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Large Extraspecial Subgroups of Widths 4 and 6

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INTRODUCTION

Extraspecial 2-subgroups which are "large" in the sense of Janko arise as an important minimal situation in the classification of finite simple groups. Determination of simple groups with such subgroups has been made possible by recent work of Timmesfeld [30, Main Theorem], which reduces the problem to the study of a few specific configurations.

All of these configurations have now been dealt with. The work of [24] shows that Timmesfeld's case (3) leads to the even-dimensional orthogonal groups over $GF(2)$. Reifart in [18, 19] and the author in [25] show that cases (5) and (8) lead to the Chevalley and twisted groups of type E over $GF(2)$. The purpose of the present paper is to determine the groups arising in cases (1) and (4), where (essentially) the extraspecial group has width 4 or 6. Certain of the subcases have already been studied by Reifart [17], Bierbrauer, and Tran van Trung. Finally, an argument of Stroth and Reifart (private communication) deals with Timmesfeld's cases (6) and (7); we have taken the liberty of reproducing their argument, for completeness.

To state our results, we introduce the "large extraspecial" situation we deal with:

GENERAL HYPOTHESIS. Let G be a finite group, with involution z. Suppose that $F^*(C_G(z))$ is an extraspecial 2-group. Assume also $O^2(G) = G$ and $z \notin O_{2',2}(G)$.

To facilitate working with this hypothesis, we adopt further conventions:

Notation. Let:

$$M = C_G(z).$$
$$Q = F^*(M).$$
$$n = \text{width of the extraspecial group } Q.$$
$$\overline{M} = M/Q \qquad \text{(bar convention for the quotient).}$$

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Now the work divides naturally into three parts. Since the work of Gorenstein and Harada [7] determines the possibilities when $n \leqslant 2$ (as noted in [2, Lemma 4.6]), Timmesfeld's case (1) with $n \leqslant 4$ really amounts to $n = 3$ or 4.

So we prove:

THEOREM A. *Assume the General Hypothesis, with* $n = 3$ *or* 4. *Then*

(i) *if* $n = 3$, $G \cong L_5(2)$, $U_5(2)$, M_{24}, He, *or* Sz.

(ii) *if* $n = 4$, $F^*(G) \cong L_6(2)$, $U_6(2)$, $\Omega_8^{\pm}(2)$, ${}^3D_4(2)$, $\Omega_8^{+}(3)$, Co.2, Co.1, F_3, *or a group of type* F_5.

Remarks. In (ii) we specify only $F^*(G)$ because of possible outer automorphisms of order 3 of the groups $\Omega_8^{+}(2)$, ${}^3D_4(2)$, $\Omega_8^{+}(3)$. The expression "of type F_5" indicates an open uniqueness problem.

Timmesfeld's case (4) in width 6 really amounts to the determination of the two sporadic groups in our next result:

THEOREM B. *Assume the General Hypothesis with* $n = 6$. *Then* $G \cong L_8(2)$, $U_8(2)$, $\Omega_{10}^{\pm}(2)$, *or a sporadic group of type* J_4 *or* F_{24}'.

Finally, Timmesfeld's cases (6) and (7) lead to the "monster" groups of Fischer:

THEOREM C (Reifart and Stroth). *Assume the General Hypothesis*

(i) *if* $n = 11$ *and* $\overline{M} \cong$ Co.2, *then* G *is a group of type* F_2,

(ii) *if* $n = 12$ *and* $\overline{M} \cong$ Co.1, *then* G *is a group of type* F_1.

Remark. The existence (as well as uniqueness) problems are open for J_4 and F_1.

We give a brief description of the proofs of these results. Certain aspects of the work are common to all three theorems. This is especially true of preliminary results of Section 1, many of which we can simply quote from [24], where they are proved essentially under the General Hypothesis. When it is necessary to check specific details, we will refer to the three situations as Hypotheses A, B, and C, respectively; or more simply, as cases A, B, and C.

In Section 2, we concentrate on the determination of the structure of $\overline{M}$, which is necessary for cases A and B. We make use of a number of crucial intermediate results of Timmesfeld in [30]; these allow us to construct the centralizer of a suitable involution in $\overline{M}$, and then the work of Gorenstein and Harada [7] may be used to determine $\overline{M}$. The arguments for widths 4 and 6 are distinct but very similar.

With $\overline{M}$ determined under all three Hypotheses A, B, C, the proof turns in the subsequent sections to analysis of individual possibilities, none of which is unduly difficult. Generally it can be said that we reduce each to some classification theorem already available. In case B, the determination of $\bar{M}$ is already sufficient, in view of work of Janko [10] and Parrott [13]. In cases A and C, a certain amount of further argument is required for each possibility. We mention in particular the method of Reifart and Stroth: When the group M (the centralizer of a 2-central involution) does not alone determine G, develop also the centralizer of a non-2-central involution to identify G.

1. Initial Reductions

We now assume the General Hypothesis and corresponding notation. Further notation will be as in [24]. In particular, if $x = z^g$ (for some $g \in G$), it is convenient to use a subscript x (rather than superscript g) to denote conjugates of M and Q, as in

$$M_x \quad \text{for} \quad M^g = C(x),$$

$$Q_x \quad \text{for} \quad Q^g = F^*(M_x).$$

We have also another bar convention,

$$\tilde{Q} \quad \text{for} \quad Q/\langle z \rangle.$$

A tilde *between* groups or elements will indicate conjugacy.

Our references to Timmesfeld's work [30] will be too frequent and diverse to allow reproduction here of all the necessary results. For convenience we will use the letter T to refer to that paper, and a reference of form T(4.5) will be to result (4.5) of [30]. Similarly we use O to refer to [24], and an expression like O(1.16) refers to result (1.16) of [24].

We now set out to establish (for all three cases A, B, C) the preliminary reductions of Section 1 of O, by checking the requirements of the axiomatization given there. We will reproduce directly only the most frequently used results, and numbering our results consistently with them. In certain exceptional situations, we may obtain the structure of $\bar{M}$; to emphasize the logical flow of the proof, of the proof, it is then convenient simply to give the groups G to which the configurations lead, through we postpone the necessary (and independent) arguments to Section 3.

The results O(1.1)–(1.4) require only the General Hypothesis. In particular we mention:

O(1.4) We may assume $Z^*(G) = 1$.

We obtain a slightly more complicated version of O(1.5):

(1.5) *Suppose* $z^G \cap Q = \{z\}$. *Then we have one of*:

(i) $n = 3$ *and* $G \cong U_5(2)$.

(ii) $n = 4$ *and* $G \cong U_6(2)$ *or* Co.2.

(iii) $n = 6$ *and* $G \cong U_8(2)$.

Thus we may now assume that $z^G \cap Q \neq \{z\}$.

Proof. Assume $z^G \cap Q = \{z\}$. Just as in O(1.5), we have $Z^*(G) = 1$ and $O^2(G) = G$, so we may apply the result of Aschbacher [2] as extended by F. Smith [21, 22] to determine $F^*(G) \cong U_{n+2}$ or Co.2 (with $n = 4$). In particular $F^*(\overline{M}) \cong U_n(2)$ or $Sp_6(2)$, so we see no group arises under Hypothesis C; and only those listed above arise under Hypotheses A, B. Now since $G = O^2(G)$ we get $G = F^*(G)$ by automorphism-group considerations, so (1.5) holds.

Now O(1.6) follows with no further checking. We must again be careful of certain possibilities in our version of O(1.7):

(1.7) *Suppose* $\overline{M}$ *acts reducibly on* $\tilde{Q}$. *Then we have one of*:

(i) $n = 3$ *and* $G \cong L_5(2)$, M_{24}, He.

(ii) $n = 4$ *and* $G \cong L_6(2)$.

(iii) $n = 6$ *and* $G \cong L_8(2)$.

So we may now assume that $\overline{M}$ *acts irreducibly on* $\tilde{Q}$.

Proof. Assume $\overline{M}$ is reducible on $\tilde{Q}$. Just as in O(1.7), we may apply the result of Dempwolff and Wong [6] to determine $F^*(G) = L_{n+2}(2)$, or possibly M_{24} or He when $n = 3$. In particular, $F^*(\overline{M}) \cong L_n(2)$, so we see no group arises under Hypothesis C; and only those listed arise for Hypotheses A, B. As in (1.5), we get $G = F^*(G)$; so (1.7) holds. Notice our assumption $z \notin O_{2,2}(G)$ eliminates the extensions $2^{n+1}L_{n+1}(2)$ that arise in [6]. In O, it was sufficient for us to assume $z \notin Z^*(G)$.

Now we observe that in cases A and B, our assumption beyond the General Hypothesis is on the width of Q only, so we have the hypothesis O(1.8)(a); and in case C, simplicity of $\overline{M}$ gives hypothesis O(1.8)(b). So we obtain:

O(1.8) We may assume G is simple.

Remark. This really avoids the case $|G : F^*(G)| = 3$ with $F^*(G) \cong \Omega_8^+(2)$, $\Omega_8^+(3)$, or ${}^3D_4(2)$.

Now O(1.8) reduces us to the situation of Timmesfeld in T, and we can make direct use of his results. A number of these deal with fusion of z in Q, and to state them we introduce:

Notation. Let a be some fixed element of $z^G \cap Q - \{z\}$, as provided by (1.5). Set L (more specifically, L_a) $= Q_a \cap M$.

We assume the notion of Suzuki type of involutions of $\bar{M}$ on the orthogonal space $\tilde{Q}$, as in Aschbacher and Seitz [4, Sect. 8]. We get a version of O(1.9):

(1.9) *Suppose an involution of $\bar{L}$ acts with type a_2 on $\tilde{Q}$. Then we have one of:*

(i) $n = 3$ *and* $G \simeq \mathrm{Sz}$.

(ii) $n = 4$ *and* $G \simeq \mathrm{Co.1}$.

Thus we may now assume no involution of $\bar{L}$ has type a_2 on $\tilde{Q}$.

Proof. If there is such an involution, we may apply T(12.16). In view of (1.7), we need consider only cases $n = 3$ with $M \simeq \Omega_6^-(2)$, and $n = 4$ with $M \simeq \Omega_8^+(2)$. (These cases arise in work of F. Smith [22].)

In particular, no groups arise under Hypotheses B and C. In case A, by Patterson and Wong [16], the former leads to $G \simeq \mathrm{Sz}$; and by Patterson's thesis [15], the latter leads to $G \simeq \mathrm{Co.1}$. So (1.9) is proved.

Now with no a_2's in $\bar{L}$, we obtain a number of important corollaries from T, which we collect in:

O(1.10) (a) $C_{\tilde{Q}}(\bar{L}) = \langle \tilde{a} \rangle$.

(b) $\bar{L}$ is weakly closed in $N_{\bar{M}}(\bar{L})$. In particular, $N_{\bar{M}}(\bar{L})$ controls $\bar{M}$-fusion in $\bar{L}$. Further $\bar{L} \cap \bar{M}' = [\bar{L}, N_{\bar{M}}(\bar{L})]$.

(c) M controls the G-fusion of elements of $Q - \langle z \rangle$.

(d) $Q \cap Q_a$ is elementary of rank $n + 1$, and $\bar{L}$ elementary of rank $n - 1$.

(e) $\tilde{Q}$ is extraspecial of $+$-type.

Remark. The assertion about the transfer in (b) comes from the result of Yoshida [31]. Part (b) shows that $N_{\bar{M}}(\bar{L})$ contains a Sylow 2-group of $\bar{M}$, and from this point of view it will be rather easy to determine $\bar{M}$ in cases A and B.

Now we get a modified version of O(1.11):

(1.11) *If $\bar{L}$ is a T.I. set in $\bar{M}$, then* $n = 4$ *and* $G \simeq F_3$ *or* ${}^3D_4(2)$. *So we may now assume that $\bar{L}$ is not a T.I. set in $\bar{M}$.*

Proof. By T(4.2), the T.I.-set assumption forces $n = 4$ and $\langle \bar{L}^{\bar{M}} \rangle \simeq L_2(8)$ or A_9. In the former case, Reifart [18] shows that $G \simeq {}^3D_4(2)$. (We provide in (3.1) an easy independent derivation of this fact.) In the latter, we see $\tilde{Q}$ does not afford the permutation representation of A_9: for if it did, a subgroup A_8 contains $\bar{L}_a$ and centralizes a nonzero vector, necessarily $\tilde{a}$ by O(1.10)(a); but $N(\bar{L}_a) = C(\tilde{a})$ and $\bar{L}_a \not\trianglelefteq A_8$. Since S_9 interchanges the other two 8-dimensional reprentations of A_9, we have $\bar{M} = A_9$ only. Then work of Parrott [14] shows G is Thompson's group F_3.

By (1.11), it makes sense to intersect $\bar{L}$ with conjugates. We follow Timmesfeld in defining for this situation:

Notation.

$$\bar{t} = \text{some involution of } \bar{L}.$$
$$\bar{R} \text{ (more specifically, } \bar{R}_t) = \langle \bar{L}^{\bar{m}} \colon \bar{m} \in \bar{M},\ \bar{t} \in \bar{L}^{\bar{m}} \rangle.$$
$$\tilde{V} = \langle \tilde{a}^{\bar{m}} \colon \bar{m} \in \bar{M},\ \bar{t} \in \bar{L}^{\bar{m}} \rangle.$$
$$\bar{N} = C_{\bar{R}}(\tilde{V}).$$

Remarks. The condition $\bar{t} \in \bar{L}^{\bar{m}}$ (which is elementary) gives $\bar{L}^{\bar{m}} \leqslant C_{\bar{M}}(\bar{t})$. In fact we see $\bar{R} \trianglelefteq C_{\bar{M}}(\bar{t})$. We will use heavily the fact that $\bar{R}$ is designed to include *all* $\bar{M}$-conjugates of $\bar{L}$ containing $\bar{t}$.

We now mention Timmesfeld's main technical result T(4.5), which asserts that when $\bar{R} > \bar{L}$, then R/N is one of the groups $L_m(\tilde{V})$ or $Sp_{2m}(\tilde{V})$ (including also $Sp_4(2)'$) or $\Omega_{2m}^{\pm}(\tilde{V})$; and in each case $LN/N = O_2(C_{R/N}(\tilde{a}))$. In cases A and B we will study more caretully Timmesfeld's intermediate results, based on T(4.5), to obtain much more specific restrictions on R/N. Indeed, much as in O and [25], we will find that the existence of certain possibilities for R/N will lead "characteristically" to corresponding structures for $\bar{M}$, and then for G.

We mention that results O(1.12)–(1.16) now follow without any further checking. Some of these results are useful when we need to determine fusion of z in $M - Q$.

This completes the verification (for all three case A, B, C) of preliminary results from O.

