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A groupoid approach to discrete inverse semigroup algebras

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Abstract

Let K be a commutative ring with unit and S an inverse semigroup. We show that the semigroup algebra KS can be described as a convolution algebra of functions on the universal étale groupoid associated to S by Paterson. This result is a simultaneous generalization of the author's earlier work on finite inverse semigroups and Paterson's theorem for the universal C^*-algebra. It provides a convenient topological framework for understanding the structure of KS, including the center and when it has a unit. In this theory, the role of Gelfand duality is replaced by Stone duality.

Using this approach we construct the finite dimensional irreducible representations of an inverse semigroup over an arbitrary field as induced representations from associated groups, generalizing the case of an inverse semigroup with finitely many idempotents. More generally, we describe the irreducible representations of an inverse semigroup S that can be induced from associated groups as precisely those satisfying a certain "finiteness condition." This "finiteness condition" is satisfied, for instance, by all representations of an inverse semigroup whose image contains a primitive idempotent.

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Contents

1. Introduction

It is by now well established that there is a close relationship between inverse semigroup C^*-algebras and étale groupoid C^*-algebras [23,8,25,24,19,9,26,32]. More precisely, Paterson assigned to each inverse semigroup S an étale (in fact, ample) groupoid $\mathscr{G}(S)$, called its universal groupoid, and showed that the universal and reduced C^*-algebras of S and $\mathscr{G}(S)$ coincide [23]. On the other hand, if $\mathscr{G}$ is a discrete groupoid and K is a unital commutative ring, then there is an obvious way to define a groupoid algebra $K\mathscr{G}$. The author showed that if S is an inverse semigroup with finitely many idempotents, then $KS \cong K\mathscr{G}_S$ for the so-called underlying groupoid $\mathscr{G}_S$ of S [30,31]; this latter groupoid coincides with the universal groupoid $\mathscr{G}(S)$ when S has finitely many idempotents. It therefore seems natural to conjecture that, for any inverse semigroup S, one has that $KS \cong K\mathscr{G}(S)$ for an appropriate definition of $K\mathscr{G}(S)$. This is what we achieve in this paper.

More precisely, given a unital commutative ring K, equipped with the discrete topology, and an ample groupoid $\mathscr{G}$ (for example $\mathscr{G}(S)$) we define a convolution algebra $K\mathscr{G}$. Our main result is to establish an isomorphism $KS \cong K\mathscr{G}(S)$, generalizing our earlier work on finite inverse semigroups [30,31]. This result has numerous applications. For instance, Paterson's theorem on the universal C^*-algebra [23] is obtained as a consequence of the case $K = \mathbb{C}$ via the Stone–Weierstrass theorem. The isomorphism of KS with $K\mathscr{G}(S)$ allows for a description of the center of an inverse semigroup algebra analogous to the group case. From this, we derive a topological proof of a result of Crabb and Munn describing the center of the algebra of a free inverse monoid [7]. Our principal application is the description of the finite dimensional irreducible representations of an inverse semigroup over a field as induced representations from associated groups. The methods and results are reminiscent of the theory developed by Munn and Ponizovsky for finite semigroups [5,22], as interpreted through [10]; see also [18]. An alternative approach to the construction of the finite dimensional irreducible representations of an arbitrary inverse semigroup over a field can be found in Munn [20].

2. Preliminaries

In this section, we summarize the background that we shall need throughout the paper.

2.1. Groupoids and inverse semigroups

Groupoids and inverse semigroups are two closely related models of partial symmetry [14]. By a *groupoid* $\mathscr{G}$, we mean a small category in which every arrow is an isomorphism. Objects will be identified with the corresponding units and the space of units will be denoted $\mathscr{G}^0$. Then, for $g \in \mathscr{G}$, the domain and range maps are given by $d(g) = g^{-1}g$ and $r(g) = gg^{-1}$, respectively. A *topological groupoid* is a groupoid whose underlying set is equipped with a topology making the product and inversion continuous, where the set of composable pairs is given the induced topology from the product topology. A subgroupoid of a groupoid is said to be *full* if it contains all the units [16].

An *inverse semigroup* is a semigroup S so that, for all $s \in S$, there exists a unique element $s^* \in S$ so that $ss^*s = s$ and $s^*ss^* = s^*$. The element s^* is termed the *inverse* of s. The set $E(S)$ of idempotents of S is a commutative subsemigroup; it is ordered by $e \leqslant f$ if and only if $ef = e$. With this ordering $E(S)$ is a meet semilattice with the meet given by the product. Hence, it is often referred to as the *semilattice of idempotents* of S. The order on $E(S)$ extends to S as the so-called *natural partial order* by putting $s \leqslant t$ if $s = et$ for some idempotent e (or equivalently $s = tf$ for some idempotent f). This is equivalent to $s = ts^*s$ or $s = ss^*t$. If $e \in E(S)$, then the set $G_e = \{s \in S \mid ss^* = e = s^*s\}$ is a group, called the *maximal subgroup* of S at e. Idempotents e, f are said to be $\mathscr{D}$-equivalent, written $e \mathrel{\mathscr{D}} f$, if there exists $s \in S$ so that $e = s^*s$ and $f = ss^*$; this is the analogue of von Neumann–Murray equivalence. An idempotent e of an inverse semigroup (and hence a semilattice) is called *primitive* if it is minimal amongst its non-zero idempotents. See the book of Lawson [14] for details.

To every inverse semigroup S, one can assign its underlying groupoid $\mathscr{G}_S$. The underlying set of $\mathscr{G}_S$ is the inverse semigroup S itself. One has $\mathscr{G}_S^0 = E(S)$ and $d(s) = s^*s$, $r(s) = ss^*$. The composition in $\mathscr{G}_S$ is just the restriction of the multiplication of S to composable pairs. The inversion operation in the groupoid coincides with the inversion of the inverse semigroup.

There is a natural action by partial bijections of S on its set of idempotents, called the *Munn representation* [14]. In order to define it, we need some further notions from the theory of posets. Let P be a poset. Then a *downset* in P is a subset X such that $y \leqslant x$ and $x \in X$ implies $y \in X$. The dual notion is called an *upset*. In any poset P, it will be convenient to use, for $p \in P$, the notation

$$p^{\uparrow} = \{q \in P \mid q \geqslant p\},$$
$$p^{\downarrow} = \{q \in P \mid q \leqslant p\}$$

for principal upsets and downsets. We will use later the notation $p^{\downharpoonleft}$ for the strict downset $\{q \in P \mid q < p\}$.

If E is a semilattice, the *Munn semigroup* T_E is the inverse semigroup of all order isomorphisms between principal downsets of E [14]. The *Munn representation* of an inverse semigroup S on its set of idempotent $E(S)$ is the homomorphism $\mu : S \to T_E$ given by sending $s \in S$ to the bijection $\mu(s) : (s^*s)^{\downarrow} \to (ss^*)^{\downarrow}$ defined by $e \mapsto ses^*$ [14].

Another important notion from inverse semigroup theory is the maximal group image homomorphism. If S is an inverse semigroup, then one can define a congruence on S by setting two elements to be equivalent if they have a common lower bound in the natural partial order. The quotient is a group G_S, called the *maximal group image* of S, and the corresponding quotient map $\sigma : S \to G_S$ is called the *maximal group image homomorphism*. It is the universal map from S to a group. See [14] for details.

2.2. Stone duality

In this paper a compact space is a *Hausdorff* space for which every open cover admits a finite subcover. However, we do not assume that a locally compact space is Hausdorff unless explicitly stated. The characteristic function of a set U will be denoted χ_U. In this paper, the role of Gelfand duality from functional analysis is played by Stone duality and so we introduce the necessary terminology and results here.

Definition 2.1 *(Generalized boolean algebra).* A *generalized boolean algebra* is a poset P admitting finite (including empty) joins and non-empty finite meets so that the meet distributes over the join and if $a \leqslant b$, then there exists $x \in P$ so that $a \wedge x = 0$ and $a \vee x = b$ where 0 is the bottom of P. Then, given $a, b \in P$ one can define the relative complement $a \setminus b$ of b in a to be the unique element $x \in P$ so that $(a \wedge b) \vee x = a$ and $a \wedge b \wedge x = 0$. Morphisms of generalized boolean algebras are expected to preserve finite joins and finite non-empty meets. A generalized boolean algebra with a maximum (i.e., empty meet) is called a *boolean algebra*.

It is well known that a generalized boolean algebra is the same thing as a boolean ring. A *boolean ring* is a ring R with idempotent multiplication. Such rings are automatically commutative of characteristic 2 [11]. The multiplicative semigroup of R is then a semilattice, which is in fact a generalized boolean algebra. The join is given by $a \vee b = a + b - ab$ and the relative complement by $a \setminus b = a - ab$. Conversely, if B is a generalized boolean algebra, we can place a boolean ring structure on it by using the meet as multiplication and the symmetric difference $a + b = (a \setminus b) \vee (b \setminus a)$ as the addition. Boolean algebras correspond in this way to unital boolean rings. For example, $\{0, 1\}$ is a boolean algebra with respect to its usual ordering. The corresponding boolean ring is the two-element field $\mathbb{F}_2$. See [11] for details.

Definition 2.2 *(Semi-boolean algebra).* A poset P is called a *semi-boolean algebra* if each principal downset $p^{\downarrow}$ with $p \in P$ is a boolean algebra.

It is immediate that every generalized boolean algebra is a semi-boolean algebra.

A key foundational tool in our paper is Stone duality [33,13]. We state the theorem here for the convenience of the reader since most textbooks [13,3] merely handle the case of boolean algebras and compact totally disconnected spaces. The book [11] covers the case of generalized boolean algebras in the exercises. Perhaps the best reference to the full result is Stone's original paper [33]. In order to state the Stone duality theorem, we need to introduce some topological notions, as well as some notation.

Definition 2.3 *(Locally compact boolean space).* A Hausdorff space X is called a *locally compact boolean space* if it has a basis of compact open sets [11].

It is easy to see that the set $B(X)$ of compact open subspaces of any Hausdorff space X is a generalized boolean algebra (and is a boolean algebra if and only if X is compact). Conversely, if A is a generalized boolean algebra and $\operatorname{Spec}(A)$ is the set of non-zero morphisms of generalized boolean algebras $A \to \{0,1\}$ endowed with the subspace topology from $\{0,1\}^A$, then $\operatorname{Spec}(A)$ is a locally compact boolean space; it is compact precisely when A is a boolean algebra. Recall that a continuous map between topological spaces is *proper* if the preimage of each compact set is compact. In particular, if $f : X \to Y$ is a proper continuous map between Hausdorff spaces, then $f^{-1} : B(Y) \to B(X)$ is a morphism of generalized boolean algebras. On the other hand, if $\varphi : A \to A'$ is a morphism of generalized boolean algebras, then $\widehat{\varphi} : \operatorname{Spec}(A') \to \operatorname{Spec}(A)$ given by $\psi \mapsto \psi\varphi$ is a proper continuous map. Thus B and Spec give contravariant functors between the categories of locally compact boolean spaces (with proper continuous maps) and generalized boolean algebras.

Theorem 2.4 *(Stone duality). The functors B and* Spec *give a contravariant equivalence between the category of locally compact boolean spaces with proper continuous maps and the category of generalized boolean algebras with morphisms of generalized boolean algebras. The natural homeomorphism $X \to \operatorname{Spec}(B(X))$ is given by $x \mapsto \varphi_x$ where $\varphi_x(U) = \chi_U(x)$. The natural isomorphism $A \to B(\operatorname{Spec}(A))$ sends $a \in A$ to the compact open set $D(a) = \{\varphi \mid \varphi(a) = 1\}$.*

2.3. The spectrum of a semilattice

Let E be a semilattice. We denote by $\widehat{E}$ the space of non-zero semilattice homomorphisms $\varphi : E \to \{0,1\}$ topologized as a subspace of $\{0,1\}^E$. Then because the inclusion of E into the semigroup algebra $\mathbb{F}_2E$ is the universal map of E into the multiplicative semigroup of a boolean ring, a homomorphism $\varphi : E \to \{0,1\}$ extends uniquely to a non-zero boolean ring homomorphism $\mathbb{F}_2E \to \mathbb{F}_2$ and hence $\widehat{E} \cong \operatorname{Spec}(\mathbb{F}_2E)$. In particular $\widehat{E}$ is a locally compact boolean space with $B(\widehat{E}) \cong \mathbb{F}_2E$ (viewed as a generalized boolean algebra). For $e \in E$, define $D(e) = \{\varphi \in \widehat{E} \mid \varphi(e) = 1\}$; this is the compact open set corresponding to e under the above isomorphism. The semilattice of subsets of the form $D(e)$ generates $B(\widehat{E})$ as a generalized boolean algebra because E generates $\mathbb{F}_2E$ as a boolean ring. In fact, the map $e \mapsto D(e)$ is the universal semilattice homomorphism of E into a generalized boolean algebra corresponding to the universal property of the inclusion $E \to \mathbb{F}_2E$. Elements of $\widehat{E}$ are often referred to as *characters*. There is an alternative approach to characters via filters.

2.3.1. Filters

A *filter* $\mathscr{F}$ on a semilattice E is a non-empty subset that is closed under pairwise meets and is an upset in the ordering. For example, if $\varphi : E \to \{0,1\}$ is a character, then $\varphi^{-1}(1)$ is a filter. Conversely, if $\mathscr{F}$ is a filter, then its characteristic function $\chi_{\mathscr{F}}$ is a non-zero homomorphism. A filter is called *principal* if it has a minimum element, i.e., is of the form $e^{\uparrow}$. A character of the form $\chi_{e^{\uparrow}}$, with $e \in E$, is called a *principal character*. Notice that every filter on a finite semilattice E is principal and in this case $\widehat{E}$ is homeomorphic to E with the discrete topology.

In general, the set of principal characters is dense in $\widehat{E}$ since $\chi_{e^{\uparrow}} \in D(e)$ and the generalized boolean algebra generated by the open subsets of the form $D(e)$ is a basis for the topology on $\widehat{E}$. Thus if $\widehat{E}$ is discrete, then necessarily each filter on E is principal and $\widehat{E}$ is in bijection with E. However, the converse is false, as we shall see in a moment. The following, assumedly well known, proposition captures when every filter is principal, when the topology on $\widehat{E}$ is discrete and when the principal characters are discrete in $\widehat{E}$.

Proposition 2.5. *Let E be a semilattice. Then:*

(1) *Each filter on E is principal if and only if E satisfies the descending chain condition;*
(2) *The topology on $\widehat{E}$ is discrete if and only if each principal downset of E is finite;*
(3) *The set of principal characters is discrete in $\widehat{E}$ if and only if, for all $e \in E$, the downset $e^{\downharpoonleft} = \{f \in E \mid f < e\}$ is finitely generated.*

Proof. Suppose first E satisfies the descending chain condition and let $\mathscr{F}$ be a filter. Then each element $e \in \mathscr{F}$ is above a minimal element of $\mathscr{F}$, else we could construct an infinite strictly descending chain. But if $e, f \in \mathscr{F}$ are minimal, then $ef \in \mathscr{F}$ implies that $e = ef = f$. Thus $\mathscr{F}$ is principal. Conversely, suppose each filter in E is principal and that $e_1 \geqslant e_2 \geqslant \cdots$ is a descending chain. Let $\mathscr{F} = \bigcup_{i=1}^{\infty} e_i^{\uparrow}$. Then $\mathscr{F}$ is a filter. By assumption, we have $\mathscr{F} = e^{\uparrow}$ for some $e \in E$. Because $e \in \mathscr{F}$, we must have $e \geqslant e_i$ for some i. On the other hand $e_j \geqslant e$ for all j since $\mathscr{F} = e^{\uparrow}$. If follows that $e = e_i = e_{i+1} = \cdots$ and so E satisfies the descending chain condition.

To prove (2), suppose first that $\widehat{E}$ is discrete. Then since $D(e)$ is compact open, it must be finite. Moreover, each filter on E is principal and so $D(e) = \{\chi_{f^{\uparrow}} \mid f \leqslant e\}$. It follows that $e^{\downarrow}$ is finite. Conversely, if each principal downset is finite, then every filter on E is principal by (1). Suppose $e^{\downarrow} \setminus \{e\} = \{f_1, \ldots, f_n\}$. Then $\{\chi_{e^{\uparrow}}\} = D(e) \setminus (D(f_1) \cup \cdots \cup D(f_n))$ is open. Thus $\widehat{E}$ is discrete.

For (3), suppose first $e^{\downharpoonleft}$ is generated by $f_1, \ldots, f_n$. Then $\chi_{e^{\uparrow}}$ is the only principal character in the open set $D(e) \setminus (D(f_1) \cup \cdots \cup D(f_n))$. This establishes sufficiency of the given condition for discreteness. Conversely, suppose that the principal characters form a discrete set. Then there is a basic neighborhood of $\chi_{e^{\uparrow}}$ of the form $U = D(e') \setminus (D(f_1) \cup \cdots \cup D(f_n))$ for certain $e', f_1, \ldots, f_n \in E$ (cf. [23]) containing no other principal character. Then $e \leqslant e'$ and $e \nleqslant f_i$, for $i = 1, \ldots, n$. In particular, $ef_1, \ldots, ef_n \in e^{\downharpoonleft}$. We claim they generate it. Indeed, if $f < e \leqslant e'$, then since $\chi_{f^{\uparrow}} \notin U$, we must have $f \leqslant f_i$ for some $i = 1, \ldots, n$ and hence $f = ef \leqslant ef_i$, for some $i = 1, \ldots, n$. This completes the proof. $\square$

For instance, consider the semilattice E with underlying set $\mathbb{N} \cup \{\infty\}$ and with order given by $0 < i < \infty$ for all $i \geqslant 1$ and all other elements are incomparable. Then E satisfies the descending chain condition but $\infty^{\downarrow}$ is infinite. If we identify $\widehat{E}$ with E as sets, then the topology is that of ∞ being a one-point compactification of the discrete space $\mathbb{N}$. The condition in (3) is called *pseudofiniteness* in [21].

