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A note on maximality of analytic crossed products

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Abstract

Let G be a compact abelian group with the totally ordered dual group $\hat{G}$ which admits the positive semigroup $\hat{G}_+$. Let N be a von Neumann algebra and $\alpha = \{\alpha_{\hat{g}}\}_{\hat{g}\in\hat{G}}$ be an automorphism group of $\hat{G}$ on N. We denote $N \rtimes_\alpha \hat{G}_+$ to the analytic crossed product determined by N and α. We show that if $N \rtimes_\alpha \hat{G}_+$ is a maximal σ-weakly closed subalgebra of $N \rtimes_\alpha \hat{G}$, then $\hat{G}_+$ induces an archimedean order in $\hat{G}$.

Keywords: von Neumann algebra; Analytic crossed product; Maximality; Semigroup

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1. Introduction

Let G be a compact abelian group with the totally ordered dual group $\hat{G}$ which has a positive semigroup $\hat{G}_+$. Let N be a von Neumann algebra and $\alpha = \{\alpha_{\hat{g}}\}_{\hat{g}\in\hat{G}}$ be an automorphism group of $\hat{G}$ on N. We are interested in the maximality of certain subalgebra of crossed product $N \rtimes_\alpha \hat{G}$ determined by N and α. This subalgebra is called an analytic crossed product. Roughly speaking, the analytic crossed products stand in the same relation to the crossed products as the Hardy algebras $H^\infty(G)$, the space of all functions of analytic type which belongs to $L^\infty(G)$, stand in relation to $L^\infty(G)$. In the case that $G = \mathbb{T}$ and $\hat{G} = \mathbb{Z}$, it is well known that $H^\infty(G)$ is a maximal weak $*$ closed subalgebra of $L^\infty(G)$. Viewing the analytic crossed products as "noncommutative H^∞ algebras" raise the following:

Question. When is the analytic crossed product $N \rtimes_\alpha \hat{G}_+$ maximal among σ-weakly closed subalgebras of the crossed product $N \rtimes_\alpha \hat{G}$?

McAsey, Muhly and the fourth author in [3] showed that, in the case that $G = \mathbb{T}$, $\hat{G} = \mathbb{Z}$ and $\hat{G}_+ = \mathbb{Z}_+$, if N is a finite von Neumann algebra, then N is a factor if and only if $N \rtimes_\alpha \mathbb{Z}_+$ is maximal as a σ-weakly closed subalgebra of $N \rtimes_\alpha \mathbb{Z}$. They also proved in [4] the same result in the case when N is an arbitrary (σ-finite) von Neumann algebra. Moreover the fourth author in [7] showed that if N is a factor, then $N \rtimes_\alpha \hat{G}_+$ is maximal where G is a compact abelian group with an archimedean totally ordered dual $\hat{G}$. Motivated by these facts, we consider the following problem:

Problem. Let N be a factor. When is the analytic crossed product $N \rtimes_\alpha \hat{G}_+$ maximal among σ-weakly closed subalgebras of $N \rtimes_\alpha \hat{G}$?

Our aim in this paper is to give the answer for this problem as follows:

Theorem 1.1. *Let N be a factor. Then an analytic crossed product $N \rtimes_\alpha \hat{G}_+$ is a maximal σ-weakly closed subalgebra of $N \rtimes_\alpha \hat{G}$ if and only if $\hat{G}_+$ induces an archimedean order in $\hat{G}$.*

In the next section we establish the notions of spectral subspaces, crossed products and its subalgebras. In Section 3, we study the structure of analytic subalgebras in a crossed product. And we shall prove Theorem 1.1, that is, we shall show that if $N \rtimes_\alpha \hat{G}_+$ is a maximal σ-weakly closed subalgebra of $N \rtimes_\alpha \hat{G}$, then $\hat{G}_+$ induces an archimedean order in $\hat{G}$ (Theorem 3.7). In Section 4, we will pay attention to the properties of semigroups in $\hat{G}$ and reconsider the maximality of analytic subalgebras.