In the remainder of the section, we produce some specific lemmas for use in Section 2 (where we determine possibilities for $\bar{M}$ in cases A and B). First, the relevant sharpening of T(4.5):

(1.17) (i) *Assume Hypothesis A. Then for* $\bar{t} \in \bar{L}^{\#}$ *we can get* $\bar{R} = \bar{L}$, *or* $R/N \in \{L_2(\tilde{V})$; *or* $\Omega_4^{\pm}(\tilde{V})$ *with* $\bar{N} = \langle \bar{t} \rangle\}$. *In particular, no further group G arises for* $n = 3$, *so we may assume* $n = 4$.

(ii) *Assume Hypothesis B. Then for* $\bar{t} \in \bar{L}^{\#}$, *we can get* $\bar{R} = \bar{L}$ *or* $R/N \in \{L_2(\tilde{V}), L_3(\tilde{V}), \Omega_4^{+}(\tilde{V})$; *or* $\Omega_6^{+}(\tilde{V})$ *with* $\bar{N} = \langle \bar{t} \rangle\}$.

Proof. We consider what may happen in T(4.5) when $\bar{R} > \bar{L}$. Suppose first that $\bar{N} = \langle \bar{t} \rangle$. Then by T(7.6) we have $n = 4$ or 6 and $R/N \simeq \Omega_n^{\pm}(2)$; or possibly $n = 4$ and $R/N \simeq Sp_4(\tilde{V})' \simeq A_6$. In this latter case, however, Timmesfeld determines $\langle L^{\bar{M}} \rangle \simeq S_3 \times A_6$; we will show in (3.2) that no group G can arise in this situation. We turn then to cases with $\bar{N} > \langle \bar{t} \rangle$. In view of O(1.12) we have $\bar{L} \cap \bar{N} > \langle \bar{t} \rangle$; we may apply T(8.4)/(9.4) to see that R/N is not symplectic or orthogonal, except possibly for $R/N \simeq \Omega_4^{+}(\tilde{V})$ when $n = 6$. (Since $\tilde{V} = [\tilde{Q}, \bar{t}]$ has rank 4, this cannot occur for $n = 3$; and for $n = 4$, we are back in case $\bar{N} = \langle \bar{t} \rangle$ by T(4.7).) Finally if R/N is linear, we see by T(6.6) that $m([\tilde{Q}, \bar{t}]) \geqslant 2 \cdot m(\tilde{V})$; so we can only get $L_2(\tilde{V})$ in case $n = 4$; and $L_2(\tilde{V})$, $L_3(\tilde{V})$ in case $n = 6$. These are now all the cases listed in (1.17).

In particular, we get $m([\tilde{Q}, \bar{t}]) \leqslant 4$ in each, so $\bar{R} > \bar{L}$ cannot arise when $n = 3$; and this contradicts our assumption in (1.11). So (1.17) is proved.

Timmesfeld also provides information about $E(\overline{M})$. We can remove (by way of postponement) one exceptional case:

(1.18) *If $E(\overline{M})$ has more than one component, then $n = 4$ and G is of type F_5. Otherwise we get one of*

(i) *$E(\overline{M})$ is quasi-simple, and $|O(\overline{M})| = 1$ or 3.*

(ii) *$n = 4$, $\overline{M}$ is solvable, and $O(\overline{M})$ is elementary of rank 3 or 4.*

Proof. By T(5.5), the assumption of several components forces $O(\overline{M}) = 1$, $E(M) \simeq A_5 \times A_5$, and $\langle \bar{L}^{\overline{M}} \rangle = A_5 \wr Z_2$. We show in (3.3) that this configuration leads, via the centralizer of a non-2-central involution, to Harada's group F_5. Otherwise by T(5.5) we get either (i), or $n \leqslant 4$ and $E(\overline{M}) = 1$ with $O(\overline{M})$ elementary of rank at most 4. Since $O_2(\overline{M}) = 1$, we have $O(\overline{M}) = F^*(\overline{M})$ in this latter case, so that $\overline{M}/O(\overline{M}) \leqslant GL_k(3)$, where $k = m(O(\overline{M}))$. We note first that $\overline{M}$ is solvable for $k = 1, 2$. Also $GL_3(3)$ has order $2^5 \cdot 3^3 \cdot 13$, and $13 \nmid |O_8{}^+(2)|$; in view of O(1.10)(e) we have $\overline{M} \leqslant O_8{}^+(\tilde{Q})$, so $\overline{M}$ must also be solvable (a $\{2, 3\}$-group) in this case. Finally let $k = 4$. Now $O_8{}^+(\tilde{Q})$ has a unique conjugacy class of elementary subgroups of rank 4: so $O(\overline{M})$ is generated by 4 distinguished subgroups $\langle \bar{\theta}_i \rangle$ $(1 \leqslant i \leqslant 4)$ satisfying $m([\tilde{Q}, \bar{\theta}_i]) = 2$. The auromizer even in $L_8(2)$ of $O(\overline{M})$ is then the "monomial" group $2^4 S_4$; where the subgroup 2^4 has the obvious diagonal action, and the S_4 the obvious permutation action. So $\overline{M}$ is solvable in this case as well. Now we may apply the work of Lundgren and Wong [12] to conclude $k = 3$ or 4. So (1.18) is proved.

Now in determining $\overline{M}$ for various configurations in Section 2, we follow the general plan of showing first that $\bar{t} \notin \overline{M}'$ for a suitable $\bar{t} \in \bar{L}^{\#}$; and then that $\overline{M}'$ has sectional 2-rank at most 4. Thus in view of (1.18)(i), we are interested in quasi-simple groups J of sectional 2-rank at most 4; it will be convenient to classify these groups J according to order and structure of a Sylow 2-group S of J. Furthermore, we do not need the entire list of Gorenstein and Harada [7]; since by O(1.10)(e) we must have $\overline{M} \leqslant O_{2n}^+(\tilde{Q})$ (now $n = 4$ or 6), simple numerical considerations will rule out many of the groups, without forcing us to study irreducible representations in degree 8 or 12. Thus we may reduce the list of [6] to:

(1.19) Assume Hypothesis B ($n = 6$). Then $H = J/O(J)$ must be among:

	$\lvert S \rvert$	*type of S*	*type of H*	*Out (H)*
(a)	2^2	elementary	$L_2(5), L_2(11)$	2
(b)	2^3	elementary	$L_2(8)$	3
		dihedral	$L_2(9); L_2(7), A_7$	2×2; 2, 2 (resp.)

	$\lvert S \rvert$	*type of S*	*type of H*	*Out (H)*
(c)	2^4	elementary	$L_2(16)$	4
		dihedral	$L_2(17)$	2
		quasi-dihedral	M_{11}	1
(d)	2^5	dihedral	$L_2(31)$	2
		wreathed	$U_3(3)$	2
(e)	2^6	"type $L_3(4)$"	$L_3(4)$	$2 \times S_3$
		$2^{1+4} \cdot 2$	$Sp_4(3)$, A_8, A_9, M_{12}	2
(f)	2^7	$2^4 D_8$	$U_4(3)$, M_{22}	D_8, 2 (resp.)
		$(Z_2 \wr Z_2) \wr Z_2$	A_{10}, A_{11}	2.

For the smaller width, we may further reduce this list to:

(1.20) Assume Hypothesis A (now $n = 4$). $H = J/O(J)$ is among:

	$\lvert S \rvert$	*type of S*	*type of H*	*Out (H)*
(a)	2^2	elementary	$L_2(5)$	2
(b)	2^3	elementary	$L_2(8)$	3
		dihedral	$L_2(9)$; $L_2(7)$, A_7	2×2; 2, 2 (resp.)
(c)	2^5	wreathed	$U_3(3)$	2
(d)	2^6	"type $L_3(4)$"	$L_3(4)$	$2 \times S_3$
		$2^{1+4} \cdot 2$	$Sp_4(3)$, A_8, A_9	2
(e)	2^7	$(Z_2 \wr Z_2) \wr Z_2$	A_{10}	2

2. Possibilities for M in Cases A and B

In this section, we determine possibilities for $\bar{M}$ under Hypotheses A and B. (As in Section 1, it is convenient also to indicate the groups G to which these choices of $\bar{M}$ lead, even though we may have to do further work in Section 3 to establish the implication.) Study of $\bar{M}$ will proceed by analysis of possibilities for centralizers of involutions $\bar{t}$, by means of the various cases for $\bar{R}_t$ listed in (1.17). We may fix on one distinctive possibility for R/N, and determine what groups M and G are possible. Then in further work we may assume that possibility for R/N can no longer arise, but some other one does; and so on, shortening at each step the list of possibilities from (1.17) that we might have to deal with.

Many of the necessary arguments for the cases of width 4 and 6 are strongly analogous. However, superficial differences will sometimes make it easier to separate arguments, at the cost of being somewhat repetitious.

We outline the course of the argument in this section:

I. Suppose there is $\bar{t} \in \bar{L}^{\#}$ with $R/N = \Omega_n^{\epsilon}(2)$, where $\epsilon = +$ or $-$. We get the cases:

(A) $\bar{t}$ 2-central in $\bar{M}$.

(i) $|O(\bar{M})| \leqslant 3^3$: *Then* $\bar{M} \cong S_3 \times \Omega_n^{\epsilon}(2)$ and G

$$G = \Omega_{n+4}^{\epsilon}(2).$$

(ii) $|O(\bar{M})| = 3^4$: Then $n = 4$, $\epsilon = +$, and

$$\langle \bar{L}^{\bar{M}} \rangle = O(\bar{M})\bar{L}; \qquad G \cong \Omega_8^{+}(3).$$

(B) $\bar{t}$ not 2-central in $\bar{M}$. Then $n = 6$ and $\epsilon = -$; $\bar{M} = \widehat{3U_4(3)} \cdot \langle \bar{t} \rangle$, and G is of type F_{24}'.

II. Assume we never get $R/N \cong \Omega_n^{\epsilon}(2)$, but do get

$$R/N \cong L_{(1/2)n}(2) \qquad \text{for some } \bar{t}.$$

Then $n = 6$, $\bar{M} \cong \widehat{3M_{22}} \cdot \langle \bar{t} \rangle$, and G is of type J_4.

III. If none of the above cases arises, then no group G is possible.

Remark. The feasibility of examining the separate cases in (1.17) (which is really just T(4.5)) shows how we are really just continuing Timmesfeld's ideas to finish the problem; as suggested in the introduction of his paper.

In the proof to follow, we will frequently consider the "Suzuki type" of the involution $\bar{t}$ acting on the orthogonal space $\tilde{Q}$—this is just the type of $\bar{t}$ as an involution of $O_{2n}^{+}(2)$, in view of O(1.10)(c). These matters are described in Aschbacher and Seitz [4, Sects. 5–8], with other information available in F. Smith [22, (2.5)] or Aschbacher [2, (4.5)]. Furthermore in our study of $C_{\bar{M}}(\bar{t})$, we can work in the centralizer of $\bar{t}$ in a suitable $O_{2n}^{+}(2)$; we may denote this possibly larger group by $C_{O_{2n}}(\bar{t})$. Information on these centralizers appears in Aschbacher and Seitz [4, (8.6)–(8.8)]; we may quote these results without direct reference.

Henceforth we drop subscripts $\bar{M}$ so as to write $C(\bar{t})$ and $N(\bar{L})$. We recall also that $N(\bar{L}_a) = C(\tilde{a})$ (in view of the definition $L_a = Q_a \cap M$, and $C_{\tilde{Q}}(\bar{L}_a) = \langle \tilde{a} \rangle$ in O(1.10)(a)).

We work first under:

Hypothesis (2.1). $\bar{t} \in \bar{L}^{\#}$ has $R/N \cong \Omega_n^{\epsilon}(2)$. ($\epsilon = +$ or $-$).

The critical first step is determination of $C(\bar{t})$. Afterwards we can settle many questions by looking inside this subgroup.

(2.2) (a) *$\bar{t}$ has type a_n on $\tilde{Q}$.*

(b) $\bar{N} = \langle \bar{t} \rangle$ *and* $\bar{R} = \langle \bar{t} \rangle \times \bar{R}_0$, *where* $\bar{R}_0 \cong \Omega_n^{\epsilon}(2)$. *Furthermore,* $C(\bar{t})/\langle \bar{t} \rangle$ *is at most* $O_n^{\epsilon}(2)$.

(c) $C(\bar{L}) = \bar{L}$. *In case* $n = 4$, *note* $\bar{M}$ *must have sectional* 2*-rank at most* 4 *so if* $E(\bar{M})$ *is quasi-simple, we may use* (1.20).

Proof. By T(4.5), $\tilde{V} = [\tilde{Q}, \bar{t}]$ has rank n, and the preimage $V = [Q, t]$ of $\tilde{V}$ is elementary; it follows that $\bar{t}$ must have type a_n. We could also check this from [4], noting that only for $\bar{t}$ of type a_n does $C_{O_{2n}}(\bar{t})$ have a section isomorphic to $\Omega_n^\epsilon(2)$. Indeed, this overlying centralizer is an elementary 2-group E of rank $\binom{n}{2}$, extended by a group $Sp_n(2)$ (split). The groups E and $Sp_n(2)$ may be represented by matrices of the forms

$$\begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix},$$

respectively, when a basis for $\tilde{Q}$ is chosen so as to put $\bar{t}$ in Suzuki form. Notice we have $\bar{N} = \langle \bar{t} \rangle$ by (1.17); it follows that $\bar{R} = \langle \bar{t} \rangle \times \bar{R}_0$ as in (b), since if we had a nonsplit extension, then the preimage $\bar{L}$ of the elementary group $LN/N = O_2(C_{R/N}(\tilde{a}))$ of order 2^{n-2} would not be elementary, a contradiction.

Now to complete the proof of (2.2), it will suffice to show that $E \cap \bar{M} = \langle \bar{t} \rangle$: For the normalizer in $Sp_n(2)$ of $\Omega_n^\epsilon(2)$ is just $O_n^\epsilon(2)$, and we would get $C(\bar{t})$ determined as in (b); with (c) following since $\bar{L}$ is self-centralizing $C(\bar{t})$. In case $n = 4$, $N(\bar{L})/\bar{L} \leqslant L_3(2)$, and this with O(1.10)(b) gives sectional 2-rank at most 4. We remark that in case $n = 4$ and $\epsilon = +$, choice of one of three representations in degree 8 determines which classes should be called c_2 and a_4.