Definition 2.6 *(Ultrafilter).* A filter $\mathscr{F}$ on a semilattice E is called an *ultrafilter* if it is a maximal proper filter.

The connection between ultrafilters and morphisms of generalized boolean algebras is well known [11].

Proposition 2.7. *Let E be a semilattice with zero.*

(1) *A principal filter $e^{\uparrow}$ is an ultrafilter if and only if e is primitive.*
(2) *Moreover, if E is a generalized boolean algebra, then a filter $\mathscr{F}$ on E is an ultrafilter if and only if $\chi_{\mathscr{F}} : E \to \{0, 1\}$ is a morphism of generalized boolean algebras.*

Proof. Evidently, if e is not a minimal non-zero idempotent, then $e^{\uparrow}$ is not an ultrafilter since it is contained in some proper principal filter. Suppose that e is primitive and $f \notin e^{\uparrow}$. Then $ef < e$ and so $ef = 0$. Thus no proper filter contains $e^{\uparrow}$. This proves (1).

To prove (2), suppose first that $\mathscr{F}$ is an ultrafilter. We must verify that $e_1 \vee e_2 \in \mathscr{F}$ implies $e_i \in \mathscr{F}$ for some $i = 1, 2$. Suppose neither belong to $\mathscr{F}$. For $i = 1, 2$, put $\mathscr{F}_i = \{e \in E \mid \exists f \in \mathscr{F}$ such that $e \geqslant e_i f\}$. Then $\mathscr{F}_i$, for $i = 1, 2$, are filters properly containing $\mathscr{F}$. Thus $0 \in \mathscr{F}_1 \cap \mathscr{F}_2$ and so we can find $f_1, f_2 \in F$ with $e_1 f_1 = 0 = e_2 f_2$. Then $f_1 f_2 \in \mathscr{F}$ and $0 = e_1 f_1 f_2 \vee e_2 f_1 f_2 = (e_1 \vee e_2) f_1 f_2 \in \mathscr{F}$, a contradiction. Thus $e_i \in \mathscr{F}_i$ some $i = 1, 2$.

Conversely, suppose that $\chi_{\mathscr{F}}$ is a morphism of generalized boolean algebras. Then $0 \notin \mathscr{F}$ and so $\mathscr{F}$ is a proper filter. Suppose that $\mathscr{F}' \supsetneq \mathscr{F}$ is a filter. Let $e \in \mathscr{F}' \setminus \mathscr{F}$ and let $f \in \mathscr{F}$. We cannot have $fe \in \mathscr{F}$ as $f \notin \mathscr{F}$. Because $fe \vee (f \setminus e) = f \in \mathscr{F}$ and $\chi_{\mathscr{F}}$ is a morphism of generalized boolean algebras, it follows that $f \setminus e \in \mathscr{F} \subseteq \mathscr{F}'$. Thus $0 = e(f \setminus e) \in \mathscr{F}'$. We conclude $\mathscr{F}$ is an ultrafilter. $\square$

It follows from the proposition that if E is a generalized boolean algebra, then the points of $\mathrm{Spec}(E)$ can be identified with ultrafilters on E. If X is a locally compact boolean space and $x \in X$, then the corresponding ultrafilter on $B(X)$ is the set of all compact open neighborhoods of x. It is not hard to see that $\mathrm{Spec}(E)$ is a closed subspace of $\widehat{E}$ for a generalized boolean algebra E. For semilattices in general, the space of ultrafilters is not closed in $\widehat{E}$, which led Exel to consider the closure of the space of ultrafilters, which he terms the space of tight filters [8].

3. Étale and ample groupoids

By a *locally compact groupoid*, we mean a topological groupoid $\mathscr{G}$ that is locally compact and whose unit space $\mathscr{G}^0$ is locally compact Hausdorff in the induced topology.

Definition 3.1 *(Étale groupoid).* A locally compact groupoid $\mathscr{G}$ is said to be *étale* (or r-discrete) if the domain map $d : \mathscr{G} \to \mathscr{G}^0$ is étale, that is, a local homeomorphism. We do not assume that $\mathscr{G}$ is Hausdorff.

For basic properties of étale groupoids, we refer to the treatises [8,23,25]. We principally follow [8] in terminology. Fix an étale groupoid $\mathscr{G}$ for this section. A basic property of étale groupoids is that their unit space is open [8, Proposition 3.2].

Proposition 3.2. *The subspace $\mathscr{G}^0$ is open in $\mathscr{G}$.*

Of critical importance is the notion of a slice (or $\mathscr{G}$-set, or local bisection).

Definition 3.3 *(Slice).* A *slice* U is an open subset of $\mathscr{G}$ such that $d|_U$ and $r|_U$ are injective (and hence homeomorphisms since d and r are open). The set of all slices of $\mathscr{G}$ is denoted $\mathscr{G}^{op}$.

One can view a slice as the graph of a partial homeomorphism between $d(U)$ and $r(U)$ via the topological embedding $U \hookrightarrow d(U) \times r(U)$ sending $u \in U$ to $(d(u), r(u))$. Notice that any slice is locally compact Hausdorff in the induced topology, being homeomorphic to a subspace of $\mathscr{G}^0$.

Proposition 3.4. *The slices form a basis for the topology of $\mathscr{G}$. The set $\mathscr{G}^{op}$ is an inverse monoid under setwise multiplication. The inversion is also setwise and the natural partial order is via inclusion. The semilattice of idempotents is the topology of $\mathscr{G}^0$.*

Proof. See [8, Propositions 3.5 and 3.8]. □

A particularly important class of étale groupoids is that of ample groupoids [23].

Definition 3.5 *(Ample groupoid).* An étale groupoid is called *ample* if the compact slices form a basis for its topology.

One can show that the compact slices also form an inverse semigroup [23]. The inverse semigroup of compact slices is denoted $\mathscr{G}^a$. The idempotent set of $\mathscr{G}^a$ is the semilattice of compact open subsets of $\mathscr{G}^0$. Notice that if $U \in \mathscr{G}^a$, then any clopen subset V of U also belongs to $\mathscr{G}^a$.

Since we shall be interested in continuous functions with compact support into discrete rings, we shall restrict our attention to ample groupoids in order to ensure that we have "enough" continuous functions with compact support. So from now on $\mathscr{G}$ is an ample groupoid. If $\mathscr{G}$ is an ample groupoid, then $\mathscr{G}^0$ is a locally compact boolean space and $B(\mathscr{G}^0) = E(\mathscr{G}^a)$. In fact, one has the following description of ample groupoids.

Proposition 3.6. *An étale groupoid $\mathscr{G}$ is ample if and only if $\mathscr{G}^0$ is a locally compact boolean space.*

Proof. If $\mathscr{G}$ is ample, we already observed that $\mathscr{G}^0$ is a locally compact boolean space. For the converse, since $\mathscr{G}^{op}$ is a basis for the topology it suffices to show that each $U \in \mathscr{G}^{op}$ is a union of compact slices. But U is homeomorphic to $d(U)$ via $d|_U$. Since $\mathscr{G}^0$ is a locally compact boolean space, we can write $d(U)$ as a union of compact open subsets of $\mathscr{G}^0$ and hence we can write U as union of compact open slices by applying $d|_U^{-1}$. □

The following proposition relates topological properties of $\mathscr{G}$ to order theoretic properties of $\mathscr{G}^a$.

Proposition 3.7. *Let $\mathscr{G}$ be an ample groupoid. Then $\mathscr{G}^a$ is a semi-boolean algebra. Moreover, the following are equivalent:*

(1) *$\mathscr{G}$ is Hausdorff;*
(2) *$\mathscr{G}^a$ is closed under pairwise intersections;*
(3) *$\mathscr{G}^a$ is closed under relative complements.*

Proof. Let $U \in \mathscr{G}^a$. Then the map $d : U \to d(U)$ gives an isomorphism between the posets $U^\downarrow$ and $B(d(U))$. Since $B(d(U))$ is a boolean algebra, this proves the first statement. Suppose that $\mathscr{G}$ is Hausdorff and $U, V \in \mathscr{G}^a$. Then $U \cap V$ is a clopen subset of U and hence belongs to $\mathscr{G}^a$. If $\mathscr{G}^a$ is closed under pairwise intersections and $U, V \in \mathscr{G}^a$, then $U \cap V$ is compact open and so $U \cap V$ is clopen in U. Then $U \setminus V = U \setminus (U \cap V)$ is a clopen subset of U and hence belongs to $\mathscr{G}^a$. Finally, suppose that $\mathscr{G}^a$ is closed under relative complements and let $g, h \in \mathscr{G}$. As $\mathscr{G}^a$ is a basis for the topology on $\mathscr{G}$, we can find slices $U, V \in \mathscr{G}^a$ with $g \in U$ and $h \in V$. If $g, h \in U$ or $g, h \in V$, then we can clearly separate them by disjoint open sets since U and V are Hausdorff.

Otherwise, $g \in U \setminus V$, $h \in V \setminus U$ and these are disjoint open sets as $\mathscr{G}^a$ is closed under relative complements. This completes the proof. □

Another important property of $\mathscr{G}^a$ is that it is *orthogonally complete*. This means that if $U, V \in \mathscr{G}^a$ such that $UV = \emptyset = VU$, then $U \cup V \in \mathscr{G}^a$.

4. The algebra of an ample groupoid

Fix for this section an ample groupoid $\mathscr{G}$. To motivate the definition of the K-algebra of $\mathscr{G}$, we begin by making the following observation. Let K be a unital commutative ring endowed with the discrete topology and let X be a locally compact Hausdorff space. Then the space of continuous K-valued functions on G with compact support is precisely the K-submodule of K^X spanned by the characteristic functions of compact open subsets of X. Indeed, if U is compact open, then trivially χ_U is continuous with compact support. Conversely, if $f : X \to K$ is continuous with compact support, then $f(X) \setminus \{0\}$ is contained in a compact subset of the discrete space K and hence is finite. Suppose that $f(X) \setminus \{0\} = \{k_1, \dots, k_m\}$. Then, each $U_i = f^{-1}(k_i)$ is compact open and $f = k_1\chi_{U_1} + \cdots + k_m\chi_{U_m}$.

Definition 4.1 *($K\mathscr{G}$).* If $\mathscr{G}$ is an ample groupoid and K is a commutative ring with unit, then we define $K\mathscr{G}$ to be the K-submodule of $K^{\mathscr{G}}$ spanned by the characteristic functions of compact open subsets of $\mathscr{G}$.

It follows from the discussion above that $K\mathscr{G}$ can be alternatively described as the K-submodule spanned by the functions $f \in K^{\mathscr{G}}$ with compact open support and which are continuous on their support, where K is equipped with the discrete topology.

Remark 4.2. One should think of $K\mathscr{G}$ as 'the algebra of continuous K-valued functions on $\mathscr{G}$ with compact support' but the reader is *cautioned* that if $\mathscr{G}$ is not Hausdorff, then $K\mathscr{G}$ *will* contain discontinuous functions. Indeed, if $\mathscr{G}$ is not Hausdorff, it will have compact open subsets that are not closed (cf. Proposition 3.7). The corresponding characteristic function of such a compact open will be discontinuous, but belong to $K\mathscr{G}$. Of course, if $\mathscr{G}$ is Hausdorff, then $K\mathscr{G}$ is precisely the space of continuous K-valued functions with compact support by the discussion before the definition. The approach we have adopted here to dealing with defining convolution algebras on non-Hausdorff groupoids is due to Connes [6].

For example, if $\mathscr{G}$ has the discrete topology, then one can identify $K\mathscr{G}$ with the vector space of all functions of finite support on $\mathscr{G}$. A basis then consists of the functions δ_g with $g \in \mathscr{G}$.

It turns out that the algebraic structure of $K\mathscr{G}$ is controlled by $\mathscr{G}^a$. We start at the level of K-modules.

Proposition 4.3. *The space $K\mathscr{G}$ is spanned by the characteristic functions of elements of $\mathscr{G}^a$.*

Proof. Evidently, if $U \in \mathscr{G}^a$, then $\chi_U \in K\mathscr{G}$. Let A be the subspace spanned by such characteristic functions. We must show that if U is a compact open subset of $\mathscr{G}$, then $\chi_U \in A$.

Since $\mathscr{G}^a$ is a basis for the topology of $\mathscr{G}$ and U is compact open, it follows $U = U_1 \cup \cdots \cup U_r$ with the $U_i \in \mathscr{G}^a$. Since $U_i \subseteq U$, for $i = 1, \dots, r$, and U is Hausdorff, it follows that any finite intersection of elements of the set $\{U_1, \ldots, U_r\}$ belongs to $\mathscr{G}^a$. The principle of inclusion–exclusion yields

$$\chi_U = \chi_{U_1 \cup \cdots \cup U_n} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} \chi_{\bigcap_{i\in I} U_i}. \tag{4.1}$$

Hence $\chi_U \in A$, as required. □

We now define the convolution product on $K\mathscr{G}$ in order to make it a K-algebra.

Definition 4.4 *(Convolution).* Let $f, g \in K\mathscr{G}$. Then their *convolution* $f * g$ is defined, for $x \in \mathscr{G}$, by

$$f * g(x) = \sum_{y \in d^{-1}d(x)} f\bigl(xy^{-1}\bigr)g(y).$$

Of course, one must show that this sum is really finite and $f * g$ belongs to $K\mathscr{G}$, which is the content of the following proposition.

Proposition 4.5. *Let $f, g \in K\mathscr{G}$. Then:*

(1) *$f * g \in K\mathscr{G}$;*
(2) *If f, g are continuous with compact support on $U, V \in \mathscr{G}^a$, respectively, then $f * g$ is continuous with compact support on UV;*
(3) *If $U, V \in \mathscr{G}^a$, then $\chi_U * \chi_V = \chi_{UV}$;*
(4) *If $U \in \mathscr{G}^a$, then $\chi_{U^{-1}}(x) = \chi_U(x^{-1})$.*

Proof. Since the characteristic functions of elements of $\mathscr{G}^a$ span $K\mathscr{G}$ by Proposition 4.3, it is easy to see that (1) and (2) are consequences of (3). We proceed to the task at hand: establishing (3).

Indeed, we have

$$\chi_U * \chi_V(x) = \sum_{y \in d^{-1}d(x)} \chi_U\bigl(xy^{-1}\bigr)\chi_V(y). \tag{4.2}$$

Suppose first $x \in UV$. Then we can find $a \in U$ and $b \in V$ so that $x = ab$. Therefore, $a = xb^{-1}$, $d(x) = d(b)$ and $\chi_U(xb^{-1})\chi_V(b) = 1$. Moreover, since U and V are slices, b is the unique element of V with $d(x) = d(b)$. Thus the right-hand side of (4.2) is 1.

Conversely, suppose $x \notin UV$ and let $y \in d^{-1}d(x)$. If $y \notin V$, then $\chi_V(y) = 0$. On the other hand, if $y \in V$, then $xy^{-1} \notin U$, for otherwise we would have $x = xy^{-1} \cdot y \in UV$. Thus $\chi_U(xy^{-1}) = 0$. Therefore, each term of the right-hand side of (4.2) is zero and so $\chi_U * \chi_V = \chi_{UV}$, as required.

Statement (4) is trivial. □

The associativity of convolution is a straightforward, but tedious exercise [8,23].

Proposition 4.6. *Let K be a commutative ring with unit and $\mathscr{G}$ an ample groupoid. Then $K\mathscr{G}$ equipped with convolution is a K-algebra.*

If $K = \mathbb{C}$, we make $\mathbb{C}\mathscr{G}$ into a $*$-algebra by defining $f^*(x) = \overline{f(x^{-1})}$.

Corollary 4.7. *The map $\varphi : \mathscr{G}^a \to K\mathscr{G}$ given by $\varphi(U) = \chi_U$ is a semigroup homomorphism.*

Remark 4.8 *(Groups).* If $\mathscr{G}^0$ is a singleton, so that $\mathscr{G}$ is a discrete group, then $K\mathscr{G}$ is the usual group algebra.

Remark 4.9 *(Locally compact boolean spaces).* In the case $\mathscr{G} = \mathscr{G}^0$, one has that $K\mathscr{G}$ is the subalgebra of $K^{\mathscr{G}}$ spanned by the characteristic functions of compact open subsets of $\mathscr{G}$ equipped with the pointwise product. If $K = \mathbb{F}_2$, then $K\mathscr{G} \cong B(\mathscr{G}^0)$ viewed as a boolean ring.

Remark 4.10 *(Discrete groupoids).* Notice that if $\mathscr{G}$ is a discrete groupoid and $g \in \mathscr{G}$, then $\{g\} \in \mathscr{G}^a$ and $\delta_g = \chi_{\{g\}}$. It follows easily that

$$\delta_g * \delta_h = \begin{cases} \delta_{gh}, & d(g) = r(h), \\ 0, & \text{else}. \end{cases}$$

Thus $K\mathscr{G}$ can be identified with the K-algebra having basis $\mathscr{G}$ and whose product extends that of $\mathscr{G}$ where we interpret undefined products as 0. This is exactly the groupoid algebra considered, for example, in [30,31].

