW, Proposition 1.1 and Theorem 4], that there exist $\varepsilon_1 \in \langle 0, 1]$ and $C_1 \geqslant 1$ such that $d^{(2)}(\exp a; e) \leqslant C_1 \|a\|^{1/r}$ for all $a \in \mathfrak{g}$ with $\|a\| \leqslant \varepsilon_1$. Therefore $|\exp a|' = d'(\exp a; e) \leqslant C_2 \|a\|^{1/M}$ for all $a \in \mathfrak{g}$ with $\|a\| \leqslant \varepsilon_1$, where $C_2 = C_1^{r/M}$.

It follows by induction from Lemma 6.3 that there exist $n \in \mathbf{N}$, $i_1, ..., i_n \in \{1, ..., d'\}$, $C_3 \geqslant 1$ and $\varepsilon \in \langle 0, 1]$ such that for all $\delta \in \langle 0, \varepsilon \rangle$ and all $a = \sum_{\alpha \in J_r^+(d')} c_\alpha a_{[\alpha]}$ with $c_\alpha \in \mathbf{R}$ and $|c_\alpha| \leqslant \delta^{\|\alpha\|}$, $\alpha \in J_r^+(d')$, there are $c'_\alpha \in \mathbf{R}$, for $\alpha \in J(d')$ with $|\alpha| \geqslant M+1$ and $s_1, ..., s_n \in \mathbf{R}$ such that $\exp(a) = \exp(b) \exp(s_1 a_{i_1}) \cdots \exp(s_n a_{i_n})$, where $b = \sum_{\alpha \in J(d');\, |\alpha| \geqslant M+1} c'_\alpha a_{[\alpha]}$ converges absolutely. Moreover, $\|b\| \leqslant C_3 \delta^{M+1}$ and $|s_j| \leqslant C_3 \delta^{w_{i_j}}$ for all $j \in \{1, ..., n\}$. But by the choice of $a_{d'+1}, ..., a_d$ there exists for all $j \in \{1, ..., d\}$ an $\alpha_j \in J_r(d')$ such that $a_j = a_{[\alpha_j]}$ and $\|\alpha_j\| = w_j$.

Now let $a \in \mathfrak{g}$ and suppose that $|a| < \min(\varepsilon, C_3^{-1} \varepsilon_1)$. Write $a = \sum_{\alpha \in J^+(d')} c_\alpha a_{[\alpha]}$ with $c_\alpha = 0$ if $\alpha \notin \{\alpha_1, ..., \alpha_d\}$. Then $|c_{\alpha_j}| \leqslant |a|^{w_j} = |a|^{\|\alpha_j\|}$ for all $j \in \{1, ..., d\}$ and hence $|c_\alpha| \leqslant |a|^{\|\alpha\|}$ for all $\alpha \in J_r^+(d')$. So for all $\alpha \in J(d')$ with $|\alpha| \geqslant M+1$ there exist a $c'_\alpha \in \mathbf{R}$ and $s_1, ..., s_n \in \mathbf{R}$ such that $\exp(a) = \exp(b) \exp(s_1 a_{i_1}) \cdots \exp(s_n a_{i_n})$, where $b = \sum_{\alpha \in J(d');\, |\alpha| \geqslant M+1} c'_\alpha a_{[\alpha]}$ converges absolutely, and, moreover, $\|b\| \leqslant C_3 \, |a|^{M+1}$ and $|s_j| \leqslant C_3 \, |a|^{w_{i_j}}$ for all $j \in \{1, ..., n\}$. Then

$$
\begin{aligned}
|\exp a|' &\leqslant |\exp b|' + |\exp(s_1 a_{i_1})|' + \cdots + |\exp(s_n a_{i_n})|' \\
&\leqslant C_2 \, \|b\|^{1/M} + C_3 \, |a| + \cdots + C_3 \, |a| \\
&\leqslant C_2 C_3 \, |a|^{(M+1)/M} + n C_3 \, |a| \\
&\leqslant (n+1) \, C_2 C_3 \, |a|.
\end{aligned}
$$

This proves the second inequality of Statement I of Proposition 6.1.

Next we prove the first inequality. By [NSW, Theorem 2], the quasi-distance $d^{(4)}$ is locally equivalent to $d^{(3)}$. So there exist $c, \varepsilon > 0$ such that $d^{(3)}(g; h) \geqslant c d^{(4)}(g; h)$ for all $g, h \in B'_\varepsilon$. In particular, $|g|' = d'(g; e) \geqslant d^{(3)}(g; e) \geqslant c \, d^{(4)}(g; e)$ for all $g \in B'_\varepsilon$.

Now let $a = \sum_{i=1}^d \xi_i a_i \in B'_\varepsilon$ and for $\delta > 0$ let $\varphi \in C^{(4)}(\delta)$. Let $b = \sum_{i=1}^d \varphi_i a_i \in \mathfrak{g}$ and B be the left invariant vector field corresponding to b. Then $\dot{\varphi}(s) = B|_{\varphi(s)}$ for all $s \in [0, 1]$. So φ is a C^∞-function and by the uniqueness theorem for integral curves (see, for example, [SaW, Theorem 2.37]) it follows that $\varphi(s) = \exp(sb)$ for all $s \in [0, 1]$, if ε is small enough. In particular, $\exp(a) = \varphi(1) = \exp(b)$. Hence, for small enough ε, it follows that $\xi_i = \varphi_i$ for all $i \in \{1, ..., d\}$. Thus $|\xi_i| < \delta^{w_i}$ for all $i \in \{1, ..., d\}$. But there is a $j \in \{1, ..., d\}$ such that $|\xi_j|^{2\bar{w}/w_j} \geqslant d^{-1} \, |a|^{2\bar{w}}$. Therefore $\delta > |\xi_j|^{1/w_j} \geqslant d^{-1/(2\bar{w})} \, |a|$ and $d^{(4)}(\exp(a); e) \geqslant d^{-1/(2\bar{w})} \, |a|$. This completes the proof of Statement I.

The proof of Statement II follows easily from Statement I and (5). ∎

COROLLARY 6.4. *There exists a* $c > 0$ *such that* $c^{-1} \, |\exp_0 a|'_0 \leqslant |\exp a|' \leqslant c \, |\exp_0 a|'_0$ *for all* $a \in \mathfrak{g}$ *with* $\|a\| \leqslant 1$.

It is also straightforward to deduce that the moduli of different bases corresponding to the same filtration are equivalent.

COROLLARY 6.5. *Let $a_1, ..., a_{d'}$ be a reduced weighted algebraic basis with weights $w_1, ..., w_{d'}$ and $b_1, ..., b_{d''}$ a second reduced weighted algebraic basis with weights $v_1, ..., v_{d''}$ such that the filtrations with respect to the two weighted algebraic bases coincide. Then the corresponding moduli $|\cdot|'_{(a)}$ and $|\cdot|'_{(b)}$ are equivalent, i.e., there exists a $c \geqslant 1$ such that*

$$c^{-1}\,|g|'_{(b)} \leqslant |g|'_{(a)} \leqslant c\,|g|'_{(b)}$$

for all $g \in G$.

Proof. By Proposition 3.1.III the two homogeneous contractions $G_0^{(a)}$ and $G_0^{(b)}$ obtained by the two reduced weighted algebraic bases are isomorphic, by an isomorphism Φ. Then $g \mapsto |\Phi(g)|'_{0(b)}$ is a homogeneous modulus on $G_0^{(a)}$, as is $|\cdot|'_{0(a)}$. Therefore there exists a $c > 0$ such that $c^{-1}\,|g|'_{0(a)} \leqslant |\Phi(g)|'_{0(b)} \leqslant c\,|g|'_{0(a)}$ for all $g \in G_0^{(a)}$. Now the corollary follows from Corollary 6.4. ∎

7. KERNELS

In this section we extend the kernel theorem of Section 5 for homogeneous groups to general groups G by exploiting the homogeneous contraction G_0 of G.

Let $(\mathcal{X}, G, U)$ be a continuous representation of a connected Lie group G and $a_1, ..., a_{d'}$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of G. Extend the algebraic basis to a vector space basis $a_1, ..., a_{d'}, ..., a_d$ as in Lemma 2.2 and adopt the notation of Section 3. Let $m \in 2w\mathbf{N}$ and $C\colon J(d') \to \mathbf{C}$ be a form of order m. In Proposition 4.5 we established that each G-weighted subcoercive form C is a G_0-weighted subcoercive form and throughout this section, we adopt the seemingly weaker assumption that C is a G_0-weighted subcoercive form.

Let $dU(C)$ be the operator on $\mathcal{X}$ corresponding to the form C. Our aim is to establish that the closure of $dU(C)$ generates a continuous semigroup S with a kernel K satisfying Gaussian type bounds. We approach this problem by first constructing a family of functions K which formally corresponds to the semigroup kernel. In the next section we verify that the K is a semigroup kernel and the generator of the semigroup is the closure of $dU(C)$.

The starting point of the construction is the observation that the kernel K, if it exists should be the fundamental solution for the heat operator $\partial_t + dL(C)$.

Precisely, if one defines $K_t = 0$ for $t \leqslant 0$ then $(t, g) \mapsto K_t(g)$ from $\mathbf{R} \times G$ into $\mathbf{C}$ should be the fundamental solution for the heat operator $\partial_t + dL(C)$, i.e.,

$$((\partial_t + dL(C))\, K_t)(g) = \delta(t)\, \delta(g) \tag{21}$$

for all $t \in \mathbf{R}$ and $g \in G$. The parametrix method expresses K as an expansion in terms of a localized version of the corresponding kernel for G_0. The expansion is a direct analogue of "time-dependent" perturbation theory.

Let W be the open neighbourhood of 0 in $\mathfrak{g}$ as in Lemma 3.3.V and $\Omega = \exp(W)$. For all $\varphi\colon \Omega \to \mathbf{C}$ define $\hat{\varphi}\colon W \to \mathbf{C}$ by $\hat{\varphi} = \varphi \circ \exp$. Let X_i and $X_i^{(0)}$ denote the vector fields on $(\mathfrak{g}, [\cdot, \cdot])$ and $(\mathfrak{g}[\cdot, \cdot]_0)$ as in Section 3. Then $X_i^{(0)}$ and X_i are very similar.

Lemma 7.1. *For all $i \in \{1, ..., d\}$ the differential operator* $\exp_{0\,*}(X_i - X_i^{(0)})$ *is of actual order N_i with $N_i < w_i$.*

Proof. Let $M, \delta, c_{\varepsilon_1, ..., \varepsilon_n}$ be as in (4). Then $X_i\big|_a - X_i^{(0)}\big|_a = \sum_{j=1}^d f_{ij}(a)\, X_j^{(0)}\big|_a$ for all $a = \sum_{l=1}^d \xi_l a_l \in W$, where

$$\begin{aligned} f_{ij}(a) &= \sum_{n=1}^{\infty} \sum_{\varepsilon_1, ..., \varepsilon_n \in \{0,1\}} c_{\varepsilon_1, ..., \varepsilon_n} \pi^j((\mathrm{ad}_{\varepsilon_1} a) \cdots (\mathrm{ad}_{\varepsilon_n} a)(a_i) - (\mathrm{ad}_0\, a)^n\, (a_i)) \\ &= \sum_{n=1}^{\infty} \sum_{\varepsilon_1, ..., \varepsilon_n \in \{0,1\}} \sum_{k=1}^{n} \sum_{i_1, ..., i_n = 1}^{d} c_{\varepsilon_1, ..., \varepsilon_n} \xi_{i_1} \cdots \xi_{i_n} \\ &\quad \times \pi^j((\mathrm{ad}_{\varepsilon_1} a_{i_1}) \cdots (\mathrm{ad}_{\varepsilon_{k-1}} a_{i_{k-1}})(\mathrm{ad}_{\varepsilon_k} a_{i_k} - \mathrm{ad}_0\, a_{i_k}) \\ &\quad \times (\mathrm{ad}_0\, a_{i_{k+1}}) \cdots (\mathrm{ad}_0\, a_{i_n})(a_i)). \end{aligned}$$

The difference of the two commutators is an element of $\mathfrak{g}$, so there exist $\mu_{l, k, \varepsilon_1, ..., \varepsilon_n} \in \mathbf{R}$, with $|\mu_{l, k, \varepsilon_1, ..., \varepsilon_n}| \leqslant 2c_1^n$ for some $c_1 > 0$, such that

$$\begin{aligned} &(\mathrm{ad}_{\varepsilon_1} a_{i_1}) \cdots (\mathrm{ad}_{\varepsilon_{k-1}} a_{i_{k-1}})(\mathrm{ad}_{\varepsilon_k} a_{i_k} - \mathrm{ad}_0\, a_{i_k})(\mathrm{ad}_0\, a_{i_{k+1}}) \cdots (\mathrm{ad}_0\, a_{i_n}) \\ &\quad = \sum_{l=1}^{d} \mu_{l, k, \varepsilon_1, ..., \varepsilon_n} a_l. \end{aligned}$$

Then

$$f_{ij}(a) = \sum_{n=1}^{\infty} \sum_{\varepsilon_1, ..., \varepsilon_n \in \{0,1\}} \sum_{k=1}^{n} \sum_{i_1, ..., i_n = 1}^{d} c_{\varepsilon_1, ..., \varepsilon_n} \xi_{i_1} \cdots \xi_{i_n} \mu_{j, k, \varepsilon_1, ..., \varepsilon_n}.$$

Now suppose $j \in \{1, ..., d\}$ and $w_j \geqslant w_i$. Since $\{\|\alpha\|: \alpha \in J(d)\}$ is a discrete set, there exist $\eta_j > 0$ such that $\|\alpha\| \geqslant w_j - w_i + \eta_j$ for all $\alpha \in J(d)$ with

$\|\alpha\| > w_j - w_i$. Let $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ be the filtration corresponding to the weighted algebraic basis $a_1, ..., a_{d'}$. Consider the element

$$(\mathrm{ad}_{\varepsilon_1} a_{i_1}) \cdots (\mathrm{ad}_{\varepsilon_{k-1}} a_{i_{k-1}})(\mathrm{ad}_{\varepsilon_k} a_{i_k} - \mathrm{ad}_0 a_{i_k})(\mathrm{ad}_0 a_{i_{k+1}}) \cdots (\mathrm{ad}_0 a_{i_n})(a_i)$$

of the Lie algebra $\mathfrak{g}$. Now $(\mathrm{ad}_0 a_{i_{k+1}}) \cdots (\mathrm{ad}_0 a_{i_n})(a_i) \in \mathfrak{g}_{w_i + w_{i_{k+1}} + \cdots + w_{i_n}}$. Therefore

$$(\mathrm{ad}_{\varepsilon_k} a_{i_k} - \mathrm{ad}_0 a_{i_k})(\mathrm{ad}_0 a_{i_{k+1}}) \cdots (\mathrm{ad}_0 a_{i_n})(a_i) \in \mathfrak{g}_{(w_i + w_{i_k} + \cdots + w_{i_n})_-}$$

by Proposition 3.1.VI if $\varepsilon_k = 1$, and clearly also if $\varepsilon_k = 0$. Thus

$$\begin{aligned}(\mathrm{ad}_{\varepsilon_1} a_{i_1}) \cdots (\mathrm{ad}_{\varepsilon_{k-1}} a_{i_{k-1}})(\mathrm{ad}_{\varepsilon_k} a_{i_k} - \mathrm{ad}_0 a_{i_k})(\mathrm{ad}_0 a_{i_{k+1}}) \cdots (\mathrm{ad}_0 a_{i_n})(a_i)\\ \in \mathfrak{g}_{(w_i + w_{i_1} + \cdots + w_{i_n})_-}.\end{aligned}$$