Now since $\bar{R} \cong C(\bar{t})$, we have $[\bar{R}, E \cap \bar{M}] \leqslant E \cap \bar{R} = \langle \bar{t} \rangle$. Thus we can ask if $\bar{R}$ centralizes any of $E/\langle \bar{t} \rangle$. In fact, except when $n = 4$ and $\epsilon = +$, we have $\bar{R}_0 = E(\bar{R})$, and we may ask about $C_E(\bar{R}_0)$. In this case the group $\bar{R}_0$ is irreducible in n dimensions, and so by Schur's lemma it centralizes among the $n \times n$ square matrices only 0 and I; which says $C_E(\bar{R}_0) = \langle \bar{t} \rangle$, giving $E \cap \bar{M} = \langle \bar{t} \rangle$.

In case $n = 4$ and $\epsilon = +$, a group $\Omega_4^+(2)$ centralizes in E a subgroup $F = \langle \bar{d} \rangle \times \langle \bar{t} \rangle$, where $\bar{d}$ and $\bar{d}\bar{t}$ are of type a_2 on $\tilde{Q}$ with product $\bar{t}$ of type a_4. The matrices A as above for $\bar{d}$ and $\bar{d}\bar{t}$ have form

$$\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

Thus in this case as well, $C(\bar{t})/\langle \bar{t} \rangle$ is at most $O_4^+(2)$. So (2.2) is proved.

It is useful also to consider the action on $\tilde{Q}$ of other involutions visible in $C(\bar{t})$.

(2.3) *Assume* $n = 6$, *and set* $\bar{K} = \bar{L} \cap \bar{R}_0$. *Then involutions of* $\bar{K}$ *are of type* a_4 *or* c_4 *on* $\tilde{Q}$, *and those of* $\bar{L} - \bar{K}$ *have type* a_6 *or* c_6. *In particular, we have* $\bar{t} \notin \bar{M}'$, *and* $\bar{K} \trianglelefteq N(\bar{L})$.

Proof. These assertions about type on $\tilde{Q}$ are obvious if our complement $\bar{R}_0$

in $C(\bar{t})$ happens to fall inside a natural complement $Sp_6(2)$ in $C_{\Omega_{12}}(\bar{t})$. For instance, we can establish this if $O(\overline{M}) = 1$ and $E(\overline{M})$ is quasi-simple. But in the absence of such knowledge, or information about first cohomology of $E/\langle \bar{t} \rangle$ extended by $\bar{R}_0$, we must make a more careful study of matrix forms considered in (2.2).

It is convenient to rearrange suitably the Suzuki form of $\bar{t}$ from [4]. We choose our orthogonal basis $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_6$ of $\tilde{Q}$ so that each $\{\tilde{v}_i, \tilde{v}_{13-i}\}$ is a hyperbolic pair. We do this in such a way that $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_6$ is an orthogonal basis for action of $\bar{R}_0$ on $\tilde{V} = [\tilde{Q}, \bar{t}]$. Thus $\bar{t}$ still has the matrix form considered earlier. Furthermore, we can arrange the basis so that $\{\tilde{v}_i, \tilde{v}_{7-i}\}$ is a hyperbolic pair for $\tilde{V}$, though in case $\epsilon = -$ we must require that $\tilde{v}_3, \tilde{v}_4$ be nonorthogonal vectors which are nonsingular for $\tilde{V}$. The elementary subgroup E discussed above consists of matrices symmetric about the "opposite" diagonal, and entries on this antidiagonal are zero. The matrices of $Sp_6(2)$ have their inverses equal to their "antitransposes."

Now the subgroup $\bar{K}$ projects onto the subgroup $J = O_2(C_{Sp_6(2)}(\tilde{a}))$. We may as well take this latter subgroup to be given by matrices whose nonzero entries away from the usual diagonal are in the left column and bottom row, as in

$$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ 1 & & & & 1 & \\ 0 & 1 & & & & 1 \end{pmatrix}$$

The E-components of elements of $\bar{K}$ must then be in $C_E(J)$. However, we see (and this requires $n = 6$) that J is self-centralizing in the set of $n \times n$ matrices defining E. We conclude that an element $\bar{u}$ of $\bar{K}$ has type 4 or 6 on $\tilde{Q}$, depending on whether the E-component of $\bar{u}$ has zeros or ones on the diagonal. Now for $\bar{u}$ of type $a_2(\bar{R}_0)$, we may find a 4-group of conjugates of $\bar{u}$ in $\bar{K}$, and it is not possible that all three of these involutions could have ones on the diagonal. Thus the elements of type $a_2(\bar{R}_0)$ must have type 4 on $\tilde{Q}$; the same assertion follows for those of type $c_2(\bar{R}_0)$, since they are products of pairs of a_2's. Furthermore we see elements of $\bar{L} - \bar{K}$ are of form $\bar{u} \cdot \bar{t}$ and so have type 6 on $\tilde{Q}$.

In particular, we see $\bar{K} \trianglelefteq N(\bar{L})$; and since $\bar{t}$ cannot be $\overline{M}$-conjugate to elements of $\bar{K}$, we get $\bar{t}^{\overline{M}} \cap \bar{L} \subseteq \bar{L} - \bar{K}$, so that $[\bar{L}, N(\bar{L})] \leqslant \bar{K}$. By O(1.10)(b) we have $\bar{t} \notin \overline{M}'$. This proves (2.3). We remark that obtaining $\bar{L} \cap E(\overline{M})$ of rank $\leqslant 4$ now gives us a hope of establishing sectional 2-rank 4 for $E(\overline{M})$.

With these preliminaries finally completed, we can proceed to determination of the groups that arise for $R/N \cong \Omega_n^{\epsilon}(2)$.

(2.4) *Suppose $\bar{t}$ is 2-central in $\overline{M}$. Then either*

(i) $M \cong S_3 \times \Omega_n^{\epsilon}(2)$ *and* $G \cong \Omega_{n+4}^{\epsilon}(2)$ *or*

(ii) $n = 4$, $\epsilon = +$ *with* $|O(\bar{M})| = 3^4$;

$$\bar{M} = O(\bar{M})\bar{L} \quad \text{and} \quad G \cong \Omega_8^+(3).$$

Proof. We consider first the case $E(\bar{M})$ quasi-simple. For $n = 6$, we see by (2.3)(i) that a Sylow 2-group of $\bar{R}_0$ is of index at most 2 in a Sylow group of $E(\bar{M})$. With (2.3)(ii) we have $E(\bar{M})$ of sectional 2-rank at most 4 in any case, and exactly 4 for $n = 6$. By (1.19)/(1.20) the only possibilities for the simple quotient $E(\bar{M})/O(\bar{M})$ with large enough 2-rank and suitable value for $|E(\bar{M})|_2$ are:

$$(n = 6) \quad A_8 = \Omega_6^+(2) \quad A_9 \quad U_4(2) = \Omega_6^-(2) \quad L_3(4) \quad M_{12},$$
$$(n = 4) \quad L_2(5) = \Omega_4^-(2) \quad L_2(7) \quad L_2(8) \quad L_2(9) \quad A_b.$$

As in (2.2), we recall that $|C(E(\bar{M}))| \leqslant 2\,|O(\bar{M})|$. For $n = 6$, we may rule out A_9 by structure of $C(\bar{t})$: For no outer automorphism centralizes a subgroup A_8, and we know $A_9 \nleqslant C(\bar{t})$ by (2.2). In a similar way we rule out M_{12} and $L_3(4)$. Thus we can only get $E(\bar{M})/O(\bar{M}) \cong \Omega_6^\epsilon(2) \cong R/N$. For $n = 4$ and $\epsilon = -$, we may rule out $L_2(7)$, $L_2(8)$, $L_2(9)$ by the requirement $\Omega_4^-(2) \leqslant C(\bar{t})$. We can obtain this condition if $\bar{t}$ acts as a transposition of $S_7 - A_7$, and $C(\bar{t}) = \langle \bar{t} \rangle \times S_5$. In fact $E(\bar{M}) \nleqslant A_7$ here, for no 3-element of $\Omega_8^+(2)$ is centralized by $3A_7$. Our group S_7 has a unique irreducible $GF(2)$-representation in degree 8, in which "short" 3-elements are fixed-point-free on $\tilde{Q}^\#$. However, such elements are visible in our $S_5 = \Omega_4^-(2)$, fixing points of $\tilde{V}^\#$. This contradiction shows only $\Omega_4^-(2)$ may arise for $\epsilon = -$. When $n = 4$ and $\epsilon = +$, we have $\bar{L} \cong E(\bar{M})$ in every case since $E(\bar{M})$ has 2-rank 2. In none of the listed simple groups does an inner or outer involution centralize a group of order 3^2; so we must have $O(\bar{M}) \neq 1$, and even $O(\bar{M}) \leqslant C(\bar{t})$. Even so, the requirement $\Omega_4^+(2) \times \langle \bar{t} \rangle \leqslant C(\bar{t})$ eliminates all but A_5, where $\bar{t}$ acts as a transposition of $S_5 - A_5$, and $\bar{M} \cong S_5 \times S_3$. However in this case, O(2.1) shows that $\bar{L} \cap S_5 \leqslant A_5$, a contradiction. So $E(\bar{M})$ is not quasi-simple for $n = 4$ and $\epsilon = +$.

So in case $n = 6$, or $n = 4$ and $\epsilon = -$, we have obtained $E(\bar{M})/O(\bar{M}) \cong \Omega_n^\epsilon(2)$. If $O(\bar{M}) = 1$, we see $\bar{M} \leqslant \langle \bar{t} \rangle$ extended by $O_n^\epsilon(2)$ is reducible on $\tilde{Q}$, contradicting (1.7). So $O(\bar{M}) \neq 1$, and it follows that $\bar{M} = S_3 \times \Omega_n^\epsilon(2)$ or $S_3 \times O_n^\epsilon(2)$. By the work of O, we have $G \cong \Omega_{n+4}^\epsilon(2)$ and $\bar{M} \cong S_3 \times \Omega_n^\epsilon(2)$.

We turn to the case $\bar{M}$ solvable; in view of (1.18), we have $n = 4$ and $\epsilon = +$. We consider first the case $|O(\bar{M})| = 3^3$. The work of Lundgren and Wong [12] shows there are three subgroups of $O(\bar{M})$ of order 3 which are fixed-point-free on $\tilde{Q}$. Since $O(\bar{M}) = F^*(\bar{M})$, we see $\bar{M}/O(\bar{M})$ must be a subgroup of the obvious monomial group 2^3S_3. (In $\mathrm{Aut}(\Omega_8^+(2))$ this extra S_3 is realized.) Because of a gap in the proof of [12, Lemma 4.5], we cannot automatically reduce to the case $O(\bar{M}) \in \mathrm{Syl}_3(\bar{M})$; to finish, we must adapt the case $n = 4^+$ of O, which is not originally treated in the present generality. We notice first that $\bar{M}$ contains a subgroup $\langle \bar{L}^{\bar{M}} \rangle = O(\bar{M})\bar{L} \cong S_3 \times \Omega_4^+(2)$. This structure induces at least the fusion discussed in O, with further possible fusion of classes if $\bar{M}$ is in fact larger. After this, there is no problem with the proof that z^G is a class of $\{3, 4\}^+$-transpositions. We then conclude that our simple group G must be $\Omega_8^+(2)$ with $\bar{M} \cong S_3 \times \Omega_4^+(2)$.

Suppose instead $|O(\bar{M})| = 3^4$. We recall our argument in (1.18), also based on [12], where we showed that $\bar{M}/O(\bar{M})$ must be a subgroup of the monomial group 2^4S_4. In fact [12] shows that $\langle \bar{L}^{\bar{M}} \rangle = O(\bar{M})\bar{L}$, where $\bar{L} = \langle \bar{\imath} \rangle \times \bar{L}_0$, with $\bar{\imath}$ inverting $O(\bar{M})$, and the elements of $\bar{L}_0^{\#}$ (the type $\bar{t}$ of (1.17)) acting as the normal 2-subgroup of the S_4. Thus $\bar{M}/O(\bar{M})\bar{L}$ is at most S_3 (this extra S_3 is realized in $\mathrm{Aut}(\Omega_8^+(3))$). We will establish the hypotheses of Aschbacher in [3]. We fix $\bar{\theta}_1$ with preimage θ_1 of order 3, and $Q_1 = [Q, \theta_1]$. The group $Q_1\langle\theta_1\rangle$ has structure $SL(2, 3)$ and is subnormal in M. Furthermore there are just 4 M-conjugates of this subgroup, and they commute pairwise. The critical condition to establish is that if k is a 4-element of Q_1 with $k^g \in M$, then in fact $k^g \in N(Q_1\langle\theta_1\rangle)$. Set $y = k^g$ and $x = y^2 = z^g$; it will be enough to show that $\bar{y} = \bar{\imath}$ (we are done if $y \in Q$). If we had $x \notin Q$, then we could take $\bar{x}$ in $\bar{L}_0$, with $C_Q(x)$ elementary of rank 5; but this group contains a conjugate of z, which we can take to be a, so that $\bar{y} \in \bar{L}_a = \bar{L}$ and $\bar{x} = 1$, a contradiction. Thus $x \in Q$, so we may assume $\tilde{x} = \tilde{a}$ and $\bar{y} \in \bar{L}$. Now we may use action of $O(\bar{M})\bar{L}$ to identify elements of $\tilde{Q}$ by the notation $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$ where $\tilde{x}_i \in \tilde{Q}$; the elements of $\widetilde{z^G \cap Q}$ are those with all four components nontrivial. We may take $\tilde{a} = (\tilde{b}\ \tilde{b}\ \tilde{b}\ \tilde{b})$, and we see $\widetilde{Q \cap Q_a}$ is generated by:

$$(\tilde{b}\ \tilde{b}\ \tilde{1}\ \tilde{1}) \qquad (\tilde{1}\ \tilde{b}\ \tilde{b}\ \tilde{1}) \qquad (\tilde{1}\ \tilde{1}\ \tilde{b}\ \tilde{b}) \qquad (\tilde{c}\ \tilde{c}\ \widetilde{bc}\ \widetilde{bc}),$$

where $\tilde{Q}_1 = \langle \tilde{b}, c\rangle$. We may assume $k = (\tilde{b}\ \tilde{1}\ \tilde{1}\ \tilde{1})$, and we see that k centralizes in $\widetilde{Q \cap Q_a}$ a hyperplane containing exactly one coset of conjugates of z, namely $\tilde{a}$. Arguing symmetrically for $\bar{y}$ $(y \in Q_a \cap M - Q)$ we see that if $\bar{y}$ interchanged pairs of $\tilde{Q}_i$, it would fix 10 conjugates (5 $\langle a\rangle$-cosets) of z in $Q \cap Q_a$. We must have $\bar{y} = \bar{\imath}$, as desired. Now we have the hypotheses of Theorems 7 and 8 of Aschbacher [3], so we conclude that G must be one of $\Omega m(q)$ $(5 \leqslant m \leqslant 8)$ or $Sp_6(2)$ (the last is clearly too small). Furthermore $Q_1\langle\theta_1\rangle$ must be of the "fundamental" type described there, so that $q = 3$. Now only $\Omega_8^+(3)$ among these groups has a large extraspecial subgroup of width 4. In particular, $\bar{M} = O(\bar{M})\bar{L}$, and (2.4) is proved.