Propositions 4.3 and 4.5 imply that $K\mathscr{G}$ is a quotient of the semigroup algebra $K\mathscr{G}^a$. Clearly $K\mathscr{G}$ satisfies the relations $\chi_{U\cup V} = \chi_U + \chi_V$ whenever $U, V \in B(\mathscr{G}^0)$ with $U \cap V = \emptyset$. We can show that these relations define $K\mathscr{G}$ as a quotient of $K\mathscr{G}^a$ in the case that $\mathscr{G}$ is Hausdorff. We conjecture that this is true in general.

Our next goal is to show that $K\mathscr{G}$ is unital if and only if $\mathscr{G}^0$ is compact.

Proposition 4.11. *The K-algebra $K\mathscr{G}$ is unital if and only if $\mathscr{G}^0$ is compact.*

Proof. Suppose first $\mathscr{G}^0$ is compact. Since it is open in the relative topology by Proposition 3.2, it follows that $u = \chi_{\mathscr{G}^0} \in K\mathscr{G}$. Now if $f \in K\mathscr{G}$, then we compute

$$f * u(x) = \sum_{y \in d^{-1}d(x)} f\big(xy^{-1}\big)u(y) = f(x)$$

since $d(x)$ is the unique element of $\mathscr{G}^0$ in $d^{-1}d(x)$. Similarly,

$$u * f(x) = \sum_{y \in d^{-1}d(x)} u\big(xy^{-1}\big)f(y) = f(x)$$

since $xy^{-1} \in \mathscr{G}^0$ implies $x = y$. Thus u is the multiplicative identity of $\mathscr{G}$.

Conversely, suppose u is the multiplicative identity. We first claim that $u = \chi_{\mathscr{G}^0}$. Let $x \in \mathscr{G}$. Choose a compact open set $U \subseteq \mathscr{G}^0$ with $d(x) \in U$. Suppose first $x \notin \mathscr{G}^0$. Then

$$0 = \chi_U(x) = u * \chi_U(x) = \sum_{y \in d^{-1}d(x)} u\bigl(xy^{-1}\bigr)\chi_U(y) = u(x)$$

since $\{d(x)\} = U \cap d^{-1}d(x)$. Similarly, if $x \in \mathscr{G}^0$, then we have

$$1 = \chi_U(x) = u * \chi_U(x) = \sum_{y \in d^{-1}d(x)} u\bigl(xy^{-1}\bigr)\chi_U(y) = u(x).$$

So we must show that $\chi_{\mathscr{G}^0} \in K\mathscr{G}$ implies that $\mathscr{G}^0$ is compact.

By Proposition 4.3, there exist $U_1, \dots, U_k \in \mathscr{G}^a$ and $c_1, \dots, c_k \in K$ so that $\chi_{\mathscr{G}^0} = c_1\chi_{U_1} + \cdots + c_k\chi_{U_k}$. Thus $\mathscr{G}^0 \subseteq U_1 \cup \cdots \cup U_k$. But then $\mathscr{G}^0 = d(U_1) \cup \cdots \cup d(U_k)$. But each $d(U_i)$ is compact, being homeomorphic to U_i, so $\mathscr{G}^0$ is compact, as required. □

The center of $K\mathscr{G}$ can be described by functions that are constant on conjugacy classes, analogously to the case of groups.

Definition 4.12 *(Class function).* Define $f \in K\mathscr{G}$ to be a *class function* if:

(1) $f(x) \neq 0$ implies $d(x) = r(x)$;
(2) $d(x) = r(x) = d(z)$ implies $f(zxz^{-1}) = f(x)$.

Proposition 4.13. *The center of $K\mathscr{G}$ is the set of class functions.*

Proof. Suppose first that f is a class function and $g \in K\mathscr{G}$. Then

$$f * g(x) = \sum_{y \in d^{-1}d(x)} f\bigl(xy^{-1}\bigr)g(y) = \sum_{y \in d^{-1}d(x) \cap r^{-1}r(x)} f\bigl(xy^{-1}\bigr)g(y) \tag{4.3}$$

since $f(xy^{-1}) = 0$ if $r(x) = r(xy^{-1}) \neq d(xy^{-1}) = r(y)$. But

$$f\bigl(xy^{-1}\bigr) = f\bigl(y\bigl(y^{-1}x\bigr)y^{-1}\bigr) = f\bigl(y^{-1}x\bigr)$$

since f is a class function and $d(y^{-1}x) = d(x) = d(y) = r(y^{-1}x)$. Performing the change of variables $z = y^{-1}x$, we obtain that the right-hand side of (4.3) is equal to

$$\sum_{z \in d^{-1}d(x) \cap r^{-1}d(x)} g\bigl(xz^{-1}\bigr)f(z) = \sum_{z \in d^{-1}d(x)} g\bigl(xz^{-1}\bigr)f(z) = g * f(x)$$

where the first equality uses that $f(z) = 0$ if $d(z) \neq r(z)$. Thus $f \in Z(K\mathscr{G})$.

Conversely, suppose $f \in Z(K\mathscr{G})$. First we consider the case $x \in \mathscr{G}$ and $d(x) \neq r(x)$. Choose a compact open set $U \subseteq \mathscr{G}^0$ so that $d(x) \in U$ and $r(x) \notin U$. Then

$$\chi_U * f(x) = \sum_{y \in d^{-1}d(x)} \chi_U\bigl(xy^{-1}\bigr)f(y) = 0$$

since $xy^{-1} \in U$ forces it to be a unit, but then $y = x$ and $xx^{-1} = r(x) \notin U$. On the other hand,

$$f * \chi_U(x) = \sum_{y \in d^{-1}d(x)} f\left(xy^{-1}\right)\chi_U(y) = f(x)$$

since $d(x)$ is the unique element of $d^{-1}d(x)$ in U. Thus $f(x) = 0$.

The remaining case is that $d(x) = r(x)$ and we have $d(z) = d(x)$. Then zx^{-1} is defined. Choose $U \in \mathscr{G}^a$ so that $zx^{-1} \in U$. Then

$$f * \chi_U(z) = \sum_{y \in d^{-1}d(z)} f\left(zy^{-1}\right)\chi_U(y) = f\left(zxz^{-1}\right)$$

since $y \in U \cap d^{-1}d(z)$ implies $y = zx^{-1}$. On the other hand,

$$\chi_U * f(z) = \sum_{y \in d^{-1}d(z)} \chi_U\left(zy^{-1}\right)f(y) = f(x)$$

since $r(zy^{-1}) = r(zx^{-1})$ and so $zy^{-1} \in U$ implies $zy^{-1} = zx^{-1}$, whence $y = x$. This shows that $f(x) = f(zxz^{-1})$, completing the proof of the proposition. $\square$

Our next proposition provides a sufficient condition for the characteristic functions of an inverse subsemigroup of $\mathscr{G}^a$ to span $K\mathscr{G}$.

Proposition 4.14. *Let $S \subseteq \mathscr{G}^a$ be an inverse subsemigroup such that:*

(1) *$E(S)$ generates the generalized boolean algebra $B(\mathscr{G}^0)$;*
(2) *$D = \{U \in \mathscr{G}^a \mid U \subseteq V \text{ some } V \in S\}$ is a basis for the topology on $\mathscr{G}$.*

Then $K\mathscr{G}$ is spanned by the characteristic functions of elements of S.

Proof. Let A be the span of the χ_V with $V \in S$. Then A is a K-subalgebra by Proposition 4.5. We break the proof up into several steps.

Step 1. The collection B of compact open subsets of $\mathscr{G}^0$ so that $\chi_U \in A$ is a generalized boolean algebra.

Proof. This is immediate from the formulas, for $U, V \in B(\mathscr{G}^0)$,

$$\chi_U * \chi_V = \chi_{UV} = \chi_{U \cap V},$$
$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V},$$
$$\chi_{U \cup V} = \chi_{U \setminus V} + \chi_{V \setminus U} + \chi_{U \cap V},$$

since A is a subalgebra. $\square$

We may now conclude by (1) that A contains χ_U for every element of $B(\mathscr{G}^0)$.

Step 2. The characteristic function of each element of D belongs to A.

Proof. If $U \subseteq V$ with $V \in S$, then $VU^{-1}U = U$ and so $\chi_U = \chi_V * \chi_{U^{-1}U} \in A$ by Step 1 since $U^{-1}U \in E(\mathscr{G}^a) = B(\mathscr{G}^0)$. $\square$

Step 3. Each χ_U with $U \in \mathscr{G}^a$ belongs to A.

Proof. Since D is a basis for $\mathscr{G}$ by hypothesis and U is compact open, we may write $U = U_1 \cup \cdots \cup U_n$ with the $U_i \in D$, for $i = 1, \ldots, n$. Since the U_i are all contained in U, any finite intersection of the U_i is clopen in U and hence belongs to $\mathscr{G}^a$. As D is a downset, in fact, any finite intersection of the U_i belongs to D. Therefore, $\chi_U \in A$ by (4.1). $\square$

Proposition 4.3 now yields the result. $\square$

5. Actions of inverse semigroups and groupoids of germs

As inverse semigroups are models of partial symmetry [14], it is natural to study them via their actions on spaces. From such "dynamical systems" we can form a groupoid of germs and hence, in the ample setting, a K-algebra.

5.1. The category of actions

Let X be a locally compact Hausdorff space and denote by I_X the inverse semigroup of all homeomorphisms between open subsets of X.

Definition 5.1 *(Action).* An *action* of an inverse semigroup S on X is a homomorphism $\varphi : S \to I_X$ written $s \mapsto \varphi_s$. As usual, if $s \in S$ and $x \in \mathrm{dom}(\varphi_s)$, then we put $sx = \varphi_s(x)$. Let us set X_e to be the domain of φ_e for $e \in E(S)$. The action is said to be *non-degenerate* if $X = \bigcup_{e \in E(S)} X_e$. If $\psi : S \to I_Y$ is another action, then a morphism from φ to ψ is a continuous map $\alpha : X \to Y$ so that, for all $x \in X$, one has that sx is defined if and only if $s\alpha(x)$ is defined, in which case $\alpha(sx) = s\alpha(x)$.

We will be most interested in what we term "ample" actions. These actions will give rise to ample groupoids via the groupoid of germs constructions.

Definition 5.2 *(Ample action).* A non-degenerate action $\varphi : S \to I_X$ of an inverse semigroup S on a space X is said to be *ample* if:

(1) X is a locally compact boolean space;
(2) $X_e \in B(X)$, for all $e \in E(S)$.

If in addition, the collection $\{X_e \mid e \in E(S)\}$ generates $B(X)$ as a generalized boolean algebra, we say the action is *boolean*.

If $\mathscr{G}$ is an ample groupoid, there is a natural boolean action of $\mathscr{G}^a$ on $\mathscr{G}^0$. Namely, if $U \in \mathscr{G}^a$, then the domain of its action is $U^{-1}U$ and the range is UU^{-1}. If $x \in U^{-1}U$, then there is a unique element $g \in U$ with $d(g) = x$. Define $Ux = r(g)$. This is exactly the partial homeomorphism whose "graph" is U. See [8,23] for details.

Proposition 5.3. *Suppose that S has ample actions on X and Y and let $\alpha : X \to Y$ be a morphism of the actions. Then:*

(1) *For each $e \in E(S)$, one has $\alpha^{-1}(Y_e) = X_e$;*
(2) *α is proper;*
(3) *α is closed.*

Proof. From the definition of a morphism, $\alpha(x) \in Y_e$ if and only if $e\alpha(x)$ is defined, if and only if ex is defined, if and only if $x \in X_e$. This proves (1). For (2), let $C \subseteq Y$ be compact. Since the action on Y is non-degenerate, we have that $C \subseteq \bigcup_{e\in E(S)} Y_e$ and so by compactness of C it follows that $C \subseteq Y_{e_1} \cup \cdots \cup Y_{e_n}$ for some idempotents $e_1, \ldots, e_n$. Thus $\alpha^{-1}(C)$ is a closed subspace of $\alpha^{-1}(Y_{e_1} \cup \cdots \cup Y_{e_n}) = X_{e_1} \cup \cdots \cup X_{e_n}$ and hence is compact since the X_{e_i} are compact. Finally, it is well known that a proper map between locally compact Hausdorff spaces is closed [2]. □

For us the main example of a boolean action is the action of an inverse semigroup S on the spectrum of its semilattice of idempotents. The reader should consult Section 2.3 for definitions and notation.

Definition 5.4 *(Spectral action).* Suppose that S is an inverse semigroup with semilattice of idempotents $E(S)$. To each $s \in S$, there is an associated homeomorphism $\beta_s : D(s^*s) \to D(ss^*)$ given by $\beta_s(\varphi)(e) = \varphi(s^*es)$. The map $s \mapsto \beta_s$ provides a boolean action $\beta : S \to I_{\widehat{E(S)}}$ [8,23], which we call the *spectral action*.

The spectral action enjoys the following universal property.

Proposition 5.5. *Let $\mathscr{C}$ be the category of boolean actions of S.*

(1) *Suppose S has a boolean action on X and an ample action on Y and let $\psi : X \to Y$ be a morphism. Then ψ is a topological embedding of X as a closed subspace of Y.*
(2) *Each homset of $\mathscr{C}$ contains at most one element.*
(3) *The spectral action $\beta : S \to I_{\widehat{E(S)}}$ is the terminal object in $\mathscr{C}$.*

Proof. By Proposition 5.3, the map ψ is proper and $\psi^{-1} : B(Y) \to B(X)$ takes Y_e to X_e for $e \in E(S)$. Since the X_e with $e \in E(S)$ generate $B(X)$ as a generalized boolean algebra, it follows that $\psi^{-1} : B(Y) \to B(X)$ is surjective. By Stone duality, we conclude ψ is injective. Also ψ is closed being proper. This establishes (1).

Again by Proposition 5.3 if $\psi : X \to Y$ is a morphism of boolean actions, then ψ is proper and $\psi^{-1} : B(Y) \to B(X)$ sends Y_e to X_e. Since the Y_e with $e \in E(S)$ generate $B(Y)$, it follows ψ^{-1} is uniquely determined by the actions of S on X and Y. But ψ^{-1} determines ψ by Stone duality, yielding (2).

Let $\alpha : S \to I_Y$ be a boolean action. By definition the map $e \mapsto Y_e$ yields a homomorphism $E(S) \to B(Y)$ extending to a surjective homomorphism $\mathbb{F}_2 E(S) \to B(Y)$. Recalling $\mathbb{F}_2 E(S) \cong B(\widehat{E(S)})$, Stone duality yields a proper continuous injective map $\psi : Y \to \widehat{E(S)}$ that sends $y \in Y$ to $\varphi_y : E(S) \to \{0, 1\}$ given by $\varphi_y(e) = \chi_{Y_e}(y)$. It remains to show that the map $y \mapsto \varphi_y$ is a morphism. Let $y \in Y$ and $s \in S$. Then sy is defined if and only if $y \in Y_{s^*s}$, if and only if

$\varphi_y(s^*s) = 1$, if and only if $\varphi_y \in D(s^*s)$. If $y \in Y_{s^*s}$, then $s\varphi_y(e) = \varphi_y(s^*es) = \chi_{Y_{s^*es}}(y)$. But Y_{s^*es} is the domain of α_{es}. Since sy is defined, $y \in Y_{s^*es}$ if and only if $sy \in Y_e$. Thus $\chi_{Y_{s^*es}}(y) = \chi_{Y_e}(sy) = \varphi_{sy}(e)$. We conclude that $\varphi_{sy} = s\varphi_y$. This completes the proof of the theorem. $\square$

Since the restriction of a boolean action to a closed invariant subspace is evidently boolean, we obtain the following description of boolean actions, which was proved by Paterson in a slightly different language [23].

Corollary 5.6. *There is an equivalence between the category of boolean actions of S and the poset of S-invariant closed subspaces of $\widehat{E(S)}$.*

5.2. Groupoids of germs

There is a well-known construction assigning to each non-degenerate action $\varphi : S \to I_X$ of an inverse semigroup S on a locally compact Hausdorff space X an étale groupoid, which we denote $S \ltimes_\varphi X$, known as the *groupoid of germs* of the action [23,8,26]. Usually, φ is dropped from the notation if it is understood. The groupoid of germs construction is functorial. It goes as follows. As a set $S \ltimes_\varphi X$ is the quotient of the set $\{(s, x) \in S \times X \mid x \in X_{s^*s}\}$ by the equivalence relation that identifies (s, x) and (t, y) if and only if $x = y$ and there exists $u \leqslant s, t$ with $x \in X_{u^*u}$. One writes $[s, x]$ for the equivalence class of (s, x) and calls it the *germ of s at x*. The associated topology is the so-called germ topology. A basis consists of all sets of the form (s, U) where $U \subseteq X_{s^*s}$ and $(s, U) = \{[s, x] \mid x \in U\}$. The multiplication is given by defining $[s, x] \cdot [t, y]$ if and only if $ty = x$, in which case the product is $[st, y]$. The units are the elements $[e, x]$ with $e \in E(S)$ and $x \in X_e$. The projection $[e, x] \mapsto x$ gives a homeomorphism of the unit space of $S \ltimes X$ with X, and so from now on we identify the unit space with X. One then has $d([s, x]) = x$ and $r([s, x]) = sx$. The inversion is given by $[s, x]^{-1} = [s^*, sx]$. The groupoid $S \ltimes X$ is an étale groupoid [8, Proposition 4.17]. The reader should consult [8,23] for details.