So if $\mu_{j,k,\varepsilon_1, ..., \varepsilon_n} \neq 0$ then $a_j \in \mathfrak{g}_{(w_i + w_{i_1} + \cdots + w_{i_n})_-}$. Hence $w_j < w_i + w_{i_1} + \cdots + w_{i_n}$ and therefore $w_{i_1} + \cdots + w_{i_n} \geqslant w_j - w_i + \eta_j$. Moreover $|\xi_l| \leqslant |a|^{w_l} \leqslant |a|$ if $l \in \{1, ..., d\}$ and $|a| \leqslant 1$. Consequently

$$\begin{aligned} f_{ij}(a) = \sum_{n=1}^{\infty} \sum_{\varepsilon_1, ..., \varepsilon_n \in \{0,1\}} \sum_{k=1}^{n} \sum_{\substack{i_1, ..., i_n = 1 \\ w_{i_1} + \cdots + w_{i_n} \geqslant w_j - w_i + \eta_j}}^{d} |c_{\varepsilon_1, ..., \varepsilon_n}| \\ \times |\xi_{i_1}| \cdots |\xi_{i_n}| \, |\mu_{j,k,\varepsilon_1, ..., \varepsilon_n}| \\ \leqslant \sum_{n=1}^{\infty} \sum_{\varepsilon_1, ..., \varepsilon_n \in \{0,1\}} \sum_{k=1}^{n} \sum_{\substack{i_1, ..., i_n = 1 \\ w_{i_1} + \cdots + w_{i_n} \geqslant w_j - w_i + \eta_j}}^{d} M \, \delta^n \, |a|^{w_{i_1} + \cdots + w_{i_n}} \, 2c_1^n. \end{aligned}$$

Let $N \in \mathbf{N}$ be such that $N > w_j - w_i + \eta_j$. Next, split the sum over n in two parts: the first over n with $n \leqslant N$ and the second over $n > N$. Then if $|a| \leqslant 1$

$$\begin{aligned} f_{ij}(a) \leqslant 2 \sum_{n=1}^{N} \sum_{\varepsilon_1, ..., \varepsilon_n \in \{0,1\}} \sum_{k=1}^{n} \sum_{\substack{i_1, ..., i_n = 1 \\ w_{i_1} + \cdots + w_{i_n} \geqslant w_j - w_i + \eta_j}}^{d} M(c_1 \delta)^n \, |a|^{w_j - w_i + \eta_j} \\ + 2 \sum_{n=N+1}^{\infty} \sum_{\varepsilon_1, ..., \varepsilon_n \in \{0,1\}} \sum_{k=1}^{n} \sum_{\substack{i_1, ..., i_n = 1 \\ w_{i_1} + \cdots + w_{i_n} \geqslant w_j - w_i + \eta_j}}^{d} M(c_1 \delta \, |a|)^n \\ \leqslant 2 \sum_{n=1}^{N} Mn(2c_1 \, d\delta)^n \, |a|^{w_j - w_i + \eta_j} + 2 \sum_{n=N+1}^{\infty} Mn(2c_1 \, d\delta \, |a|)^n. \end{aligned}$$

Now the lemma follows if one takes $a \in W$, $|a| \leqslant 1$ and $|a| < (3c_1 \, d\delta)^{-1}$. ∎

Set $H = dL(C)$ and let H_X be the elliptic operator constructed from H with the vector fields X_i of Section 3 replacing the generators A_i. It follows immediately from this lemma together with the remark following Lemma 5.8 that

$$H_X = P_{X^{(0)}} + H' \tag{22}$$

where $P_{X^{(0)}} = \sum_{\|\alpha\| = m} c_\alpha X^{(0)\alpha}$ and $\exp_{0*}(H') = \sum_\alpha f_\alpha A^\alpha$ is an operator with actual order N with $N < m$ and the f_α are C^∞-functions on $\exp_0(W)$. Moreover, $dL_{G_0}(P) = \exp_{0*}(P_{X^{(0)}})$ where P is the principal part of C. The results of Section 5 apply to $dL_{G_0}(P)$ and one verifies the transformation property

$$(H^*(\psi \circ \log), \varphi \circ \log) = (dL_{G_0}(P^\dagger)((\sigma\psi) \circ \log_0), \varphi \circ \log_0)$$
$$+ \sum_\alpha (-1)^{|\alpha|} (A^\alpha(f_\alpha(\sigma\psi \circ \log_0)), \varphi \circ \log_0) \tag{23}$$

for all $\psi \in C_c^\infty(W)$ and $\varphi \in L_1(\mathfrak{g})$ with $\operatorname{supp} \varphi \subset W$ where σ is as in (5).

Let $\tilde{K}$ be the kernel on G_0 corresponding to the operator $dL_{G_0}(P)$. If $t \leqslant 0$ we define $\tilde{K}_t = 0$ as before. Now let $\chi, \chi' \in C_c^\infty(\Omega)$ be real with $\chi(e) = 1$ and $\chi' = 1$ on $\operatorname{supp} \chi$. We will identify a function τ on G with the function $1 \otimes \tau$ on $\mathbf{R} \times G$. For $t \in \mathbf{R}$ define the function $K_t^{(0)}$ on G with compact support in Ω, by $\widehat{K_t^{(0)}} = (\tilde{K}_t \circ \exp_0) \cdot \hat{\chi}$. Then the function $(t, g) \mapsto K^{(0)}(t, g) = K_t^{(0)}(g)$ is locally integrable, so $K^{(0)}$ is a distribution. We shall prove that it is an approximation of the fundamental solution of $\partial_t + H$. Let $\psi \in C_c^\infty(\mathbf{R} \times G)$. Then with $\tau(t, g) = \psi(t, \exp \log_0(g))\, \chi'(\exp \log_0(g))\, \sigma(\log_0(g))$ for all $g \in \exp_0(W)$ it follows from (23) that

$$\begin{aligned}
&(\psi, (\partial_t + H)\, K^{(0)}) \\
&\quad = ((-\partial_t + H^*)(\psi\chi'), K^{(0)}) \\
&\quad = \int_{\mathbf{R}} dt \int_{G_0} dg\, \overline{((-\partial_t + dL_{G_0}(P^\dagger))\, \tau)(t, g)}\, \tilde{K}_t(g)\, (\hat{\chi} \circ \log_0)(g) \\
&\qquad + \sum_\alpha (-1)^{|\alpha|} \int_{\mathbf{R}} dt \int_{G_0} dg\, \overline{(A^\alpha(f_\alpha \tau))(t, g)}\, \tilde{K}_t(g)\, (\hat{\chi} \circ \log_0)(g).
\end{aligned}$$

We consider the two terms on the right hand side separately. For the first term one uses Proposition 5.6 to deduce that

$$\begin{aligned}
&\int_{\mathbf{R}} dt \int_{G_0} dg\, \overline{((-\partial_t + dL_{G_0}(P^\dagger))\, \tau)(t, g)}\, \tilde{K}_t(g)\, (\hat{\chi} \circ \log_0)(g) \\
&\qquad = \psi(0, e) + \sum_i (\psi, (L_i \tilde{K}_t \circ \exp_0 \circ \log) \cdot \chi_i)
\end{aligned}$$

where the sum is finite, the operators L_i are operators with actual order less than m and the $\chi_i \in C_c^\infty(\Omega)$.

The second term can be handled similarly and one deduces that

$$((\partial_t + H)(K_t^{(0)}))(g) = \delta(t)\,\delta(g) + M_t(g)$$

as distributions on $\mathbf{R} \times G$, where $M_t = \sum_i (L_i \widetilde{K}_t \circ \exp_0 \circ \log) \cdot \chi_i$, the sum is finite, the operators L_i have actual order less than m and the $\chi_i \in C_c^\infty(\Omega)$.

Proposition 5.5 gives estimates for $\widetilde{K}$. Specifically, for all $\alpha \in J(d')$ there exist $b, c > 0$ such that

$$|(A^\alpha \widetilde{K}_t)(g)| \leqslant c t^{-(D' + \|\alpha\|)/m} e^{-b((|g|_0')^m t^{-1})^{1/(m-1)}}$$

uniformly for all $t > 0$ and $g \in G_0$. Consequently, one has by Corollary 5.7

$$|(L\widetilde{K}_t)(g)| \leqslant c t^{-(D' + N)/m} e^{\omega t} e^{-b((|g|_0')^m t^{-1})^{1/(m-1)}}$$

uniformly for all $t > 0$ and all g in a compact subset on which the operator L of actual order N is defined. It then follows from the estimates of Corollary 6.4 that for each $\alpha \in J(d')$ there are $b, c, \eta > 0$ and $\omega \geqslant 0$ such that

$$\begin{aligned} |(A^\alpha K_t^{(0)})(g)| &\leqslant c t^{-(D' + \|\alpha\|)/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}}, \\ |(A^\alpha M_t)(g)| &\leqslant c t^{-(D' + \|\alpha\| + m - \eta)/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}} \end{aligned} \tag{24}$$

uniformly for all $t > 0$ and $g \in G$, since $K_t^{(0)}$ has compact support. For $n \in \mathbf{N}$ define $K_t^{(n)}$ inductively by $K_t^{(n)} = -(K^{(n-1)} \hat{*} M)_t$, where the convolution product $\hat{*}$ is given by

$$\begin{aligned} (\chi \hat{*} \psi)_t(g) &= \int_{\mathbf{R}} ds \int_G dh\, \chi_s(h)\, \psi_{t-s}(h^{-1}g) \\ &= \int_{\mathbf{R}} ds \int_G dh\, \chi_{t-s}(h)\, \psi_s(h^{-1}g). \end{aligned}$$

The main theorem of this section is the following.

Theorem 7.2. *The series*

$$K_t = \sum_{n \geqslant 0} K_t^{(n)} \tag{25}$$

is L_p-convergent to a limit $K_t \in L_{p;\infty}$ for all $p \in [1, \infty]$ and $t > 0$. The limit K satisfies the heat equation (21), with the convention $K_t = 0$ for $t \leqslant 0$. Moreover, $t \mapsto K_t$ is continuous from $\langle 0, \infty \rangle$ into $L_1^\rho(G)$ for all $\rho \geqslant 0$, *where* $L_1^\rho(G)=L_1(G; e^{\rho\,|g|'}\,dg)$ *is the weighted space with norm* $\|\varphi\|_1^\rho=\int dg\, e^{\rho\,|g|'}\,|\varphi(g)|$. *Finally, for each* $\alpha\in J(d')$ *there exist* $b, c>0$ *and* $\omega\geqslant 0$ *such that*

$$|K_t(g)|\leqslant ct^{-D'/m}e^{\omega t}e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

$$|(A^\alpha K_t)(g)|\leqslant ct^{-(D'+\|\alpha\|)/m}e^{\omega t}e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

for all $g\in G$ *and* $t>0$.

Proof. Variations of this theorem have been proved in [ElR6, Theorem 4.1] for unweighted subcoercive operators and in [ElR5, Theorem 4.1] for weighted strongly elliptic operators. We refer to these two papers for the proof of the Gaussian bounds.

The only new part is the continuity of $t\mapsto K_t$ from $\langle 0,\infty\rangle$ into $L_1^\rho(G)$. It follows from the Gaussian bounds (24) that $K_t^{(0)}\in L_1^\rho$ and $M_t\in L_1^\rho$ for all $\rho\geqslant 0$ by a quadrature estimate, using the volume estimates of Proposition 6.1.II. Specifically, one has bounds

$$\|K_t^{(0)}\|_1^\rho\leqslant ce^{\omega(1+\rho^m)\,t},\qquad \|M_t\|_1^\rho\leqslant ct^{-(m-\eta)/m}e^{\omega(1+\rho^m)\,t}$$

for some $c, \omega, \eta>0$, uniformly for all $t>0$ and $\rho\geqslant 0$. Then, by induction on n, one has

$$\|K_t^{(n)}\|_1^\rho\leqslant c\,\frac{(bt)^{n\eta/m}}{\Gamma(1+n\eta/m)}\,e^{\omega(1+\rho^m)\,t}$$

for some $b>0$, uniformly for all $n\in\mathbf{N}_0$, $t>0$ and $\rho\geqslant 0$. Therefore (25) converges in L_1^ρ for all $t>0$. Obviously the map $t\mapsto K_t^{(n)}$ is continuous from $\langle 0,\infty\rangle$ into L_1^ρ for all $n\in\mathbf{N}_0$, so the map $t\mapsto K_t$ is also continuous from $\langle 0,\infty\rangle$ into L_1^ρ. ∎

In the next section the continuity of $t\mapsto K_t$ is used to deduce that the closure of the operator $dU(C)$ generates a continuous semigroup in each representation.

8. WEIGHTED SUBCOERCIVE OPERATORS

In this section we extend the generator theorem of Section 5 to a general group G in an arbitrary continuous representation and show that the "kernel" K constructed in Section 7 is indeed the kernel of the semigroup.

Adopt the notation of the previous section. So C is a G_0-weighted subcoercive form. Let $(\mathcal{X}, G, U)$ be a continuous representation of G. Then one has bounds $\|U(g)\|\leqslant Me^{\rho\,|g|'}$ with $M\geqslant 1$ and $\rho\geqslant 0$. But $K_t\in L_1^\rho(G)$, because of the Gaussian bounds, and once can define bounded operators S_t on $\mathscr{X}$ by

$$S_t x = U(K_t)\, x = \int_G dg\, K_t(g)\, U(g)\, x.$$

Note that $t \mapsto S_t x$ is continuous from $\langle 0, \infty\rangle$ into $\mathscr{X}$ for all $x \in \mathscr{X}$, since $t \mapsto K_t$ is continuous from $\langle 0, \infty\rangle$ into $L_1^\rho(G)$ (see Theorem 7.2). Because of the bounds $\|K_t^{(n)}\|_1^\rho \leqslant c(b^n t^n/n!)^{\eta/m}\, e^{\omega(1+\rho^m)\, t}$ it follows that $\lim_{t\downarrow 0} S_t x = \lim_{t\downarrow 0} U(K_t^{(0)})\, x$, if one of the two limits exists. But $(\tilde{K}_t)_{t>0}$ is a bounded approximation of the identity (cf. the proof of Lemma 3.3 in [AER]) and hence

$$\lim_{t\downarrow 0} U(K_t^{(0)})\, x = \lim_{t\downarrow 0} \int_W da\, \sigma(a)\, \tilde{K}_t(\exp_0(a))\, \hat{\chi}(a)\, U(\exp(a))\, x = x.$$

Therefore $\lim_{t\downarrow 0} S_t x = x$ strongly if U is strongly continuous and weakly* if U is weakly* continuous.

We first apply this to the L_p^ρ-, and $L_{p;\,n}^{\rho\prime}$-, spaces with respect to the left regular representation. Then $S_t \varphi = K_t * \varphi$ and hence $S_t L_p^\rho \subseteq L_{p;\,\infty}^\rho \subseteq D(H)$. Moreover, if $p \in [1, \infty\rangle$ and $q \in \langle 1, \infty]$ is conjugate to p then

$$-\int_{\mathbf{R}} dt\, (\partial_t \tau)(t)\, (\psi, S_t \varphi) + \int_{\mathbf{R}} dt\, \tau(t)\, (\psi, H S_t \varphi)$$

$$= -\int_{\mathbf{R}} dt (\partial_t \tau)(t)\, (\psi, S_t \varphi) + \int_{\mathbf{R}} dt\, \tau(t)\, (H^* \psi, S_t \varphi) = 0$$

for all $\varphi \in L_p^\rho$, $\tau \in C_c^\infty(\langle 0, \infty\rangle)$ and $\psi \in C_c^\infty(G)$. But by continuity and density ([ElR1, Theorem 2.4]) this is valid for all $\psi \in L_q^\rho$. On the other hand the map $t \mapsto H S_t \varphi$ is continuous if $\varphi \in L_{p;\,m}^{\rho\prime}$. Therefore it follows from the lemma of Du Bois–Reymond that $t \mapsto (\psi, S_t \varphi)$ is differentiable and $(d/dt)(\psi, S_t \varphi) + (\psi, H S_t \varphi) = 0$ for all $\varphi \in L_{p;\,m}^{\rho\prime}$, $\psi \in L_q^\rho$ and $t > 0$. Then

$$\frac{d}{dt} S_t \varphi + H S_t \varphi = 0 \tag{26}$$

strongly for all $\varphi \in L_{p;\,m}^{\rho\prime}$ by the mean value theorem and the continuity of $t \mapsto H S_t \varphi$.

The family $S = (S_t)_{t>0}$ forms a semigroup if, and only if, K is a convolution semigroup. But the definition of K seems unsuited to direct verification of this property. We argue that it follows from the lower semiboundedness of Re H on L_2.