(2.5) *Suppose $\bar{t}$ is not 2-central in $\bar{M}$. Then $n = 6$ and $\epsilon = -$.* We have $\bar{M} \cong 3\widehat{U_4(3)} \cdot \langle \bar{t} \rangle$ *and G of type F_{24}'.*

Proof. We begin by considering the cases with $n = 4$. In view of O(1.10)(b), our assumption $|\bar{M} : C(\bar{t})|$ even means $|\bar{t}^{\bar{M}} \cap \bar{L}| = 2$ or 4 or 6.

Consider first the case $\epsilon = -$. Here the involutions of $\bar{K} = \bar{L} \cap \bar{R}_0$ are already conjugate in $N(\bar{L}) \cap C(\bar{t})$, so we can only have $|\bar{t}^{\bar{M}} \cap \bar{L}| = 4$. Thus $|N(\bar{L})/\bar{L}| = 12$ or 24, and we see that $N(\bar{L})/\bar{L} = A_4$ or S_4 (actually, part of the parabolic stabilizing the hyperplane $\bar{K}$ of $\bar{L}$). In particular we get $\bar{t} \notin \bar{M}'$ by O(1.10)(b), and so $\bar{K} = \bar{L} \cap E(\bar{M})$. We see that $|E(\bar{M})|_2 = 2^4$ or 2^5; and by action a 3-element of $N(\bar{L})$ we see $E(\bar{M})$ must contain either a subgroup $\mathbb{Z}_4 \times \mathbb{Z}_4$, or an elementary 2^4. The only candidate in (1.20) is $U_3(3)$. But this group has no 5-elements whereas $C(\bar{t})$ does, a contradiction.

Suppose now $\epsilon = +$. In the case $\bar{M}$ solvable, we obtain from Lundgren and Wong [12] that $\bar{t}$ is 2-central, whether $O(\bar{M})$ has order 3^3 or 3^4. So we may assume $E(\bar{M})$ is quasi-simple. In case $\bar{L} \leqslant E(\bar{M})$, we have 2-rank at least 3 with $2^4 \leqslant |\bar{M}|_2 \leqslant 2^6$, so that $E(\bar{M})/O(\bar{M}) \cong Sp_4(3)$, A_8, A_9 by (1.20). However, we do not get $C(\bar{t}) \leqslant \langle \bar{t} \rangle \times O_4^+(2)$ for these groups. Thus $|\bar{L} \cap E(\bar{M})| \leqslant 2^2$ and $|E(\bar{M})|_2 \leqslant 2^5$. Since all possibilities in (1.20)(a)(b)(c) have a single class of involutions, and $\bar{t}$ is not 2-central, we must in fact have $\bar{t} \notin E(\bar{M})$. Suppose then that $|\bar{L} \cap E(\bar{M})| = 2^2$. Then $\bar{t}^{\bar{M}}$ does not meet $\bar{L}$, so $|\bar{t}^{\bar{M}} \cap \bar{L}| = 2$ or 4. It follows that $N(\bar{L})/\bar{L}$ is a 2-group, since $\bar{L}$ is self-normalizing in $\Omega_4^+(2)$. But then $\bar{L} \cap E(\bar{M})$ must by O(1.10)(b) meet at least two involution classes, a contradiction. We must have $|\bar{L} \cap E(\bar{M})| = 2$. Since $\bar{L} \cdot E(\bar{M})/E(\bar{M})$ is a 4-group, we must have $|\bar{t}^{\bar{M}} \cap \bar{L}| = 2$ (and not 4 or 6). Thus $|E(\bar{M})|_2 \leqslant 2^3$. Among the possibilities in (1.20) we may rule out $L_2(8)$ since it has no involutory outer automorphism. In the other cases, except possibly $L_2(9)$, the restriction $|C(E(\bar{M}))| \leqslant 2\,|O(\bar{M})|$ forces $O(\bar{M}) \neq 1$ and $\bar{M} \cong S_3 \times E(\bar{M})\langle \bar{t} \rangle$. We rule out A_5 since outer automorphisms are 2-central in S_5. Outer automorphisms of A_7 and $L_2(7)$ do not give $C(\bar{t}) \leqslant \langle \bar{t} \rangle \times O_4^+(2)$. The same restriction eliminates $L_2(9)$, even if $O(\bar{M}) = 1$. This takes care of all possibilities in (1.20). The contradiction eliminates the case $n = 4$ for Hypothesis (2.1) with $\bar{t}$ not 2-central.

We turn to the more difficult case $n = 6$. With $|\bar{M} : C(\bar{t})|$ even we have $|\bar{M}|_2 \geqslant 2^8$ and $E(\bar{M})|_2 \geqslant 2^7$. Eventually we will show these are in fact equalities. The main tool will be analysis of R_i/N_i for $\bar{i} \in \bar{K}^{\#}$, where $\bar{K}$ is the group $\bar{L} \cap \bar{R}_0$ of (2.3)(i).

It is useful to observe first that: $|\bar{t}^{\bar{M}} \cap \bar{L}|$ is either 10 (resp. 6) or 16 when $\epsilon = +$ (resp. $-$). For $\bar{t}^{\bar{K}} \cap \bar{L}$ lies in $\bar{t}^{\#}$ by (2.3), and action of an $\Omega_4^{\epsilon}(2)$ in $N(\bar{L}) \cap C(\bar{t})$ gives orbits of size 9 and 6 (resp. 5, 10 for $\epsilon = -$) on $\bar{K}^{\#}$. Since $|\bar{t}^{\bar{M}} \cap \bar{L}|$ is even, only the above even possibilities may arise.

From now on we fix $\bar{i}$ of type a_2 in $\bar{L} \cap \bar{R}_0$. Since $\bar{K} \trianglelefteq N(\bar{L})$ by (2.3), we know $\bar{K}$ must contain involutions 2-central in $\bar{M}$, and by action of an $\Omega_4^{\epsilon}(2)$ as above, we may as well assume $\bar{i}$ is 2-central. We consider the possibilities for $\bar{R}_i/\bar{N}_i$. Certainly $\bar{R}_i > \bar{L}$, from conjugacy already in $C(\bar{t}) \cap C(\bar{i})$. Since by (2.3) $\bar{i}$ has type a_4 or c_4 on $\tilde{Q}$, we know from $C_{\Omega_{12}}(\bar{t})$ as in [2] that $R_i/N_i \not\supseteq \Omega_6^{\pm}(2), L_3(2)$. The only possibilities left in (1.17) are $L_2(\tilde{V}_i) \cong S_3$ and $\Omega_4^+(\tilde{V}_i) \cong S_3 \times S_3$. We shall see that these are parallel possibilities corresponding to $O(\bar{M}) = 1$ and $O(\bar{M}) \neq 1$; and only the latter in fact will arise. We give first the argument eliminating S_3, repeating only in outline those ideas which recur in the case of $\Omega_4^+(2)$.

We suppose by the way of contradiction that $\bar{R}_i/\bar{N}_i \cong S_3$. We see that $\bar{R}_t \cap \bar{R}_i$ already induces the full S_3 on the 3 $\bar{M}$-conjugates of $\bar{L}$ in $\bar{R}_i$. Since these conjugates intersect at $\langle \bar{t}, \bar{\imath} \rangle$, and they generate $\bar{R}_i$, we get:

(a) $Z(\bar{R}_i) = \langle \bar{t} \rangle \times \langle \bar{\imath} \rangle$ and $\bar{R}_i \leqslant \bar{R}_t$. This forces $\bar{R}_i = \langle \bar{t} \rangle \times 2^3 S_3$ by considering $\bar{R}_t$-conjugates of $\bar{K}$ containing $\bar{\imath}$. Now $\bar{R}_i = C(\bar{t}) \cap C(\bar{\imath})$ and $\bar{R}_i \trianglelefteq C(\bar{\imath})$; in view of (a), it follows that:

(b) $| C(\bar{\imath}) : \bar{R}_i | = 2$ (thus $| \bar{M} |_2 = 2^8$ and $| E(\bar{M})|_2 = 2^7$). For certainly we can only have $\bar{t} \sim \overline{ti}$ in $C(\bar{\imath})$, and this must occur since $C(\bar{\imath})$ contains a Sylow group, and $C(\bar{t})$ does not. Now we may refine our previous calculation to:

(c) $| \bar{t}^{\bar{M}} \cap \bar{L} | = 10$ (resp. 6) as $\epsilon = +$ (resp. $-$). This now forces $| N(\bar{L})/\bar{L} | = 2^3 3^2 5$ in both cases. We recall that $\bar{K} \trianglelefteq N(\bar{L})$. We claim in fact:

(d) $C(\bar{K}) = \bar{L}$. For $C(\bar{K})$ centralizes $\bar{\imath}$ and a conjugate $\bar{\imath}'$ in $\bar{K}$, and thus by (a) acts on $\langle \bar{t} \rangle \times \langle \bar{\imath} \rangle$ and $\langle \bar{t} \rangle \times \langle \bar{\imath}' \rangle$. We conclude $C(\bar{K}) \leqslant C(\bar{t})$, as desired for (d). We see also that $N(\bar{L})/\bar{L} \leqslant L_4(2)$. We have $5 \mid | C(\bar{\imath})|$ so that $N(\bar{K})$ is transitive on $\bar{K}^\#$. We see easily that $F(N(\bar{L})/\bar{L}) = 1$. We must have:

(e) $N(\bar{L})/\bar{L} \cong A_6$, acting as $Sp_4(2)'$ on $\bar{K}$. Now from O(1.10)(b), we know a Sylow 2-group of $E(\bar{M})$ is contained in $N(\bar{L})$, so we see $E(\bar{M})$ has sectional 2-rank 4, and Sylow groups of order 2^7. From (1.19)(f) we might only have $E(\bar{M})/O(\bar{M}) \cong U_4(3)$, M_{22}, A_{10}, A_{11}. We eliminate the alternating groups, since they have no elementary groups 2^4 consisting entirely of 2-central involutions. We eliminate M_{22}, since its elementary subgroups 2^4 normalized by A_6 are in fact T.I. sets. So we have:

(f) $\epsilon = -$ and $E(\bar{M})/O(\bar{M}) = U_4(3)$. Note we cannot have $\epsilon = +$, for no outer involution of $U_4(3)$ centralizes $\Omega_6^+(2)$. We see $O(\bar{M}) = 1$ also: for if $O(\bar{M}) \neq 1$, we see $\bar{t} \notin C(O(\bar{M}))$, and this forces involutions of type $\bar{\imath}$ to centralize $O(\bar{M})$, against (b). We claim finally that $U_4(3)\langle \bar{t} \rangle$ has no irreducible $GF(2)$-representation of degree 12. To see this, consider a subgroup 3^4. $\Omega_4^-(3) = 3^4 A_6$ of $U_4(3)$. The subgroup 3^4 would be generated by four commuting conjugates of a 3-element fixing only a subspace of $\tilde{Q}$ of dimension 4 and type $+$. (In $\widehat{3U_4(3)}$, of course, this 3^4 becomes an extraspecial 3^{1+4}, and there is no problem with its maximal elementary subgroups of order 3^3.) This contradiction shows $R_i/N_i \not\cong S_3$.

So we have $R_i/N_i \cong \Omega_4^+(2)$. Then $| \bar{L} \cap \bar{N}_i | = 2^3$. We choose $\tilde{b}$ so that $\{\tilde{a}, \tilde{b}\}$ is a hyperbolic pair for $\tilde{V}_i$; then $\bar{R}_i = \langle \bar{L}_a, \bar{L}_b \rangle$. As in T(4.7) we get $\bar{N}_i = (\bar{L}_a \cap \bar{N}_i)(\bar{L}_b \cap \bar{N}_i)$, and so $2^3 \leqslant | \bar{N}_i | \leqslant 2^5$. Also by T(4.7) we have $\langle \bar{\imath} \rangle = Z(\bar{R}_i) \cap \bar{L}_a \geqslant \bar{L}_a \cap \bar{L}_b \cap \bar{N}_i \geqslant \langle \bar{\imath} \rangle$, so in fact $| \bar{N}_i | = 2^5$. We note that with $\bar{R}_0 = E(\bar{R}_t)$, the subgroup $\bar{L}_a \cap O_2(C_{\bar{R}_0}(\bar{\imath}))$ centralizes $\tilde{V}$ and so lies in $\bar{N}_i$; we conclude in fact $\bar{N}_i = O_2(C_{\bar{R}_0}(\bar{\imath}))$, extraspecial of width 2 and type $+$. Then $\bar{R}_i \simeq \langle \bar{t}, \bar{\theta} \rangle \times \bar{N}_i S_3$, where $\bar{\theta}$ is a suitable 3-element of $\bar{R}_i$ inverted by $\bar{t}$, so that $\bar{\theta}$ also centralizes N_i. Notice also that

$$| C(\bar{\imath}) \cap \langle \bar{\theta} \rangle C(\bar{t}) : \bar{R}_i | = 1 \text{ (resp. 3)}, \qquad \text{when} \quad \epsilon = + \text{ (resp. } -).$$

We may now adapt parts of the proof in the previous case. We obtain the analogous results:

(a) $Z^*(\bar{R}_i) = \langle \bar{t}, \bar{\theta} \rangle \times \langle \bar{\imath} \rangle$ with $\bar{R}_i \leqslant \bar{R}_t \langle \bar{\theta} \rangle$.

(b) $| \bar{t}^{C(\bar{\imath})} | = 6$ with $| C(\bar{\imath}) : \bar{R}_i | = 2$.

(c) $| \bar{t}^{\bar{M}} \cap \bar{L} | = 10$ (resp. 6) as $\epsilon = +$(resp. $-$).

(d) $C(\bar{K}) = \langle \bar{\theta} \rangle \bar{L} = \langle \bar{t}, \bar{\theta} \rangle \times \bar{K}$.

(e) $N(\bar{L})/\bar{L} \simeq A_6$.

(f) $\epsilon = -$ and $E(\bar{M})/O(\bar{M}) \simeq U_4(3)$.

In this case we must have $O(\bar{M}) \neq 1$ by the condition $3^3 \mid | \bar{R}_i |$. In view of our previous remarks about representation of $U_4(3)$, we must have $E(\bar{M}) \simeq \widehat{3U_4(3)}$ and $\bar{M} = \widehat{3U_4(3)} \cdot \langle \bar{t} \rangle$. These are the hypotheses of Parrott in [13], so we have $| G | = | F'_{24} |$, which is what we mean by "G is of type F'_{24}." Thus (2.5) is proved, leaving open the question of whether F'_{24} is the unique simple group of its order (even with such a centralizer M). It is possible now to compute much further local information in G, to agree with F'_{24}. Presumably it is not unduly difficult to characterize the group by some permutation representation.