Observe that if $\mathscr{B}$ is a basis for the topology of X, then a basis for $S \ltimes X$ consists of those sets of the form (s, U) with $U \in \mathscr{B}$, as is easily verified. Let us turn to some enlightening examples.

Example 5.7 *(Transformation groupoids).* In the case that S is a discrete group, the equivalence relation on $S \times X$ is trivial and the topology is the product topology. The resulting étale groupoid is consequently Hausdorff and is known in the literature as the *transformation groupoid* associated to the transformation group (S, X) [25,23].

Example 5.8 *(Maximal group image).* An easy example is the case when X is a one-point set on which S acts trivially. It is then straightforward to see that $S \ltimes X$ is the maximal group image G_S of S. Indeed, elements of the groupoid are equivalence classes of elements of S where two elements are considered equivalent if they have a common lower bound.

Example 5.9 *(Underlying groupoid).* Another example is the case $X = E(S)$ with the discrete topology. The action is the so-called Munn representation $\mu : S \to I_{E(S)}$ given by putting $X_e = e^{\downarrow}$ and defining $\mu_s : X_{s^*s} \to X_{ss^*}$ by $\mu_s(e) = ses^*$ [14]. The spectral action is the dual of the (right) Munn representation. We observe that each equivalence class $[s, e]$ contains a unique element of the form (t, e) with $t^*t = e$, namely $t = se$. Then e is determined by t. Thus arrows of $S \ltimes X$ are in bijection with elements of S via the map $s \mapsto [s, s^*s]$. One has $d(s) = s^*s$, $r(s) = ss^*$ and if s, t are composable, their composition is st. The inverse of s is s^*. Hence the groupoid of germs is the underlying groupoid $\mathscr{G}_S$ of S. The slice $(s, \{s^*s\})$ contains just the arrow s, so the topology is discrete.

The next proposition establishes the functoriality of the germ groupoid.

Proposition 5.10. *Let S act on X and Y and suppose $\psi : X \to Y$ is a morphism. Then there is a continuous functor $\Psi : S \ltimes X \to S \ltimes Y$ given by $[s, x] \mapsto [s, \psi(x)]$. If the actions are boolean, then Ψ is an embedding of groupoids and the image consists precisely of those arrows of $S \ltimes Y$ between elements of $\psi(X)$.*

Proof. We verify that Ψ is well defined. First note that $x \in X_{s^*s}$ if and only if $\psi(x) \in Y_{s^*s}$ for any $s \in S$ by the definition of a morphism. Suppose that (s, x) is equivalent to (t, x). Then we can find $u \leqslant s, t$ with $x \in X_{u^*u}$. It then follows that $\psi(x) \in Y_{u^*u}$ and so $(s, \psi(x))$ is equivalent to $(t, \psi(x))$. Thus Ψ is well defined. The details that Ψ is a continuous functor are routine and left to the reader. In the case the actions are boolean, the fact that Ψ is an embedding follows easily from Proposition 5.5. The description of the image follows because if $[s, y]$ satisfies $y, sy \in \psi(X)$ and $y = \psi(x)$, then $[s, y] = \Psi([s, x])$. $\square$

In [8,23], the term *reduction* is used to describe a groupoid obtained by restricting the units to a closed subspace and taking the full subgroupoid of all arrows between these objects.

The following is [8, Proposition 4.18].

Proposition 5.11. *The basic open set (s, U) with $U \subseteq X_{s^*s}$ is a slice of $S \ltimes X$ homeomorphic to U.*

From the proposition, it easily follows that if $\varphi : S \to I_X$ is an ample action and $\mathscr{G} = S \ltimes_\varphi X$, then each open set of the form (s, U) with $U \in B(X)$ belongs to $\mathscr{G}^a$ and the collection of such open sets is a basis for the topology on $\mathscr{G}$. Thus $S \ltimes_\varphi X$ is an ample groupoid. Moreover, the map $s \mapsto (s, X_{s^*s})$ in this setting is a homomorphism $\theta : S \to \mathscr{G}^a$, as the following lemma shows in the general case.

Lemma 5.12. *Let S have a non-degenerate action on X. If (s, U) and (t, V) are basic neighborhoods of $\mathscr{G} = S \ltimes X$, then*

$$(s, U)(t, V) = \bigl(st, t^*(U \cap tV)\bigr)$$

and $(s, U)^{-1} = (s^*, sU)$. Consequently, the map $s \mapsto (s, X_{s^*s})$ is a homomorphism from S to $\mathscr{G}^{op}$ and furthermore if the action is ample, then it is a homomorphism to $\mathscr{G}^a$.

Proof. First observe that

$$\begin{aligned} t^*(X_{s^*s} \cap tX_{t^*t}) &= t^*(X_{s^*s} \cap X_{tt^*}) = t^*(X_{s^*stt^*}) = t^*s^*stt^*(X_{s^*stt^*}) \\ &= t^*s^*s(X_{s^*stt^*}) = X_{t^*s^*st} = X_{(st)^*(st)}. \end{aligned}$$

The final statement now follows from the first.

By definition $U \subseteq X_{s^*s}$ and $V \subseteq X_{t^*t}$. Therefore, we have $t^*(U \cap tV) \subseteq t^*(X_{s^*s} \cap tX_{t^*t}) = X_{(st)^*st}$ and so $(st, t^*(U \cap tV))$ is well defined. Suppose $x \in U$ and $y \in V$ with $ty = x$. Then

$x \in U \cap tV$ and so $y \in t^*(U \cap tV)$. Moreover, $[s, x] \cdot [t, y] = [st, y] \in (st, t^*(U \cap tV))$. This shows $(s, U)(t, V) \subseteq (st, t^*(U \cap tV))$. For the converse, assume $[st, y] \in (st, t^*(U \cap tV))$. Then $y \in t^*tV = V$ and if we set $x = ty$, then $x \in tt^*U \subseteq U$. We conclude $[st, y] = [s, x] \cdot [t, y] \in (s, U)(t, V)$. The equality $(s, U)^{-1} = (s^*, sU)$ is trivial. □

Notice that the action of the slice (s, X_{s^*s}) on $\mathscr{G}^0 = X$ is exactly the map $x \mapsto sx$. Indeed, the domain of the action of (s, X_{s^*s}) is X_{s^*s} and if $x \in X_{s^*s}$, then $[s, x]$ is the unique element of the slice with domain x. But $r([s, x]) = sx$.

In the case of the trivial action on a one-point space, the map from Lemma 5.12 is the maximal group image homomorphism σ. In the case of the Munn representation, the map is $s \mapsto \{t \in S \mid t \leqslant s\} = s^{\downarrow}$. It is straightforward to verify that in this case the homomorphism is injective. The reader should compare with [30,31].

Summarizing, we have the following proposition.

Proposition 5.13. *Let $\varphi : S \to I_X$ be an ample action and put $\mathscr{G} = S \ltimes_{\varphi} X$. Then:*

(1) *$\mathscr{G}$ is an ample groupoid;*
(2) *There is a homomorphism $\theta : S \to \mathscr{G}^a$ given by $\theta(s) = (s, X_{s^*s})$;*
(3) *$\bigcup \theta(S) = \mathscr{G}$;*
(4) *$\{U \in \mathscr{G}^a \mid U \subseteq \theta(s) \text{ some } s \in S\}$ is a basis for the topology on $\mathscr{G}$;*
(5) *There is a homomorphism $\Theta : S \to K\mathscr{G}$ given by*

$$\Theta(s) = \chi_{\theta(s)} = \chi_{(s, X_{s^*s})},$$

which is a $*$-homomorphism when $K = \mathbb{C}$.

Moreover, if φ is a boolean action, then:

(6) *$E(\theta(S))$ generates $B(\mathscr{G}^0) \cong B(X)$ as a generalized boolean algebra;*
(7) *$\Theta(S)$ spans $K\mathscr{G}$.*

Proof. Item (4) is a consequence of the fact that if $U \subseteq X_{s^*s}$, then the basic set (s, U) is contained in $\theta(s) = (s, X_{s^*s})$. Item (5) follows from Proposition 4.5 and Lemma 5.12. Item (7) is a consequence of (4), (6) and Proposition 4.14. The remaining statements are clear. □

The universal groupoid of an inverse semigroup was introduced by Paterson [23] and has since been studied by several authors [8,15,26,32].

Definition 5.14 *(Universal groupoid).* Let S be an inverse semigroup. The groupoid of germs $\mathscr{G}(S) = S \ltimes_{\beta} \widehat{E(S)}$ is called the *universal groupoid* of S [23,8]. It is an ample groupoid. Notice that if $E(S)$ is finite (or more generally if each principal downset of $E(S)$ is finite), then the underlying groupoid of S is the universal groupoid.

Propositions 5.5 and 5.10 immediately imply the following universal property of $\mathscr{G}(S)$, due to Paterson [23].

Proposition 5.15. *Let $\varphi : S \to I_X$ be a boolean action. Then there is a unique continuous functor $\Phi : S \ltimes X \to \mathscr{G}(S)$ so that $\Phi((s, X_{s^*s})) = (s, D(s^*s))$. Moreover, Φ is an embedding of topological groupoids with image a reduction of $\mathscr{G}(S)$ to a closed S-invariant subspace of $\widehat{E(S)}$.*

Examples of universal groupoids of inverse semigroups can be found in [23, Chapter 4]. It is convenient at times to use the following algebraic embedding of the underlying groupoid into the universal groupoid.

Lemma 5.16. *Let $s \in S$. Then $[s, \chi_{(s^*s)\uparrow}] = [t, \chi_{(s^*s)\uparrow}]$ if and only if $s \leqslant t$. Consequently, $[s, \chi_{(s^*s)\uparrow}] \in (t, D(t^*t))$ if and only if $s \leqslant t$. Moreover, the map $s \mapsto [s, \chi_{(s^*s)\uparrow}]$ is an injective functor from the underlying groupoid of S onto a dense subgroupoid of $\mathscr{G}(S)$.*

Proof. We verify the first two statements. The final statement is straightforward from the previous ones and can be found in [23, Proposition 4.4.6]. If $s \leqslant t$, the germs of s and t at $\chi_{(s^*s)\uparrow}$ clearly coincide. Assume conversely, that the germs are the same. By definition there exists $u \leqslant s, t$ so that $\chi_{(s^*s)\uparrow} \in D(u^*u)$, i.e., $u^*u \geqslant s^*s$. But then $u = su^*u = ss^*su^*u = ss^*s = s$. Thus $s \leqslant t$. The second statement follows since $[s, \chi_{(s^*s)\uparrow}] \in (t, D(t^*t))$ if and only if $[s, \chi_{(s^*s)\uparrow}] = [t, \chi_{(s^*s)\uparrow}]$. □

We end this section with a sufficient condition for the groupoid of germs of an action to be Hausdorff, improving on [8, Proposition 6.2] (see also [23,15] where they assume that $S \setminus \{0\}$ is a semilattice). For the universal groupoid, the condition is shown also to be necessary and is the converse to [23, Corollary 4.3.1]. As a consequence we obtain a much easier proof that the universal groupoid of a certain commutative inverse semigroup considered in [23, Appendix C] is not Hausdorff.

Observe that a poset P is a semilattice if and only if the intersection of principal downsets is again a principal downset. We say that a poset is a *weak semilattice* if the intersection of principal downsets is finitely generated as a downset. The empty downset is considered to be generated by the empty set.

Theorem 5.17. *Let S be an inverse semigroup. Then the following are equivalent:*

(1) *S is a weak semilattice;*
(2) *The groupoid of germs of any non-degenerate action $\theta : S \to I_X$ such that X_e is clopen for all $e \in E(S)$ is Hausdorff;*
(3) *$\mathscr{G}(S)$ is Hausdorff.*

In particular, every groupoid of germs for an ample action of S is Hausdorff if and only if S is a weak semilattice.

Proof. First we show that (1) implies (2). Suppose $[s, x] \neq [t, y]$ are elements of $\mathscr{G}$. If $x \neq y$, then choose disjoint neighborhoods U, V of x and y in X, respectively. Clearly, $(s, U \cap X_{s^*s})$ and $(t, V \cap X_{t^*t})$ are disjoint neighborhoods of $[s, x]$ and $[t, y]$, respectively.

Next assume $x = y$. If $s^\downarrow \cap t^\downarrow = \emptyset$, then (s, X_{s^*s}) and (t, X_{t^*t}) are disjoint neighborhoods of $[s, x]$ and $[t, x]$, respectively, since if $[s, z] = [t, z]$ then there exists $u \leqslant s, t$. So we are left with the case $s^\downarrow \cap t^\downarrow \neq \emptyset$. Since S is a weak semilattice, we can find elements $u_1, \ldots, u_n \in S$ so that $u \leqslant s, t$ if and only if $u \leqslant u_i$ for some $i = 1, \ldots, n$. Let $V = X \setminus (\bigcup_{i=1}^n X_{u_i^* u_i})$; it is an open set by hypothesis. If $x \in X_{u_i^* u_i}$ for some i, then since $u_i \leqslant s, t$, it follows $[s, x] = [t, x]$, a contradiction. Thus $x \in V$. Define $W = V \cap X_{s^*s} \cap X_{t^*t}$. We claim (s, W) and (t, W) are disjoint neighborhoods of $[s, x]$ and $[t, x]$, respectively. Indeed, if $[r, z] \in (s, W) \cap (t, W)$, then $[s, z] = [r, z] = [t, z]$ and hence there exists $u \leqslant s, t$ with $z \in X_{u^*u}$. But then $u \leqslant u_i$ for some $i = 1, \dots, n$ and so $z \in X_{u_i^* u_i}$, contradicting that $z \in W \subseteq V$.

Trivially (2) implies (3). For (3) implies (1), let $s, t \in S$ and suppose $s^\downarrow \cap t^\downarrow \neq \emptyset$. Proposition 3.7 implies that $(s, D(s^*s)) \cap (t, D(t^*t))$ is compact. But clearly

$$\bigl(s, D(s^*s)\bigr) \cap \bigl(t, D(t^*t)\bigr) = \bigcup_{u \in s^\downarrow \cap t^\downarrow} \bigl(u, D(u^*u)\bigr)$$

since $[s, x] = [t, x]$ if and only if there exists $u \leqslant s, t$ with $x \in D(u^*u)$. By compactness, we may find $u_1, \dots, u_n \in s^\downarrow \cap t^\downarrow$ so that

$$\bigl(s, D(s^*s)\bigr) \cap \bigl(t, D(t^*t)\bigr) = \bigl(u_1, D\bigl(u_1^* u_1\bigr)\bigr) \cup \cdots \cup \bigl(u_n, D\bigl(u_n^* u_n\bigr)\bigr).$$

We claim that $u_1, \dots, u_n$ generate the downset $s^\downarrow \cap t^\downarrow$. Indeed, if $u \leqslant s, t$, then we have $[u, \chi_{(u^*u)^\uparrow}] \in (s, D(s^*s)) \cap (t, D(t^*t))$ and so $[u, \chi_{(u^*u)^\uparrow}] \in (u_i, D(u_i^* u_i))$ for some i. But then $u \leqslant u_i$ by Lemma 5.16. This completes the proof. □

Examples of semigroups that are weak semilattices include E-unitary and 0-E-unitary inverse semigroups [14,8,23,15]. Notice that if $s \in S$, then the map $x \mapsto x^*x$ provides an order isomorphism between $s^\downarrow$ and $(s^*s)^\downarrow$; the inverse takes e to se. Recall that a poset is called *Noetherian* if it satisfies ascending chain condition on downsets, or equivalent every downset is finitely generated. If each principal downset of $E(S)$ is Noetherian, it then follows from the above discussion that S is a weak semilattice. This occurs of course if $E(S)$ is finite, or if each principal downset of $E(S)$ is finite. More generally, if each principal downset of $E(S)$ contains no infinite ascending chains and no infinite anti-chains, then S is a weak semilattice.

Example 5.18 *(A non-Hausdorff groupoid).* The following example can be found in [23, Appendix C]. Define a commutative inverse semigroup $S = \mathbb{N} \cup \{\infty, z\}$ by inflating ∞ to a cyclic group $\{\infty, z\}$ of order 2 in the example just after Proposition 2.5. Here $\mathbb{N} \cup \{\infty\}$ is the semilattice considered there with $0 < i < \infty$ for $i \geqslant 1$ and all other elements incomparable. One has $\{\infty, z\}$ is a cyclic group of order 2 with non-trivial element z. The element z acts as the identity on $\mathbb{N}$. Then $\infty^\downarrow \cap z^\downarrow = \mathbb{N}$ is not finitely generated as a downset, in fact the positive naturals are an infinite anti-chain of maximal elements. It follows S is not a weak semilattice and so $\mathscr{G}(S)$ is not Hausdorff. Moreover, the compact open slice $(z, D(\infty))$ is not closed and hence $\chi_{(z, D(\infty))}$ is a discontinuous element of $K\mathscr{G}(S)$.