2. Preliminaries

Throughout this paper, G will denote a compact abelian group with the operation written additively. Elements of G will be denoted by lowercase Roman letters and the normalized Haar measure on G will be denoted by m. The dual of G will be written $\hat{G}$ and the elements of $\hat{G}$ will be distinguished from those of G by a caret. The pairing between G and $\hat{G}$ will be written $\langle g, \hat{h}\rangle$ ($\forall g \in G$, $\forall \hat{h} \in \hat{G}$) and the Fourier transform will take this form:

$$\hat{f}(\hat{h}) = \int_G \langle g, \hat{h}\rangle f(g)\, dm(g) \quad \bigl(\forall f \in L^1(G),\ \forall \hat{h} \in \hat{G}\bigr).$$

Suppose that $\hat{G}$ has a positive semigroup $\hat{G}_+$, that is, $\hat{G}_+$ satisfies the conditions

$$\text{(i)} \quad \hat{G}_+ \cap (-\hat{G}_+) = \{\hat{0}\}, \qquad \text{(ii)} \quad \hat{G}_+ \cup (-\hat{G}_+) = \hat{G}.$$

Under these conditions, $\hat{G}_+$ induces an order in $\hat{G}$. That is, if we define $\hat{g} \geqslant \hat{h}$ to mean that $\hat{g} - \hat{h} \in \hat{G}_+$, then $\hat{G}$ is a totally ordered set with the order $\geqslant$. The most important example is the case that $G = \mathbb{T}$, $\hat{G} = \mathbb{Z}$ and $\hat{G}_+ = \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. A given group $\hat{G}$ may have many different orders. An order is said to be archimedean if it has the following property: to every pair of elements $\hat{g}, \hat{h}$ of $\hat{G}$ such that $\hat{g} > \hat{0}$ and $\hat{h} > \hat{0}$, there corresponds a positive integer n such that $n\hat{g} \geqslant \hat{h}$. For example, it is clear that $\mathbb{Z}_+$ has an archimedean order in $\mathbb{Z}$. As a non-archimedean order, there is a lexicographic order. For example, in the case that $G = \mathbb{T}^2$ and $\hat{G} = \mathbb{Z}^2$, we consider the positive semigroup $\hat{G}_+$ as $\{(m, n) \in \mathbb{Z}^2 \mid m > 0, \text{ or } m = 0 \text{ and } n \geqslant 0\}$. Then $\hat{G}_+$ induces the lexicographic order $\leqslant_L$ in $\hat{G}$. In this case, $(1, 0)$ and $(0, 1)$ belong to $\hat{G}_+$, but, for any positive integer n, $n(0, 1) - (1, 0) = (-1, n) \notin \hat{G}_+$, that is $(1, 0) \nleqslant_L n(0, 1)$. Therefore this order is non-archimedean.

Let M be a von Neumann algebra and $\beta = \{\beta_g\}_{g \in G}$ be an automorphism group of G on M. For each $f \in L^1(G)$, we define $\beta(f)$ by the integral

$$\beta(f)(x) = \int_G f(g)\beta_g(x)\, dm(g) \quad (\forall x \in M).$$

For each fixed $x \in M$, the set

$$I_\beta(x) = \bigl\{ f \in L^1(G) \bigm| \beta(f)(x) = 0 \bigr\}$$

is a closed ideal of $L^1(G)$. The Arveson spectrum $\mathrm{Sp}_\beta(x)$ of x with respect to the automorphism group $\{\beta_g\}_{g \in G}$ is defined to be the hull of $I_\beta(x)$ as follows:

$$\mathrm{Sp}_\beta(x) = \bigl\{ \hat{h} \in \hat{G} \bigm| \hat{f}(\hat{h}) = 0 \ \bigl(\forall f \in I_\beta(x)\bigr) \bigr\},$$

where $\hat{f}$ is the Fourier transform of f. For each subset $E \subset \hat{G}$, the spectral subspace $M^\beta(E)$ is defined to be the set

$$M^\beta(E) = \bigl\{ x \in M \mid \mathrm{Sp}_\beta(x) \subseteq E \bigr\}.$$

It is known that, for each subset E of $\hat{G}$, $M^\beta(E)$ is β-invariant, that is, $\beta_g(M^\beta(E)) = M^\beta(E)$ $(\forall g \in G)$.