The greater part of our work is now done, and there is just one more group to be located. We have completed our analysis of the consequences of Hypothesis (2.1), so from now on we work under the assumption:

HYPOTHESIS (2.6). For $\bar{t} \in \bar{L}^{\#}$ we never get $R/N \simeq \Omega_n^{\epsilon}(2)$, but do get $R/N \simeq L_m(2)$, where $m = \frac{1}{2}n$.

Remark. In view of (1.7)(i) and (1.11), this is the last case to consider for $n = 4$.

As before, we first wish to determine $C(\bar{t})$:

(2.7) (a) *$\bar{t}$ has type a_n on $\tilde{Q}$.*

(b) $C(\bar{t}) = \langle \bar{t} \rangle \times \bar{S} \cdot L_m(2)$, *where $\bar{S}$ is a natural module for $\bar{R}/\bar{N}$.*

Proof. We can see that $\tilde{U} = [\tilde{Q}, \bar{t}]$ has rank n by T(6.6); and $\tilde{U}$ is a sum of two natural R/N-modules, so that $[\tilde{Q}, \bar{t}]$ is elementary, and $\bar{t}$ has type a_6. Furthermore, as in (2.2) above (by analogy with T(4.5)), we may put a nondegenerate symplectic product $(\cdot, \cdot)$ on $\tilde{U}$, preserved by a quotient $Sp_n(2)$ of $C_{\Omega_{2n}}(\bar{t})$.

We consider first the nature of $\bar{N}$. Since $\frac{1}{2}n = 2$ or 3, we may find $\tilde{a}, \tilde{b} \in \tilde{V}$ with $\langle \bar{L}_a, \bar{L}_b \rangle$ covering $\bar{R}/\bar{N}$; it follows as in T(4.7) that $\bar{R} = \langle \bar{L}_a, \bar{L}_b \rangle$ and

$\bar{N} = (\bar{L}_a \cap \bar{N})(\bar{L}_b \cap \bar{N})$. We have $\bar{L}_a \cap \bar{N}$ of rank $\frac{1}{2}n$ by O(1.10)(d), so $2^m \leqslant |\bar{N}| \leqslant 2^{n-1}$. Suppose we had $\bar{L}_a \cap \bar{N} = \bar{N}$; that is, $\bar{L}_a \cap \bar{N} = \bar{L}_b \cap \bar{N}$ for all $\tilde{a}, \tilde{b} \in \tilde{V}^{\#}$. Then $[\bar{L}_a \cap \bar{L}_b, \widetilde{Q_a \cap Q_b}] \leqslant \langle \tilde{a} \rangle \cap \langle \tilde{b} \rangle = 1$. Now by T(6.6) we have $\tilde{V} \leqslant \widetilde{Q \cap Q_b}$ for all $\tilde{b} \in \tilde{V}^{\#}$; further each $\widetilde{Q \cap Q_b}$ has rank n. As in T(4.5) we find that with respect to the inner product $(\cdot, \cdot)$ on $\tilde{U}$ we have $\tilde{b}^{\perp} = \widetilde{Q_b \cap U}$ of rank $n-1$. Now the situation $\bar{L}_a \cap \bar{L}_b$ of rank at least m centralizing $\widetilde{Q_a \cap Q_b}$ of rank at least $\frac{1}{2}n + 1$ contradicts the fact that $\bar{L}_a$ (of rank $n-1$) acts as transvections according to $\tilde{a}$ on $\widetilde{Q_a \cap Q}$ (of rank n). (Compare T(3.1)(2)). Thus $\bar{L}_a \cap \bar{N} < \bar{N}$.

In case $n = 4$, we have $|\bar{N}| = 2^3$, and $\bar{N}/\langle \bar{t} \rangle$ is then a natural module for $\bar{R}/\bar{N}$. Since involutions of R/N centralize hyperplanes of $\bar{N}$, we get $\bar{R} = \langle \bar{t} \rangle \times \bar{S} \cdot L_2(2)$, as desired. (Compare T(2.5).) If $n = 6$, we get $|\bar{N}| = 2^4$ or 2^5. In any case $\bar{N}/\langle \bar{t} \rangle$ must contain only a simple natural (or dual), and possibly a trivial, $L_3(2)$-composition factor. By T(2.5) we get $C_{\bar{N}}(L_3(2))$ of rank 1 or 2, respectively. Since now each $\bar{L}_a \cap \bar{N}$ has rank 3, the latter case would involve $\bar{L}_a \cap \bar{L}_b = \langle \bar{t} \rangle$, and so the rest of $\bar{L}_a \cap \bar{N}$ cannot be contained in the natural submodule for $\bar{R}/\bar{N}$. This forces $\bar{L}_a \cap \bar{N} \geqslant C_{\bar{N}}(L_3(2))$, a contradiction. We must then have $\bar{R} = \langle \bar{t} \rangle \times \bar{S} \cdot L_3(2)$. Notice that the parabolic subgroup $2^2S_3 = C_{R/N}(\tilde{a})$ is also the stabilizer of the hyperplane $\bar{L}_a \cap \bar{S}$ of $\bar{S}$.

It remains now to show that $\bar{R}$ is all of $C(\bar{t})$. Unfortunately this is rather tedious. We will need to consider matrix forms as in (2.2)/(2.3). As in those arguments, we choose a basis for $\tilde{Q}$ through $\tilde{U}$, so that $\{\tilde{v}_i, \tilde{v}_{2n+1-i}\}$ is a hyperbolic pair for $\tilde{Q}$; and with respect to the product $(\cdot, \cdot)$ on $\tilde{U}$, $\{\tilde{v}_i, \tilde{v}_{n+1-i}\}$ is a hyperbolic pair. We may even choose the maximal isotropic subspace $\tilde{V}$ of $\tilde{U}$ to have basis $\{\tilde{v}_1, ..., \tilde{v}_m\}$.

Now $C_{O_{2n}}(\bar{t})$ will be given by $E \cdot Sp_n(2)$, where $E = C(\tilde{U}) \cap C(\bar{t})$ and $Sp_n(2)$ are represented by matrices of form

$$\begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix}, \quad \text{where} \quad A = A^{\perp}; \qquad \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \quad \text{where} \quad X^{-1} = X^{\perp},$$

where the inverted T represents reflection about the antidiagonal. The matrices A have zero entries on the antidiagonal. We note that $\bar{R} \leqslant C(\bar{t})$ acts on $\bar{L}^{C(\bar{t})}$ and so on the maximal isotropic subspace $\tilde{V}$ of $\tilde{U}$; and thus on the series $\tilde{Q} \geqslant \widetilde{C_Q(V)} \geqslant \tilde{U} \geqslant \tilde{V} \geqslant 1$. It follows that the matrices X in the above form can be taken in the parabolic $F \cdot L_m(2)$ of $Sp_n(2)$, where the matrices of F and $L_m(2)$ (resp.) are given by

$$\begin{pmatrix} I_m & 0 \\ B & I_m \end{pmatrix}, \quad \text{where} \quad B = B^{\perp}; \qquad \begin{pmatrix} Y & 0 \\ 0 & (Y^{-1})^{\perp} \end{pmatrix}.$$

We must consider how our $\bar{R}$ can be embedded in this situation.

Clearly our group $\bar{R}$ covers the $L_m(2)$ of $Sp_n(2)$. We note also than $C(\bar{t}) = \bar{K} \cdot \bar{R}$, where $\bar{K}$ is the kernel of the action on $\bar{L}^{C(\bar{t})}$. For in case $n = 4$, $\bar{R}$ already induces a full S_3 on these 3 conjugates; and for $n = 6$ our subgroup $L_3(2)$ acting on these 7 conjugates is maximal in A_7, and not normalized by an element of $S_7 - A_7$. Our argument above showing $\bar{L}_a \cap \bar{N} \leqslant \bar{N}$ in fact showed $\bar{N} \nleqslant C(\tilde{U})$; we see then that $\bar{N} \cap C(\tilde{U}) = \langle \bar{t} \rangle$, and $\bar{S}$ must project onto a subgroup or rank m of F. And the unique minimal nontrivial $L_m(2)$-submodule in F is given by matrices as above with B of form

$$n = 4\colon \begin{pmatrix} a & b \\ c & a \end{pmatrix}, \qquad n = 6\colon \begin{pmatrix} a & b & 0 \\ c & 0 & b \\ 0 & c & a \end{pmatrix}$$

with $a + b + c = 0$. Now we see $[\bar{K} \cap E, \bar{R}] \leqslant \bar{R} \cap E = \langle \bar{t} \rangle$; indeed, since $\bar{S} \cdot L_m(2) = \bar{R}'$ we get $\bar{K} \cap E \leqslant C(\bar{R})$. Now we may compute in both cases that $C_E(\bar{R}) = \langle \bar{t} \rangle$, so that in fact $\bar{K} \cap E = \langle \bar{t} \rangle$. For $n = 6$ we see $Z(2^6 \cdot L_3(2)) = 1$ so that $\bar{K} \cap C(\bar{t}) = \langle \bar{t} \rangle$ and $C(\bar{t}) = \bar{R}$. For $n = 4$, $Z(2^3 \cdot L_2(2))$ is given by the matrix B of form

$$\begin{pmatrix} I_2 & 0 \\ I_2 & I_2 \end{pmatrix}.$$

Suppose such an element $\bar{u}$ falls into $C(\bar{t})$ (hence, into $\bar{K}$). Then we see $C(\bar{t})/\langle \bar{t} \rangle$ covers the full parabolic $2^3 L_2(2)$ of $Sp_4(2)$. We claim in fact a subgroup J covering the 2^3 of this parabolic may be taken inside our complement $Sp_4(2)$. For $C_E(J)$ is defined by matrices B of the form

$$\begin{pmatrix} aI_2 & 0 \\ bI_2 & aI_2 \end{pmatrix}.$$

So the E-part of elements of J must have this form. Since u actually commutes with $\bar{R}'$ and $C_E(\bar{R}') = \langle \bar{t} \rangle$, we get $b = 0$ and may assume $a = 0$ on replacing by $\bar{u}\bar{t}$ if necessary. Now the value of $a = 0$ or 1 determines whether involutions of J have type a_4 or c_4 on $\tilde{Q}$; since they are all conjugate by $L_2(2)$ and form a 4-group, they cannot all have $a = 1$; so they have $a = 0$. Furthermore, not all three can have $b = 1$; for some $\bar{v}$ with $b = 0$ we see $\overline{uv}$ is a transvection of $Sp_4(2)$; by conjugacy under $L_2(2)$ of the three elements of this type in J we get $b = 0$ for all elements of J; so we have $J \leqslant Sp_4(2)$.

Suppose in particular we have an element $\bar{d} = \overline{uv}$ as above which is a transvection in $Sp_4(2)$; it must then be of type a_2 on $\tilde{Q}$, and we argue to eliminate this case. Now $\bar{d}^{\bar{M}}$ must be a class of 3- or $\{3, 4\}^+$-transposition in $\bar{M} \leqslant O_8^-(2)$. Notice we could not have $\bar{d} \in Z^*(\bar{M})$ with $\bar{d}$ inverting $O(\bar{M})$: for by (1.7), $O(\bar{M})$ must be fixed-point-free on $\tilde{Q}$, so that $[\tilde{Q}, \bar{d}]$ would have rank $n = 4$. Thus we have $E(\langle \bar{d}^{\bar{M}} \rangle) = E(\bar{M})$, and we may use the fundamental work of Fischer [32] and Timmesfeld [28] to determine $E(\bar{M})/O(\bar{M})$. Suppose first we have a class of nondegenerated $\{3, 4\}^+$-transpositions. Then $\bar{d}^{\bar{M}}$ is a uniquely determined class of 2-central involutions in a suitable Chevalley or twisted group over $GF(2)$. Since $|O(\overline{M})| \leqslant 3$ we get $|C(O(\overline{M}))| \leqslant 2 \, |O(\overline{M})|$; and now since $\bar{L}$ has rank 3, we see that $\bar{L} \cap E(\overline{M})$ must meet the 2-central class of $\bar{d}$, contradicting (1.9). So we suppose now that $\bar{d}$ determines a class of 3-transpositions. In the case of a symplectic or unitary group over $GF(2)$ (except possibly $Sp_4(2) = S_6 = O_4^-(3)$) the normality of $\bar{L} \cap E(\overline{M})$ in a Sylow 2-group of $E(\overline{M})$ forces this intersection to meet the class of $\bar{d}$, a contradiction as before. Now of the remaining possible 3-transposition groups, only a few could fit inside $O_8^+(2)$, and we consider them individually. The condition $\bar{L}$ weakly closed of rank 3 eliminates the possibilities: $\bar{d}$ a transvection acting on $\Omega_6^-(2) = U_4(2)$, $\bar{d}$ a transposition of S_7 or S_{10}. The case of a transposition acting on $A_8 = \Omega_6^+(2)$ or A_9 is ruled out by the representation theory, which would force the 2-central involutions in $\bar{L} \cap E(\overline{M})$ to centralize more than a 4-dimensional subspace of $\tilde{Q}$, contradicting what we already know. The case of A_6 leads via irreducibility on $\tilde{Q}$ in (1.7) to the situation $\overline{M} = S_3 \times S_6$, eliminated in (1.17) (actually, considered in (3.2)). The only possibility left is for $\bar{d}$ to be a transvection of $O_4^-(2) = S_5$; but here we use irreducibility to get $\overline{M} = S_3 \times O_4^-(2)$—and this is the case of Hypothesis (2.1), contradicting Hypothesis (2.6). We conclude finally that there is no such $\bar{d} = \overline{uv}$ in $\overline{M}$.

This completes the proof that $C(\bar{t}) = \bar{R}$ when $n = 4$. We also get $C(\bar{L}) = \bar{L}$ in both cases because we have determined this inside $C(\bar{t})$. So (2.7) holds.

We get also the exact analog of (2.3):

(2.8) *Assume $n = 6$, and $\bar{K} = \bar{L} \cap \bar{R}'$ (recall $\bar{R}' = \bar{S} \cdot L_3(2)$). Then involutions of $\bar{K}$ are of type a_4 or c_4 on $\tilde{Q}$, and those of $\bar{L} - \bar{K}$ have type a_6 or c_6. In particular $\bar{t} \notin \overline{M}'$ and $\bar{K} \cong N(\bar{L})$.*

Proof. Once again $\bar{K}$, elementary of order 2^4 and normal in $C_{\bar{R}}(\tilde{a})$, projects onto $J = O_2(C_{Sp_6(2)}(\tilde{a}))$ in our complement $Sp_6(2)$. The assertions of (2.8) follow just as in (2.3).