6. The isomorphism of algebras

The main theorem of this section says that if K is any unital commutative ring endowed with the discrete topology and S is an inverse semigroup, then $KS \cong K\mathscr{G}(S)$. The idea is to combine Paterson's proof for C^*-algebras [23] with the author's proof for inverse semigroups with finitely many idempotents [30,31]. Recall that the semigroup algebra KS is the free K-module with basis S equipped with the usual convolution product

$$\sum_{s\in S} c_s s \cdot \sum_{t\in S} d_t t = \sum_{s,t\in S} c_s d_t st.$$

In the case that $K = \mathbb{C}$, we make $\mathbb{C}S$ into a $*$-algebra by taking

$$\left(\sum_{s\in S} c_s s\right)^* = \sum_{s\in S} \overline{c_s} s^*.$$

We begin with a lemma that is an easy consequence of Rota's theory of Möbius inversion [28,27].

Lemma 6.1. *Let P be a finite poset. Then the set $\{\chi_{p\downarrow} \mid p \in P\}$ is a basis for K^P.*

Proof. The functions $\{\delta_p \mid p \in P\}$ form the standard basis for K^P. With respect to this basis

$$\chi_{p\downarrow} = \sum_{q\leqslant p} \delta_q.$$

Thus by Möbius inversion,

$$\delta_p = \sum_{q\leqslant p} \chi_{q\downarrow}\mu(q, p)$$

where μ is the Möbius function of P. This proves the lemma. $\square$

Alternatively, one can order $P = \{p_1, \ldots, p_n\}$ so that $p_i \leqslant p_j$ implies $i \leqslant j$. The linear transformation $p_i \mapsto \sum_{p_j\leqslant p_i} p_j$ is given by a unitriangular integer matrix and hence is invertible over any commutative ring with unit.

As a corollary, we obtain the following infinitary version.

Corollary 6.2. *Let P be a poset. Then the set $\{\chi_{p\downarrow} \mid p \in P\}$ in K^P is linearly independent.*

Proof. It suffices to show that, for any finite subset $F \subseteq P$, the set $F' = \{\chi_{p\downarrow} \mid p \in F\}$ is linearly independent. Consider the projection $\pi : K^P \to K^F$ given by restriction. Lemma 6.1 implies that $\pi(F')$ is a basis for K^F. We conclude that F' is linearly independent. $\square$

We are now ready for one of our main theorems, which generalizes the results of [30,31] for the case of an inverse semigroup with finitely many idempotents.

Theorem 6.3. *Let K be a commutative ring with unit and S an inverse semigroup. Then the homomorphism $\Theta : S \to K\mathscr{G}(S)$ given by $\Theta(s) = \chi_{(s,D(s^*s))}$ extends to an isomorphism of KS with $K\mathscr{G}(S)$. Moreover, when $K = \mathbb{C}$ the map Θ extends to a $*$-isomorphism.*

Proof. Proposition 5.13 establishes everything except the injectivity of the induced homomorphism $\Theta : KS \to K\mathscr{G}(S)$. This amounts to showing that the set of elements $\{\Theta(s) \mid s \in S\}$ is linearly independent. The key idea is to exploit the dense embedding of the underlying groupoid of S as a subgroupoid of $\mathscr{G}(S)$ from Lemma 5.16. More precisely, Lemma 5.16 provides an injective mapping $S \to \mathscr{G}(S)$ given by $s \mapsto [s, \chi_{(s^*s)\uparrow}] = \widehat{s}$.

Define a K-linear map $\psi : K\mathscr{G} \to K^S$ by $\psi(f)(s) = f(\widehat{s})$. Then, if $t \in S$, one has that $\psi(\Theta(t)) = \chi_{t\downarrow}$ by Lemma 5.16. Corollary 6.2 now implies that $\psi(\Theta(S))$ is linearly independent and hence $\Theta(S)$ is linearly independent, completing the proof. □

In the case that $E(S)$ is finite, one has that $\mathscr{G}(S)$ is the underlying groupoid and so we recover the following result of the author [30,31] (the final statement of which a proof can be found in these references).

Corollary 6.4. *Let S be an inverse semigroup so that $E(S)$ is finite and suppose K is a commutative ring with unit. Let $\overline{S} = \{\bar{s} \mid s \in S\}$ be a disjoint copy of S. Endow $K\overline{S}$ with a multiplicative structure by putting*

$$\bar{s} \cdot \bar{t} = \begin{cases} \overline{st}, & s^*s = tt^*, \\ 0, & \textit{else}. \end{cases}$$

Then there is an isomorphism from KS to $K\overline{S}$ sending s to $\sum_{t \leqslant s} \bar{t}$. Hence KS is isomorphic to a finite direct product of finite dimensional matrix algebras over the group algebras of maximal subgroups of S.

The special case where $S = E(S)$ was first proved by Solomon [27]. As a consequence of Theorem 6.3 and Proposition 4.11, we obtain the following topological criterion for an inverse semigroup algebra to have a unit as well as a characterization of the center of KS.

Corollary 6.5. *Let K be a commutative ring with unit and S an inverse semigroup. Then KS has a unit if and only if $\widehat{E(S)}$ is compact. The center of KS is the space of class functions on $\mathscr{G}(S)$.*

Let $FI(X)$ be the free inverse monoid on a set X with $|X| \geqslant 1$. Crabb and Munn described the center of $KFI(X)$ in [7]. We give a topological proof of their result using that $KFI(X) \cong K\mathscr{G}(FI(X))$ by describing explicitly the class functions on $\mathscr{G}(FI(X))$. The reader should consult [14] for the description of elements of $FI(X)$ as Munn trees.

Theorem 6.6. *Let X be a non-empty set. Then if $|X| = \infty$, the center of $KFI(X)$ consists of the scalar multiples of the identity. Otherwise, $Z(KFI(X))$ is a subalgebra of $KE(FI(X))$ isomorphic to the algebra of functions $f : FI(X)/\mathscr{D} \to K$ spanned by the finitely supported functions and the constant map to* 1.

Proof. The structure of $\mathscr{G}(FI(X))$ is well known cf. [23, Chapter 4] or [32]. Let $F(X)$ be the free group on X and denote its Cayley graph by Γ. Let $\mathscr{T}$ be the space of all subtrees of Γ containing 1. Viewing a subtree as the characteristic function of a map $V(\Gamma) \cup E(\Gamma) \to \{0, 1\}$, we give $\mathscr{T}$ the topology of pointwise convergence. It is easy to see that $\mathscr{T}$ is a closed subspace of $\{0, 1\}^{V(\Gamma)\cup E(\Gamma)}$. The space $\mathscr{T}$ is homeomorphic to the character space of $E(FI(X))$. The groupoid $\mathscr{G} = \mathscr{G}(FI(X))$ consists of all pairs $(w, T) \in F(X) \times \mathscr{T}$ so that $w \in V(T)$. The topology is the product topology. In particular, the pairs (w, T) with T finite are dense in $\mathscr{G}$. One has $d(w, T) = w^{-1}T$, $r(w, T) = T$ and the product is defined by $(w, T)(w', T') = (ww', T)$ if $w^{-1}T = T'$. The inverse is given by $(w, T)^{-1} = (w^{-1}, w^{-1}T)$. The groupoid $\mathscr{G}$ is Hausdorff and so $K\mathscr{G}$ consists of continuous functions with compact support in the usual sense.

Let f be a class function. We claim that the support of f is contained in $\mathscr{G}^0 = \mathscr{T}$. Since f is continuous with compact support and K is discrete, it follows that $f^{-1}(K \setminus \{0\})$ is compact open and hence the support of f. Thus $f^{-1}(K \setminus \{0\})$ is of the form $(\{w_1\} \times C_1) \cup \cdots \cup (\{w_m\} \times C_m)$ where the C_i are compact open subsets of $\mathscr{T}$. Suppose that $(w, T) \in \{w_i\} \times C_i$ with $w \neq 1$, and so in particular $w = w_i$. As $\{w_i\} \times C_i$ is open and the finite trees are dense, there is a finite tree T' containing 1 and w so that (w, T') belongs $\{w_i\} \times C_i$. But no finite subtree of Γ is invariant under a non-trivial element of $F(X)$, so $d(w, T') = w^{-1}T' \neq T' = r(w, T')$ and hence $f(w, T') = 0$ as f is a class function. This contradiction shows that the support of f is contained in $\mathscr{G}^0$.

Thus we may from now on view f as a continuous function with compact support on $\mathscr{T}$. Next observe that if $f(T) = c$ for a tree T and $u \in V(T)$, then $f(u^{-1}T) = c$. Indeed, $d(u, T) = u^{-1}T$ and $r(u, T) = T$. Thus $f(u^{-1}T) = f((u, T)^{-1}(1, T)(u, T)) = f(T) = c$ as f is a class function.

Let f be a class function. Since K is discrete, $f = c_1\chi_{U_1} + \cdots + c_k\chi_{U_k}$ where $U_1, \ldots, U_k$ are non-empty disjoint compact open subsets of $\mathscr{T}$ and $c_1, \ldots, c_k$ are distinct non-zero elements of K. It is easy to see then that $\chi_{U_1}, \ldots, \chi_{U_k}$ must then be class functions. In other words, the class functions are spanned by the characteristic functions χ_U of compact open subsets U of $\mathscr{T}$ so that $T \in U$ implies $u^{-1}T \in U$ for all vertices u of T.

Suppose first that X is infinite. We claim that no proper non-empty compact open subset U of $\mathscr{T}$ has the above property. Suppose this is not the case. Then there is a subtree T_0 that does not belong to U. Since X is infinite and U is determined by a boolean formula which is a finite disjunction of allowing/disallowing finitely many vertices and edges of Γ, there is a letter $x \in X$ so that no vertex or edge in the boolean formula determining U involves the letter x. Let $T \in U$. Then $T \cup xT_0 \in U$ since the edges and vertices appearing in xT_0 are irrelevant in the definition of U and $T \in U$. Thus $x^{-1}(T \cup xT_0) = x^{-1}T \cup T_0 \in U$. But since the edges and vertices appearing $x^{-1}T$ again are irrelevant to the boolean formula defining U, we must have $T_0 \in U$, a contradiction.

Next suppose that X is finite. First note that the finite trees form a discrete subspace of $\mathscr{T}$. Indeed, if T is a finite subtree of Γ containing 1, then since X is finite there are only finitely many edges of Γ incident on T that do not belong to it. Then the neighborhood of T consisting of all subtrees containing T but none of these finitely many edges incident on T consists only of $\{T\}$. So if T is a finite tree, then $U_T = \{v^{-1}T \mid v \in T\}$ is a finite open subset of $\mathscr{T}$ and hence its characteristic function belongs to the space of class functions. We claim that the space of class functions has basis the functions of the form χ_{U_T} with T a finite subtree of Γ containing 1 and of the identity $\chi_{\mathscr{G}^0}$. This will prove the theorem since the sets of the form U_T are in bijection with the $\mathscr{D}$-classes of S.

So let U be a compact open set so that $T \in U$ and $u \in T$ implies $u^{-1}T \in U$. Suppose that U contains only finite trees. Since the finite trees are discrete in $\mathscr{T}$ by the above argument, it follows that U is finite. The desired claim now follows from the above case. So we may assume that U contains an infinite tree T. Since X is finite, it is easy to see that there exists $N > 0$ so that U consists of those subtrees of Γ whose closed ball of radius N around 1 belongs to a certain subset F of the finite set of possible closed balls of radius N of an element of $\mathscr{T}$. We claim that

U contains all infinite trees. Then applying the previous case to the complement of U proves the theorem.

Suppose first $|X| = 1$. Then some translate of the infinite subtree T has closed ball of radius N around 1 a path of length $2N$ centered around 1 and so this closed ball belongs to F. However, all infinite subtrees of Γ have this closed ball as the ball of radius N around 1 for some translate. Thus U contains all the infinite subtrees.

Next suppose $|X| \geqslant 2$. Let T' be an infinite tree with closed ball B of radius N around 1 and let v be a leaf of B at distance N from 1 (such exists since T' is infinite and X is finite). Then there is a unique edge of B with terminal vertex v, let us assume it is labeled by x^ϵ with $x \in X$ and $\epsilon = \pm 1$. Since T is infinite, we can find a vertex u of T at distance N from 1. Let T_0 be the closed ball of radius N in T around 1. Then in T_0, the vertex u is the endpoint of a unique edge of Γ. If this edge is not labeled by x^ϵ, then put $T_1 = T_0 \cup uv^{-1}B$. Otherwise, choose $y \in X$ with $y \neq x$ and put $T_1 = T_0 \cup \{u \xrightarrow{y} uy\} \cup uyv^{-1}B$. In either case, the closed balls of radius N around 1 in T_1 and T coincide, and so $T_1 \in U$. But there is a translate T_2 of T_1 so that the closed ball of radius N about 1 in T_2 is exactly B. Thus $B \in F$ and so $T' \in U$. This completes the proof of the theorem. $\square$

Let $\mathscr{G}$ be an ample groupoid and $C_c(\mathscr{G})$ be the usual algebra of continuous (complex-valued) functions with compact support on $\mathscr{G}$ [8,23]. Notice that $\mathbb{C}\mathscr{G}$ is a subalgebra of $C_c(\mathscr{G})$ since any continuous function to $\mathbb{C}$ with respect to the discrete topology is continuous in the usual topology. Let $\|\cdot\|$ be the C^*-norm on $C_c(\mathscr{G})$ [8,23]. The following is essentially [23, Proposition 2.2.7].

Proposition 6.7. *Let $\mathscr{G}$ be an ample groupoid with $\mathscr{G}^0$ countably based. Then $C^*(\mathscr{G})$ is the completion of $\mathbb{C}\mathscr{G}$ with respect to its own universal C^*-norm.*

Proof. To prove the theorem, it suffices to verify that any non-degenerate $*$-representation $\pi : \mathbb{C}\mathscr{G} \to \mathscr{B}(H)$ with H a (separable) Hilbert space extends uniquely to $C_c(\mathscr{G})$. Indeed, this will show that $\mathbb{C}\mathscr{G}$ is dense in the C^*-norm on $C_c(\mathscr{G})$ and that the restriction of the C^*-norm on $C_c(\mathscr{G})$ to $\mathbb{C}\mathscr{G}$ is its own C^*-norm. Suppose V is an open neighborhood in $\mathscr{G}$ and $f \in C_c(V)$. Then since $\mathscr{G}$ has a basis of compact open subsets, we can cover the compact support of f by a compact open subset U. Thus it suffices to define the extension of π for any $f \in C(U)$ where U is a compact open subset of $\mathscr{G}$. Since U has a basis of compact open subsets, the continuous functions on U with respect to the discrete topology separate points. The Stone–Weierstrass theorem implies that we can find a sequence $f_n \in \mathbb{C}\mathscr{G}$ with support in U so that $\|f_n - f\|_\infty \to 0$. Now the argument of [8, Proposition 3.14] shows that if $g \in C(U) \cap \mathbb{C}\mathscr{G}$, then $\|\pi(g)\| \leqslant \|g\|_\infty$. It follows that $\pi(f_n)$ is a Cauchy sequence and so has a limit that we define to be $\pi(f)$. It is easy to check that $\pi(f)$ does not depend on the Cauchy sequence by a simple interweaving argument.

Routine verification shows that π is a representation of $C_c(\mathscr{G})$ (cf. [23]). For instance, if $f \in C(U)$ and $g \in C(V)$ with U and V compact open and if $f_n \to f$ and $g_n \to g$, where $\{f_n\}, \{g_n\} \subseteq \mathbb{C}\mathscr{G}$ are supported on U and V, respectively, then $\pi(f+g)$ is by definition

$$\lim \pi(f_n + g_n) = \lim \pi(f_n) + \lim \pi(g_n) = \pi(f) + \pi(g).$$

Similarly, $\pi(fg) = \pi(f)\pi(g)$. Continuity follows from the definition. It is the unique extension since if π' is another extension, then [8, Proposition 3.14] implies $\|\pi'(g)\| \leqslant \|g\|_\infty$ for $g \in C(U)$. Thus $\|\pi(f_n) - \pi'(f)\| \to 0$ and so $\pi'(f) = \pi(f)$. $\square$

We now recover an important result of Paterson [23].

Corollary 6.8 *(Paterson). Let S be a countable inverse semigroup. Then there is an isomorphism $C^*(S) \cong C^*(\mathscr{G}(S))$ of universal of C^*-algebras.*

Paterson also established an isomorphism of reduced C^*-algebras [23].

7. Irreducible representations

Our aim is to construct the finite dimensional irreducible representations of an arbitrary inverse semigroup over a field K and determine when there are enough such representations to separate points. Our method can be viewed as a generalization of the groupoid approach of the author [30,31] to the classical theory of Munn and Ponizovsky [5] for inverse semigroups with finitely many idempotents. See also [10]. We begin by describing all the finite dimensional irreducible representations of an ample groupoid. The desired result for inverse semigroups is deduced as a special case via the universal groupoid.

In fact, much of what we do works over an arbitrary commutative ring with unit K, which shall remain fixed for the section. Let A be a K-algebra. We say that an A-module M is *non-degenerate* if $AM = M$. We consider here only the category of non-degenerate A-modules. So when we write the words "simple module," this should be understood as meaning non-degenerate simple module. Note that if A is unital, then an A-module is non-degenerate if and only if the identity of A acts as the identity endomorphism. A representation of A is said to be *non-degenerate* if the corresponding module is non-degenerate. As usual, there is a bijection between isomorphism classes of (finite dimensional) simple A-modules and equivalence classes of non-degenerate (finite dimensional) irreducible representations of A by K-module endomorphisms.

7.1. Irreducible representations of ample groupoids

Fix an ample groupoid $\mathscr{G}$. Then one can speak about the orbit of an element of $\mathscr{G}^0$ and its isotropy group.