We shall next define crossed products and analytic crossed products. Let N be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\alpha = \{\alpha_{\hat{g}}\}_{\hat{g} \in \hat{G}}$ be an automorphism group of $\hat{G}$ on N. The crossed product $N \rtimes_\alpha \hat{G}$ of N by α is the von Neumann algebra acting on a Hilbert space $\ell^2(\hat{G}, \mathcal{H}) = \{\xi : \hat{G} \to \mathcal{H} \mid \sum_{\hat{h} \in \hat{G}} \|\xi(\hat{h})\|^2 < \infty\}$ generated by the operators $\pi_\alpha(x)$ and $\lambda(\hat{g})$ defined by the equations, for each $x \in N$,

$$\bigl\{\pi_\alpha(x)\xi\bigr\}(\hat{g}) = \alpha_{-\hat{g}}(x)\xi(\hat{g}) \quad \bigl(\forall \xi \in \ell^2(\hat{G}, \mathcal{H}),\ \forall \hat{g} \in \hat{G}\bigr)$$

and

$$\bigl\{\lambda(\hat{g})\xi\bigr\}(\hat{h}) = \xi(\hat{h} - \hat{g}) \quad \bigl(\forall \xi \in \ell^2(\hat{G}, \mathcal{H}),\ \forall \hat{h}, \hat{g} \in \hat{G}\bigr).$$

For simplicity, we write $M = N \rtimes_\alpha \hat{G}$ and $\mathcal{K} = \ell^2(\hat{G}, \mathcal{H})$. The analytic crossed product $N \rtimes_\alpha \hat{G}_+$ determined by N and α is defined to be the σ-weakly closed subalgebra of M generated by $\pi_\alpha(N)$ and $\{\lambda(\hat{g})\}_{\hat{g} \in \hat{G}_+}$, that is,

$$N \rtimes_\alpha \hat{G}_+ = \overline{\mathrm{alg}\bigl\{\pi_\alpha(N), \bigl\{\lambda(\hat{h})\bigr\}_{\hat{h} \in \hat{G}_+}\bigr\}}^{\sigma - w}.$$

For each $g \in G$, we define

$$(W_g\xi)(\hat{h}) = \langle g, \hat{h}\rangle \xi(\hat{h}) \quad (\forall \xi \in \mathcal{K},\ \forall \hat{h} \in \hat{G}).$$

The automorphism group $\{\tilde{\alpha}_g\}_{g \in G}$ of G on M which is dual to $\{\alpha^{\hat{h}}\}_{\hat{h} \in \hat{G}}$ in the sense of Takesaki [11] is implemented by the unitary operator W_g, that is,

$$\tilde{\alpha}_g(x) = W_g x W_g^* \quad (\forall x \in M,\ \forall g \in G).$$

It is elementary to check that the spectral resolution of $\{W_g\}_{g \in G}$ is given by the formula

$$W_g = \sum_{\hat{h} \in \hat{G}} \langle g, \hat{h}\rangle E_{\hat{h}},$$

where $E_{\hat{h}}$ is the projection on $\mathcal{K}$ defined by the formula

$$(E_{\hat{h}}\xi)(\hat{k}) = \begin{cases} \xi(\hat{h}) & \hat{h} = \hat{k}, \\ 0 & \hat{h} \neq \hat{k} \end{cases} \quad (\forall \xi \in \mathcal{K}).$$

Then it is easy to check that the projection $E_{\hat{h}}$ can be calculated as the (Bochner) integral

$$E_{\hat{h}}(\xi) = \int_G \overline{\langle g, \hat{h}\rangle} W_g \xi \, dm(g) \quad (\forall \xi \in \mathcal{K}).$$

Moreover, for each $\hat{h} \in \hat{G}$, we define a σ-weakly continuous linear map $\varepsilon_{\hat{h}}$ on M by the integral

$$\varepsilon_{\hat{h}}(x) = \int_G \overline{\langle g, \hat{h}\rangle} \tilde{\alpha}_g(x) \, dm(g) \quad (\forall x \in M).$$

3. Proof of Theorem 1.1

In this section, we concentrate to prove Theorem 1.1.