(2.9) *We have $n = 6$, $\overline{M} \cong \widehat{3 \cdot M_{22}} \cdot \langle \bar{t} \rangle$, and C of type J_4.*

Proof. First we assume $n = 4$, to eliminate this case. As noted under Hypothesis (2.1), we may assume by [12] that we are in the case $E(\overline{M})$ quasi-simple. Recall as usual that $|C(E(\overline{M}))| \leqslant 2 \, |O(\overline{M})|$. If we had $\bar{x} \in \bar{L}^\# \cap C(E(\overline{M}))$, then R_x/N_x contains $E(\overline{M})$, against our assumption (2.9) in view of (1.17); so $\bar{L} \cap C(E(\overline{M})) = 1$. Suppose first $\bar{t}$ is 2-central in $\overline{M}$. From O(1.10)(b), $\overline{M}$ has sectional 2-rank 3 and $|\overline{M}|_2 = 2^4$. In (1.20)(a)(b) we could only get 2-rank 3 (even with outer automorphisms) from $L_2(8)$, but this would contradict $|\overline{M}| = 2^4$. Now with $\bar{t}$ not 2-central, we have $|\bar{t}^{\overline{M}} \cap \bar{L}| = 2$ or 4 or 6. Since $N_{\bar{R}}(\bar{L}) = \overline{LN}$, this means $|N(\bar{L})/\bar{L}| = 4$ or 8 or 12, so $|\overline{M}|_2 = 2^5$ or 2^6. Suppose first $|\overline{M}|_2 = 2^6$ so that $N(\bar{L})/\bar{L}$ is of order 8, and Sylow in $L_3(2)$. Then $N(\bar{L})$ fixes a flag of $\bar{L}$; in particular by O(1.10)(b) we get $\bar{L} \cap \overline{M}' < \bar{L}$, so $|E(\overline{M})|_2 \leqslant 2^5$.

Furthermore $|\bar{t}^{\bar{M}} \cap \bar{L}| = 4$ forces $|\bar{L} \cap \bar{M}'| = 2^2$. However, action of $N(\bar{L})$ then forces $\bar{L}$ to meet at least two involution classes in $E(\bar{M})$, whereas the groups in (1.20)(a)–(c) have just one class. So we may assume $|\bar{M}|_2 = 2^5$. In particular $\bar{t} \notin E(\bar{M})$ since $\bar{t}$ is not 2-central, and all possibilities in (1.20)(a)–(c) have a single involution class; we get in fact $|E(\bar{M})|_2 \leqslant 2^3$ from (1.20). Now in case $|\bar{L} \cap E(\bar{M})| = 2^2$ we can only get $|\bar{t}^{\bar{M}} \cap \bar{L}| = 2$ (not 4, since $|\bar{M}|_2 \leqslant 2^5$). But then as above $\bar{L}$ meets two involution classes of $E(\bar{M})$, a contradiction. Finally we might get $|\bar{L} \cap E(\bar{M})| = 2$ only for $E(\bar{M})/O(\bar{M}) = L_2(9)$. In this case, the involution $\bar{x}$ of $\bar{L} \cap E(\bar{M})$ must satisfy either $\bar{L} \trianglelefteq C(\bar{x})$ or $\bar{R}_x/\bar{N}_x \cong L_2(2)$. The latter is ruled out by the work above, showing $\bar{t}$ is not 2-central. The former is not the case in $\mathrm{Aut}(L_2(9))$. This contradiction finally eliminates the case $n = 4$.

So we now assume $n = 6$. We look first at the case $\bar{t}$ is 2-central in $\bar{M}$. Here in view of (2.8) a Sylow group of $\bar{M}'$ is found in $\bar{R}' = \bar{S} \cdot L_3(2)$, so that $E(\bar{M})$ has sectional 2-rank 4 and $|E(\bar{M})|_2 = 2^6$. As in (2.4) we could only get $E(\bar{M})/O(\bar{M}) = \Omega_6^{\pm}(2)$. We rule out the case $O(\bar{M}) \neq 1$, since $\Omega_6^{\pm}(2) \ntrianglelefteq C(\bar{t})$ by (2.7). When $O(\bar{M}) = 1$, we get a contradiction since no outer automorphism of $\Omega_6^{\pm}(2)$ centralizes a subgroup $2^3 \cdot L_3(2)$.

Now we know $\bar{t}$ is not 2-central. The rest of the proof will follow closely that of (2.5). First we note that $N(\bar{L}) \cap C(\bar{t})$ already has orbits of size 3, 6, 6 on $\bar{K}^{\#}$. Since $|\bar{t}^{\bar{M}} \cap \bar{L}|$ is even and this set lies in the coset $\bar{t}\bar{K}$, we get

$$|\bar{t}^{\bar{M}} \cap \bar{L}| = 4 \text{ or } 10 \text{ or } 16.$$

We consider now involutions of $\bar{K}^{\#}$; in particular, we fix $\bar{\imath} \in \bar{K} \cap O_2(\bar{R}')$. Of the classes visible to us, at least this one must be 2-central in $\bar{M}$. Again the proof divides into two parallel cases, depending on the structure of $\bar{R}_i/\bar{N}_i$, and we show only $\bar{R}_i/\bar{N}_i \cong \Omega_4^+(2)$ may occur, by eliminating first the case of $L_2(2)$.

Thus we assume first $\bar{R}_i/\bar{N}_i \cong L_2(2)$. We establish the following results exactly as in (2.5):

(a) $Z(\bar{R}_i) = \langle \bar{t} \rangle \times \langle \bar{\imath} \rangle$ with $\bar{R}_i \leqslant \bar{R}_t$.

(b) $|C(\bar{\imath}) : \bar{R}_i| = 2$, so $|\bar{M}| = 2^8$ and $|E(\bar{M})|_2 = 2^7$.

(c) $|\bar{t}^{\bar{M}} \cap \bar{L}| = 10$ and $|N(\bar{L})/\bar{L}| = 2^3 \cdot 3 \cdot 5$.

(Note the possibilities $|\bar{t}^{\bar{M}} \cap \bar{L}| = 4$ and 16 are ruled out by (b), since $\bar{\imath}$ is 2-central.)

(d) $C(\bar{K}) = \bar{L}$.

We also obtain:

(e) $N(\bar{L})/\bar{L} \cong S_5$.

For as before we get $F(N(\bar{L})/\bar{L}) = 1$, and by order we can only get $E(N(\bar{L})/ \cong A_5$. Indeed we see $N(\bar{L})/\bar{L}$ must be S_5, acting in its 4-dimensional representation on $\bar{K}$ with 3-elements fixed-point-free. We see now in $N(\bar{L})$ that $E(\bar{M})$ has Sylow groups of order 2^7 and sectional 2-rank 4. We obtain

(f) $E(\bar{M})/O(\bar{M}) \simeq M_{22}$.

For the possibilities are given in (1.19)(f); with A_{10}, A_{11} eliminated as in (2.5). Furthermore the elementary subgroups of $U_4(3)$ which are weakly closed are in fact normalized by A_6 rather that S_5 (as we saw earlier). So only M_{22} is possible.

Now, just as in (2.5), we get $O(\bar{M}) = 1$ from the assumption that $\bar{R}_i/\bar{N}_i \simeq L_2(2)$. It is easily checked from a character table that the group $M_{22} \cdot \langle \bar{t} \rangle$ has no 12-dimensional $GF(2)$-representation, and this contradicts (1.7).

So we may now assume $R_i/N_i \simeq \Omega_4^+(2)$. As in (2.5) we may choose $\tilde{a}$, $\tilde{b}$ to be a hyperbolic pair so that $\bar{R}_i = \langle \bar{L}_a, \bar{L}_b \rangle$ and $\bar{N}_i = (\bar{L}_a \cap \bar{N}_i)(\bar{L}_b \cap \bar{N}_i)$, and obtain $\bar{N}_i$ extraspecial of width 2 and type $+$. We get $\bar{R}_i = \langle \bar{t}, \bar{\theta} \rangle \times \bar{N}_i \cdot S_3$, where $\bar{\theta}$ is a suitable 3-element of $\bar{R}_i$ inverted by $\bar{t}$. Now we may argue as above to obtain the analogous results:

(a) $Z^*(\bar{R}_i) = \langle \bar{t}, \bar{\theta} \rangle \times \langle \bar{t} \rangle$ with $\bar{R}_i \leqslant \bar{R}_t\langle \bar{\theta} \rangle$.

(b) $|\bar{t}^{C(\bar{t})}| = 6$ with $|C(\bar{t}) : \bar{R}_i| = 2$.

(c) $|\bar{t}^{\bar{M}} \cap \bar{L}| = 10$ and $|N(\bar{L})/\bar{L}| = 2^3 \cdot 3 \cdot 5$.

(d) $C(\bar{K}) = \langle \bar{\theta} \rangle \bar{L} = \langle \bar{t}, \bar{\theta} \rangle \times \bar{K}$.

(e) $N(\bar{L})/L \simeq S_5$.

(f) $E(\bar{M})/O(\bar{M}) \simeq M_{22}$.

As above, but now with $R_i/N_i = \Omega_4^+(2)$, we can only get $|O(\bar{M})| = 3$ with $\bar{M} \simeq \widehat{3M_{22}} \cdot \langle \bar{t} \rangle$. It follows from work of Janko [10] that G is of type J_4. Now (2.9) is proved. Notice that the question of uniqueness (and also existence) is again left open.

We have now considered all possibilities in (1.17) for $n = 4$. There is essentially one more case to eliminate for $n = 6$.

(2.10) *In case $n = 6$, suppose for $\bar{t} \in \bar{L}^\#$ we never get $R/N \simeq \Omega_6^\epsilon(2)$ or $L_3(2)$. Then no simple group G is possible.*

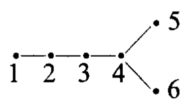
Proof. In view of (1.17), we must get some $R/N \simeq L_2(2)$ or $\Omega_4^+(2)$. By T(11.9), the case $\Omega_4^+(2)$ must arise; so we fix $\bar{t}$ with such R/N.

We obtain certain results on $\bar{R}$ as in the analogous parts of (2.5)/(2.9). We choose $[\tilde{a}, \tilde{b}]$ to be a hyperbolic pair of $\tilde{V}$ and get $\bar{R} = \langle \bar{L}_a, \bar{L}_b \rangle$ with $\bar{N} = (\bar{L}_a \cap \bar{N})(\bar{L}_b \cap \bar{N})$. We have $\langle \bar{t} \rangle = Z(\bar{R}) \cap \bar{L}$ and $\bar{N}' \leqslant \langle \bar{t} \rangle$ from T(4.7). Since $\bar{N}$ is generated by at least three $\bar{R}$-conjugates of $\bar{L}_a \cap \bar{N}$, each centralized by a Sylow 2-group of R/N, we see $\langle \bar{t} \rangle = \bar{L}_a \cap \bar{L}_b \cap \bar{N}$. It follows since $|\bar{L}_a \cap \bar{N}| = 2^3$ that $|\bar{N}| = 2^5$. Note also that $\tilde{V} = [\tilde{Q}, \bar{t}]$ has rank 4 with elementary preimage V, so $\bar{t}$ has type a_4 on $\tilde{Q}$.

By way of further explanation: Recall that in the actual groups G already determined, having $n = 6$ with some $R/N = \Omega_4^+(2)$, we found R/N acting naturally on $\bar{N}/\langle \bar{t} \rangle$. We will be able, with some effort, to establish this in the present case. However, the real groups G also have other, "larger" choices of R/N available, which is not true in the present case. The absence of such under Hypothesis (2.10) will allow us to contradict this result on the action of R/N.

For we notice first that a Sylow 2-group of R/N centralizes a plane of $\bar{N}/\langle \bar{t} \rangle$, so this cannot be a natural module of $R/N = \Omega_4^+(2)$. To contradict this—that is, to establish the analog of the situation in (2.5)/(2.9)—we must make a closer analysis of $C(\bar{t})$.

Since now t has type a_4 rather than a_6, it will be more complicated to investigate $C(\bar{t})$ inside $O_{12}(2)$. This is most conveniently done by means of Chevalley-group theory. We adopt a numbering for the root system



Our element t of type a_4 can be taken to the product of involutions corresponding to the roots

$$1\ \ 1\ \ 1\ \ 2\ \begin{smallmatrix}1\\1\end{smallmatrix} \qquad \text{and} \qquad 0\ \ 1\ \ 2\ \ 2\ \begin{smallmatrix}1\\1\end{smallmatrix}.$$

Then $O_2(C_{\Omega_{12}}(\bar{t}))$ is the unipotent radical U of the parabolic subgroup defined by the fourth fundamental root. We have $Z(U)$ elementary of rank 6, and 6, and $U/Z(U)$ elementary of rank 16. A complement to U in the full parabolic is provided by the group $L \times \Omega$; where L is the $L_4(2)$ generated by root subgroups for the first three roots, and Ω the $\Omega_4^+(2)$ defined by the fifth and sixth. A complement to U in $C_{\Omega_{12}}(t)$ is provided by $Sp \times \Omega$, where Sp is a subgroup $Sp_4(2)$ provided by $x_\alpha(1)\, x_\beta(1)$ and $x_\gamma(1)$, where

$$\alpha = 1\ \ 0\ \ 0\ \ 0\ \begin{smallmatrix}0\\0\end{smallmatrix} \qquad \beta = 0\ \ 0\ \ 1\ \ 0\ \begin{smallmatrix}0\\0\end{smallmatrix} \qquad \gamma = 0\ \ 1\ \ 0\ \ 0\ \begin{smallmatrix}0\\0\end{smallmatrix}.$$

We observe that $U/Z(U)$ is a tensor product of the natural modules for L and Ω, and hence for Sp and Ω. In particular, $U/Z(U)$ is a sum of four natural modules for L and for Ω; a decomposition for L is provided by the natural L-conjugates of

$$\left\{0\ \ 0\ \ 0\ \ 1\ \begin{smallmatrix}0\\0\end{smallmatrix},\ 0\ \ 0\ \ 0\ \ 1\ \begin{smallmatrix}1\\0\end{smallmatrix},\ 0\ \ 0\ \ 0\ \ 1\ \begin{smallmatrix}0\\1\end{smallmatrix},\ 0\ \ 0\ \ 0\ \ 1\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}.$$

A decomposition for Ω is provided by the natural Ω-conjugates of

$$\left\{0\ \ 0\ \ 0\ \ 1\ \begin{smallmatrix}0\\0\end{smallmatrix},\ 0\ \ 0\ \ 1\ \ 1\ \begin{smallmatrix}0\\0\end{smallmatrix},\ 0\ \ 1\ \ 1\ \ 1\ \begin{smallmatrix}0\\0\end{smallmatrix},\ 1\ \ 1\ \ 1\ \ 1\ \begin{smallmatrix}0\\0\end{smallmatrix}\right\}.$$

Now we are in position to do some computing inside $C_{\Omega_{12}}(\bar{t})$.