Definition 7.1 *(Orbit).* Define an equivalence relation on $\mathscr{G}^0$ by setting $x \sim y$ if there is an arrow $g \in \mathscr{G}$ such that $d(g) = x$ and $r(g) = y$. An equivalence class will be called an *orbit*. If $x \in \mathscr{G}^0$, then

$$G_x = \{g \in \mathscr{G} \mid d(g) = x = r(g)\}$$

is called the *isotropy group* of $\mathscr{G}$ at x. It is well known and easy to verify that up to conjugation in $\mathscr{G}$ (and hence isomorphism) the isotropy group of x depends only on the orbit of x. Thus we may speak unambiguously of the isotropy group of the orbit.

To motivate the terminology, if G is a group acting on a space X, then the orbit of $x \in X$ in the groupoid $G \ltimes X$ is exactly the orbit of x in the usual sense. Moreover, the isotropy group of $G \ltimes X$ at x is isomorphic to the stabilizer in G of x (i.e., the usual isotropy group).

Remark 7.2 *(Underlying groupoid).* If S is an inverse semigroup and $\mathscr{G}$ is its underlying groupoid, then the orbit of $e \in E(S)$ is precisely the set of idempotents of S that are $\mathscr{D}$-equivalent to e and the isotropy group G_e is the maximal subgroup at e [14].

In an ample groupoid, the orbit of a unit is its orbit under the action of $\mathscr{G}^a$ described earlier. Indeed, given $d(g) = x$ and $r(g) = y$, choose a slice $U \in \mathscr{G}^a$ containing g. Clearly, we have $Ux = y$. Conversely, if U is a slice with $Ux = y$, then we can find $g \in U$ with $d(g) = x$ and $r(g) = y$.

The following lemma seems worth noting, given the importance of finite orbits in what follows. One could give a topological proof along the lines of [29] since this is essentially the same argument used in computing the fundamental group of a cell complex.

Lemma 7.3. *Let S be an inverse semigroup with generating set A acting non-degenerately on a space X and let $\mathscr{O}$ be the orbit of $x \in X$. Let $\mathscr{G} = S \ltimes X$ be the groupoid of germs of the action. Fix, for each $y \in \mathscr{O}$, an element $p_y \in S$ so that $p_y x = y$ where we choose p_x to be an idempotent. For each pair $(a, y) \in A \times \mathscr{O}$ such that $ay \in \mathscr{O}$, define $g_{a,y} = [p^*_{ay} a p_y, x] \in \mathscr{G}$. Then the isotropy group G_x of $\mathscr{G}$ at x is generated by the set of elements $\{g_{a,y} \mid a \in A,\ y, ay \in \mathscr{O}\}$. In particular, if A and $\mathscr{O}$ are finite, then G_x is finitely generated.*

Proof. First note that if $ay \in \mathscr{O}$, then $p^*_{ay} a p_y x = p^*_{ay} ay = x$ and so $g_{a,y} \in G_x$. Let us define, for $a \in A$ and $y \in \mathscr{O}$ with $a^* y \in \mathscr{O}$, the element $g_{a^*,y} = [p^*_{a^*y} a^* p_y, x] \in \mathscr{G}^a$. Notice that $g_{a^*,y} = g^{-1}_{a,a^*y}$.

Suppose that $[s, x] \in G_x$ and write $s = a_n \cdots a_1$ with the $a_i \in A \cup A^*$. Define $x_i = a_i \cdots a_1 x$, for $0 = 1, \ldots, n$ (so $x_0 = x = x_n$) and consider the element $t = (p^*_{x_n} a_n p_{x_{n-1}}) \cdots (p^*_{x_2} a_2 p_{x_1})(p^*_{x_1} a_1 p_{x_0})$ of S. Then $t \leqslant s$ and $tx = sx = x$. Hence $[s, x] = [t, x] = g_{a_n,x_{n-1}} \cdots g_{a_1,x_0}$, as required. □

Applying this to the Munn representation and the underlying groupoid, we obtain the following folklore result (a simple topological proof of which can be found in [29]).

Corollary 7.4. *Let S be a finitely generated inverse semigroup and e an idempotent whose $\mathscr{D}$-class contains only finitely many idempotents. Then the maximal subgroup G_e of S at e is finitely generated.*

We remark that if we consider the spectral action of S on $\widehat{E(S)}$ and $e \in E(S)$, then the orbit of $\chi_{e\uparrow}$ is $\{\chi_{f\uparrow} \mid f \mathrel{\mathscr{D}} e\}$ and $G_{\chi_{e\uparrow}} = G_e$.

Fix $x \in \mathscr{G}^0$. Define $L_x = d^{-1}(x)$ (inverse semigroup theorists should think of this as the $\mathscr{L}$-class of x). The isotropy group G_x acts on the right of L_x and L_x/G_x is in bijection with the orbit of x via the map $tG_x \mapsto r(t)$. Indeed, if $s, t \in L_x$, then $r(s) = r(t)$ implies $t^{-1}s \in G_x$ and of course $s = t(t^{-1}s)$. Conversely, every element of tG_x evidently has range $r(t)$.

There is also a natural action of $\mathscr{G}^a$ on the left of L_x that we shall call, in analogy with the case of inverse semigroups [5], the *Schützenberger representation* of $\mathscr{G}^a$ on L_x. If $U \in \mathscr{G}^a$, then we define a map

$$U\cdot : L_x \cap r^{-1}\bigl(U^{-1}U\bigr) \to L_x \cap r^{-1}\bigl(UU^{-1}\bigr)$$

by putting $Ut = st$ where s is the unique element of U with $d(s) = r(t)$ (or equivalent, $Ut = y$ where $yt^{-1} \in U$). We leave the reader to verify that this is indeed an action of $\mathscr{G}^a$ on L_x by partial bijections.

There is an alternative construction of L_x and G_x which will be quite useful in what follows. Let $\widetilde{L}_x = \{U \in \mathscr{G}^a \mid x \in U^{-1}U\}$ and put $\mathscr{G}^a_x = \{U \in \mathscr{G}^a \mid Ux = x\}$. Notice that $\widetilde{L}_x = \{U \in \mathscr{G}^a \mid$

$U \cap L_x \neq \emptyset\}$ and $\mathscr{G}^a_x = \{U \in \mathscr{G}^a \mid U \cap G_x \neq \emptyset\}$. It is immediate that $\mathscr{G}^a_x$ is an inverse subsemigroup of $\mathscr{G}^a$ acting on the right of $\widetilde{L}_x$. An element of $\widetilde{L}_x$ intersects L_x in exactly one element by the definition of a slice.

Lemma 7.5. *Define a map $\nu : \widetilde{L}_x \to L_x$ by $U \cap L_x = \{\nu(U)\}$. Then:*

(1) *ν is surjective;*
(2) *$\nu(U) = \nu(V)$ if and only if U and V have a common lower bound in $\widetilde{L}_x$;*
(3) *$\nu : \mathscr{G}^a_x \to G_x$ is the maximal group image homomorphism.*

Proof. If $t \in L_x$ and $U \in \mathscr{G}^a$ with $t \in U$, then $U \in \widetilde{L}_x$ with $\nu(U) = t$. This proves (1). For (2), trivially, if $W \subseteq U, V$ is a common lower bound in $\widetilde{L}_x$, then $\nu(U) = \nu(W) = \nu(V)$. Conversely, suppose that $\nu(U) = \nu(V) = t$. Then $U \cap V$ is an open neighborhood of t. Since $\mathscr{G}^a$ is a basis for the topology on $\mathscr{G}$, we can find $W \in \mathscr{G}^a$ with $t \in W \subseteq U \cap V$. As $W \in \widetilde{L}_x$, this yields (2).

Evidently, ν restricted to $\mathscr{G}^a_x$ is a group homomorphism. By (2), it is the maximal group image since any common lower bound in $\widetilde{L}_x$ of elements of $\mathscr{G}^a_x$ belongs to $\mathscr{G}^a_x$. This proves (3). □

Remark 7.6. In fact, ν gives a morphism from the right action of $\mathscr{G}^a_x$ on $\widetilde{L}_x$ to the right action of G_x on L_x.

Consider a free K-module KL_x with basis L_x. The right action of G_x on L_x induces a right KG_x-module structure on KL_x. Let T be a transversal for L_x/G_x. We assume $x \in T$.

Proposition 7.7. *The isotropy group G_x acts freely on the right of L_x and hence KL_x is a free right KG_x-module with basis T.*

Proof. It suffices to show that G_x acts freely on L_x. But this is clear since if $t \in L_x$ and $g \in G_x$, then $tg = t$ implies $g = t^{-1}t = x$. □

We now endow KL_x with the structure of a left $K\mathscr{G}$-module by linearly extending the Schützenberger representation. Formally, suppose $f \in K\mathscr{G}$ and $t \in L_x$. Define

$$ft = \sum_{y \in L_x} f\left(yt^{-1}\right)y. \tag{7.1}$$

Proposition 7.8. *If $U \in \mathscr{G}^a$ and $t \in L_x$, then*

$$\chi_U t = \begin{cases} Ut, & r(t) \in U^{-1}U, \\ 0, & \textit{else}. \end{cases} \tag{7.2}$$

Consequently, KL_x is a well-defined $K\mathscr{G}$-KG_x bimodule.

Proof. Since $K\mathscr{G}$ is spanned by characteristic functions of elements of $\mathscr{G}^a$ (Proposition 4.3) in order to show that (7.1) is a finite sum, it suffices to verify (7.2). If $r(t) \notin U^{-1}U$, then $\chi_U(yt^{-1}) = 0$ for all $y \in L_x$. On the other hand, suppose $r(t) \in U^{-1}U$ and say $r(t) = d(s)$ with $s \in U$. Then $Ut = st$ and $y = st$ is the unique element of L_x with $yt^{-1} \in U$. Hence

$\chi_U t = y = st = Ut$. Since the Schützenberger representation is an action, it follows that (7.1) gives a well-defined left module structure to KL_x.

To see that KL_x is a bimodule, it suffices to verify that if $f \in \mathscr{G}^a$, $g \in G_x$ and $t \in L_x$, then $(ft)g = f(tg)$. This is shown by the following computation:

$$(ft)g = \left(\sum_{y \in L_x} f\left(yt^{-1}\right)y\right)g = \sum_{y \in L_x} f\left(yt^{-1}\right)yg,$$

$$f(tg) = \sum_{z \in L_x} f\left(zg^{-1}t^{-1}\right)z = \sum_{y \in L_x} f\left(yt^{-1}\right)yg$$

where the final equality of the second equation is a consequence of the change of variables $y = zg^{-1}$. $\square$

We are now prepared to define induced modules.

Definition 7.9 *(Induction).* For $x \in \mathscr{G}^0$ and V a KG_x-module, we define the corresponding *induced $K\mathscr{G}$-module* to be $\mathrm{Ind}_x(V) = KL_x \otimes_{KG_x} V$.

The induced modules coming from elements of the same orbit coincide. More precisely, if y is in the orbit of x with, say, $d(s) = x$ and $r(s) = y$ and if V is a KG_x-module, then V can be made into a KG_y-module by putting $gv = s^{-1}gsv$ for $g \in G_y$ and $v \in V$. Then $\mathrm{Ind}_x(V) \cong \mathrm{Ind}_y(V)$ via the map $t \otimes v \mapsto ts^{-1} \otimes v$ for $t \in L_x$ and $v \in V$.

The following result, and its corollary, will be essential to studying induced modules.

Proposition 7.10. *Let $t, u, s_1, \ldots, s_n \in L_x$ with $s_1, \ldots, s_n \notin tG_x$. Then there exists $U \in \mathscr{G}^a$ so that $\chi_U t = u$ and $\chi_U s_i = 0$ for $i = 1, \ldots, n$.*

Proof. The set $B(\mathscr{G}^0)$ is a basis for the topology of $\mathscr{G}^0$. Hence we can find $U_0 \in B(\mathscr{G}^0)$ so that $U_0 \cap \{r(t), r(s_1), \ldots, r(s_n)\} = \{r(t)\}$. Choose $U \in \mathscr{G}^a$ so that $ut^{-1} \in U$. Replacing U by UU_0 if necessary, we may assume that $r(s_i) \notin U^{-1}U$ for $i = 1, \ldots, n$. Then $Ut = ut^{-1}t = u$ and so $\chi_U t = u$ by Proposition 7.8. On the other hand, Proposition 7.8 provides $\chi_U s_i = 0$, for $i = 1, \ldots, n$. This completes the proof. $\square$

An immediate corollary of the proposition is the following.

Corollary 7.11. *The module KL_x is cyclic, namely $KL_x = K\mathscr{G} \cdot x$. Consequently, if V is a KG_x-module, then $KL_x \otimes_{KG_x} V = K\mathscr{G} \cdot (x \otimes V)$.*

It is easy to see that Ind_x is a functor from the category of KG_x-modules to the category of $K\mathscr{G}$-modules. We now consider the restriction functor from $K\mathscr{G}$-modules to KG_x-modules, which is right adjoint to the induction functor.

Definition 7.12 *(Restriction).* For $x \in \mathscr{G}^0$, let $\mathscr{N}_x = \{U \in B(\mathscr{G}^0) \mid x \in U\}$. If V is a $K\mathscr{G}$-module, then define $\mathrm{Res}_x(V) = \bigcap_{U \in \mathscr{N}_x} UV$ where we view V as a $K\mathscr{G}^a$-module via $Uv = \chi_U v$ for $U \in \mathscr{G}^a$ and $v \in V$.

In order to endow $\mathrm{Res}_x(V)$ with the structure of a KG_x-module, we need the following lemma.

Lemma 7.13. *Let V be a $K\mathscr{G}$-module and put $W = \mathrm{Res}_x(V)$. Then:*

(1) *$K\mathscr{G}_x^a \cdot W = W$;*
(2) *If $U \notin \widetilde{L}_x$, then $UW = \{0\}$;*
(3) *Let $U, U' \in \widetilde{L}_x$ be such that $\nu(U) = \nu(U')$. Then $Uw = U'w$ for all $w \in W$.*

Proof. To prove (1), first observe that $\mathscr{N}_x \subseteq \mathscr{G}_x^a$ so $W \subseteq K\mathscr{G}_x^a \cdot W$. For the converse, suppose that $U \in \mathscr{G}_x^a$ and $w \in W$. Let $U_0 \in \mathscr{N}_x$. Then $U_0Uw = U(U^{-1}U_0U)w = Uw$ since $x \in U^{-1}U_0U$ and $w \in W$. Since U_0 was arbitrary, we conclude that $Uw \in W$.

Turning to (2), suppose that $w \in W$ and $Uw \neq 0$. Then $U^{-1}Uw \neq 0$. Suppose $U_0 \in \mathscr{N}_x$. Then $U_0U^{-1}Uw = U^{-1}UU_0w = U^{-1}Uw$. Hence the stabilizer in $B(\mathscr{G}^0)$ of $U^{-1}Uw$ is a proper filter containing the ultrafilter $\mathscr{N}_x$ and the element $U^{-1}U$. We conclude that $U^{-1}U \in \mathscr{N}_x$ and so $U \in \widetilde{L}_x$.

Next, we establish (3). If $\nu(U) = \nu(U')$, then U and U' have a common lower bound $U_0 \in \widetilde{L}_x$ by Lemma 7.5. Hence, for any $w \in W$, we have $Uw = UU_0^{-1}U_0w = U_0w = U'U_0^{-1}U_0w = U'w$ as $U_0^{-1}U_0 \in \mathscr{N}_x$. This completes the proof. □

As a consequence of Lemmas 7.5 and 7.13, for $t \in L_x$ and $w \in \mathrm{Res}_x(V)$, there is a well-defined element tw obtained by putting $tw = Uw$ where $U \in \mathscr{G}^a$ contains t. Trivially, the map $w \mapsto tw$ is linear. Moreover, if $g \in G_x$ and $g \in U \in \mathscr{G}^a$, then $U \in \mathscr{G}_x^a$. Hence this definition gives W the structure of a KG_x-module since the action of $\mathscr{G}_x^a$ on W factors through its maximal group image G_x by the aforementioned lemmas. In particular, $xw = w$ for $w \in W$. Let us now prove that if V is a simple $K\mathscr{G}$-module and $\mathrm{Res}_x(V) \neq 0$, then it is simple.

Lemma 7.14. *Let V be a simple $K\mathscr{G}$-module. Then the KG_x-module $\mathrm{Res}_x(V)$ is either zero or a simple KG_x-module.*

Proof. Set $W = \mathrm{Res}_x(V)$ and suppose that $0 \neq w \in W$. We need to show that $KG_x \cdot w = W$. Let $w' \in W$. Viewing V as a $K\mathscr{G}^a$-module, we have $K\mathscr{G}^a \cdot w = V$ by simplicity of V. Hence $w' = (c_1U_1 + \cdots + c_nU_n)w$ with $U_1, \ldots, U_n \in \mathscr{G}^a$. Moreover, by Lemma 7.13 we may assume $U_1, \ldots, U_n \in \widetilde{L}_x$. Let $t_i = \nu(U_i)$. Choose $U \in \mathscr{N}_x$ so that $r(t_i) \in U$ implies $r(t_i) = x$. Then $w' = Uw' = (c_1UU_1 + \cdots + c_nUU_n)w$. But $UU_i \in \widetilde{L}_x$ if and only if $r(t_i) = x$, in which case $UU_i \in \mathscr{G}_x^a$. Thus $w' \in K\mathscr{G}_x^a \cdot w = KG_x \cdot w$. It follows that W is simple. □

Next we establish the adjunction between Ind_x and Res_x. Because the functor $KL_x \otimes_{KG_x} (-)$ is left adjoint to $\mathrm{Hom}_{K\mathscr{G}}(KL_x, -)$, it suffices to show the latter is isomorphic to Res_x.