Lemma 3.1. *Keep the notation as in Section* 2. *Let $M = N \rtimes_\alpha \hat{G}$ and Γ be a subset in $\hat{G}$. Then Γ is a semigroup if and only if $M^{\tilde{\alpha}}(\Gamma)$ is a σ-weakly closed subalgebra of M.*

Proof. Applying [10, Proposition 15.3], for each semigroup Γ in $\hat{G}$, we have

$$M^{\tilde{\alpha}}(\Gamma) M^{\tilde{\alpha}}(\Gamma) \subseteq M^{\tilde{\alpha}}(\Gamma + \Gamma) \subseteq M^{\tilde{\alpha}}(\Gamma).$$

Conversely, we assume that Γ is not a semigroup. Then there exist $\hat{h}, \hat{k} \in \Gamma$ such that $\hat{h} + \hat{k} \notin \Gamma$. Since, for each $g \in G$, $\tilde{\alpha}_g(\lambda(\hat{h})) = \langle g, \hat{h}\rangle \lambda(\hat{h})$ and $\tilde{\alpha}_g(\lambda(\hat{k})) = \langle g, \hat{k}\rangle \lambda(\hat{k})$, we see that $\lambda(\hat{h})$ and $\lambda(\hat{k})$ belong to $M^{\tilde{\alpha}}(\{\hat{h}\})$ and $M^{\tilde{\alpha}}(\{\hat{k}\})$, respectively, and so $\lambda(\hat{h})$ and $\lambda(\hat{k})$ belong to $M^{\tilde{\alpha}}(\Gamma)$. Since $M^{\tilde{\alpha}}(\Gamma)$ is a subalgebra of M, $\lambda(\hat{h})\lambda(\hat{k})$ is in $M^{\tilde{\alpha}}(\Gamma)$. This implies that $\mathrm{Sp}_{\tilde{\alpha}}(\lambda(\hat{h})\lambda(\hat{k})) \subseteq \Gamma$. However, by [10, Proposition 15.3], we have

$$\mathrm{Sp}_{\tilde{\alpha}}\big(\lambda(\hat{h})\lambda(\hat{k})\big) \subseteq \mathrm{Sp}_{\tilde{\alpha}}\big(\lambda(\hat{h})\big) + \mathrm{Sp}_{\tilde{\alpha}}\big(\lambda(\hat{k})\big) \subseteq \{\hat{h} + \hat{k}\} \nsubseteq \Gamma.$$

This contradicts the assumption and hence Γ is a semigroup. This completes the proof. □

Using [2, Corollary 4.3.2], we have the following:

Proposition 3.2. *Let Γ be a semigroup in $\hat{G}$ which contains $\hat{0}$. Then the σ-weakly closed subalgebra of M generated by $\pi_\alpha(N)$ and $\{\lambda(\hat{h})\}_{\hat{h}\in\Gamma}$ coincides with $M^{\tilde{\alpha}}(\Gamma)$. In particular, $N \rtimes_\alpha \hat{G}_+ = M^\alpha(\hat{G}_+)$.*

Lemma 3.3. *Keep the notation as in Section 2. Then, for each $x \in M$, we have*

$$\mathrm{Sp}_{\tilde{\alpha}}(x) = \{\hat{h} \in \hat{G} \mid \varepsilon_{\hat{h}}(x) \neq 0\}.$$

Moreover, for each semigroup Γ in $\hat{G}$, we have

$$M^{\tilde{\alpha}}(\Gamma) = \{x \in M \mid \varepsilon_{\hat{h}}(x) = 0 \ (\forall \hat{h} \notin \Gamma)\}.$$

Proof. Take any $x \in M$. By a simple calculation, we note that, for each $\hat{h} \in \Gamma$ and $f \in L^1(G)$, the following equation holds:

$$\varepsilon_{\hat{h}}\big(\tilde{\alpha}(f)(x)\big) = \hat{f}(\hat{h})\varepsilon_{\hat{h}}(x). \tag{3.1}$$