We claim first that $\bar{L} \cap Z(U) = \langle \bar{t} \rangle$. For $Z(U)$ consists of involutions of type a_2 and a_4. In view of (1.9) such a $\bar{u} \in \bar{L} \cap Z(U)$ would have type a_4 with $[\tilde{Q}, \bar{u}] = \tilde{V} = [\tilde{Q}, \bar{t}]$. But then we would get a contradiction as in T(4.7). Thus $\bar{L} \cap Z(U) = \langle \bar{t} \rangle$, and by action of R/N it follows that $\bar{N} \cap Z(U) = \langle \bar{t} \rangle$. Thus $U\bar{N}/\langle \bar{t} \rangle$ must be an R/N-submodule of $U/Z(U)$. (We see $\bar{N} \leqslant U$ since $\bar{N}$ centralizes the chain $Q \geqslant C_Q(V) \geqslant \tilde{V}$.) We consider R/N next. Since our complement Ω is trivial on $\tilde{V}$, and Sp acts naturally, we see our group R/N must cover (modulo Ω) a natural subgroup $\Omega_4^+(2)$ of Sp. Now the restriction of $U/Z(U)$ to four natural modules for Sp (and hence for a natural subgroup $\Omega_4^+(2)$) forces R/N to act naturally on $\bar{N}/\langle \bar{t} \rangle$. This contradicts our earlier assertion, and establishes (2.10).

In view of (1.17), the results (2.5)/(2.9)/(2.10) establish Theorem B. Furthermore the results (2.5)/(2.9) will establish Theorem A once we have dealt with certain special cases for $\bar{M}$ in Section 3.

3. Further Analysis of $\bar{M}$ for $n = 4$

We now take up various cases in width 4 which we left aside earlier, once we had determined $\bar{M}$. Thus we assume $n = 4$ throughout this section, and we may be able to assume certain of the other preliminary results, depending on further assumptions.

(3.1) (Reifart) *Assume $\bar{L}$ is a T.I. set and $\langle \bar{L}^{\bar{M}} \rangle \cong L_2(8)$. Then $G \cong {}^3D_4(2)$.*

Proof. We give a short independent proof, using the viewpoint of Section 3 of O. We observe first that the representation of $L_2(8) = E(\bar{M}) = O^2(\bar{M})$ on $\tilde{Q}$ is afforded by the Steinberg module. Its Brauer character is provided by the unique ordinary character of defect 0 (degree 8). For involution $\bar{t}$, we notice that $[\tilde{Q}, \bar{t}]$ has rank 4 (since the Steinberg module is the tensor product of the three algebraically conjugate natural representations). Furthermore $\bar{t}$ inverts a 3-element $\bar{\theta}$ which must centralize exactly a quaternion subgroup of order 8 in Q; it follows that $\bar{t}$ must be of type c_4 on $\tilde{Q}$. In particular for a preimage t, we have $C_Q(t)$ Abelian but not elementary. If we have $t \in z^G$, then by symmetry $C_{Q_t}(z) = Q_t \cap M$ is not elementary, and t has square roots in this group; however Sylow 2-groups of $\bar{M}$ are elementary, a contradiction. Thus $z_R \cap M \subseteq Q$. We may apply the result of Timmesfeld [29, Corollary C]; in the list only ${}^3D_4(2)$ has a large extraspecial subgroup. So (3.1) is proved.

Remark. In [27] Thomas also gives a characterization of ${}^3D_4(2^n)$ by centralizer of involution. To use that work, we would need to know a priori that $\bar{M} = E(\bar{M})$ and the extension of Q by $L_2(8)$ is split.

Our next result takes care of a pathological case in (1.17):

(3.2) *Suppose for some $\bar{t} \in \bar{L}^{\#}$ we have $R/N \simeq A_6$, and $\langle \bar{L}^{\bar{M}} \rangle = S_3 \times A_6$. Then no simple group G is possible.*

We fix $\bar{t}$ as above; we have $\bar{t} \in C(E(\bar{M})) \simeq S_3$. (We may use preliminary results through (1.16), as well as (1.18) to get $E(\bar{M})$ quasi-simple.) The idea of the proof is as follows: We will show suitable involution-cosets must contain conjugates of z. This will lead quickly to contradiction.

We begin with some facts about the representation of $\bar{M}$ on $\tilde{Q}$, as in Section 2 of O:

(3.2.1) (a) *We have $\tilde{Q} = \tilde{W} \times \tilde{W}^{\bar{t}}$, where $\tilde{W}$ is a natural $E(\bar{M})$-module. $\tilde{V} = [\tilde{Q}, \bar{t}] = \{\tilde{w}\tilde{w}^{\bar{t}} : \tilde{w} \in \tilde{W}\}$. The group $C(E(M)) \simeq S_3$ permutes $\{\tilde{W}, \tilde{W}^{\bar{t}}, \tilde{V}\}$ in the obvious way. Further $\bar{M} = \langle \bar{L}^{\bar{M}} \rangle$.*

(b) *$\bar{t}$ has type a_4 on $\tilde{Q}$; so do other involutions of $\bar{M}$.*

(c) *$\widetilde{z^G \cap Q}$ is provided by the nonzero vectors of $\tilde{W}$, $\tilde{W}^{\bar{t}}$, $\tilde{V}$.*

Proof. The assertions of (a) are easily established as in O(2.2)/(2.3); in particular, $N(\tilde{V})/\langle \bar{t} \rangle$ is at most $Sp_4(\tilde{V}) = S_6$. Type of involutions of $\bar{M}$ on $\tilde{Q}$ is handled as in O(3.1). We cannot have $\bar{M} \simeq S_3 \times S_6$, since transvections $Sp_4(2)$ would be of type a_2 on $\tilde{Q}$, against (1.18). Finally $z^G \cap Q$ is determined as in O(2.5), since $E(\bar{M})$ is transitive on $\tilde{V}^{\#}$.

(3.2.2) *Involution-cosets of $\langle \bar{t} \rangle$ and $E(\bar{M})$ contain conjugates of z.*

Proof. We consider first the canonical situation of (1.17) with $\tilde{a} \in \tilde{Q}$. We alter notation slightly so that $\tilde{a} \in \tilde{W}$—that is, $\tilde{a}$ is centralized not by $\bar{t}$ but by an $O(\bar{M})$-conjugate of $\bar{t}$. Here by O(1.10)(d) we have $Q \cap Q_a$ elementary of rank 5. The orthogonal complement $\tilde{a}^{\perp}$ to $\tilde{a}$ in $\tilde{W}$ (for the scalar product on $\tilde{W}$ defined as in T(4.5)) has rank 4. As in O(2.7), we see for a preimage $\tilde{W}$ of $\tilde{W}$ that $C_W t(a)$ has rank 4; and intersects Q_a in a subgroup of rank 2. We observe by (3.2.1)(a) that $C_W t(a)$ consists of conjugates of z. A particular involution of $C_W t(a) - Q_a$ may be taken to lie in the centralizer of a 3-element which is not fixed-point-free on $Q_a/\langle a \rangle$. Now if we reverse the roles of z and a, we see $C_W t(a)$ becomes a subgroup projecting onto a hyperlane of $\bar{L}$. The hyperlane must intersect $E(\bar{M})$. Also the particular involution we chose centralizes a 3-element fixing points of $\tilde{Q}$ and so lying in $E(\bar{M})$; this involution may be taken to give $\bar{t}$. So (3.2.2) is established.

We may consider the consequences of (3.2.2) in the light of the methods of Section 3 of O. So let $t \in z^G \cap (M - Q)$ with $\bar{t} \in C(E(\bar{M}))$. We show first that $t \sim tz$. Since $\bar{M}$ has no elementary groups of rank 5, we get $Q \cap Q_t \neq 1$ by O(1.15). Furthermore by O(1.13) (using (3.2.1)(b)) we see $C_{\bar{M}}(\bar{t})$ contains a quotient A_6, which forces $\widetilde{Q \cap Q_t} = \tilde{V}$. Thus $(Q \cap Q_t)\langle z \rangle$ is an elementary group of rank 5, consisting of conjugates of z by (3.2.1)(c); and $t \in Q_a$ for all

$\tilde{a} \in (\widetilde{Q \cap Q_t})^{\#}$. If $\tilde{a} = \tilde{w}\tilde{w}^{\bar{t}}$ for $\tilde{w} \in \tilde{W}$ chosen as in (3.3.1)(a), we obtain a decomposition:

$$\widetilde{Q \cap Q_a} = \langle \tilde{v}\tilde{v}^t : \tilde{v} \in \tilde{w}^{\perp} \rangle \times \langle \tilde{w} \rangle$$

just as in O(2.7). Thus we may write $Q_a \cap Q \cap Q_t = \langle a \rangle \times \langle b, c \rangle$ for suitable $\tilde{b}, \tilde{c} \in \tilde{a}^{\perp}$ (inside $\tilde{V}$) but $\tilde{b} \not\perp \tilde{c}$. Now we have $\langle t, z \rangle \leqslant Q_a \cap Q_d$ for all $d \in \langle b, c \rangle^{\#}$. The condition "$\langle z, b, c \rangle$ consists entirely of conjugates of z" means (in view of (3.2.1)(c)) that $\langle z, b, c \rangle$ lies in some $V_{t'} = [Q_a, t']$ for suitable $t' \in C(E(M_a/Q_a))$. This forces $\bigcap_{d \in \langle b,c \rangle^{\#}} (Q_a \cap Q_d)$ to lie in $V_{t'}$ also. Thus we get $\langle t, z \rangle \leqslant V_{t'}$, and so $t \sim z \sim tz$ in view of (3.2.1)(c), as desired.

It follows also from the above that all involutions in the coset $\bar{t}$ are conjugate; also that $\overline{Q \cap M_t} = \langle \bar{t} \rangle$, since $Q \cap M_t$ has rank 5 and $Q \cap Q_t$ has rank 4.

Now we consider $x \in z^G \cap (M - Q)$ with $\bar{x} \in E(\bar{M})$. We get $Q \cap Q_x \neq 1$ as above using O(1.15). And using (1.13), action of a 3-element of $C_M(x)$ projecting onto $O(\bar{M})$ forces $Q \cap Q_x$ to have rank 2 or 4. The case of rank 2 would force $\overline{Q_x \cap M}$ to have rank 3 and so (we may assume) contain $\bar{t}$. But then a suitable preimage $t \in z^G$ lies in $Q_x \cap M$, contradicting our previous determination $\overline{Q_t \cap M} = \langle \bar{t} \rangle$. (Recall $t \in Q_x$ iff $x \in Q_t$ by O(1.6)(a).) We then get $\widetilde{Q \cap Q_x} = [\tilde{Q}, \bar{x}]$ of rank 4, and $\overline{Q_x \cap M} = \langle \bar{x} \rangle$.

Now we may obtain our contradiction. For just as in O(2.8) we see the extension of Q by A_6 splits; it follows that we may choose x in a complement A_6, and then $C_M(x)$ is $C_Q(x)$ extended by a group $S_3 \times D_8$; where we take $t \in z^G \cap (M - Q)$ with $S_3 = \langle t, \theta \rangle$ and $\langle \bar{t}, \bar{\theta} \rangle = C(E(\bar{M}))$; and $D_8 = C_{A_6}(x)$. We see that $Q_x \cap M$ is just the subgroup $\langle x \rangle$ $(Q_x \cap Q)$ not intersecting $\langle t, z \rangle$. Thus in M_x/Q_x we get an elementary subgroup of rank 4 provided by the images of $\langle t, z \rangle \times D_8/\langle x \rangle$. This contradicts $\bar{M} \cong S_3 \times A_6$. So (3.2) is established.

Finally we come to the bizarre case corresponding to Harada's group F_5. Here we may proceed as in the argument of Reifart and Stroth in the following section: Beginning with the 2-central centralizer M, we construct $C(x)$ for an involution x not 2-central in G; then G may be identified by work of Harada [9].

(3.3) *Suppose* $\langle \bar{L}^{\bar{M}} \rangle \cong A_5 \wr Z_2$ *and* $E(M) \cong A_5 \times A_5$. *Then* G *is of type* F_5.

Notice we may use preliminary results through (1.17).

Notation. Set $\bar{E} = E(\bar{M}) = \bar{A} \times \bar{B}$, where $\bar{A}$ and $\bar{B}$ are the components. Let $\bar{A}_1 \in \mathrm{Syl}_2(\bar{A})$.

As in Timmesfeld's construction in T(5.2) corresponding to this case, we may fix $\bar{t} \in \bar{L} - \bar{E}$, so that $\bar{B} = \bar{A}^{\bar{t}}$. Then we may suppose $\bar{L} = \langle \bar{t} \rangle \times \langle \bar{x}\bar{x}^{\bar{t}} : \bar{t} \in \bar{A}_1 \rangle$. In particular in (1.17) we have $R/N \cong \Omega_4^-(2)$: Indeed as in (2.2) we see $\bar{N} = \langle \bar{t} \rangle$ and $\bar{R} = \langle \bar{t} \rangle \times \bar{R}'$, where $\bar{R}' = \langle \bar{x}\bar{x}^{\bar{t}} : \bar{x} \in \bar{A} \rangle$. In the usual way we see $\bar{t}$ has type $a_4(\tilde{Q})$, since $\tilde{V} = [\tilde{Q}, \bar{t}]$ has rank 4 and elementary preimage. In particular, $\bar{t}$

could not centralize $O(\bar{M})$ if $O(\bar{M}) \neq 1$, by structure of $C_{\Omega_8}(\bar{t})$; and $O(\langle \bar{L}^{\bar{M}} \rangle) = 1$ then forces $O(\bar{M}) = 1$, $\bar{E} = F^*(\bar{M})$. It follows that $\bar{M}$ is at most $S_5 \wr Z_2$. Indeed, our assumption on $\langle \bar{L}^{\bar{M}} \rangle$ forces $[\bar{t}, \bar{M}] \simeq \bar{R}'$, so that $\bar{M}$ is at most $\bar{E}\langle f \rangle \langle \bar{t} \rangle$, where f acts as an outer automorphism of $\bar{A}$ and $\bar{B}$, and $C(\bar{t}) = \bar{R}'\langle f \rangle \times \langle \bar{t} \rangle = \langle \bar{t} \rangle \times O_4^-(2)$. We will be able to do the necessary analysis for $\bar{E}\langle \bar{t} \rangle$, even through $\bar{M}$ might be slightly larger.