PROPOSITION 8.1. *Each symmetric operator $H = dL_G(C)$ on $L_2(G)$, where C is a G_0-weighted subcoercive form, is essentially self-adjoint and lower semibounded.*

Proof. It suffices to establish that the range of $(\lambda I + \bar{H})$ is equal to L_2 and its inverse is bounded for all large positive λ. For this we use a resolvent version of the foregoing parametrix techniques.

Let $\chi, \chi' \in C_c^\infty(G)$, $\operatorname{supp} \chi' \subset \Omega$, $\chi(e) = 1$ and $\chi' = 1$ on $\operatorname{supp} \chi$. Then for all $\varphi \in C_c^\infty(G)$ and $\psi \in L_2(G)$ one has for all $r \in C_c^\infty(G)$ with $\operatorname{supp} r \subseteq \operatorname{supp} \chi$

$$\begin{aligned}\int_G dr\, r(g)\,(\psi, (\lambda I + H)\, L(g)\, \varphi) \\ &= (\psi, (\lambda I + H)(r * \varphi)) \\ &= \int_G dg\, ((\lambda I + H)\, r)(g)(\varphi, L(g)\, \varphi)\, \chi'(g) \\ &= \int_G dg\, r(g)((\lambda I + H)\, \tau)(g) \qquad (27)\end{aligned}$$

where $\tau(g) = (\psi, L(g)\, \varphi)\, \chi'(g)$. Since $C_c^\infty(G)$ is dense in $L_1(G)$ it follows by continuity that (27) is valid for all $r \in L_1(G)$ with $\operatorname{supp} r \subseteq \operatorname{supp} \chi$. Now let r_λ be the function on G with support contained in Ω such that $\hat{r}_\lambda = (\tilde{R}_\lambda \circ \exp_0) \cdot \hat{\chi}$ where $\tilde{R}_\lambda$ denotes the kernel of the resolvent $(\lambda I + dL_{G_0}(P))^{-1}$ on G_0. Then using (22) one readily calculates that

$$\begin{aligned}(\psi, (\lambda I + H)(r_\lambda * \varphi)) \\ &= \int_W da\, \sigma(a)(\tilde{R}_\lambda \circ \exp_0)(a)\, \hat{\chi}(a)((\lambda I + P_{X^{(0)}} + H')\, \hat{\tau})(a) \\ &= \int_W da\, \sigma(a)\, \delta(a)\, \hat{\chi}(a)\, \hat{\tau}(a) + \int_W da\, \sigma(a)\, \hat{s}_\lambda(a)(\psi, L(\exp a)\, \varphi),\end{aligned}$$

in the sense of distributions where $\hat{s}_\lambda$ has the form $\hat{s}_\lambda = \sum_i ((L^{(i)} \tilde{R}_\lambda) \circ \exp_0) \cdot \hat{\chi}_i$. Once again the $\hat{\chi}_i \in C_c^\infty(W)$ and the $L^{(i)}$ are operators of actual order less than m. But the estimates of Lemma 5.9 imply that $\|r_\lambda\|_1 \leqslant c\lambda^{-1}$ and $\|s_\lambda\|_1 \leqslant c\lambda^{-\eta/m}$ for some $\eta > 0$ and large λ. Therefore, if R_λ and S_λ denote the operators of convolution with r_λ and s_λ, respectively, then $\|R_\lambda\|_{2\to 2} \leqslant c\lambda^{-1}$ and $\|S_\lambda\|_{2\to 2} \leqslant c\lambda^{-\eta/m}$. So

$$(\psi, (\lambda I + H)(r_\lambda * \varphi)) = (\psi, \varphi) + (\psi, s_\lambda * \varphi)$$

and

$$(\lambda I + \bar{H})\, R_\lambda \varphi = \varphi + S_\lambda \varphi \qquad (28)$$

for all $\varphi \in C_c^\infty(G)$. By density it follows that $R_\lambda L_2 \subseteq D(\bar{H})$ and (28) is valid for all $\varphi \in L_2$. Thus if $c\lambda^{-\eta/m} < 1$ then $(I + S_\lambda)$ has a bounded inverse and

$$\varphi = (\lambda I + \bar{H})\, R_\lambda (I + S_\lambda)^{-1}\, \varphi$$

for all $\varphi \in L_2(G)$. This establishes that the range of $(\lambda I + \bar{H})$ is equal to $L_2(G)$ for all large λ and hence $\bar{H}$ is self-adjoint. But it then follows that

$$\varphi = (I + S_\lambda^*)^{-1}\, R_\lambda^* (\lambda I + \bar{H})\, \varphi$$

and hence

$$\|\varphi\|_2 \leqslant c\lambda^{-1}(1 - c\lambda^{-\eta/m})^{-1}\, \|(\lambda I + \bar{H})\, \varphi\|_2.$$

Therefore $(\lambda I + \bar{H})$ has a bounded inverse. Thus $\bar{H}$ is lower semibounded by spectral theory. ∎

Now it is straightforward to prove that K is a convolution semigroup.

Since $\operatorname{Re} H$ is a symmetric weighted subcoercive operator on $L_2(G)$ it follows from Proposition 8.1 that it is lower semibounded on L_2, i.e., $\operatorname{Re}(\varphi, H\varphi) \geqslant -\nu\, \|\varphi\|_2^2$ for some $\nu \geqslant 0$ and all $\varphi \in L'_{2;\, m}$. Next observe that if $\varphi_t \in D(H)$ satisfies the Cauchy equation

$$\frac{d}{dt}\, \varphi_t + H\varphi_t = 0 \tag{29}$$

for all $t > 0$ then

$$\frac{d}{dt}\, \|\varphi_t\|_2^2 = -2\operatorname{Re}(\varphi_t, H\varphi_t) \leqslant 2\nu\, \|\varphi_t\|_2^2.$$

Therefore $t \mapsto e^{-\nu t}\, \|\varphi_t\|_2$ is a decreasing function. Now suppose $\varphi_t^{(1)}$ and $\varphi_t^{(2)}$ both satisfy (29) and $\varphi_t^{(1)} \to \varphi$, $\varphi_t^{(2)} \to \varphi$ as $t \to 0$. Then $\varphi_t^{(1)} - \varphi_t^{(2)}$ also satisfies the equation but $\varphi_t^{(1)} - \varphi_t^{(2)} \to 0$ as $t \to 0$. Therefore, as a consequence of the foregoing decrease property, $\varphi_t^{(1)} = \varphi_t^{(2)}$, i.e., the solution of (29) is uniquely determined by the initial data $\varphi = \varphi_0$.

Now let $\varphi \in L'_{2;\, m}$. Then $\varphi_t = S_{t+s}\varphi = K_{t+s} * \varphi$ satisfies (29) with initial data $\varphi_0 = S_s\varphi$ (see (26)). Moreover, $\varphi_t = S_t S_s \varphi$ satisfies the equation with the same initial data. Therefore $(S_{t+s} - S_t S_s)\varphi = 0$ for all $\varphi \in L'_{2;\, m}$ and then, by continuity, for all $\varphi \in L_2$. This establishes that S is a semigroup on L_2. But this implies that K_t is a convolution semigroup. Therefore S is also a semigroup on the other L_p^ρ-spaces or in any Banach space representation.

It follows from (26) that the generator H_S of S is an extension of H on L_p^ρ. Now $L_{p;\,\infty}^\rho$ is a dense S-invariant subspace and hence a core of H_S. Therefore H_S must be the closure of H.

At this point we have essentially established the main result for the left regular representation in the L'_p-spaces if $p \in [1, \infty\rangle$.

THEOREM 8.2. *Let* $(\mathcal{X}, G, U)$ *be a continuous representation,* $a_1, ..., a_{d'}$ *a reduced weighted algebraic basis in the Lie algebra* $\mathfrak{g}$ *of* G *and* C *a* G_0*-weighted subcoercive form of order* m *where* G_0 *is the homogeneous contraction of* G. *Let* $H = dU(C)$ *be the associated operator. Then one has the following.*

I. *The closure* $\bar{H}$ *of* H *generates a continuous semigroup* S *and* S *has* K *as kernel.*

II. *The semigroup* S *is holomorphic in a sector* $\Lambda(\theta) = \{z \in \mathbf{C} : |\arg z| < \theta\}$ *where the angle of holomorphy* θ *satisfies the bounds* $\theta \geqslant \theta_{C, G_0}$.

III. $\bar{H} = H^{\dagger *}$, *where* $H^\dagger = dU_*(C^\dagger)$ *is the dual operator.*

Proof. Since the kernel K is a convolution semigroup it now follows that $(S_t)_{t>0} = (U(K_t))_{t>0}$ is a continuous semigroup. One then deduces as in Theorem 3.4 of [AER] that $\bar{H}$ is the generator and S is holomorphic, with the holomorphy sector containing at least $\Lambda(\theta_{C, G_0})$. ∎

As a consequence of the bounds on the kernel we can compare the domain of powers of the operator $\overline{dU(C)}$ and the differential structure of the representation associated with the weighted algebraic basis $a_1, \cdots, a_{d'}$, i.e., the spaces $\mathcal{X}'_n$.

COROLLARY 8.3. *Let* $(\mathcal{X}, G, U)$ *be a continuous representation,* $a_1, ..., a_{d'}$ *a reduced weighted algebraic basis in the Lie algebra* $\mathfrak{g}$ *of* G *and* C *a* G_0*-weighted subcoercive form of order* m. *Let* S *be the semigroup generated by the closure of the operator* $H = dU(C)$.

I. *The semigroup* S *maps into the smooth* C^∞*-elements, i.e.,* $S_t \mathcal{X} \subseteq \mathcal{X}_\infty$ *for all* $t > 0$.

II. *If* $k \in [0, \infty\rangle$ *then there exist* $c > 0$ *and* $\omega \geqslant 0$ *such that*

$$\|S_t x\|'_k \leqslant c t^{-k/m} e^{\omega t} \|x\|$$

for all $t > 0$ *and* $x \in \mathcal{X}$.

III. *If* $n \in \mathbf{N}$ *and* $k \in [0, nm\rangle$ *then* $D(\bar{H}^n) \subseteq \mathcal{X}'_k$ *and there exists* $c > 0$ *such that*

$$\|x\|'_k \leqslant \varepsilon^{mn-k} \|\bar{H}^n x\| + c\varepsilon^{-k} \|x\|$$

for all $x \in D(\bar{H}^n)$ *and* $\varepsilon \in \langle 0, 1]$. *In particular*

$$\mathscr{X}_\infty = \bigcap_{n=1}^{\infty} D(\bar{H}^n),$$

so the spaces of C^∞*-elements of* $(\mathscr{X}, G, U)$ *and of the operator* $\bar{H}$ *coincide.*

Proof. Statements I and II follow immediately from the fact that the kernel K_t is smooth and, together with its derivatives, satisfies Gaussian bounds.

If $n \in \mathbf{N}$, $k \in [0, nm\rangle$ and $\lambda \geqslant 0$ is large enough then

$$(\lambda I + \bar{H})^{-n} = (n-1)!^{-1} \int_0^\infty dt\, e^{-\lambda t} t^{n-1} S_t.$$

Therefore $D(\bar{H}) = R((\lambda I + \bar{H})^{-n}) \subseteq \mathscr{X}'_k$ by Statement II. Moreover,

$$\begin{aligned}\|A^\alpha(\lambda I + \bar{H})^{-n}\| &\leqslant (n-1)!^{-1}\, c \int_0^\infty dt\, e^{-(\lambda-\omega)\,t} t^{n-1-\|\alpha\|/m} \\ &= c'(\lambda-\omega)^{(mn-\|\alpha\|)/m} \leqslant c''\lambda^{(mn-\|\alpha\|)/m}\end{aligned}$$

if λ is large enough. Taking ε proportional to $\lambda^{-1/m}$ and rearranging it follows that

$$\|A^\alpha x\| \leqslant \varepsilon^{mn-\|\alpha\|}\, \|\bar{H}^n x\| + c'''\varepsilon^{-\|\alpha\|}\, \|x\|$$

for all $x \in D(\bar{H}^n)$ and for small positive values of ε. Statement III then follows. ∎

Remark. Note that the constants c in Corollary 8.3 depend on the kernel only though the constants M and ω in the bounds $\|A^\alpha K_t\|_1^\rho \leqslant Mt^{-\|\alpha\|/m} e^{\omega t}$ if $\rho \geqslant 0$ is such that $\|U(g)\| \leqslant Me^{\rho\,|g|'}$.

9. REGULARITY

The bounds on the semigroup in the previous section enable the deduction of several regularity results for the operators $dU(C)$ associated with a representation $(\mathscr{X}, G, U)$, a reduced weighted algebraic basis and a G_0-weighted subcoercive form of order m. Recall that $w = \min\{x \in [1, \infty\rangle : x \in w_i\mathbf{N}$ for all $i \in \{1, ..., d'\}\}$. We adopt the notation of [BuB] for the real interpolation spaces. We need two special interpolation spaces associated with the representation U and the distance corresponding to the weighted algebraic basis.

Let $\mathcal{O}$ be a bounded open neighbourhood of the identity e of G, $p \in [1, \infty]$ and $n \in \mathbf{N}$. Then for each $\gamma \in \langle 0, n\lambda_1 \rangle$, with λ_1 the smallest weight of the algebraic basis, define $\|\cdot\|_\gamma^{n,p,U} \colon \mathcal{X} \to [0, \infty]$ by

$$\|x\|_\gamma^{n,p,U} = \|x\| + \left(\int_{\mathcal{O}^n} d\mu_n(\mathbf{g}) \, (|\mathbf{g}|^{-\gamma} \, \|(I_U(g_1)) \cdots (I - U(g_n)) \, x\|)^p \right)^{1/p},$$

where $\mathbf{g} = (g_1, ..., g_n)$ and $|\mathbf{g}| = |g_1|' + \cdots + |g_n|'$. Moreover, μ_n is the absolutely continuous measure with respect to the left Haar measure on G^n with density $\mathbf{g} \mapsto |\mathbf{g}|^{-nD'}$. The usual changes are needed in the case $p = \infty$. Then the Lipschitz space $\mathcal{X}_\gamma^{n,p}(U)$ is defined by

$$\mathcal{X}_\gamma^{n,p}(U) = \{x \in \mathcal{X} \colon \|x\|_\gamma^{n,p,U} < \infty\}.$$

It is a Banach space with respect to the norm $\|\cdot\|_\gamma^{n,p,U}$. Note that as the space is independent of the choice of $\mathcal{O}$, up to equivalence of norms, we have omitted it from the notation.

Next we introduce a uniform version of the Lipschitz spaces. First, for each $x \in \mathcal{X}$ and $n \in \mathbf{N}_0$ define $\omega_x^{(n)} \colon \langle 0, \infty \rangle \to [0, \infty \rangle$ by $\omega_x^{(0)}(t) = \|x\|$ and

$$\omega_x^{(n)}(t) = \sup_{\substack{g_1, ..., g_n \in G \\ |g_j|' \leq t}} \|(I - U(g_1)) \cdots (I - U(g_n)) \, x\|$$

for $n \in \mathbf{N}$. Secondly, for $\gamma \in \langle 0, n\lambda_1 \rangle$ define $\|\cdot\|_\gamma^{n,p,\omega} \colon \mathcal{X} \to [0, \infty]$ by

$$\|x\|_\gamma^{n,p,\omega} = \|x\| + \left(\int_0^1 dt \, t^{-1} (t^{-\gamma} \omega_x^{(n)}(t))^p \right)^{1/p}.$$

Then the space

$$\mathcal{X}_\gamma^{n,p,\omega} = \{x \in \mathcal{X} \colon \|x\|_\gamma^{n,p,\omega} < \infty\}$$

is a Banach space with respect to the norm $\|\cdot\|_\gamma^{n,p,\omega}$.

Finally we also use $\|\cdot\|_{\mathcal{Y}}$ to denote the norm on a Banach space $\mathcal{Y}$.