If $\hat{h} \in \Gamma$ satisfies $\varepsilon_{\hat{h}}(x) \neq 0$, then we have

$$\hat{f}(\hat{h})\varepsilon_{\hat{h}}(x) = \varepsilon_{\hat{h}}\big(\tilde{\alpha}(f)(x)\big) = 0 \quad \big(\forall f \in I_{\tilde{\alpha}}(x)\big).$$

Since $\varepsilon_{\hat{h}}(x) \neq 0$, we have $\hat{f}(\hat{h}) = 0$. This implies that $\hat{h} \in \mathrm{Sp}_{\tilde{\alpha}}(x)$.

Conversely, we assume that $\hat{k} \in \hat{G}$ such that $\varepsilon_{\hat{k}}(x) = 0$. Putting $p_{\hat{k}}(g) = \overline{\langle g, \hat{k}\rangle}$ $(\forall g \in G)$, then it is clear that $p_{\hat{k}} \in L^1(G)$ and

$$\hat{p}_{\hat{k}}(\hat{d}) = \int_G \langle g, \hat{d}\rangle p_{\hat{k}}(g)\,dm(g) = \begin{cases} 1 & (\hat{d} = \hat{k}), \\ 0 & (\hat{d} \neq \hat{k}). \end{cases}$$

By Eq. (3.1), we have

$$\varepsilon_{\hat{d}}\big(\tilde{\alpha}(p_{\hat{k}})(x)\big) = \hat{p}_{\hat{k}}(\hat{d})\varepsilon_{\hat{d}}(x) = 0 \quad (\forall \hat{d} \in \hat{G}).$$

Then we obtain $\tilde{\alpha}(p_{\hat{k}})(x) = 0$. Since $\hat{p}_{\hat{k}}(\hat{k}) = 1 \neq 0$, we have $\hat{k} \notin \mathrm{Sp}_{\tilde{\alpha}}(x)$. This completes the proof. □

The next result follows immediately from Lemma 3.3.

Lemma 3.4. *Let $M = N \rtimes_\alpha \hat{G}$. For each semigroups Σ and Γ in $\hat{G}$, Γ contains Σ properly if and only if the subalgebra $M^{\tilde{\alpha}}(\Gamma)$ contains $M^{\tilde{\alpha}}(\Sigma)$ properly.*

In the case of $M = N \rtimes_\alpha \hat{G}$, the dual action $\tilde{\alpha}$ and the generators $\{\lambda(\hat{h})\}_{\hat{h}\in\hat{G}}$ of M satisfy the equation

$$\tilde{\alpha}_g\big(\lambda(\hat{h})\big) = \langle g, \hat{h}\rangle\lambda(\hat{h}) \quad (\forall g \in G,\ \forall \hat{h} \in \hat{G}).$$

This equation is satisfied precisely when $\mathrm{Sp}_{\tilde{\alpha}}(\lambda(\hat{h})) = \{\hat{h}\}$ $(\forall \hat{h} \in \hat{G})$. Hence the assertion of Lemma 3.4 is natural. But, in general, Lemma 3.4 is not necessary.

Example 3.5. Let M be a von Neumann algebra and P_n $(n = 0, 1)$ be orthogonal projections in M such that $P_0 + P_1 = I$. Putting $u_t = P_0 + e^{2\pi i 2t} P_1$ $(\forall t \in \mathbb{R})$, then $\{u_t\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group of M and so we can define the automorphism group $\{\alpha_t\}_{t\in\mathbb{R}}$ of $\mathbb{R}$ on M which is implemented by u_t. Then, for each $x \in M$, we have

$$\alpha_t(x) = P_0xP_0 + P_1xP_1 + e^{-2\pi i2t}P_0xP_1 + e^{2\pi i2t}P_1xP_0.$$