Proposition 7.15. *The functors Res_x and $\mathrm{Hom}_{K\mathscr{G}}(KL_x, -)$ are naturally isomorphic. Thus Ind_x is the left adjoint of Res_x.*

Proof. Let V be a $K\mathscr{G}$-module and put $W = \mathrm{Res}_x(V)$. Then we define a homomorphism $\psi : \mathrm{Hom}_{K\mathscr{G}}(KL_x, V) \to W$ by $\psi(f) = f(x)$. First note that if $U \in \mathscr{N}_x$, then $Uf(x) = f(Ux) = f(x)$ and so $f(x) \in W$. Clearly ψ is K-linear. To see that it is G_x-equivariant, let $g \in G_x$ and choose $U \in \mathscr{G}_x^a$ with $g \in U$. Then observe, using Proposition 7.8, that $\psi(gf) = (gf)(x) =$

$f(xg) = f(g) = f(Ux) = Uf(x) = gf(x) = g\psi(f)$. This shows that ψ is a KG_x-morphism. If $\psi(f) = 0$, then $f(x) = 0$ and so $f(KL_x) = 0$ by Corollary 7.11. Thus ψ is injective. To see that ψ is surjective, let $w \in W$ and define $f : KL_x \to V$ by $f(t) = tw$, for $t \in L_x$, where tw is as defined after Lemma 7.13. Then $f(x) = xw = w$. It thus remains to show that f is a $K\mathscr{G}$-morphism. To achieve this, it suffices to show that $f(Ut) = Utw$ for $U \in \mathscr{G}^a$. Choose $U' \in \widetilde{L}_x$ with $t \in U'$; so $Utw = UU'w$ by definition. If $r(t) \notin U^{-1}U$, then $Ut = 0$ (by Proposition 7.8) and $UU' \notin \widetilde{L}_x$. Thus $f(Ut) = 0$, whereas $Utw = UU'w = 0$ by Lemma 7.13. On the other hand, if $r(t) \in U^{-1}U$ and say $d(s) = r(t)$ with $s \in U$, then $Ut = st$ and $st \in UU' \in \widetilde{L}_x$. Thus $f(Ut) = f(st) = (st)w$, whereas $Utw = UU'w = (st)w$. This completes the proof that f is a $K\mathscr{G}$-morphism and hence ψ is onto. It is clear that ψ is natural. □

It turns out that $\mathrm{Res}_x \mathrm{Ind}_x$ is naturally isomorphic to the identity functor.

Proposition 7.16. *Let V be a KG_x-module. Then* $\mathrm{Res}_x \mathrm{Ind}_x(V) = x \otimes V$ *is naturally isomorphic to V as a KG_x-module.*

Proof. Let T be a transversal for L_x/G_x with $x \in T$. Because T is a KG_x-basis for KL_x, it follows that $KL_x \otimes_{KG_x} V = \bigoplus_{t\in T}(t \otimes V)$. We claim that $\mathrm{Res}_x \mathrm{Ind}_x(V) = x \otimes V$. Indeed, if $U \in \mathscr{N}_x$, then $U(x \otimes v) = Ux \otimes v = x \otimes v$. Conversely, suppose $w = t_1 \otimes v_1 + \cdots + t_n \otimes v_n$ belongs to $\mathrm{Res}_x \mathrm{Ind}_x(V)$. Choose $U \in B(\mathscr{G}^0)$ so that $x \in U$ and $U \cap \{r(t_1), \ldots, r(t_n)\} \subseteq \{x\}$. Then, by Proposition 7.8, we have $w = Uw \in x \otimes V$, establishing the desired equality.

Now $x \otimes V$ is naturally isomorphic to V as a KG_x-module via the map $x \otimes v \mapsto v$ since if $g \in G$ and $U \in \mathscr{G}_x^a$ with $g \in U$, then $g(x \otimes v) = U(x \otimes v) = Ux \otimes v = g \otimes v = x \otimes gv$. □

A useful fact is that the induction functor is exact. In general, Res_x is left exact but it need not be right exact.

Proposition 7.17. *The functor* Ind_x *is exact, whereas* Res_x *is left exact.*

Proof. Since KL_x is a free KG_x-module, it is flat and hence Ind_x is exact. Clearly $\mathrm{Res}_x = \mathrm{Hom}_{K\mathscr{G}}(KL_x, -)$ is left exact. □

Our next goal is to show that if V is a simple KG_x-module, then the $K\mathscr{G}$-module $\mathrm{Ind}_x(V)$ is simple with a certain "finiteness" property, namely it is not annihilated by Res_x. Afterwards, we shall prove that all simple $K\mathscr{G}$-modules with this "finiteness" property are induced modules; this class of simple $K\mathscr{G}$-modules contains all the finite dimensional ones when K is a field. This is exactly what is done for inverse semigroups with finitely many idempotents in [30,31]. Here the proof becomes more technical because the algebra need not be unital. Also topology is used instead of finiteness arguments in the proof. The main idea is in essence that of [10]: to exploit the adjunct relationship between induction and restriction.

The following definition will play a key role in constructing the finite dimensional irreducible representations of an inverse semigroup.

Definition 7.18 *(Finite index).* Let us say that an object $x \in \mathscr{G}^0$ has *finite index* if its orbit is finite.

Proposition 7.19. *Let $x \in \mathscr{G}^0$ and suppose that V is a simple KG_x-module. Then* $\mathrm{Ind}_x(V)$ *is a simple $K\mathscr{G}$-module. Moreover, if K is a field, then* $\mathrm{Ind}_x(V)$ *is finite dimensional if and only* *if x has finite index and V is finite dimensional. Finally, if V and W are non-isomorphic KG_x-modules, then* $\mathrm{Ind}_x(V) \ncong \mathrm{Ind}_x(W)$.

Proof. We retain the notation above. Let T be a transversal for L_x/G_x with $x \in T$. Since L_x/G_x is in bijection with the orbit $\mathscr{O}$ of x, the set T is finite if and only if x has finite index. Because T is a KG_x-basis for KL_x, it follows that $KL_x \otimes_{KG_x} V = \bigoplus_{t\in T}(t \otimes V)$. In particular, $\mathrm{Ind}_x(V)$ is finite dimensional when K is a field if and only if T is finite and V is finite dimensional. This establishes the second statement. We turn now to the proof of simplicity.

Suppose that $0 \neq W$ is a $K\mathscr{G}$-submodule. Then $\mathrm{Res}_x(W)$ is a KG_x-submodule of $\mathrm{Res}_x \mathrm{Ind}_x(V) \cong V$. We claim that it is non-zero. Let $0 \neq w \in W$. Then $w = t_1 \otimes v_1 + \cdots + t_n \otimes v_n$ for some $v_1, \ldots, v_n \in V$ and $t_1, \ldots, t_n \in T$. Moreover, $v_j \neq 0$ for some j. By Proposition 7.10, we can find $U \in \mathscr{G}^a$ so that $\chi_U t_j = x$ and $\chi_U t_i = 0$ for $i \neq j$. Then $\chi_U w = x \otimes v_j \neq 0$ belongs to $\mathrm{Res}_x(W)$. Simplicity of V now yields $\mathrm{Res}_x \mathrm{Ind}_x(V) = \mathrm{Res}_x(W) \subseteq W$. Corollary 7.11 then yields

$$\mathrm{Ind}_x(V) = K\mathscr{G} \cdot (x \otimes V) = K\mathscr{G} \cdot \mathrm{Res}_x \mathrm{Ind}_x(V) \subseteq K\mathscr{G} \cdot W \subseteq W,$$

establishing the simplicity of $\mathrm{Ind}_x(V)$.

The final statement follows because $\mathrm{Res}_x \mathrm{Ind}_x$ is naturally equivalent to the identity functor. □

Next we wish to show that modules of the above sort obtained from distinct orbits are non-isomorphic.

Proposition 7.20. *Suppose that x, y are elements in distinct orbits. Then induced modules of the form* $\mathrm{Ind}_x(V)$ *and* $\mathrm{Ind}_y(W)$ *are not isomorphic.*

Proof. Put $M = \mathrm{Ind}_x(V)$ and $N = \mathrm{Ind}_y(W)$. Proposition 7.16 yields $\mathrm{Res}_x(M) \cong V \neq 0$. On the other hand, if $w = t_1 \otimes v_1 + \cdots + t_n \otimes v_n \in M$ is non-zero, then since $y \notin \{r(t_1), \ldots, r(t_n)\}$, we can find $U \in B(\mathscr{G}^0)$ so that $y \in U$ and $r(t_i) \notin U$, for $i = 1, \ldots, n$. Then $Uw = 0$ by Proposition 7.8. Thus we have $\mathrm{Res}_y(M) = 0$. Applying a symmetric argument to N shows that $M \ncong N$. □

To obtain the converse of Proposition 7.19, we shall use Stone duality. Also the inverse semigroup $\mathscr{G}^a$ will play a starring role since each $K\mathscr{G}$-module gives a representation of $\mathscr{G}^a$. What we are essentially doing is imitating the theory for finite inverse semigroups [5], as interpreted through [30,31], for ample groupoids; see also [10]. The type of simple modules which can be described as induced modules are what we shall term spectral modules.

Definition 7.21 *(Spectral module).* Let V be a non-zero $K\mathscr{G}$-module. We say that V is a *spectral module* if there is a point $x \in \mathscr{G}^0$ so that $\mathrm{Res}_x(V) \neq 0$.

Remark 7.22. It is easy to verify that $\mathrm{Res}_x(K\mathscr{G}) \neq 0$ if and only if x is an isolated point. On the other hand, the cyclic module KL_x satisfies $\mathrm{Res}_x(KL_x) = Kx$. This shows that in general Res_x is not exact. However, if x is an isolated point of $\mathscr{G}^0$, then $\mathrm{Res}_x(V) = \delta_x V$ and so the restriction functor is exact in this case.

Every induced module from a non-zero module is spectral due to the isomorphism $\mathrm{Res}_x \mathrm{Ind}_x(V) \cong V$. Let us show that the spectral assumption is not too strong a condition. In particular, we will establish that all finite dimensional modules over a field are spectral. Recall that if A is a commutative K-algebra, then the idempotent set $E(A)$ of A is a generalized boolean algebra with respect to the natural partial order. The join of e, f is given by $e \vee f = e + f - ef$ and the relative complement by $e \setminus f = e - ef$.

Proposition 7.23. *Let V be a $K\mathscr{G}$-module with associated representation $\varphi : K\mathscr{G} \to \mathrm{End}_K(V)$. Let $\alpha : \mathscr{G}^a \to \mathrm{End}_K(V)$ be the representation given by $U \mapsto \varphi(\chi_U)$. Assume that $B = \alpha(B(\mathscr{G}^0))$ contains a primitive idempotent. Then V is spectral. This occurs in particular if B is finite or more generally satisfies the descending chain condition.*

Proof. Let A be the subalgebra spanned by $\alpha(B(\mathscr{G}^0))$; so $A = \varphi(K\mathscr{G}^0)$. Then the map $\alpha : B(\mathscr{G}^0) \to E(A)$ is a morphism of generalized boolean algebras. Indeed, we compute $\alpha(U \cup V) = \varphi(\chi_{U \cup V}) = \varphi(\chi_U) + \varphi(\chi_V) - \varphi(\chi_{U \cap V}) = \alpha(U) + \alpha(V) - \alpha(U)\alpha(V) = \alpha(U) \vee \alpha(V)$. Thus B is a generalized boolean algebra. Stone duality provides a proper continuous map $\widehat{\alpha} : \mathrm{Spec}(B) \to \mathscr{G}^0$. So now let e be a primitive idempotent of B. Then $e^{\uparrow}$ is an ultrafilter on B and $\chi_{e^{\uparrow}} \in \mathrm{Spec}(B)$ by Proposition 2.7. Let $x = \widehat{\alpha}(\chi_{e^{\uparrow}})$. Then

$$\mathrm{Res}_x(V) = \bigcap_{f \in e^{\uparrow}} fV = eV \neq 0$$

completing the proof. □

The above proposition will be used to show that every finite dimensional representation over a field is spectral. Denote by $M_n(K)$ the algebra of $n \times n$-matrices over K. The following lemma is classical linear algebra.

Lemma 7.24. *Let K be a field and $F \leqslant M_n(K)$ a semilattice. Then we have $|F| \leqslant 2^n$.*

Proof. We just sketch the argument. If $e \in F$, then $e^2 = e$ and so the minimal polynomial of e divides $x(x-1)$. Thus e is diagonalizable. But commutative semigroups of diagonalizable matrices are easily seen to be simultaneously diagonalizable, so $F \leqslant K^n$. But the idempotent set of K is $\{0, 1\}$, so $F \leqslant \{0, 1\}^n$ and hence $|F| \leqslant 2^n$. □

Corollary 7.25. *Let $\varphi : K\mathscr{G} \to M_n(K)$ be a finite dimensional representation over a field K. Then $\alpha(B(\mathscr{G}^0))$ is finite where $\alpha : B(\mathscr{G}^0) \to M_n(K)$ is given by $\alpha(U) = \varphi(\chi_U)$. Consequently, every finite dimensional (non-zero) $K\mathscr{G}$-module is spectral.*

Now we establish the main theorem of this section.

Theorem 7.26. *Let $\mathscr{G}$ be an ample groupoid and fix $D \subseteq \mathscr{G}^0$ containing exactly one element from each orbit. Then there is a bijection between spectral simple $K\mathscr{G}$-modules and pairs (x, V) where $x \in D$ and V is a simple KG_x-module (taken up to isomorphism). The corresponding simple $K\mathscr{G}$-module is $\mathrm{Ind}_x(V)$. When K is a field, the finite dimensional simple $K\mathscr{G}$-modules correspond to those pairs (x, V) where x is of finite index and V is a finite dimensional simple KG_x-module.*

Proof. Propositions 7.19 and 7.20 yield that the modules described in the theorem statement form a set of pairwise non-isomorphic spectral simple $K\mathscr{G}$-modules. It remains to show that all spectral simple $K\mathscr{G}$-modules are of this form. So let V be a spectral simple $K\mathscr{G}$-module and suppose $\mathrm{Res}_x(V) \neq 0$. Then $\mathrm{Res}_x(V)$ is a simple KG_x-module by Lemma 7.14. By the adjunction between induction and restriction, the identity map on $\mathrm{Res}_x(V)$ gives rise to a non-zero $K\mathscr{G}$-morphism $\psi : \mathrm{Ind}_x \mathrm{Res}_x(V) \to V$. Since $\mathrm{Ind}_x \mathrm{Res}_x(V)$ is simple by Proposition 7.19 and V is simple by hypothesis, it follows that ψ is an isomorphism by Schur's lemma. This completes the proof of the first statement since the induced modules depend only on the orbit up to isomorphism. The statement about finite dimensional simple modules is a consequence of Proposition 7.19 and Corollary 7.25. □

7.2. Irreducible representations of inverse semigroups

Fix now an inverse semigroup S and let $\mathscr{G}(S)$ be the universal groupoid of S. If $\varphi \in \widehat{E(S)}$ has finite index in $\mathscr{G}(S)$, then we shall call φ a *finite index character* of $E(S)$ in S. This notion of index really depends on S. Notice that the orbit of φ in $\mathscr{G}(S)$ is precisely the orbit of φ under the spectral action of S on $\widehat{E(S)}$. If $E(S)$ is finite, then of course $\widehat{E(S)} = E(S)$ and all characters have finite index. If $\varphi \in \widehat{E(S)}$, then $S_\varphi = \{s \in S \mid s\varphi = \varphi\}$ is an inverse subsemigroup of S and one easily checks that the isotropy group G_φ of φ in $\mathscr{G}(S)$ is precisely the maximal group image of S_φ since if $s, s' \in S_\varphi$ and $t \leqslant s, s'$ with $\varphi \in D(t^*t)$ and $t \in S$, then $t \in S_\varphi$. This allows us to describe the finite dimensional irreducible representations of an inverse semigroup without any explicit reference to $\mathscr{G}(S)$. So without further ado, we state the classification theorem for finite dimensional irreducible representations of inverse semigroups, thereby generalizing the classical results for inverse semigroups with finitely many idempotents [5,30,31,22].

Theorem 7.27. *Let S be an inverse semigroup and K a field. Fix a set $D \subseteq \widehat{E(S)}$ containing exactly one finite index character from each orbit of finite index characters under the spectral action of S on $\widehat{E(S)}$. Let S_φ be the stabilizer of φ and set G_φ equal to the maximal group image of S_φ. Then there is a bijection between finite dimensional simple KS-modules and pairs (φ, V) where $\varphi \in D$ and V is a finite dimensional simple KG_φ-module (considered up to isomorphism).*

Proof. This is immediate from Theorem 7.26 and the above discussion. □

Remark 7.28. That there should be a theorem of this flavor was first suggested in unpublished joint work of S. Haatja, S.W. Margolis and the author from 2002.

Let us draw some consequences. First we give necessary and sufficient conditions for an inverse semigroup to have enough finite dimensional irreducible representations to separate points. Then we provide examples showing that the statement cannot really be simplified.