THEOREM 9.1. *Let $(\mathcal{X}, G, U)$ be a continuos representation, $a_1, ..., a_{d'}$ a reduced weighted algebraic basis in the Lie algebra $\mathfrak{g}$ of G, λ_1 the smallest weight and C a G_0-weighted subcoercive form of order m, where G_0 is the homogeneous contraction of G. Let S be the semigroup generated by the closure of the operator $H = dU(C)$.*

I. *If* $p \in [1, \infty]$, $\gamma > 0$, $\underline{n} = \min\{n \in \mathbf{N}; \gamma < nw\}$, $\underline{k} = \min\{k \in \mathbf{N}: k \geqslant \underline{n}w/\lambda_1\}$ *and* $k, n \in \mathbf{N}$ *are such that* $k \geqslant \underline{k}$, $n \geqslant \underline{n}$ *then*

$$(\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K}} = (\mathscr{X}, D(\bar{H}^n))_{\gamma/(nm),\, p;\, \mathrm{K}} = \mathscr{X}^{k,\, p,\, \omega}_{\gamma} = \mathscr{X}^{k,\, p}_{\gamma}(U)$$

as Banach spaces.

II. *Let* $p \in [1, \infty]$. *If* $n_1, n_2 \in \mathbf{N}$ *and* $0 < \gamma < n_1 \wedge n_2$ *then*

$$(\mathscr{X}, \mathscr{X}'_{n_1 w})_{\gamma/(n_1 w),\, p;\, \mathrm{K}} = (\mathscr{X}, \mathscr{X}'_{n_2 w})_{\gamma/(n_2 w),\, p;\, \mathrm{K}}.$$

III. *If* $l, n \in \mathbf{N}_0$ *and* $k \in \langle lw, nw \rangle$ *then there exists a* $c > 0$ *such that*

$$\|x\|'_k \leqslant \varepsilon^{nw-k} \|x\|'_{nw} + c\varepsilon^{-(k-lw)} \|x\|'_{lw}$$

for all $\varepsilon > 0$ *and* $x \in \mathscr{X}'_{nw}$.

IV. *If* $l, n \in \mathbf{N}_0$ *and* $k \in \langle lw, nw \rangle$ *then there exists a* $c > 0$ *such that*

$$N'_k(x) \leqslant \varepsilon^{nw-k} N'_{nw}(x) + c\varepsilon^{-(k-lw)} \|x\|'_{lw}$$

for all $\varepsilon > 0$ *and* $x \in \mathscr{X}'_{nw}$.

V. *If* $n, k \in \mathbf{N}$, $\gamma \in \langle 0, nw \rangle$ *and* $p \in [1, \infty]$ *then*

$$\{x \in D(\bar{H}^k): \bar{H}^k x \in (\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K}}\} \subseteq (\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K};\, km}.$$

Moreover, if λ *is large enough then there exists a* $c > 0$ *such that*

$$\|x\|_{(\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K};\, km}} \leqslant c\, \|(\bar{H} + \lambda I)^k x\|_{(\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K}}}$$

for all $x \in \{x \in D(\bar{H}^k): \bar{H}^k x \in (\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K}}\}$ *where* $(\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K};\, km}$ *denotes the space of (weighted) km-times differentiable vectors for the Lipschitz space* $(\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K}}$.

Proof. The proofs are very similar to those in Section 5 of [ElR5], so we only indicate the differences. The equality $(\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K}} = (\mathscr{X}, D(\bar{H}^n))_{\gamma/(nm),\, p;\, \mathrm{K}}$ follows as in Proposition 5.1 of [ElR5] and therefore Statement II is valid. So $(\mathscr{X}, \mathscr{X}'_{nw})_{\gamma/(nw),\, p;\, \mathrm{K}} = (\mathscr{X}, \mathscr{X}'_{\underline{n}w})_{\gamma/(\underline{n}w),\, p;\, \mathrm{K}}$. Now it follows as on pp. 581–582 in [ElR5] that

$$\omega^{\underline{k}}_{x_\infty} \leqslant c \sum_{\substack{\alpha \in J(d) \\ |\alpha| = \underline{k}}} t^{\|\alpha\|} \, \|A^\alpha x_\infty\|$$

for some $c > 0$, uniformly for all $t \in \langle 0, 1]$ and $x_\infty \in \mathscr{X}_\infty$. Since $\|\alpha\| \geqslant \lambda_1 \underline{k} \geqslant \underline{n}w$ for all α with $|\alpha| = \underline{k}$ one can argue as in the proof of Theorem 5.7 in

[ElR5] to deduce that $(\mathcal{X}, \mathcal{X}'_{\underline{n}w})_{\gamma/(\underline{n}w),\, p;\, \mathrm{K}} \subseteq \mathcal{X}^{k,\, p,\, \omega}_{\underline{\gamma}}$. The inclusion $\mathcal{X}^{k,\, p,\, \omega}_{\underline{\gamma}} \subseteq \mathcal{X}^{k,\, p,\, \omega}_{\gamma}$ follows by definition and the local boundedness of the representation. Next the inclusions $\mathcal{X}^{k,\, p,\, \omega}_{\gamma} \subseteq \mathcal{X}^{k,\, p}_{\gamma}(U) \subseteq (\mathcal{X}, D(\bar{H}^k))_{\gamma/(km),\, p;\, \mathrm{K}}$ can be proved precisely as in Steps 2 and 3 of the proof of Theorem 3.2 in [ElR2]. Statement I follows by an application of the reiteration theorem (see [BuB]).

Since $\mathcal{X}'_{lw}$ is continuously embedded in $(\mathcal{X}, \mathcal{X}'_{nm})_{lw/(nm),\, \infty;\, \mathrm{K}}$, by the proof of Proposition 5.1 in [ElR5], Statement III follows from Corollary 8.3.III and Statement I as in [ElR2] Proposition 4.3. Statement IV is an easy consequence of Statement III. ∎

Next we turn to unitary representations.

THEOREM 9.2. *Let $(\mathcal{X}, G, U)$ be a unitary representation, $a_1, ..., a_{d'}$ a reduced weighted algebraic basis in the Lie algebra $\mathfrak{g}$ of G and C a G_0-weighted subcoercive form of order m, where G_0 is the homogeneous contraction of G.*

I. *The operator $H = dU(C)$ is closed.*

II. *For all $n \in \mathbf{N}$ and all large $\lambda > 0$*

$$D((\lambda I + H)^{nw/m}) = \mathcal{X}'_{nw}$$

with equivalent norms.

III. *For each $\varepsilon > 0$ there exists a $\nu \in \mathbf{R}$, independent of the representation U, such that*

$$\mathrm{Re}(x, Hx) \geqslant (\mu_{C,\, G_0} - \varepsilon)(\|x\|'_{m/2})^2 - \nu\, \|x\|^2$$

for all $x \in \mathcal{X}_\infty$.

IV. *If $n \in \mathbf{N}$ then*

$$\mathcal{X}'_{nw} = \bigcap_{i=1}^{d'} D(A_i^{nw/w_i}).$$

V. *For each $\theta \in \langle 0, \theta_{C,\, G_0} \rangle$ there exists an $\omega > 0$ such that $\|S_z\| \leqslant e^{\omega |z|}$ uniformly for all $z \in \Lambda(\theta)$ where S is the holomorphic semigroup generated by H.*

Proof. The proofs of Statements I, II and IV are as in the proof of Theorem 5.8 in [ElR5] and the proof of Statement V is similar to the proof in [BGJR]. Since Statement III is stronger than Statement III of Theorem 5.8 in [ElR5] we give a new proof.

Let C_0 be the weighted subcoercive form such that

$$dV(C_0) = \sum_{\substack{\alpha \in J(d') \\ \|\alpha\| = m/2}} (-1)^{|\alpha|}\, A^{\alpha *} A^{\alpha}$$

in any continuous representation $(\mathcal{Y}, G, V)$. Let M be the number of multi-indices $\alpha \in J(d')$ with $\|\alpha\| = m/2$. Then

$$(N'_{V, m/2}(x))^2 \leqslant (x, dV(C_0)\, x) \leqslant M(N'_{V, m/2}(x))^2$$

for all $x \in \mathcal{Y}_\infty(V)$ if V is unitary. Next, let $\alpha_0 \in J(d')$ with $\|\alpha_0\| = m/2$ and $\varepsilon \in \langle 0, (2M)^{-1} \mu_{C, G_0} \rangle$. Further let C_1 be the homogeneous form such that

$$dV(C_1) = (\mu_{C, G_0} - 2M\varepsilon)(-1)^{|\alpha_0|}\, A^{\alpha_0 *} A^{\alpha_0} + \varepsilon\, dV(C_0).$$

Then

$$\mathrm{Re}(\varphi, dL_{G_0}(\Re C - C_1)\, \varphi) \geqslant M\varepsilon(N'_{2, L_{G_0}}(\varphi))^2$$

for all $\varphi \in L_{2; \infty}(G_0)$. So $\Re C - C_1$ is a G_0-weighted subcoercive form for which the corresponding operator is essentially self-adjoint and its closure generates a semigroup. Hence $\overline{dU(\Re C - C_1)}$ is lower semibounded by spectral theory, with lower bound $-\nu \leqslant 0$. Therefore

$$\begin{aligned}\mathrm{Re}(dU(C)\, x, x) = (dU(\Re C)\, x, x) &\geqslant (x, dU(C_1)\, x) - \nu\, \|x\|^2 \\ &\geqslant (\mu_{C, G_0} - 2M\varepsilon)\, \|A^{\alpha_0} x\|^2 - \nu\, \|x\|^2.\end{aligned}$$

Since the number of multi-indices α_0 with $\|\alpha_0\| = m/2$ is finite the theorem follows. ∎

It is also possible to obtain regularity results for the left regular representation on the L_p-spaces with respect to left Haar measure if $p \in \langle 1, \infty \rangle$. These are basically a result of the good kernel bounds and the regularity on L_2.

Corollary 9.3. *Let G be a connected Lie group, $a_1, \ldots, a_{d'}$ a reduced algebraic basis of the Lie algebra $\mathfrak{g}$ of G and C a G_0-weighted subcoercive form of order m. Let L be the left regular representation on L_p, where $p \in \langle 1, \infty \rangle$. Then*

I. *The operator $H = dL(C)$ is closed.*

II. *For all $n \in \mathbf{N}$ one has*

$$D((\lambda I + H)^{nw/m}) = L'_{p; nw}$$

with equivalent norms, if $\lambda > 0$ is large enough.

III. *If $n \in \mathbf{N}$ then*

$$L'_{p; nw} = \bigcap_{i=1}^{d'} D(A_i^{nw/w_i}).$$

Similar statements are valid on the space L_p-spaces with respect to right Haar measure, $L_{\hat{p}}$.

Proof. The proof is precisely the same as for the unweighted operators in [BER]. ∎

COROLLARY 9.4. *Let G be a connected Lie group, $a_1, ..., a_{d'}$ a reduced algebraic basis of the Lie algebra $\mathfrak{g}$ of G and C a G_0-weighted subcoercive form of order m. Let L be the left regular representation on L_p, where $p \in \langle 1, \infty \rangle$, and $H = dL(C)$. If $\theta \in \langle 0, \theta_C \rangle$ then there is a $\nu_0 \geqslant 0$, independent of p, such that the operators $\nu I + H$, $\nu > \nu_0$, have a bounded functional calculus over the bounded functions holomorphic in a sector $\Lambda(\varphi)$ with $\varphi \in \langle \pi/2 - \theta, \pi]$.*

Proof. The proof is precisely the same as in [ElR4]. ∎

Note that in the next section we establish that C is G_0-weighted subcoercive if, and only if, it is G-weighted subcoercive so the last two results could be phrased entirely in terms of G.

10. WEIGHTED SUBCOERCIVE FORMS: PART II

In this section we prove that all conditions of Proposition 4.5 concerning the Gårding inequality are equivalent. Moreover, we give other characterizations in the spirit of the characterization of hypoelliptic operators by Rockland operators on a homogeneous group.

THEOREM 10.1. *Let G be a connected Lie group, $a_1, ..., a_{d'}$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of G and G_0 the corresponding homogeneous contraction of G. Further let $m \in 2w\mathbf{N}$ and C be an m-th order form with principal part P. The following conditions are equivalent.*

I. *The form C is G-weighted subcoercive.*

II. *The form C is G_0-weighted subcoercive.*

III. *For all non-trivial irreducible unitary representations $(\mathcal{X}, G_0, U)$ of G_0 one has*

$$\mathrm{Re}(x, dU(P)\, x) > 0$$

for all $x \in \mathcal{X}_\infty(U)$ with $x \neq 0$, where P is the principal part of C.

IV. *The operator $dL_{G_0}(\Re P)$ is a positive Rockland operator.*

Moreover, if these conditions are valid then $\mu_{C,G} = \mu_{C,G_0}$ and $\theta_{C,G} = \theta_{C,G_0}$.

Proof. The implication I ⇒ II has been established in Proposition 4.5 together with the inequality $\mu_{C,G} \leqslant \mu_{C,G_0}$. The converse implication II ⇒ I follows from Theorem 9.2.III and $\mu_{C,G} \geqslant \mu_{C,G_0}$. Therefore I and II are equivalent and $\theta_{C,G} = \theta_{C,G_0}$.

The implication II ⇒ III is trivial since $N'_{U,m/2}(x) \neq 0$ if U is a non-trivial irreducible unitary representation and $x \in \mathscr{X}_\infty(U)$ is non-zero.

If III is valid then $dL_{G_0}(\Re P)$ is hypoelliptic by the Helffer–Nourrigat theorem. Moreover, the Plancherel formula, [Kir, Proposition 4], gives $(dL_{G_0}(\Re P)\,\varphi, \varphi) \geqslant 0$ for all $\varphi \in C_c^\infty(G_0)$, and hence by continuity, for all $\varphi \in L_{2;\infty}(G_0)$. So $\Re P$ is a positive Rockland form.

The implication IV ⇒ II follows from Theorem 2.5 of [ElR7]. ▮

It now follows that all conditions of Proposition 4.5 are equivalent.

Remark. If $G_0 = \mathbf{R}^d$ then the equivalence II ⇔ III in Theorem 10.1 states that a form C is G_0-weighted subcoercive if, and only if, $\mathrm{Re} \sum_{\|\alpha\| = m} c_\alpha (i\xi)^\alpha > 0$ for all $\xi \in \mathbf{R}^d$ with $\xi \neq 0$. This gives new proofs for Example 4.1.

The implication $1' \Rightarrow 4$ in Proposition 4.5 states that

$$\mathrm{Re}(\varphi, dL_{G_0}(P)\,\varphi) \geqslant \mu(N'_{2;m/2}(\varphi))^2 - \nu\,\|\varphi\|_2^2$$

for all $\varphi \in L_{2;\infty}(G_0)$ if C is a G-weighted subcoercive form, where P is the principal part of C. This clearly implies that for all $\gamma \in \langle 0, m/2 \rangle$, $p \in [1, \infty]$ and $n \in \mathbf{N}$ with $n > m$ there exist $\mu > 0$ and $\nu \in \mathbf{R}$ such that

$$\mathrm{Re}(\varphi, dL_{G_0}(P)\,\varphi) \geqslant \mu(\|\varphi\|_\gamma^{n,p,L_{G_0}})^2 - \nu\,\|\varphi\|_2^2$$

for all $\varphi \in L_{2;\infty}(G_0)$. We next show that this seemingly weaker inequality also characterizes weighted subcoercivity.

Proposition 10.2. *Let G be a connected Lie group, $a_1, \ldots, a_{d'}$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of G and G_0 the corresponding homogeneous contraction of G. Further let $m \in 2w\mathbf{N}$ and C be an m-th order form with principal part P. The following conditions are equivalent.*

I. *The form C is G-weighted subcoercive.*

II. *There exist $\gamma \in \langle 0, m/2 \rangle$, $p \in [1, \infty]$, $n \in \mathbf{N}$ with $n > m$, $\mu > 0$ and $\nu \in \mathbf{R}$ such that*

$$\mathrm{Re}(\varphi, dL_{G_0}(P)\,\varphi) \geqslant \mu(\|\varphi\|_\gamma^{n,p,L_{G_0}})^2 - \nu\,\|\varphi\|_2^2$$

for all $\varphi \in L_{2;\infty}(G_0)$.