(3.3.1) $N_{\bar{E}}(\bar{L}) = (\bar{A}_1 \times \bar{A}_1^{\bar{t}})\langle \bar{\theta}\bar{\theta}^{\bar{t}} \rangle$, *where* $\bar{\theta}$ *is a* 3-*element of* $N_{\bar{A}}(\bar{A}_1)$. *In particular, we have* $| \tilde{a}^{\bar{M}} | = 75$.

Proof. Determination of $N_{\bar{E}}(\bar{L})$ is easy. Since $N(\bar{L}) = C(\tilde{a})$, it follows in any case that $| \bar{M} : C(\tilde{a})| = 75$.

We obtain further information about the representation of $\bar{M}$ on $\tilde{Q}$ by considering, in contrast to $\tilde{a}$, a vector $\tilde{x}$ of nonsingular type in the orthogonal space $\tilde{V}$.

(3.3.2) (a) $C_{\bar{E}\langle \bar{t} \rangle}(\tilde{x}) \simeq S_5$. *This quotient centralizes in* $\widetilde{C_Q(x)/\langle \tilde{x} \rangle}$ *only a group* $\langle \tilde{x}, \tilde{y} \rangle$ *where* y *is a* 4-*element.*

(b) $\tilde{Q}$ *is a sum of two modules for* $\bar{A}$ *represented as* $SL_2(4)$. *Similar remarks hold for* $\bar{B}$.

Proof. Since there are just 135 cosets in Q containing involutions, (3.3.1) forces $| \tilde{x}^{\bar{M}} | \leqslant 60$ so that $| C_{\bar{E}\langle \bar{t} \rangle}(\tilde{x})| \geqslant 120$. Furthermore this centralizer intersects $\bar{R}'$ in a subgroup S_3. We see that 2^3 and 5^2 do not divide $| C_{\bar{E}}(\tilde{x})|$; but but since this order is at least 60, 5 must divide it. Now any 5-element of $C_{\bar{E}}(\tilde{x})$ lies outside $\bar{A}$, $\bar{B}$, $\bar{R}'$ and so centralizes no 3-element of $C_{\bar{E}}(\tilde{x})$. Using Sylow's theorem, we can only get $C_n(\tilde{x})$ of order 60 and structure A_5; so $C_{\bar{E}\langle \bar{t} \rangle}(\tilde{x}) \simeq S_5$. In particular, $| \tilde{x}^{\bar{M}} | = 60$. In view of (3.4.1) involutions of Q are conjugate to x or a (hence z). In particular, structure of $C(\tilde{x})$ and $C(\tilde{a})$ shows that the former does not centralize another involution-coset in $\widetilde{C_Q(x)/\langle \tilde{x} \rangle}$, though it must fix a point of this 6-dimensional space. So (a) is proved. For (b), let $\tilde{W}$ be a minimal $\bar{A}$-submodule of $\tilde{Q}$, and so of dimension 4. If we had also $\bar{B} \leqslant N(\tilde{W})$, then we would have $\bar{B} \leqslant C(\tilde{W})$, and centralizers of involutions of type $\tilde{a}$ or $\tilde{x}$ in $\tilde{W}$ would be too large for (3.3.1) or (3.3.2)(a). So $\bar{B} \nleqslant N(\tilde{W})$. Indeed for suitable involutions $\bar{y} \in \bar{B}$ we get $Q \simeq \tilde{W} \times \tilde{W}^{\bar{y}}$, a decomposition of $\bar{A}$-submodules. To see that the representation of $\bar{A}$ on $\tilde{W}$ is not that of $\Omega_4^-(2)$, note in (3.3.1) and (3.3.2)(a) that no 3-element of $\bar{A}$ centralizes an involution of $Q - \langle z \rangle$. Now (3.4.2)(b) is proved also.

We may now establish the local information that will lead us to structure of $C(x)$. We let an asterisk denote images in the quotient $C(x)^* = C(x)/\langle x \rangle$.

(3.3.3) (a) $C_Q(x)^*$ *is extraspecial of width* 3 *and type* $+$.

(b) $C_M(x)^*$ *normalizes a maximal subgroup of* $C_Q(x)^*$: *a central product of*

$\langle f^* \rangle$ and F^*, where f^* is a 4-element, and F^* is extraspecial of width 2 and type $+$.

Proof. Since x is an involution, $C_Q(x) = \langle x \rangle \times D$, where D is extraspecial of width 3 and type $+$, giving (a). Now since x and xz are not conjugate to z, we have $C(x)^* \cap C(z^*) = C_M(x)^*$. By (3.3.2)(a) this group is just $C_Q(x)^*$ extended by a group S_5 (and possibly an extra involution $\bar{a} \notin \bar{E}\langle \bar{t} \rangle$ as mentioned earlier). Now the A_5-module $C_Q(x)^*/\langle z^* \rangle$ of dimension 6 has just one nontrivial composition factor, of type $SL_2(4)$—since a 3-element θ of $C_M(x)$ centralizes a subgroup of rank 4 of $\tilde{Q}$, and so of rank 2 in $\widetilde{C_Q(x)}/\langle \tilde{x} \rangle$. In particular $F^* = [C_Q(x)^*, \theta^*]$ is extraspecial of width 2 and type $+$. Now if we let f be the 4-element y of (3.3.2)(a), then $C_M(x)^*$ normalizes the central product of $\langle f^* \rangle$ and F^*. (Compare T(2.3).) So (3.3.3) is proved.

(3.3.4) $C(x) — \widehat{2 \cdot Hs} \cdot 2$.

Proof. First we consider a four group $\langle z, a \rangle^*$, where $a \in z^G \cap C_Q(x)$. Since $O(C_{M_b}(x)^*) = 1$ for all $b \in \langle z, a \rangle^\#$ the usual generation lemma shows $O(C(x)^*) = 1$. Since our reasoning for z applies equally well to any $z' \in z^G$ with $x \in Q_{0'}$, we see each such z' brings into $C(x)$ a different group $C_{M_{z'}}(x)$ of the structure $\langle x \rangle \times 2^{1+6} \cdot S_5$. We get $z'^* \notin Z(C(x)^*)$. As in O(1.8), conjugates of z^* generate the unique minimal normal subgroup of $C(x)^*$, and it follows that $Z^*(C(x)^*) = 1$.

Now we claim that $O^2(C(x)^*)$ contains at least $\langle f^* \rangle F^*$ extended by A_5. This might only fail if the A_5-module $\langle f^* \rangle F^*/\langle z^* \rangle$ were decomposable. Since the action of A_5 on $C_Q(x)^*/\langle z^* \rangle$ is self-dual (with $A_5 \leqslant \Omega_6^+(2)$) this would force the A_5 to centralize a subgroup of rank 2 of $C_Q(x)^*/\langle z \rangle^*$, contradicting (3.4.2)(a).

We let u be an involution such that $\langle u^* \rangle$ is a complement to $\langle f^* \rangle F^*$ in $C_Q(x)^*$. We suppose $u^* \in O^2(C(x)^*)$. Then $F^*(O^2(C(x)^*)) = C_Q(x)^*$ is extraspecial of width 3 and type $+$. In view of (1.17) we may apply Theorem A for width 3 to conclude $O^2(C(x^*)) \cong U_5(2), L_5(2), M_{24}$, or He. In particular we would get $C_M(x)^*/C_Q(x)^* \cong U_3(2)$ or $L_3(2)$, a contradiction.

So $u^* \notin O^2(C(x)^*)$, and $F^*(O^2(C(x)^*) = \langle f^* \rangle F^*$ is symplectic but not extraspecial. It follows from Aschbacher's classification theorem [1] that $O^2(C(x)^*) \cong HS$; and with u^* we get $C(x)^* \cong \mathrm{Aut}(HS)$. Furthermore we see x has square roots in $C_M(x)$ such as tf. It follows that $C(x) \cong \widehat{2 \cdot HS} \cdot 2$, proving (3.3.4).

Now by work of Harada [9] we get G of type F_5, proving (3.3).

The results (3.1)–(3.3) fill in the gaps left in (1.11)/(1.17)/(1.18). Thus Theorem A is now fully proved.

4. Proof of Theorem C

We now assume the hypotheses of Theorem C. We have all the results of Section 1; in particular we may assume G is simple. In both cases the proof will proceed by going from M to structure of $C(x)$ where x is a non-2-central involution. We describe first the representation of $\bar{M}$ on $\tilde{Q}$, including 2-central involutions $a \in z^+ \cap Q - \{z\}$.

It is convenient to begin by considering the larger case $n = 12$. The 24-dimensional irreducible $\tilde{Q}$ for $\bar{M} \cong$ Co.1 must be given by $\Lambda/2\Lambda$, where Λ is the Leech lattice, as shown by Griess in [8]. For Λ, we will use the notation of Conway's article [5]. The weakly closed subgroup $\bar{L}$ of rank 11 must be given by the images of the transformation ϵ_C for C a set in the Golay code $\mathscr{C}$; and then $N(\bar{L}) = C(\tilde{a})$ is the image of the group N. In particular $N(\bar{L})$ has the structure $\bar{L} \cdot M_{24}$ (split). This is the group fixing a "coordinate frame" in Λ; of course all 48 vectors of a coordinate frame are identified in $\Lambda/2\Lambda$, since $8v_i - 8v_j \in 2\Lambda$. Thus $\tilde{a}$ is the image of a lattice vector of type Λ_4.

We may consider instead a involution $x \in Q - z^G$, such that $\tilde{x}$ is the image of a lattice vector $4v_0 - 4v_\infty$ of type Λ_2. Then $C_M(x)/\langle x\rangle$ is an extraspecial group of width 11 and type $+$, extended by a subgroup Co.2 of Co.1.

Now in case $n = 11$, the 22-dimensional irreducible module for $\bar{M} \cong$ Co.2 is provided by the section $\tilde{x}^\perp/\langle\tilde{x}\rangle$ of the orthogonal $\tilde{Q}$ just described. The weakly closed subgroup $\bar{L}$ of rank 10 is again provided by images of transformations ϵ_8, where now $\mathscr{C}$ is a set in the Golay code orthogonal to the set $\{0, \cong\}$. Then if $\bar{L} = L_a$ for $a \in z^G \cap Q - \{z\}$ we have $N(\bar{L}) = C(\tilde{a})$ of structure $\bar{L} \cdot M_{22}$ (split). Indeed if we take $\tilde{a}$ to be the image of $8v_0$, we see that $8v_0 = (4v_0 + 4v_\infty) + (4v_0 - 4v_\infty)$ and so $N(\bar{L})$ is the subgroup .422 of Conway.

If instead we take $\tilde{x}$ to be the image of $4v_0 - 4v_1$, we notice $4v_0 - 4v_1 = (4v_0 - 4v_\infty) + (4v_\infty - 4v_1)$; we see then that $C(\tilde{x})/\langle\tilde{x}\rangle$ is an extraspecial group of width 10 and type $+$, extended by a subgroup .222 of structure $U_6(2) \cdot 2$ inside $\bar{M} =$ Co.2.

Now in both cases, we use an asterisk to denote images in the quotient $C(x)^* = C(x)/\langle x\rangle$. We obtain a number of easy results just as in (3.4.4):

$$C(x)^* \cap C(z^*) = C_M(x)^*.$$
$$C_Q(x)^* = F^*(C_M(x)^*) \text{ is extraspecial of width 10 or 11 (respectively).}$$
$$Z^*(C_M(x)^*) = 1.$$

We consider the case $n = 11$. Here $C_M(x)^*$ has the configuration $2^{1+20} \cdot U_6(2) \cdot 2$. By the result of Reifart [17] or the author [24], we see that $O^2(C(x)^*) = {}^2E_6(2)$ and $C(x)^* \cong \mathrm{Aut}({}^2E_6(2))$. Since we can find square roots of x in $C_M(x)$, we get $C(x) \cong \widehat{2 \cdot {}^2E_6(2)} \cdot 2$. Now it follows from the result of Stroth [29] that G is of type F_2.

TABLE I

Finite Simple Groups G with Large Extraspecial Subgroup Q

G	Width of Q	Type	$\bar{M} = M/Q$
$L_{n+2}(2)$	n	+	$L_n(2)$
$U_{n+2}(2)$	$n \geqslant 2$	+	$U_n(2)$
$\Omega^+_{n+4}(2)$	n even $\geqslant 2$	+	$S_3 \times \Omega_n^+(2)$
$\Omega^-_{n+4}(2)$	n even $\geqslant 2$	+	$S_3 \times \Omega_n^-(2)$
$A_6 = L_2(9)$	1	+	trivial
M_{11}	1	−	$S_3 = O_2^-(2)$
$HaJ = J_2, J_3$	2	−	$A_5 = \Omega_4^-(2)$
M_{12}, A_9	2	+	S_3
$G_2(3)$	2	+	$S_3 \times 3$
$U_4(3), L_4(3)$	2	+	$S_3 \times S_3 = \Omega_4^+(2)$
M_{24}, He	3	+	$L_3(2)$
Sz	3	−	$\Omega_6^-(2)$
${}^3D_4(2)$	4	+	$L_2(8)$
$\Omega_8^+(3)$	4	+	solvable $3^4 \cdot 2^3$
Co. 2	4	+	$Sp_6(2)$
Co. 1	4	+	$\Omega_8^+(2)$
F_5	4	+	$A_5 \wr Z_2$
F_3	4	+	A_9
[a] F'_{24}	6	+	$\widehat{3U_4(3)} \cdot 2$
[b] J_4	6	+	$\widehat{3M_{22}} \cdot 2$
${}^2E_6(2)$	10	+	$U_6(2)$
$E_6(2)$	10	+	$L_6(2)$
[a] F_2	11	+	Co. 2
[b] F_1	12	+	Co. 1
$E_7(2)$	16	+	$\Omega^+_{12}(2)$
$E_8(2)$	28	+	$E_7(2)$

[a] Uniqueness problem remains.
[b] Existence and uniqueness problems remain.

We turn to the case $n = 12$. Here $C_M(x)^*$ has the configuration $2^{1+22} \cdot \text{Co.2}$. By the result just proved we must have $C(x)^* \simeq F_2$, and similarly $C(x) \simeq \widehat{2 \cdot F_2}$. But now with both centralizers $M = C(z)$ and $C(x) \simeq \widehat{2 \cdot F_2}$, we have the assumptions of Griess [8] for a group of type F_1. Now Theorem C is proved. We have not addressed the uniqueness problems for F_1 and F_2 (or existence for F_1). The uniqueness proof of Leon and Sims [11] requires further conditions which should not be unduly difficult to establish.

Remarks. This paper, with the earlier work [17, 18, 23, 24] completes the determination of possibilities in the exceptional cases of the main theorem of Timmesfeld [30]. Thus simple groups with large extraspecial subgroups are determined. For the convenience of the reader, we include these simple groups in Table I.

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