Corollary 7.29. *An inverse semigroup S has enough finite dimensional irreducible representations over K to separate points if and only if:*

(1) *The characters of $E(S)$ of finite index in S separate points of $E(S)$;*
(2) *For each $e \in E(S)$ and each $e \neq s \in S$ so that $s^*s = e = ss^*$, there is a character φ of finite index in S so that $\varphi(e) = 1$ and either:*

(a) $s\varphi \neq \varphi$; *or*
(b) $s\varphi = \varphi$ *and there is a finite dimensional irreducible representation* ψ *of* G_φ *so that* $\psi([s,\varphi]) \neq 1$.

Proof. Suppose first that S has enough finite dimensional irreducible representations to separate points and that $e \neq f$ are idempotents of S. Choose a finite dimensional simple KS-module $W = KL_\varphi \otimes_{KG_\varphi} V$ with φ a finite index character and such that e and f act differently on W. Recalling that $x \mapsto \chi_{D(x)}$ for $x \in E(S)$ under the isomorphism $KS \to K\mathscr{G}$, it follows from Proposition 7.8 that, for $t \in L_\varphi$,

$$xt = \begin{cases} t, & r(t)(x) = 1, \\ 0, & r(t)(x) = 0. \end{cases} \tag{7.3}$$

Therefore, in order for e and f to act differently on W, there must exist $t \in L_\varphi$ with $r(t) = \rho$ a finite index character such that $\rho(e) \neq \rho(f)$.

Next suppose that $e \neq s$ and $s^*s = e = ss^*$. By assumption there is a finite dimensional simple KS-module $W = KL_\varphi \otimes_{KG_\varphi} V$, with φ a finite index character, where s and e act differently. By (7.3) there must exist $t \in L_\varphi$ and $v \in V$ so that $\rho = r(t)$ satisfies $\rho \in D(e)$ and $s(t \otimes v) \neq e(t \otimes v) = t \otimes v$. Since ρ has finite index, if $s\rho \neq \rho$ then we are done. So assume $s\rho = \rho$.

Recall that under the isomorphism of algebras $KS \to K\mathscr{G}(S)$, we have that $s \mapsto \chi_{(s,D(e))}$. Since $et \neq 0$ implies $st \neq 0$ (as $s^*s = e$), there must exist $y \in L_\varphi$ so that $yt^{-1} \in (s, D(e))$ and moreover $st = y$ in KL_φ by Proposition 7.8. We must then have $yt^{-1} = [s,\rho] \in G_\rho$ as $s\rho = \rho$. Now $st = y = t(t^{-1}y) = t(t^{-1}[s,\rho]t)$ and so $t \otimes v \neq s(t \otimes v) = t \otimes t^{-1}[s,\rho]tv$. Thus if we make V a (simple) KG_ρ-module via $gv = (t^{-1}gt)v$, then $[s,\rho]$ does not act as the identity on this module. This completes the proof of necessity.

Let us now proceed with sufficiency. First we make an observation. Let φ be a character of finite index with associated finite orbit $\mathscr{O}$. Let V be the trivial KG_φ-module. It is routine to verify using Proposition 7.8 that $KL_\varphi \otimes_{KG_\varphi} V$ has a basis in bijection with $\mathscr{O}$ and S acts on the basis by restricting the action of S on $\widehat{E(S)}$ to $\mathscr{O}$. We call this the *trivial representation associated to* $\mathscr{O}$.

Suppose $s, t \in S$ with $s \neq t$. Assume first that $s^*s \neq t^*t$ and let φ be a finite index character with $\varphi(s^*s) \neq \varphi(t^*t)$. Then in the trivial representation associated to the orbit of φ, exactly one of s and t is defined on φ and so this finite dimensional irreducible representation separates s and t. A dual argument works if $ss^* \neq tt^*$.

So let us now assume that $s^*s = t^*t$ and $ss^* = tt^*$. Then it suffices to separate s^*s from t^*s in order to separate s and t. So we are left with the case that $s^*s = e = ss^*$ and $s \neq e$. We have two cases. Suppose first we can find a finite index character φ with $s\varphi \neq \varphi$. Again, the trivial representation associated to the orbit of φ separates s and e.

Suppose now that there is a finite index character φ with $\varphi(e) = 1$ and $s\varphi = \varphi$ and a finite dimensional simple KG_φ-module V so that $[s,\varphi]$ acts non-trivially on V. It is then easy to see using Proposition 7.8 that $s(\varphi \otimes v) = [s,\varphi] \otimes v = \varphi \otimes [s,\varphi]v$ since $[s,\varphi] \in (s, D(e))$. Thus s acts non-trivially on $KL_\varphi \otimes_{KG_\varphi} V$, completing the proof. □

An immediate consequence of this corollary is the following folklore result.

Corollary 7.30. *Let S be an inverse semigroup with finitely many idempotents and K a field. Then there are enough finite dimensional irreducible representations of S over K to separate* *points if and only if each maximal subgroup of S has enough finite dimensional irreducible representations to separate points.*

As a first example, consider the bicyclic inverse monoid, presented by $B = \langle x \mid x^*x = 1\rangle$. Any non-degenerate finite dimensional representation of B must be by invertible matrices since left invertibility implies right invertibility for matrices. Hence one cannot separate the idempotents of B by finite dimensional irreducible representations of B over any field. To see this from the point of view of Corollary 7.29, we observe that $\widehat{E(B)}$ is the one-point compactification of the natural numbers. Namely, if F is a filter on $E(B)$, then either it has a minimum element $x^n(x^*)^n$, and hence is a principal filter, or it contains all the idempotents (which is the one-point compactification). All the principal filters are in a single (infinite) orbit. The remaining filter is in a singleton orbit with isotropy group $\mathbb{Z}$. It obviously separates no idempotents.

Let us next give an example to show that there can be enough finite dimensional irreducible representations of an inverse semigroup to separate points, and yet there can be a finite index character φ so that the isotropy group G_φ does not have enough irreducible representations to separate points. Let $K = \mathbb{C}$. Then any finite inverse semigroup has enough finite dimensional irreducible representations to separate points, say by the above corollary. Hence any residually finite inverse semigroup has enough finite dimensional irreducible representations over $\mathbb{C}$ to separate points. On the other hand, the maximal group image G of an inverse semigroup S is the isotropy group of the trivial character that sends all idempotents to 1, which is a singleton orbit of $\widehat{E(S)}$. Let us construct a residually finite inverse semigroup whose maximal group image does not have any non-trivial finite dimensional representations.

A famous result of Mal'cev [17] says that a finitely generated group G with a faithful finite dimensional representation over $\mathbb{C}$ is residually finite. Since any representation of a simple group is faithful and an infinite simple group is trivially not residually finite, it follows that finitely generated infinite simple groups have no non-trivial finite dimensional representations over $\mathbb{C}$. An example of such a group is the famous Thompson's group V, which is a finitely presented infinite simple group [4].

In summary, if we can find a residually finite inverse semigroup whose maximal group image is a finitely generated infinite simple group, then we will have found the example we are seeking. To construct our example, we make use of the Birget–Rhodes expansion [1]. Let G be any group and let E be the semilattice of finite subsets of G ordered by reverse inclusion (so the meet is union). Let G act on E by left translation, so $gX = \{gx \mid x \in X\}$, and form the semidirect product $E \rtimes G$. Let S be the inverse submonoid of $E \rtimes G$ consisting of all pairs (X, g) so that $1, g \in X$. This is an E-unitary (in fact F-inverse) monoid with maximal group image G and identity $(\{1\}, 1)$. It is also residually finite. To see this, we use the well-known fact that an inverse semigroup all of whose $\mathscr{R}$-classes are finite is residually finite (indeed, the right Schützenberger representations on the $\mathscr{R}$-classes separate points). Hence it suffices to observe that $(X, g)(X, g)^* = (X, 1)$ and so the $\mathscr{R}$-class of (X, g) consists of all elements of the form (X, h) with $h \in X$, which is a finite set.

Let us observe that Mal'cev's result immediately implies that a finitely generated group has enough finite dimensional irreducible representations over $\mathbb{C}$ to separate points if and only if it is residually finite. One direction here is trivial. For the non-trivial direction, suppose G has enough finite dimensional irreducible representations over $\mathbb{C}$ to separate points and suppose $g \neq 1$. Then G has a finite dimensional irreducible representation $\varphi : G \to GL_n(\mathbb{C})$ so that $\varphi(g) \neq 1$. But $\varphi(G)$ is a finitely generated linear group and so residually finite by Mal'cev's theorem. Thus we can find a homomorphism $\psi : \varphi(G) \to H$ with H a finite group and $\psi(\varphi(g)) \neq 1$. Mal'cev's theorem immediately extends to inverse semigroups.

Proposition 7.31. *Let S be a finitely generated inverse subsemigroup of $M_n(\mathbb{C})$. Then S is residually finite.*

Proof. Set $V = \mathbb{C}^n$. We know that $E(S)$ is finite by Lemma 7.24 and hence each maximal subgroup is finitely generated by Corollary 7.4. It follows that each maximal subgroup is residually finite by Mal'cev's theorem since the maximal subgroup G_e is a faithful group of linear automorphisms of eV for $e \in E(S)$. But it is well known that if S is an inverse semigroup with finitely many idempotents, then S is residually finite if and only if all its maximal subgroups are residually finite [12]. Indeed, for the non-trivial direction observe that the right Schützenberger representations of S separate points into partial transformation wreath products of the form $G \wr T$ with T a transitive faithful inverse semigroup of partial permutations of a finite set and G a maximal subgroup of S. But such a wreath product is trivially residually finite when G is residually finite. $\square$

Now the exact same proof as the group case establishes the following result.

Proposition 7.32. *Let S be a finitely generated inverse semigroup. Then S has enough finite dimensional irreducible representations over $\mathbb{C}$ to separate points if and only if S is residually finite.*

For the remainder of the section we take K to be a commutative ring with unit. We now characterize the spectral KS-modules in terms of S. In particular, we shall see that if $E(S)$ satisfies the descending chain condition, then all non-zero KS-modules are spectral and so we have a complete parametrization of all simple KS-modules.

Proposition 7.33. *Let S be an inverse semigroup and let V be a non-zero KS-module. Then V is a spectral $K\mathscr{G}(S)$-module if and only if there exists $v \in V$ so that $fv = v$ for some idempotent $f \in E(S)$ and, for all $e \in E(S)$, one has $ev \neq 0$ if and only if $ev = v$. In particular, if $\varphi : S \to \mathrm{End}_K(V)$ is the corresponding representation and $\varphi(E(S))$ contains a primitive idempotent (for instance, if it satisfies the descending chain condition), then V is spectral.*

Proof. Recall that $e \mapsto \chi_{D(e)}$ under the isomorphism of KS with $K\mathscr{G}(S)$. Suppose first V is spectral and let $\theta \in \widehat{E(S)}$ so that $\mathrm{Res}_\theta(V) \neq 0$. Fix $0 \neq v \in \mathrm{Res}_\theta(V)$. If $\theta(f) \neq 0$, then $D(f) \in \mathscr{N}_\theta$ and so $fv = v$. Suppose that $e \in E(S)$ with $ev \neq 0$. Then $\{U \in B(\widehat{E(S)}) \mid Uev = ev\}$ is a proper filter containing $\mathscr{N}_\theta$ and $D(e)$. Since $\mathscr{N}_\theta$ is an ultrafilter, we conclude $D(e) \in \mathscr{N}_\theta$ and so $ev = D(e)v = v$.

Conversely, suppose there is an element $v \in V$ so that $fv = v$ some $f \in E(S)$ and $ev \neq 0$ if and only if $ev = v$ for all $e \in E(S)$; in particular $v \neq 0$. Let $A = \varphi(KE(S))$ where $\varphi : KS \to \mathrm{End}_k(V)$ is the associated representation. We claim that the set B of elements $e \in E(A)$ so that $ev \neq 0$ implies $ev = v$ is a generalized boolean algebra containing $\varphi(E(S))$. It clearly contains 0. Suppose $e, f \in B$ and $efv \neq 0$. Then $ev \neq 0 \neq fv$ so $efv = v$. On the other hand, assume $(e + f - ef)v = (e \vee f)v \neq 0$. Then at least one of ev or fv is non-zero. If $ev \neq 0$ and $fv = 0$, then we obtain $(e \vee f)v = ev + fv - efv = v$. A symmetric argument applies if $ev = 0$ and $fv \neq 0$. Finally, if $ev \neq 0 \neq fv$, then $(e \vee f)v = ev + fv - efv = v$. To deal with relative complements, suppose $e, f \in B$ and $(e - ef)v = (e \setminus f)v \neq 0$. Then $ev \neq 0$ and so $ev = v$. Therefore, $(e - ef)v = v - fv$. If $fv \neq 0$, then $fv = v$ and so $(e - ef)v = 0$, a contradiction. Thus $fv = 0$ and $(e \setminus f)v = v$. Since $E(S)$ generates $B(\widehat{E(S)})$ as a generalized boolean algebra via the map $e \mapsto D(e)$, it follows that if $U \in B(\widehat{E(S)})$ and $Uv \neq 0$, then $Uv = v$. Let $\mathscr{F} = \{U \in B(\widehat{E(S)}) \mid Uv = v\}$. Clearly, $\mathscr{F}$ is a proper filter. We claim that it is an ultrafilter. Indeed, suppose that $U' \notin \mathscr{F}$. Then $U'v = 0$ and so $(U \setminus U')v = Uv - UU'v = v$. Thus $U \setminus U' \in \mathscr{F}$ and so $\emptyset = U' \cap (U \setminus U')$ shows that the filter generated by U' and $\mathscr{F}$ is not proper. Thus $\mathscr{F}$ is an ultrafilter on $B(\widehat{E(S)})$ and hence is of the form $\mathscr{N}_\theta$ for a unique element $\theta \in \widehat{E(S)}$. It follows $v \in \mathrm{Res}_\theta(V)$.

For the final statement, suppose that $\varphi(f) \in \varphi(E(S))$ is primitive and $0 \neq v \in fV$. Then, for all $e \in E(S)$, $efv = ev \neq 0$ implies $\varphi(ef) \neq 0$ and so $\varphi(ef) = \varphi(f)$ by primitivity. Thus $ev = efv = fv = v$. □

It turns out that if every idempotent of an inverse semigroup is central, then every simple KS-module is spectral.

Proposition 7.34. *Let S be an inverse semigroup with central idempotents. Then every simple KS-module is spectral.*

Proof. Let V be a simple KS-module and suppose $e \in E(S)$. Since e is central, it follows that eV is KS-invariant and hence $eV = V$, and so e acts as the identity, or $eV = 0$, whence e acts as 0. Thus V is spectral by Proposition 7.33. □

There are other classes of inverse semigroups all of whose modules are spectral (and hence for which we have a complete list of all simple modules).

Proposition 7.35. *Let S be an inverse semigroup such that $E(S)$ is isomorphic to $(\mathbb{N}, \geqslant)$. Then every non-zero KS-module is spectral.*

Proof. Suppose $E(S) = \{e_i \mid i \in \mathbb{N}\}$ with $e_i e_j = e_{\max\{i,j\}}$. Let V be a KS-module. If $eV = V$ for all $e \in E(S)$, then trivially V is spectral. Otherwise, we can find $n > 0$ minimum so that $e_n V \neq V$. Then $V = e_n V \oplus (1 - e_n)V$ and $(1 - e_n)V \neq 0$. Choose a non-zero vector v from $(1 - e_n)V$. We claim

$$e_i v = \begin{cases} v, & i < n, \\ 0, & i \geqslant n. \end{cases}$$

It will then follow that V is a spectral KS-module by Proposition 7.33. Indeed, if $i < n$, then e_i acts as the identity on V by choice of n. On the other hand, if $i \geqslant n$, then $e_i(1 - e_n) = e_i - e_i e_n = e_i - e_i = 0$. This completes the proof. □

Putting it all together we obtain the following theorem.

Theorem 7.36. *Let S be an inverse semigroup and K a commutative ring with unit. Fix a set $D \subseteq \widehat{E(S)}$ containing exactly one character from each orbit of the spectral action of S on $\widehat{E(S)}$.*

Let S_φ be the stabilizer of φ and set G_φ equal to the maximal group image of S_φ. Then there is a bijection between simple KS-modules V so that there exists $v \in V$ with

$$\emptyset \neq \{e \in E(S) \mid ev = v\} = \{e \in E(S) \mid ev \neq 0\}$$

and pairs (φ, W) where $\varphi \in D$ and W is a simple KG_φ-module (considered up to isomorphism). This in particular, describes all simple KS-modules if the idempotents of S are central or from a descending chain isomorphic to $(\mathbb{N}, \geqslant)$.

For example, if B is the bicyclic monoid, then the simple KB-modules are the simple $K\mathbb{Z}$-modules and the representation of B on the polynomial ring $K[x]$ by the unilateral shift. At the moment we do not have an example of an inverse semigroup S and a simple KS-module that is not spectral. By specializing to inverse semigroups with descending chain condition on idempotents, we obtain the following generalization of Munn's results [5,22].

Corollary 7.37. *Let S be an inverse semigroup satisfying descending chain condition on idempotents and let K be a commutative ring with unit. Fix a set $D \subseteq E(S)$ containing exactly one idempotent from each $\mathscr{D}$-class. Then there is a bijection between simple KS-modules and pairs (e, V) where $e \in D$ and V is a simple KG_e-module (considered up to isomorphism). The corresponding KS-module is finite dimensional if and only if the $\mathscr{D}$-class of e contains finitely many idempotents and V is finite dimensional.*

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