Proof. We only need to prove the implication II $\Rightarrow$ I. The proof is a modification of the reduction theorem in Section 2 in [HeN]. We show that Condition III of Theorem 10.1 is valid using a scaling argument and a refinement of the proof of Lemma 5.1. We may assume that $p=2$ and $\gamma<1$ by an application of the reiteration theorem [BuB, Proposition 3.2.18]. Moreover, we may assume that $G=G_0$.

If U is a bounded representation in $\mathscr{X}$ on G we define $N_\gamma^U\colon \mathscr{X}\to[0,\infty]$ by

$$N_\gamma^U(x)=\left(\int_{G^n} d\mu_n(\mathbf{g})\,(|\mathbf{g}|^{-\gamma}\,\|(I-U(g_1))\cdots(I-U(g_n))\,x\|)^2\right)^{1/2},$$

where $\mathbf{g}=(g_1,...,g_n)$, $|\mathbf{g}|=|g_1|'+\cdots+|g_n|'$ and μ_n is the absolutely continuous measure with respect to the left Haar measure on G^n with density $\mathbf{g}\mapsto|\mathbf{g}|^{-nD'}$ as before. Then there exists a constant $c>0$ such that

$$\|x\|_\gamma^{n,2,U}\leqslant\|x\|+N_\gamma^U(x)\leqslant\|x\|_\gamma^{n,2,U}(x)+c\,\|x\|$$

for all $x\in\mathscr{X}$. So one has

$$\mathrm{Re}(\varphi,dL_G(P)\,\varphi)\geqslant\mu(N_\gamma^{L_G}(\varphi))^2-\nu\,\|\varphi\|_2^2$$

for all $\varphi\in L_{2;\infty}(G)$. Therefore, by scaling,

$$\delta^m\,\mathrm{Re}(\varphi,dL_G(P)\,\varphi)\geqslant\mu\delta^{2\gamma}(N_\gamma^{L_G}(\varphi))^2-\nu\,\|\varphi\|_2^2$$

uniformly for all $\delta>0$ and $\varphi\in L_{2;\infty}(G)$.

Next we need some details about standard induced representations of G. We follow [HeN, Section 2] and [CoG]. Let $\mathfrak{m}$ be subalgebra of $\mathfrak{g}$ and let $b_1,...,b_k\in\mathfrak{g}$ be such that $k=\operatorname{codim}\mathfrak{m}$ and $\mathfrak{m}+\operatorname{span}\{b_1,...,b_i\}$ is a subalgebra of $\mathfrak{g}$ for all $i\in\{1,...,k\}$. Such elements exist by [CoG, Theorem 1.1.13]. Define $\alpha\colon\mathbf{R}^k\to G$ by

$$\alpha(s_1,...,s_k)=\exp(s_1b_1)\cdots\exp(s_kb_k).$$

For every $g\in G$ there exist (unique) $E_{\mathfrak{m}}(g)\in\mathfrak{m}$ and $F_{\mathfrak{m}}(g)\in\mathbf{R}^k$ such that

$$g=\exp(E_{\mathfrak{m}}(g))\,\alpha(F_{\mathfrak{m}}(g))$$

(see [CoG, Theorem 1.2.12]). Assume that the elements $b_1,...,b_k$ are normalized such that

$$\int_G dg\,\varphi(g)=\int_{\mathfrak{m}}dm\int_{\mathbf{R}^k}ds\,\varphi((\exp m)\,\alpha(s))$$

for all $\varphi \in C_c(G)$. Let $l \in \mathfrak{g}^*$ and suppose that $l([\mathfrak{m}, \mathfrak{m}]) = \{0\}$. Then $U_{l,\mathfrak{m}}: L_2(\mathbf{R}^k) \to L_2(\mathbf{R}^k)$ defined by

$$(U_{l,\mathfrak{m}}(g)\,\varphi)(s) = e^{il(E_{\mathfrak{m}}(\alpha(s)\,g))}\varphi(F_{\mathfrak{m}}(\alpha(s)\,g))$$

is unitary and $U_{l,\mathfrak{m}}$ is a unitary representation of G in $L_2(\mathbf{R}^k)$ which depends on the choice of $b_1, ..., b_k$. If $\mathfrak{m}$ is a polarizing subalgebra for l then the representation $U_{l,\mathfrak{m}}$ is irreducible, and all irreducible unitary representations of G are of this form, up to unitary equivalence (see [GoG, Chapter 2]).

We also need some results on reduction of variables. Let $\mathfrak{n} \subset \mathfrak{m}$ be subalgebras with $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{n}$ and let $b_1, ..., b_p, ..., b_q \in \mathfrak{g}$, where $q = \operatorname{codim} \mathfrak{n}$, such that $\mathfrak{n} + \operatorname{span}\{b_1, ..., b_i\}$ is a subalgebra of $\mathfrak{g}$ for all $i \in \{1, ..., q\}$ and $\mathfrak{m} = \mathfrak{n} + \operatorname{span}\{b_1, ..., b_p\}$. Set $k = q - p = \operatorname{codim} \mathfrak{m}$. Now we define $\alpha: \mathbf{R}^q \to G$ by $\alpha(s_1, ..., s_q) = \exp(s_1 b_1) \cdots \exp(s_q b_q)$ and also introduce $\beta: \mathbf{R}^k \to G$ by $\beta(s_1, ..., s_k) = \exp(s_1 b_{p+1}) \cdots \exp(s_k b_{p+k})$. For $\xi \in \mathbf{R}^p$ define $l_\xi \in \mathfrak{g}^*$ by

$$l_\xi\left(a + \sum_{i=1}^{p} t_i b_i + \sum_{i=1}^{k} s_i b_{p+i}\right) = \sum_{i=1}^{p} \xi_i s_i$$

for all $a \in \mathfrak{n}$, $t \in \mathbf{R}^p$ and $s \in \mathbf{R}^k$. Let $l \in \mathfrak{g}^*$ and suppose that $l([\mathfrak{m}, \mathfrak{m}]) = \{0\}$ and, moreover, $l(b_i) = 0$ for all $i \in \{1, ..., p\}$. We give a relation between $U_{l,\mathfrak{n}}$ and $U_{l+l_\xi,\mathfrak{m}}$. Note that $U_{l+l_\xi,\mathfrak{n}} = U_{l,\mathfrak{n}}$. Let $\mathbf{F}$ denote the (partial) Fourier transform on $L_2(\mathbf{R}^p \times \mathbf{R}^k)$ with respect to the first p variables. If $\varphi \in \mathcal{S}(\mathbf{R}^p \times \mathbf{R}^k)$ and $\xi \in \mathbf{R}^p$ define $(\mathbf{F}\varphi)_\xi \in \mathcal{S}(\mathbf{R}^k)$ by $(\mathbf{F}\varphi)_\xi\,(s) = (\mathbf{F}\varphi)(\xi, s)$.

Lemma 10.3. *If $\varphi \in \mathcal{S}(\mathbf{R}^p \times \mathbf{R}^k)$ then*

$$(\mathbf{F} U_{l,\mathfrak{n}}(g)\,\varphi)_\xi = U_{l+l_\xi,\mathfrak{m}}(g)(\mathbf{F}\varphi)_\xi$$

for all $\xi \in \mathbf{R}^p$ and $g \in G$.

Proof. Let $s \in \mathbf{R}^k$ and $t \in \mathbf{R}^p$. Then

$$\begin{aligned}(U_{l,\mathfrak{n}}(g)\,\varphi)(t, s) &= e^{il(E_{\mathfrak{n}}(\alpha(t,s)\,g))}\varphi(F_{\mathfrak{n}}(\alpha(t, s)\,g))\\ (U_{l+l_\xi,\mathfrak{m}}(g)(\mathbf{F}\varphi)_\xi)(s) &= e^{i(l+l_\xi)(E_{\mathfrak{m}}(\beta(s)\,g))}(\mathbf{F}\varphi)_\xi\,(F_{\mathfrak{m}}(\beta(s)\,g)).\end{aligned} \tag{30}$$

Now

$$\begin{aligned}\alpha(t, s)\,g = \alpha(t, 0)\,\beta(s)\,g &= (\alpha(t, 0)\exp E_{\mathfrak{m}}(\beta(s)\,g))\,\beta(F_{\mathfrak{m}}(\beta(s)\,g))\\ &= \exp E_{\mathfrak{n}}(\alpha(t, 0)\exp E_{\mathfrak{m}}(\beta(s)\,g))\\ &\quad\times \alpha(F_{\mathfrak{n}}(\alpha(t, 0)\exp E_{\mathfrak{m}}(\beta(s)\,g)))\cdot\beta(F_{\mathfrak{m}}(\beta(s)\,g)).\end{aligned}$$

So

$$E_{\mathfrak{n}}(\alpha(t, s)\, g) = E_{\mathfrak{n}}(\alpha(t, 0) \exp E_{\mathfrak{m}}(\beta(s)\, g))$$

and

$$F_{\mathfrak{n}}(\alpha(t, s)\, g) = (\pi_1(F_{\mathfrak{n}}(\alpha(t, 0) \exp E_{\mathfrak{m}}(\beta(s)\, g))), F_{\mathfrak{m}}(\beta(s)\, g))$$

where π_1 is the projection from $\mathbf{R}^p \times \mathbf{R}^k$ onto $\mathbf{R}^p$. Since $l([\mathfrak{m}, \mathfrak{m}]) = \{0\}$ it follows from the Campbell–Baker–Hausdorff formula that $l(\log(\exp a \exp b)) = l(a) + l(b)$ for all $a, b \in \mathfrak{m}$. Therefore

$$l(E_{\mathfrak{n}}(\alpha(t, s)\, g)) = l(t_1 b_1) + \cdots + l(t_p b_p) + l(E_{\mathfrak{m}}(\beta(s)\, g)) = l(E_{\mathfrak{m}}(\beta(s)\, g)).$$

If one uses $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{n}$ and the Campbell–Baker–Hausdorff formula once again one sees that $F_{\mathfrak{n}}(\exp a \exp b) = F_{\mathfrak{n}}(\exp a) + F_{\mathfrak{n}}(\exp b)$ for all $a, b \in \mathfrak{m}$. So

$$F_{\mathfrak{n}}(\alpha(t, 0) \exp E_{\mathfrak{m}}(\beta(s)\, g)) = (t, 0) + F_{\mathfrak{n}} \exp E_{\mathfrak{m}}(\beta(s)\, g).$$

Therefore

$$\varphi(F_{\mathfrak{n}}(\alpha(t, s)\, g)) = \varphi(t + \pi_1 F_{\mathfrak{n}} \exp E_{\mathfrak{m}}(\beta(s)\, g), F_{\mathfrak{m}}(\beta(s)\, g))$$

for all $t \in \mathbf{R}^p$. Using the identity $l_\xi(a) = \xi \cdot \pi_1 F_{\mathfrak{n}} \exp a$ for all $a \in \mathfrak{m}$ one establishes that

$$\begin{aligned}(\mathbf{F} U_{l,\mathfrak{n}}(g)\, \varphi)(\xi, s) &= e^{il(E_{\mathfrak{m}}(\beta(s)\, g))} e^{i\xi \cdot \pi_1 F_{\mathfrak{n}} \exp E_{\mathfrak{m}}(\beta(s)\, g)} (\mathbf{F}\varphi)(\xi, F_{\mathfrak{m}}(\beta(s)\, g)) \\ &= e^{i(l + l_\xi)(E_{\mathfrak{m}}(\beta(s)\, g))} (\mathbf{F}\varphi)_\xi\, (F_{\mathfrak{m}}(\beta(s)\, g)) \\ &= (U_{l + l_\xi, \mathfrak{m}}(g)(\mathbf{F}\varphi)_\xi)(s)\end{aligned}$$

and the lemma has been proved. ▮

This relation is the key to obtaining a connection between $N_\gamma^{U_{l+l_\xi}, \mathfrak{m}}$ and $N_\gamma^{U_{l,\mathfrak{n}}}$.

LEMMA 10.4. *Let* $\xi_0 \in \mathbf{R}^p$ *and* $\tau \in C_c^\infty(\mathbf{R}^p)$ *be positive with* $\|\tau\|_2 = 1$. *For* $j \in \mathbf{N}$ *define* $\tau_j \in \mathcal{S}(\mathbf{R}^p)$ *by* $\tau_j(\xi) = j^{p/2} \tau(j(\xi - \xi_0))$. *Let* $\psi \in \mathcal{S}(\mathbf{R}^k)$ *and set* $\varphi_j = (\mathbf{F}^{-1} \tau_j) \otimes \psi$. *Then*

I. $N_\gamma^{U_{l+l_{\xi_0}}, \mathfrak{m}}(\psi) = \lim_{j \to \infty} N_\gamma^{U_{l,\mathfrak{n}}}(\varphi_j)$,

II. $(\psi, U_{l + l_{\xi_0}, \mathfrak{m}}(a^\alpha)\, \psi) = \lim_{j \to \infty} (\varphi_j, dU_{l,\mathfrak{n}}(a^\alpha)\, \varphi_j)$ *for all* $\alpha \in J(d)$.

Proof. For all $j \in \mathbf{N}$ one has

$$(N_\gamma^{U_{l,\mathfrak{n}}}(\varphi_j))^2$$

$$= \int_{G^n} d\mu_n(\mathbf{g}) \, (|\mathbf{g}|^{-\gamma} \, \|\mathbf{F}(I - U_{l,\mathfrak{n}}(g_1)) \cdots (I - U_{l,\mathfrak{n}}(g_n)) \, \varphi_j\|)^2$$

$$= \int_{G^n} d\mu_n(\mathbf{g}) \, (|\mathbf{g}|^{-\gamma} \, \|\tau_j \otimes (I - U_{l+l_\xi,\mathfrak{m}}(g_1)) \cdots (I - U_{l+l_\xi,\mathfrak{m}}(g_n)) \, \psi\|)^2$$

$$= \int_{G^n} d\mu_n(\mathbf{g}) \int_{\mathbf{R}^k} ds \int_{\mathbf{R}^p} d\xi \, |\mathbf{g}|^{-2\gamma} \, |\tau_j(\xi)|^2$$

$$\times |((I - U_{l+l_\xi,\mathfrak{m}}(g_1)) \cdots (I - U_{l+l_\xi,\mathfrak{m}}(g_n)) \, \psi)(s)|^2$$

$$= \int_{G^n} d\mu_n(\mathbf{g}) \int_{\mathbf{R}^k} ds \, |\mathbf{g}|^{-2\gamma} \, \psi_j(\mathbf{g}, s),$$

where

$$\psi_j(\mathbf{g}, s) = \int_{\mathbf{R}^p} d\xi \, |\tau_j(\xi)|^2 \, |((I - U_{l+l_\xi,\mathfrak{m}}(g_1)) \cdots (I - U_{l+l_\xi,\mathfrak{m}}(g_n)) \, \psi)(s)|^2.$$

Obviously

$$\lim_{j \to \infty} \psi_j(\mathbf{g}, s) = |((I - U_{l+l_{\xi_0},\mathfrak{m}}(g_1)) \cdots (I - U_{l+l_{\xi_0},\mathfrak{m}}(g_n)) \, \psi)(s)|^2$$

for all $\mathbf{g} \in G^n$ and $s \in \mathbf{R}^k$, by (30), so if we can show that $\int_{G^n} d\mu_n(\mathbf{g}) \times \int_{\mathbf{R}^k} ds \, |\mathbf{g}|^{-2\gamma} \, \psi_j(\mathbf{g}, s)$ is uniformly bounded in j then the first statement follows from the Lebesgue dominated convergence theorem.

Clearly

$$\int_{\mathbf{R}^k} ds \, \psi_j(\mathbf{g}, s) = \int_{\mathbf{R}^p} d\xi \, |\tau_j(\xi)|^2 \, \|(I - U_{l+l_\xi,\mathfrak{m}}(g_1)) \cdots (I - U_{l+l_\xi,\mathfrak{m}}(g_n)) \, \psi\|^2$$

$$\leqslant \int_{\mathbf{R}^p} d\xi \, |\tau_j(\xi)|^2 \, 2^{2n} \, \|\psi\|^2 = 2^{2n} \, \|\psi\|^2$$

for all $\mathbf{g} \in G^n$. So

$$\int_{\{\mathbf{g}: \, |\mathbf{g}| \geqslant 1\}} d\mu_n(\mathbf{g}) \, |\mathbf{g}|^{-2\gamma} \, \psi_j(\mathbf{g}, s) \leqslant 2^{2n} \, \|\psi\|^2 \int_{\{\mathbf{g}: \, |\mathbf{g}| \geqslant 1\}} d\mu_n(\mathbf{g}) \, |\mathbf{g}|^{-2\gamma} < \infty$$

for all $j \in \mathbf{N}$. Finally, let $\mathbf{g} = (g_1, ..., g_n) \in G^n$. Then

$$\int_{\mathbf{R}^k} ds\, \psi_j(\mathbf{g}, s) \leqslant 2^{2(n-1)} \int_{\mathbf{R}^p} d\xi\, |\tau_j(\xi)|^2 \, \|(I - U_{l+l_\xi, \mathfrak{m}}(g_n))\, \psi\|^2$$

Now suppose $g_n = \exp(a)$. Then

$$\begin{aligned} \|(I - U_{l+l_\xi, \mathfrak{m}}(g_n))\, \psi\| &\leqslant \|a\| \left(\sum_{i=1}^{d} \|dU_{l+l_\xi, \mathfrak{m}}(a_i)\, \psi\|^2 \right)^{1/2} \\ &\leqslant d^{1/2}\, \|a\| \sum_{i=1}^{d} \|dU_{l+l_\xi, \mathfrak{m}}(a_i)\, \psi\|. \end{aligned}$$

For all $i \in \{1, ..., d\}$ and $s \in \mathbf{R}^k$ let $P_i(s) \in \operatorname{span}\{b_1, ..., b_p\}$ be such that

$$\frac{d}{dt} E_{\mathfrak{m}}(\beta(s) \exp(ta_i))|_{t=0} = P_i(s) + b$$

for some $b \in \mathfrak{n}$. Then P_i is a polynomial function and

$$(dU_{l+l_\xi, \mathfrak{m}}(a_i)\, \psi)(s) = (dU_{l, \mathfrak{m}}(a_i)\, \psi)(s) + i \sum_{i=1}^{d} l_\xi(P_i(s))\, \psi(s)$$

for all $\xi \in \mathbf{R}^p$. Since $\int d\xi\, |\tau_j(\xi)|^2\, |\xi_{i_1} \xi_{i_2}|^2$ is uniformly bounded for all $i_1, i_2 \in \{1, ..., d\}$ one deduces that

$$\int_{\mathbf{R}^k} ds\, \psi_j(\mathbf{g}, s) \leqslant c\, \|a\|^2$$

for some $c > 0$, uniformly for all $j \in \mathbf{N}$ and $\mathbf{g} \in G^n$, where $a = \log g_n$. Now $\|a\| \leqslant c'\, |g_n|$ if $|g_n| \leqslant 1$. Therefore $\|a\| \leqslant c'\, |g_n|' \leqslant c'\, |\mathbf{g}|$ if $|\mathbf{g}| \leqslant 1$. Since $\gamma < 1$ one then establishes that

$$\int_{\{\mathbf{g} :\, |\mathbf{g}| \leqslant 1\}} d\mu_n(\mathbf{g})\, |\mathbf{g}|^{-2\gamma}\, \psi_j(\mathbf{g}, s) \leqslant c(c')^2 \int_{\{\mathbf{g} :\, |\mathbf{g}| \leqslant 1\}} d\mu_n(\mathbf{g})\, |\mathbf{g}|^{2(1-\gamma)} < \infty$$

uniformly for all $j \in \mathbf{N}$ and Statement I follows.

One can establish Statement II by a similar argument (see also the proof of Lemma 2.2 in [HeN]). ∎

COROLLARY 10.5. *If*

$$\delta^m \operatorname{Re}(\varphi, dU_{l, \mathfrak{n}}(P)\, \varphi) \geqslant \mu \delta^{2\gamma} (N_\gamma^{U_{l, \mathfrak{n}}}(\varphi))^2 - \nu\, \|\varphi\|^2$$

for all $\varphi \in \mathcal{S}(\mathbf{R}^q)$ *then*

$$\delta^m \operatorname{Re}(\psi, dU_{l+l_{\xi_0}, \mathfrak{m}}(P)\, \psi) \geqslant \mu \delta^{2\gamma}(N_\gamma^{U_{l+l_{\xi_0}, \mathfrak{m}}}(\psi))^2 - \nu \, \|\psi\|^2$$

for all $\psi \in \mathcal{S}(\mathbf{R}^k)$ *and* $\xi_0 \in \mathbf{R}^p$.

Corollary 10.6. *If* $\mathfrak{n} \subseteq \mathfrak{m}$ *are subalgebras of* $\mathfrak{g}$ *with codimensions* k *and* q *and* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{n}$ *and* $l \in \mathfrak{g}^*$ *is such that* $l([\mathfrak{m}, \mathfrak{m}]) = \{0\}$ *then*

$$\delta^m \operatorname{Re}(\varphi, dU_{l, \mathfrak{n}}(P)\, \varphi) \geqslant \mu \delta^{2\gamma}(N_\gamma^{U_{l, \mathfrak{n}}}(\varphi))^2 - \nu \, \|\varphi\|^2$$

for all $\varphi \in \mathcal{S}(\mathbf{R}^q)$ *implies*

$$\delta^m \operatorname{Re}(\psi, dU_{l, \mathfrak{m}}(P)\, \psi) \geqslant \mu \delta^{2\gamma}(N_\gamma^{U_{l, \mathfrak{m}}}(\varphi))^2 - \nu \, \|\psi\|^2$$

for all $\psi \in \mathcal{S}(\mathbf{R}^k)$.

Now we finish the proof of Proposition 10.2. Let $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ be the filtration of $\mathfrak{g}$ and $\lambda_1 < \cdots < \lambda_k$ the weights of the filtration. Let $(\gamma_t)_{t>0}$ be the family of dilations on the homogeneous Lie algebra $\mathfrak{g}$. Let $l \in \mathfrak{g}^*$ and $\mathfrak{m}$ a polarizing subalgebra of $\mathfrak{g}$ for l. For $j \in \{1, ..., k\}$ set

$$\mathfrak{m}_j = \mathfrak{m} \cap \operatorname{span}\{a \in \mathfrak{g} \colon \exists_{\lambda \geqslant \lambda_j} \forall_{t>0} [\gamma_t(a) = t^\lambda a]\}$$

and set $\mathfrak{m}_{k+1} = \{0\}$. Then $\mathfrak{m}_{k+1} \subset \mathfrak{m}_k \subset \cdots \subset \mathfrak{m}_2 \subset \mathfrak{m}_1 = \mathfrak{m}$ are subalgebras of $\mathfrak{g}$ and $[\mathfrak{m}_j, \mathfrak{m}_j] \subset \mathfrak{m}_{j+1}$ for all $j \in \{1, ..., k\}$. The representation $U_{l, \mathfrak{m}_{k+1}}$ is unitarily equivalent with the left regular representation L_G of G in $L_2(G)$, so

$$\delta^m \operatorname{Re}(\varphi, dU_{l, \mathfrak{m}_{k+1}}(P)\, \varphi) \geqslant \mu \delta^{2\gamma}(N_\gamma^{U_{l, \mathfrak{m}_{k+1}}}(\varphi))^2 - \nu \, \|\varphi\|^2$$

for all $\varphi \in \mathcal{S}(\mathbf{R}^d)$ and $\delta > 0$. Hence by downward induction on j it follows from Corollary 10.6 that

$$\delta^m \operatorname{Re}(\varphi, dU_{l, \mathfrak{m}_j}(P)\, \varphi) \geqslant \mu \delta^{2\gamma}(N_\gamma^{U_{l, \mathfrak{m}_j}}(\varphi))^2 - \nu \, \|\varphi\|^2$$

for all $j \in \{1, ..., k\}$ and $\varphi \in \mathcal{S}(\mathbf{R}^{n_j})$, where $n_j = \operatorname{codim} \mathfrak{m}_j$. But $U_{l, \mathfrak{m}_1} = U_{l, \mathfrak{m}}$, so

$$\delta^m \operatorname{Re}(x, dU(P)\, x) \geqslant \mu \delta^{2\gamma}(N_\gamma^U(x))^2 - \nu \, \|x\|^2$$

for any irreducible unitary representation U of G, $x \in \mathcal{X}_\infty(U)$ and $\delta > 0$. Now suppose U is a non-trivial irreducible unitary representation of G and $x \in \mathcal{X}_\infty(U)$ is non-trivial. Then $N_\gamma^U(x) \neq 0$ since otherwise $(I - U(g_1)) \cdots (I - U(g_n))\, x = 0$ for all $g_1, ..., g_n \in G \backslash \{e\}$ and therefore $A^\alpha x = 0$ for all $\alpha \in J(d)$ with $|\alpha| = m$. Choose $\delta > 0$ so large that $\mu \delta^{2\gamma}(N_\gamma^U(x))^2 - \nu \, \|x\|^2 > 0$. Then $\delta^m \operatorname{Re}(x, dU(P)\, x) > 0$ and $\operatorname{Re}(x, dU(P)\, x) > 0$. Now the proposition follows from Theorem 10.1.III. ∎

The next proposition gives a necessary and sufficient condition for a form P to be a Rockland form on the homogeneous contraction group.

PROPOSITION 10.7. *Let G be a connected Lie group, $a_1, ..., a_{d'}$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of G and $m \in 2w\mathbf{N}$. Let C be a form of order m_0 with $m_0 \leqslant m$. The following conditions are equivalent.*

I. *The order of the form C equals m and the operator $dL_{G_0}(P)$ is hypoelliptic, where P is the principal part of C.*

II. *There exists a $c > 0$ such that*

$$\|A_i \varphi\|_2 \leqslant \varepsilon^{m-w_i} \|dL_G(C)\varphi\|_2 + c\varepsilon^{-w_i} \|\varphi\|_2$$

uniformly for all $\varepsilon \in \langle 0, 1]$, $\varphi \in C_c^\infty(G)$ and $i \in \{1, ..., d'\}$.

III. *There exist $c > 0$ and a neighbourhood V of the identity of G such that*

$$\|A_i \varphi\|_2 \leqslant \varepsilon^{m-w_i} \|dL_G(C)\varphi\|_2 + c\varepsilon^{-w_i} \|\varphi\|_2$$

uniformly for all $\varepsilon \in \langle 0, 1]$, $\varphi \in C_c^\infty(V)$ and $i \in \{1, ..., d'\}$.

Proof. I $\Rightarrow$ II. Suppose $dL_{G_0}(P)$ is hypoelliptic and $m = m_0$. Consider the form $C_1 = C^\dagger C$. The principal part of C_1 is $P^\dagger P$ and clearly $dL_{G_0}(\Re(P^\dagger P)) = dL_{G_0}(P^\dagger P)$ is a positive Rockland operator on $L_2(G_0)$. Hence the form C_1 is a weighted subcoercive form by Theorem 10.1. So by Theorems 9.1.III and 9.2.II there exist $c, \lambda > 0$ such that

$$\|A_i \varphi\|_2 \leqslant \varepsilon^{m-w_i} \|(\lambda I + dL_G(C_1))^{1/2} \varphi\|_2 + c\varepsilon^{-w_i} \|\varphi\|_2$$

uniformly for all $\varepsilon > 0$, $\varphi \in C_c^\infty(G)$ and $i \in \{1, ..., d'\}$. Since $dL_G(C_1)$ is the generator of a bounded semigroup it follows from [Rob, Lemma II.3.2], that there exists a $c' > 0$ such that $\|(\lambda I + dL_G(C_1))^{1/2} \varphi\|_2 \leqslant \|(dL_G(C_1))^{1/2} \varphi\|_2 + c' \|\varphi\|_2$, uniformly for all $\varphi \in C_c^\infty(G)$. Then

$$\begin{aligned}\|A_i \varphi\|_2 &\leqslant \varepsilon^{m-w_i} \|(dL_G(C_1))^{1/2} \varphi\|_2 + (c\varepsilon^{-w_i} + c'\varepsilon^{m-w_i}) \|\varphi\|_2 \\ &= \varepsilon^{m-w_i} \|dL_G(C)\varphi\|_2 + (c\varepsilon^{-w_i} + c'\varepsilon^{m-w_i}) \|\varphi\|_2\end{aligned}$$

from which Condition II follows.

The implication II $\Rightarrow$ III is trivial, so it remains to prove III $\Rightarrow$ I. Temporarily, define the form $P: J(d') \to \mathbf{C}$ by

$$P(\alpha) = \begin{cases} C(\alpha) & \text{if } \|\alpha\| = 0, \\ 0 & \text{if } \|\alpha\| < m. \end{cases}$$

Then P is the principal part of the form C if $m = m_0$, but $P = 0$ if $m_0 < m$. We use the notation of Section 3. In particular, W is the set constructed in Lemma 3.3.V. We may assume that $\exp W \subset V$.

First, the bounds in Condition III can be rephrazed as

$$\varepsilon^{w_i} \left(\int_W da\, \sigma(a)\, |(X_i \psi)(a)|^2 \right)^{1/2} \leqslant \varepsilon^m \left(\int_W da\, \sigma(a) \left| \sum_{\|\alpha\| \leqslant m} c_\alpha (X^\alpha \psi)(a) \right|^2 \right)^{1/2} + c \left(\int_W da\, \sigma(a)\, |\psi(a)|^2 \right)^{1/2}$$

for all $\psi \in C_c^\infty(W)$, $\varepsilon \in \langle 0, 1]$ and $i \in \{1, ..., d'\}$. Next fix $\psi \in C_c^\infty(W)$. Let $t \in \langle 0, 1]$. Replacing ε by εt and ψ by $\psi_{t^{-1}}$ in the previous inequality gives

$$\varepsilon^{w_i} \left(\int_W da\, \sigma(a)\, t^{w_i} |(X_i \psi_{t^{-1}})(a)|^2 \right)^{1/2} \leqslant \varepsilon^m \left(\int_W da\, \sigma(a) \left| \sum_{\|\alpha\| \leqslant m} t^{m - \|\alpha\|} c_\alpha t^{\|\alpha\|} (X^\alpha \psi_{t^{-1}})(a) \right|^2 \right)^{1/2} + c \left(\int_W da\, \sigma(a)\, |\psi_{t^{-1}}(a)|^2 \right)^{1/2}$$

for all $\varepsilon \in \langle 0, 1]$ and $i \in \{1, ..., d'\}$. (Note that the integrals only need to be carried out over $\gamma_t(W)$.) Changing variables and dividing by $t^{D'/2}$ then gives the estimates

$$\varepsilon^{w_i} \left(\int_W da\, \sigma(\gamma_t(a))\, |t^{D'/2} t^{w_i} (X_i \psi_{t^{-1}})(\gamma_t(a))|^2 \right)^{1/2} \leqslant \varepsilon^m \left(\int_W da\, \sigma(\gamma_t(a)) \left| \sum_{\|\alpha\| \leqslant m} t^{m - \|\alpha\|} c_\alpha t^{D'/2} t^{\|\alpha\|} (X^\alpha \psi_{t^{-1}})(\gamma_t(a)) \right|^2 \right)^{1/2} + c \left(\int_W da\, \sigma(\gamma_t(a))\, |t^{D'/2} \psi_{t^{-1}}(\gamma_t(a))|^2 \right)^{1/2}$$

for all $\varepsilon \in \langle 0, 1]$ and $i \in \{1, ..., d'\}$. Therefore by Corollary 3.7 one deduces that

$$\varepsilon^{w_i} \left(\int_W da\, |(X_i^{(0)} \varphi)(a)|^2 \right)^{1/2} \leqslant \varepsilon^m \left(\int_W da \left| \sum_{\|\alpha\| = m} c_\alpha (X^{(0)\alpha} \psi)(a) \right|^2 \right)^{1/2} + c \left(\int_W da\, |\psi(a)|^2 \right)^{1/2}$$

for all $\psi \in C_c^\infty(W)$, $\varepsilon \in \langle 0, 1]$ and $i \in \{1, ..., d'\}$.

Next let $\psi \in C_c^\infty(\mathfrak{g})$. There exists $r \geqslant 1$ such that $\operatorname{supp} \psi_r \subset W$. Then applying the previous inequality to ψ_r, gives

$$(\varepsilon r)^{w_i} \left(\int_{\mathfrak{g}} da \, |(X_i^{(0)} \psi)(a)|^2 \right)^{1/2}$$

$$\leqslant (\varepsilon r)^m \left(\int_{\mathfrak{g}} da \left| \sum_{\|\alpha\| = m} c_\alpha (X^{(0)\alpha} \psi)(a) \right|^2 \right)^{1/2} + c \left(\int_{\mathfrak{g}} da \, |\psi(a)|^2 \right)^{1/2}$$

and choosing $\varepsilon = r^{-1}$ finally gives

$$\|A_i^{(0)} \varphi\| \leqslant \|dL_{G_0}(P) \, \varphi\| + c \, \|\varphi\|$$

uniformly for all $\varphi \in C_c^\infty(G)$ and by density, for all $\varphi \in L_{2;\infty}(G_0)$.

Now on can argue as in the proof of Proposition 10.2. By scaling one has

$$\delta^{w_i} \, \|A_i^{(0)} \varphi\| \leqslant \delta^m \, \|dL_{G_0}(P) \varphi\| + c \, \|\varphi\|$$

for all $\delta > 0$ and $\varphi \in L_{2;\infty}(G_0)$ and by reduction (Lemma 10.4.II)

$$\delta^{w_i} \, \|dU(a_i) x\| \leqslant \delta^m \, \|dU(P) \, x\| + c \, \|x\|$$

for each irreducible unitary representation U of G_0, $x \in \mathscr{X}_\infty(U)$ and $\delta > 0$. Suppose U is non-trivial, $dU(P) \, x = 0$ and $x \neq 0$. Then $\delta^{w_i} \, \|dU(a_i) x\| \leqslant c \, \|x\|$ and hence $\|dU(a_i) \, x\| = 0$ for all $i \in \{1, ..., d'\}$. Since $a_1, ..., a_{d'}$ is an algebraic basis, this implies $x = 0$, which is a contradiction. So $dL_{G_0}(P)$ is a Rockland operator, and hypoelliptic by the Helffer–Nourrigat theorem. In particular, $P \neq 0$ and the order of the form C equals m. ∎

This proposition has immediate implications for subcoercive forms.

THEOREM 10.8. *Let G be a connected Lie group, $a_1, ..., a_{d'}$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of G. Suppose $m \in 2w\mathbf{N}$ and let C be an m-th order form with principal part P. Then the following conditions are equivalent.*

I. *The form C is G-weighted subcoercive.*

II. *There are $c, \mu > 0$ and an open neighbourhood V of the identity of G such that*

$$\mu \varepsilon^{2w_i} \, \|A_i \varphi\|_2^2 \leqslant \varepsilon^m \operatorname{Re}(\varphi, dL_G(C) \, \varphi) + c \, \|\varphi\|_2^2$$

for all $\varphi \in C_c^\infty(V)$, all $\varepsilon \in \langle 0, 1]$ and all $i \in \{1, ..., d'\}$,

III. *The closure of $dL_G(C)$ generates a holomorphic semigroup S on $L_2(G)$ which is quasi-contractive in an open sector $\Lambda(\theta) \subset \mathbf{C}$ with $\theta \in \langle 0, \pi/2 \rangle$.*

Moreover, S_t *maps* $L_2(G)$ *into* $D(A_i)$ *for all* $i \in \{1, ..., d'\}$ *and there exist* $c, \omega > 0$ *such that* $\|A_i S_t\|_{2\to 2} \leqslant c t^{-w_i/m} e^{\omega t}$ *for all* $t > 0$.

Proof. The implication I ⇒ II has been proved in Section 1 and the implication I ⇒ III follows from Corollary 8.3.II. The converse implication II ⇒ I follows from

$$\begin{aligned}\mu\varepsilon^{2w_i} \|A_i\varphi\|_2^2 &\leqslant \varepsilon^m \|\varphi\|_2 \|dL_G(\Re C)\varphi\|_2 + c\|\varphi\|_2^2\\ &\leqslant \varepsilon^m(\varepsilon^{-m}\|\varphi\|_2^2 + \varepsilon^m\|dL_G(\Re C)\varphi\|_2^2) + c\|\varphi\|_2^2\\ &\leqslant (\varepsilon^m\|dL_G(\Re C)\varphi\|_2 + (1+c)\|\varphi\|_2)^2.\end{aligned}$$

One then deduces from Proposition 10.7 that $dL_{G_0}(\Re P)$ is hypoelliptic, where P is the principal part of C. But the contraction process also shows that $dL_{G_0}(\Re P)$ is a positive operator. Therefore C is G-weighted subcoercive by Theorem 10.1.

It remains to prove III ⇒ I. It follows from the bounds on the derivatives of the semigroup, by Laplace transformation, that there exists a $c > 0$ such that

$$\|A_i\varphi\|_2 \leqslant \varepsilon^{m-w_i}\|dL_G(C)\varphi\|_2 + c\varepsilon^{-w_i}\|\varphi\|_2$$

uniformly for all $\varepsilon \in \langle 0, 1]$, $\varphi \in C_c^\infty(G)$ and $i \in \{1, ..., d'\}$. Hence $dL_{G_0}(P)$ is hypoelliptic by Proposition 10.7. But it follows from quasi-contractivity that $e^{i\alpha}(\overline{dL_G(C)} - \omega I)$ generates a contraction semigroup, if ω is large enough, uniformly for all $\alpha \in \langle -\theta, \theta\rangle$. Hence, by the Lumer–Phillips theorem, $\mathrm{Re}(\varphi, e^{i\alpha}(dL_G(C) - \omega I)\varphi) \geqslant 0$ for all $\varphi \in L'_{2;m}(G)$. Applying the contraction process it follows that $\mathrm{Re}(\varphi, e^{i\alpha}\, dL_{G_0}(P)\varphi) \geqslant 0$ for all $\varphi \in L'_{2;m}(G_0)$ and $\alpha \in \langle -\theta, \theta\rangle$. The proof of this implication is a variation of the proofs used in Propositions 4.5 and 10.7. Then $|(\varphi, dL_{G_0}(\Im P)\varphi)| \leqslant M(\varphi, dL_{G_0}(\Re P)\varphi)$ for all $\varphi \in L'_{2;m}(G_0)$, where $M = \cot\theta$. Hence, by reduction, Lemma 10.4.II, it follows that

$$|(x, dU(\Im P)x)| \leqslant M(x, dU(\Re P)x)$$

for each unitary irreducible representation U of G_0 and all $x \in \mathscr{X}_\infty(U)$, and then, by density, for all $x \in \mathscr{X}'_m(U)$. But it follows from [Sch, Lemma XII.3.1], that

$$|(y, dU(\Im P)\,x)| \leqslant M(x, dU(\Re P)\,x)^{1/2}\,(y, dU(\Re P)\,y)^{1/2} \tag{31}$$

for all $x, y \in \mathscr{X}'_m(U)$.

We shall prove that $\Re P$ is a Rockland form. Let U be a non-trivial irreducible unitary representation of G_0, $x \in \mathscr{X}_\infty(U)$ and suppose $dU(\Re P)\,x = 0$. Then it follows from (31) that $(y, dU(\Im P)\,x = 0$ for all $y \in \mathscr{X}'_m(U)$. Since $\mathscr{X}'_m(U)$

is dense in $\mathcal{X}$ one establishes that $dU(\Im P)\, x = 0$. Therefore $dU(P)\, x = 0$ and thus $x = 0$ since P is a Rockland form. Hence $\Re P$ is positive Rockland form and C is G-weighted coercive by Theorem 10.1. This completes the proof of the theorem. ∎

This corollary shows that Conditions I, II and III in Theorem 1.1 are equivalent in case of a reduced weighted algebraic basis.

If the principal part of the form C is symmetric one can weaken the assumptions of the previous theorem. One only needs quasi-contractivity of the semigroup on the positive real line.

THEOREM 10.9. *Let G be a connected Lie group, $a_1, ..., a_{d'}$ a reduced weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of G and G_0 the corresponding homogeneous contraction of G. Suppose $m \in 2w\mathbf{N}$ and let C be an m-th order form with symmetric principal part P, i.e., $P = P^\dagger$. The following conditions are equivalent.*

I. *The form C is G-weighted subcoercive.*

II. *The closure of $dL_G(C)$ generates a continuous, quasi-contraction, semigroup S on $L_2(G)$ which maps into $D(A_i)$ for all $i \in \{1, ..., d'\}$. Moreover, there exist $c, \omega > 0$ such that $\|A_i S_t\|_{2\to 2} \leqslant c t^{-w_i/m} e^{\omega t}$ for all $t > 0$.*

Proof. It follows as in the proof of Theorem 10.8 that $\Re P = P$ is hypoelliptic and $\operatorname{Re} dL_{G_0}(P) \geqslant 0$. So $\operatorname{Re} dL_{G_0}(P)$ is a positive Rockland operator. ∎

11. GENERAL ALGEBRAIC BASES

In Section 2 we passed from a weighted algebraic basis to a reduced weighted algebraic basis and the subsequent results have been largely formulated in terms of reduced bases. In this section we examine the passage from the reduced basis back to the original basis and the extension of the foregoing results to general weighted bases.

Let $a_1, ..., a_{d'}$ be a weighted algebraic basis with weights $w_1, ..., w_{d'}$ and filtration $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$. Assume $\bigcap_{i=1}^{d'} w_i\mathbf{N} \neq \emptyset$. We can define a distance $d(\,\cdot\,;\,\cdot\,)$ and modulus $|\cdot|'_{(a)} = |\cdot|'$ on G similarly to the definitions with respect to a reduced weighted algebraic basis in the beginning of Section 6.

Next, Proposition 2.1 established that if the elements of the algebraic basis $a_1, ..., a_{d'}$ are suitably ordered then there exists a reduced weighted algebraic basis $b_1, ..., b_{d''}$ with weights $v_1, ..., v_{d''}$ such that $b_i = a_i$ and $v_i = w_i$ for all $i \in \{1, ..., d''\}$ and $a_i \in \mathfrak{g}_{\underline{w_i}}$ for all $i \in \{d''+1, ..., d'\}$. Moreover, the filtrations corresponding to the algebraic basis $a_1, ..., a_{d'}$ and the reduced basis $b_1, ..., b_{d''}$ coincide. The reduced basis is a subset of the original basis obtained by eliminating those directions a_j such that $a_j \in \mathfrak{g}_{\underline{w_j}}$. But the moduli $|\cdot|'_{(a)}$ and $|\cdot|'_{(b)}$ are equivalent.

LEMMA 11.1. *There exists a* $c \geqslant 1$ *such that*

$$c^{-1} |g|'_{(b)} \leqslant |g|'_{(a)} \leqslant c\, |g|'_{(b)}$$

for all $g \in G$.

Proof. Obviously $|g|'_{(a)} \leqslant |g|'_{(b)}$ for all $g \in G$. Next, for all $i \in \{1, ..., d'\}$ let $w'_i = \min\{\lambda > 0 \colon a_i \in \mathfrak{g}_\lambda\}$. Then one easily proves by induction on the weights of the filtration that the filtration $(\mathfrak{g}_\lambda)_{\lambda \geqslant 0}$ equals the filtration corresponding to the weighted algebraic basis $a_1, ..., a_{d'}$ with weights $w'_1, ..., w'_{d'}$. So $a_1, ..., a_{d'}$ with weights $w'_1, ..., w'_{d'}$ is a reduced weighted algebraic basis. Let $|\cdot|'_{(a')}$ denote the modulus with respect to this weighted algebraic basis. Then obviously $|g|'_{(a')} \leqslant |g|'_{(a)}$ for all $g \in G$ with $|g|'_{(a)} < 1$. But the moduli $|\cdot|'_{(a')}$ and $|\cdot|'_{(b)}$ are equivalent by Corollary 6.5. Therefore the lemma follows for small g. For large g the distances are comparable by [VSC, Proposition III.4.2]. ▮

COROLLARY 11.2. *There exists a* $c \geqslant 1$ *such that*

$$c^{-1} \delta^{D'} \leqslant |B'_\delta| \leqslant c \delta^{D'}$$

for all $\delta \in \langle 0, 1]$, *where* $D' = \sum_{\lambda > 0} \lambda \dim(\mathfrak{g}_\lambda / \mathfrak{g}_{\underline{\lambda}})$ *is the local dimension and* B'_δ *is the ball with radius* δ.

Proof. This follows from Proposition 6.1.II and the previous lemma. ▮

The reduced weighted algebraic basis $b_1, ..., b_{d''}$ is constructed from the weighted algebraic basis by deleting the "overweight" directions (see Proposition 2.1). But these directions have a representation

$$a_j = \sum_{\substack{\alpha \in J^+(d'') \\ \|\alpha\|_v \leqslant w_j}} c_{j\alpha} b_{[\alpha]} \tag{32}$$

where we have used $\|\cdot\|_v$ to denote the length of the multi-index with respect to the weights v_i of the reduced basis. On the other hand, such a representation also exists if a_j is an element of $\{b_1, ..., b_{d''}\}$. Hence in a continuous representation $(\mathcal{X}, G, U)$ of the group

$$A_j = \sum_{\substack{\alpha \in J^+(d'') \\ \|\alpha\|_v \leqslant w_j}} c_{j\alpha} B_{[\alpha]}$$

where $B_j = dU(b_j)$. Therefore, expanding the commutators, there are $c'_{j\alpha} \in \mathbf{R}$ such that

$$A_j = \sum_{\substack{\alpha \in J^+(d'') \\ \|\alpha\|_v \leq w_j}} c'_{j\alpha} B^\alpha. \tag{33}$$

If $C: J(d') \to \mathbf{C}$ is an mth order form (33) implies there exist $c'_\beta \in \mathbf{C}$ such that

$$dU(C) = \sum_{\substack{\alpha \in J(d') \\ \|\alpha\|_w \leq m}} c_\alpha A^\alpha = \sum_{\substack{\beta \in J(d'') \\ \|\beta\|_v \leq m}} c'_\beta B^\beta \tag{34}$$

where we have now used $\|\cdot\|_w$ to denote the weighted length of the multi-indices with respect to the weights w_i of the original basis. The form C is an mth order form with respect to the weighted algebraic basis $a_1, ..., a_{d'}$ and we use the notation $C = C_a$ to denote the dependence on the basis. Further let $C_b: J(d'') \to \mathbf{C}$ be the mth order form with the coefficients c'_β entering on the right hand side of (34). Then (34) states that $dU(C_a) = dU(C_b)$. The form C_b has order less than or equal to m with respect to the weighted algebraic basis $b_1, ..., b_{d''}$ and weights $v_1, ..., v_{d''}$.

We temporarily add a subscript a and b to the spaces $\mathscr{X}'_n(U)$ and the (semi)norms $\|\cdot\|'_{U,n}$ and $N'_{U,n}$ to denote the dependence of the weighted algebraic basis. Obviously $\mathscr{X}'_{a,n}(U) \subseteq \mathscr{X}'_{b,n}(U)$, $N'_{b,U,n}(x) \leq N'_{a,U,n}(x)$ and $\|x\|'_{b,U,n} \leq \|x\|'_{a,U,n}$ for all $n \in [0, \infty\rangle$ and $x \in \mathscr{X}'_{a,n}(U)$, since the b_i are a subset of the a_i with the same weight. Next suppose $m \in w_i\mathbf{N}$ for all $i \in \{1, ..., d'\}$ and set

$$v = \min\{x \in [1, \infty\rangle : x \in v_i\mathbf{N} \text{ for all } i \in \{1, ..., d''\}\}$$
$$w = \min\{x \in [1, \infty\rangle : x \in w_i\mathbf{N} \text{ for all } i \in \{1, ..., d'\}\}.$$

Then $w \in v\mathbf{N}$. Let $k \in \mathbf{N}$. It follows from (34) that $\mathscr{X}'_{b,kv}(U) \subseteq \mathscr{X}'_{a,kv}(U)$, $N'_{a,U,kv}(x) \leq c\, \|x\|'_{b,U,kv}$ and hence $\|x\|'_{a,U,kv} \leq c'\, \|x\|'_{b,U,kv}$ for some $c, c' > 0$, uniformly for all $x \in \mathscr{X}'_{b,kv}(U)$. So the spaces $\mathscr{X}'_{a,kv}(U)$ and $\mathscr{X}'_{b,kv}(U)$ are equal, with equivalent norms. Moreover, it follows from Theorem 9.1.IV that there exists a $c > 0$ such that

$$N'_{a,U,kv}(x) \leq c(N'_{b,U,kv}(x) + \|x\|) \tag{35}$$

for all $x \in \mathscr{X}'_{a,kv}(U)$.

Now Theorem 1.1 of the introduction follows as a corollary of the results we have established for reduced weighted bases.

Proof of Theorem 1.1. Let C be an mth order form and assume that the weights w_i satisfy $m/w_i \in 2\mathbf{N}$.

If the weighted algebraic basis $a_1, ..., a_{d'}$ is a reduced weighted algebraic basis then the theorem follows from Proposition 4.5, $1' \Rightarrow 3$ and Theorems 7.2, 8.2 and 10.8.

If, however, $a_1, ..., a_{d'}$ is not a reduced weighted basis one can proceed as above and introduce the reduced weighted subbasis $b_1, ..., b_{d''}$. Then $C = C_a$ is the given mth order form. Let C_b be the associated form of order less than or equal to m with respect to the weighted algebraic basis $b_1, ..., b_{d''}$. We say that C_b satisfies Condition I of Theorem 1.1 if there are $\mu, \nu > 0$ and an open neighbourhood V of the identity of G such that

$$\operatorname{Re}(\varphi, dL_G(C_b)\,\varphi) \geqslant \mu(N'_{b,2;m/2}(\varphi))^2 - \nu\,\|\varphi\|_2^2$$

for all $\varphi \in C_c^\infty(V)$. Similarly, we say that the form C_b satisfies Conditions II, III, or IV of Theorem 1.1 if the particular condition is valid for the form C_b, the algebraic basis $b_1, ..., b_{d''}$, weights $v_1, ..., v_{d''}$ and infinitesimal generators $B_1, ..., B_{d''}$.

We first show that the order of the form C_b equals m if the form C_b satisfies one of the Conditions I–IV of Theorem 1.1. Obviously one has the implications I $\Rightarrow$ II and IV $\Rightarrow$ III for the form C_b. The proof is the same as in Section 1. But if the form C_b satisfies Condition II or III then there exist $c > 0$ and a neighbourhood V of the identity of G such that

$$\|B_i\varphi\|_2 \leqslant \varepsilon^{m-w_i}\,\|dL_G(C_b)\,\varphi\|_2 + c\varepsilon^{-w_i}\,\|\varphi\|_2$$

uniformly for all $\varepsilon \in \langle 0, 1]$, $\varphi \in C_c^\infty(V)$ and $i \in \{1, ..., d''\}$. This follows as in the proof of Theorem 10.8. Therefore the order of the form C_b equals m by Proposition 10.7. Hence the Conditions I–IV are all equivalent for the form C_b.

Now we prove Theorem 1.1 for the form C_a. If C_a satisfies Condition I, i.e., C_a is a G-weighted subcoercive form, then in the left regular representation on $L_2(G)$ one has

$$\begin{aligned}\operatorname{Re}(\varphi, dL_G(C_b)\,\varphi) &= \operatorname{Re}(\varphi, dL_G(C_a)\,\varphi)\\ &\geqslant \mu(N'_{a,2;m/2}(\varphi))^2 - \nu\,\|\varphi\|_2^2\\ &\geqslant \mu(N'_{b,2;m/2}(\varphi))^2 - \nu\,\|\varphi\|_2^2\end{aligned}$$

for all $\varphi \in C_c^\infty(V)$, with V the open neighbourhood of the identity occurring in the definition of the subcoercivity of C_a. Hence C_b satisfies Condition I and C_b is an mth order weighted subcoercive form. Since the b_i are a subset of the a_i with the same weight Conditions II, III, and IV for C_a obviously imply the same condition for the form C_b.

Conversely, if C_b is an mth order weighted subcoercive form then it follows from (35) that the form C_a is weighted subcoercive. Then Condition II is also valid for C_a, as we have proved already in Section 1. In any representation $(\mathcal{X}, G, U)$ the closure of $dU(C_b)$ generates a semigroup which is holomorphic in an open sector containing $\Lambda(\theta_{C_b, G})$. Moreover, it has a representation independent kernel. Since $dU(C) = dU(C_a) = dU(C_b)$ this establishes the generator property for $dU(C)$. The Gaussian bounds for the semigroup kernel follow from (33) and the bounds on the derivatives of the kernel with respect to the B^β. The bounds on the derivatives of the semigroup in Condition III for C_a follow again by a quadrature estimate. This completes the proof of Theorem 1.1. ∎

It should again be emphasized that Theorem 1.1 is valid for any Lie group G and any weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of G. Although most of the foregoing material involves reduced weighted algebraic bases and the corresponding homogeneous contraction G_0 the final result is independent of these concepts.

In Section 1 we defined $\|x\|'_n = 0$ if $n \notin \{ \|\alpha\|_w : \alpha \in J(d')\}$ to avoid complications in various proofs. We now drop this condition for the weighted algebraic basis $a_1, ..., a_{d'}$. For $n \in [0, \infty\rangle$ define $|||\cdot|||'_n : \mathcal{X}'_{a,n}(U) \to [0, \infty\rangle$ by

$$|||x|||'_n = \max_{\substack{\alpha \in J(d') \\ \|\alpha\| \leqslant n}} \|A^\alpha x\|.$$

Then $(\mathcal{X}'_{a,n}(U), |||\cdot|||'_n)$ is a normed space and the two spaces $(\mathcal{X}'_{a,n}(U), |||\cdot|||'_n)$ and $(\mathcal{X}'_{b,n}(U), \|\cdot\|'_{b,n})$ are equal, with equivalent norms, if $n \in v\mathbf{N}$. Hence all conclusions of Theorems 9.1 and 9.2 and Corollaries 8.3, 9.3 and 9.4 are valid if C is a G-weighted subcoercive form with respect to the weighted algebraic basis and the norms $|||\cdot|||'_n$ on the space $\mathcal{X}'_{a,n}(U)$. Most statements follow directly from the comparable statement for the reduced weighted algebraic basis $b_1, ..., b_{d''}$, so we indicate the differences. In Corollary 8.3.II one fixes $k \in [0, \infty\rangle$. Let $k_0 = \max\{ \|\alpha\|_v : \alpha \in J(d''), \|\alpha\|_v \leqslant k\}$. Then it follows from (34) that $\mathcal{X}'_{b,k_0}(U) \subseteq \mathcal{X}'_{a,b}(U)$ and $|||x|||'_k \leqslant c' \, \|x\|'_{b,U,k_0}$ for some $c' > 0$. Therefore

$$|||S_t|||'_k \leqslant c' \, \|S_t\|'_{b,U,k_0} \leqslant cc' t^{-k_0/m} e^{\omega t} \, \|x\| \leqslant cc' t^{-k/m} e^{\omega' t} \, \|x\|$$

for a suitable $\omega' \geqslant \omega$. Next, in Corollary 8.3.III one has $\|x\|'_{b,U,k_0} \leqslant \varepsilon^{mn-k_0} \|\bar{H}^n x\| + c\varepsilon^{-k_0} \|x\|$, which is equivalent to the J-interpolation inclusion $(\mathcal{X}, D(\bar{H}^n))_{k_0/(mn), 1; \mathrm{J}} \subseteq \mathcal{X}'_{b,k_0}(U)$ (see [Tri, Lemma 1.10.1(a)]). Therefore one has the following continuous inclusions

$$(\mathcal{X}, D(\bar{H}^n))_{k/(mn), 1; \mathrm{J}} \subseteq (\mathcal{X}, D(\bar{H}^n))_{k_0/(mn), 1; \mathrm{J}} \subseteq \mathcal{X}'_{b,k_0}(U) \subseteq \mathcal{X}'_{a,k}(U),$$

from which the new version of Corollary 8.3.III for the algebraic basis $a_1, ..., a_{d'}$ and the new norm follows. Theorem 9.1.III can be proved similarly.

Earlier work [ElR6] on unweighted bases and subcoercive operators was based on the assumption of $\tilde{G}$-coercivity where $\tilde{G}$ denotes the Rothschild–Stein local approximant of G, i.e., $\tilde{G}$ is the nilpotent Lie group with d' generators which is free of step r where d' and r are the number of elements and the rank of the algebraic basis, respectively. Thus $\tilde{G} = G(d', r, 1, ..., 1)$ (see Example 2.7).

The next proposition establishes that the earlier results [ElR6] are a corollary of Theorem 1.1.

PROPOSITION 11.3. *Let G be a connected Lie group and $a_1, ..., a_{d'}$ a weighted algebraic basis of the Lie algebra $\mathfrak{g}$ of G with weights $w_1, ..., w_{d'}$. Let λ be larger than the largest weight in the filtration corresponding to the weighted algebraic basis and larger than w_i for all $i \in \{1, ..., d'\}$. Let $C: J(d') \to \mathbf{C}$ be a form of order m with $m \in 2w\mathbf{N}$. If C is a $G(d', \lambda, w_1, ..., w_{d'})$-weighted subcoercive form then C is a G-weighted subcoercive form.*

Proof. Let $b_1, ..., b_{d''}$ be the reduced weighted algebraic basis as in the beginning of this section. Again we indicate with a subscript the dependence of the algebraic basis. Then $a_i \in \mathfrak{g}_{\underline{w_i}}$ if, and only if, $i > d''$. Let G_0 be the homogeneous contraction of G. Define the form $C'_b: J(d'') \to \mathbf{C}$ by $C'_b(\alpha) = C_a(\alpha)$ for all $\alpha \in J(d'')$. We first prove that C'_b is a G_0-weighted subcoercive form, hence a G-weighted subcoercive form, and secondly that $C = C_a$ is G-weighted subcoercive.

Let $\tilde{\mathfrak{g}} = \mathfrak{g}(d', \lambda, w_1, ..., w_{d'}) = \mathfrak{G}/I$ be the weighted nilpotent Lie algebra with generators $\tilde{a}_1, ..., \tilde{a}_{d'}$ and weights $w_1, ..., w_{d'}$ which is free of step λ and $\tilde{G} = G(d', \lambda, w_1, ..., w_{d'})$ the corresponding connected simply connected Lie group (see Example 2.7). There exists a unique Lie algebra homomorphism $\tilde{T}; \tilde{\mathfrak{g}} \to \mathrm{Hom}(L_{2;\infty}(G_0))$ such that

$$\tilde{T}(\tilde{a}_i) = \begin{cases} dL_{G_0}(b_i)|_{L_{2;\infty}(G_0)} & \text{if} \quad i \in \{1, ..., d''\}, \\ 0 & \text{if} \quad i \in \{d''+1, ..., d'\}. \end{cases}$$

Then $\tilde{T}$ is a representation of $\tilde{\mathfrak{g}}$ in the Hilbert space $L_2(G_0)$ by skew-adjoint operators such that the (dense) set of analytic vectors for L_{G_0} is a set of analytic vectors for $\tilde{T}(\tilde{a}_i)$ for all $i \in \{1, ..., d'\}$. So, by [Sim, Corollary 2], there exists a unitary representation U of $\tilde{G}$ in $L_2(G_0)$ such that $L_{2;\infty}(G) \subseteq ((L_2(G_0))_\infty)(U)$ and $\tilde{T}(\tilde{a}_i)$ is the restriction of $dU(\tilde{a}_i)$ to $L_{2;\infty}(G_0)$ for all

$i \in \{1, ..., d'\}$. Then, by Theorem 9.2.III, it follows that there exist $\mu, \nu > 0$ such that

$$\mathrm{Re}(\varphi, dL_{G_0}(C'_b)\,\varphi) = \mathrm{Re}(\varphi, dU(C_a)\,\varphi) \geqslant \mu(\|\varphi\|'_{a,\,U,\,m/2})^2 - \nu\,\|\varphi\|_2^2$$
$$= \mu(\|\varphi\|'_{b,\,2;\,m/2})^2 - \nu\,\|\varphi\|_2^2$$

for all $\varphi \in ((L_2(G_0))_\infty)(U)$ and, in particular, for all $\varphi \in L_{2;\,\infty}(G_0)$, where $\|\cdot\|'_{b,\,2;\,m/2}$ is the norm on $L'_{2;\,m/2}(G_0)$ with respect to the algebraic basis $b_1, ..., b_{d''}$. So C'_b is a G_0-weighted subcoercive form and hence by Theorem 10.1 a G-weighted subcoercive form.

Arguing as in the beginning of this section, but now using (1) instead of (32) for all $j \in \{d''+1, ..., d'\}$, it follows that there exists a form $C''_b: J(d'') \to \mathbf{C}$ of order strictly less than m, such that $dL_G(C_a) = dL_G(C'_b) + dL_G(C''_b)$. Then the principal parts of C'_b and $C'_b + C''_b$ coincide, so $C'_b + C''_b$ is a G-weighted subcoercive form by Proposition 4.5. Finally, since the norms $\|\cdot\|'_{a,\,2;\,m/2}$ and $\|\cdot\|'_{b,\,2;\,m/2}$ on $L'_{2;\,m/2}(G)$ are equivalent, it then follows that $C = C_a$ is a G-weighted subcoercive form. ∎

One can now immediately recover all the main results of [ElR6].

Corollary 11.4. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $a_1, ..., a_{d'}$ an (unweighted) algebraic basis of rank r of $\mathfrak{g}$. Let $C: J(d') \to \mathbf{C}$ be a subcoercive form of order m and step r (see [ElR3]). Let $(\mathcal{X}, G, U)$ be a representation of G. Then the closure of the operator $dU(C)$ generates a holomorphic semigroup S which is holomorphic in an open sector containing $\Lambda(\theta_{C,\,\tilde{G}})$, where $\tilde{G} = G(d', r, 1, ..., 1)$. Moreover, S has a representation independent kernel which satisfies Gaussian type bounds of order m.*

Proof. If C is a subcoercive form of order m and step r then C is a $G(d', r, 1, ..., 1)$-weighted subcoercive form (see Example 4.3). Hence it is a G-weighted subcoercive form. The corollary follows immediately. ∎

The final example shows that the assumptions of [ElR6] are strictly stronger then those of weighted subcoercivity.

Example 11.5. Consider the five-dimensional Heisenberg group G. Thus one has a Lie algebra basis $a_1, ..., a_5$ with $[a_1, a_2] = [a_3, a_4] = a_5$. Take the weighted algebraic basis $a_1, ..., a_4$ with all weights equal to one. Then $\tilde{G} = G(4, 2, 1, 1, 1, 1)$ has dimension 10 and the Lie algebra $\tilde{\mathfrak{g}}$ has a basis $\{\tilde{a}_1, ..., \tilde{a}_4\} \cup \{\tilde{a}_{ij}: 1 \leqslant i < j \leqslant 4\}$. The commutation relations are $[\tilde{a}_i, \tilde{a}_j] = \tilde{a}_{ij}$ if $1 \leqslant i < j \leqslant 4$. Let C_1 be the form such that

$$dU(C_1) = -A_1^2 - A_2^2 - A_3^2 - A_4^2$$

for any representation. Further let C_2 be the form such that

$$dU(C_2) = dU(C_1) - \lambda i^{-1}([A_1, A_2] - [A_3, A_4]),$$

where λ is an eigenvalue of the operator $dV(C_1)$, and V is an irreducible unitary representation of $\tilde{G}$ with $dV(\tilde{a}_{12} - \tilde{a}_{34}) = iI$. Then C_2 is not a subcoercive form of step 2 since $(x_\lambda, dV(C_2)\, x_\lambda) = 0$, where x_λ is an eigenvector of $dV(C_1)$ with eigenvalue λ (see [ElR6, Corollary 3.5]). On the other hand, $dl_G(C_2) = dL_G(C_1)$, so C_2 is a G-weighted subcoercive form.

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