We note that $\alpha_t(x) = e^{2\pi int}x$ if and only if $\mathrm{Sp}_\alpha(x) = \{n\}$. Thus if we put $\Gamma = \{2n \mid n \in \mathbb{Z}_+\}$, then Γ is a semigroup in $\mathbb{Z}$ which is contained in $\mathbb{Z}_+$ properly. However it easily see that

$$M^\alpha(\Gamma) = \{x \in M \mid P_0xP_1 = 0\} = M^\alpha(\mathbb{Z}_+).$$

The fourth author in [7] studied the structure of invariant subspaces and cocycles for $N \rtimes_\alpha \hat{G}_+$ when $\hat{G}_+$ induces an archimedean order in $\hat{G}$, and showed the following:

Theorem 3.6 [7, Theorem 6.3]. *If N is a factor, then $N \rtimes_\alpha \hat{G}_+$ is a maximal σ-weakly closed subalgebra of M.*

We note that this result was obtained under the assumption that N admits a trace. However, considering a non-commutative L^2-space in the sense of Haagerup [1], we may rewrite it without this assumption.

To prove Theorem 1.1, we need the converse assertion. Indeed, we shall show it without the assumption that N is a factor. Let us say that Γ is a maximal semigroup in $\hat{G}$ if $\hat{G}$ is the only semigroup in $\hat{G}$ which contains Γ as a proper subset.

Theorem 3.7. *Let $M = N \rtimes_\alpha \hat{G}$. If $M^{\tilde{\alpha}}(\hat{G}_+)(= N \rtimes_\alpha \hat{G}_+)$ is a maximal σ-weakly closed subalgebra of M, then $\hat{G}_+$ induces an archimedean order in $\hat{G}$.*

Proof. Assume that $M^{\tilde{\alpha}}(\hat{G}_+)$ is a maximal σ-weakly closed subalgebra of M. If $\hat{G}_+$ is not maximal, then there exists a semigroup Σ in $\hat{G}$ such that $\hat{G}_+ \subsetneq \Sigma \subsetneq \hat{G}$. By Lemmas 3.1 and 3.3, $M^{\tilde{\alpha}}(\Sigma)$ is a σ-weakly closed subalgebra of M satisfying

$$M^{\tilde{\alpha}}\big(\hat{G}_+\big) \subsetneq M^{\tilde{\alpha}}(\Sigma) \subsetneq M^{\tilde{\alpha}}(\hat{G}) = M.$$

This contradiction shows that $\hat{G}_+$ is a maximal semigroup in $\hat{G}$. Moreover, $\hat{G}_+$ satisfies $\hat{G}_+ \cap (-\hat{G}_+) = \{\hat{0}\}$ and $\hat{G}_+ \cup (-\hat{G}_+) = \hat{G}$. Thus, by [6, Theorem 8.1.3], $\hat{G}_+$ induces an archimedean order in $\hat{G}$. This completes the proof. □

4. Some remarks on maximality

In this section, we shall give some remarks on maximality of $M^{\tilde{\alpha}}(\Gamma)$, where Γ is a semigroup of $\hat{G}$. First we modify Theorem 3.7 as follows:

Theorem 4.1. *Let $M = N \rtimes_\alpha \hat{G}$ and Γ be a semigroup in $\hat{G}$. If $M^{\tilde{\alpha}}(\Gamma)$ is a maximal σ-weakly closed subalgebra of M, then Γ is a maximal semigroup in $\hat{G}$. Moreover, let Γ be a semigroup which satisfies the conditions*

$$\Gamma \cap (-\Gamma) = \{\hat{0}\} \quad \textit{and} \quad \Gamma \neq \{\hat{0}\}. \tag{4.1}$$

If N is a factor, then $M^{\tilde{\alpha}}(\Gamma)$ is maximal if and only if Γ is maximal. In this case Γ induces an archimedean order in $\hat{G}$.

Proof. Let Γ be a maximal semigroup of $\hat{G}$ which satisfies (4.1). By [6, Theorem 8.1.3] and its proof, we see that $\hat{G} = \Gamma \cup (-\Gamma)$ and Γ induces an archimedean order in $\hat{G}$. Thus $M^{\tilde{\alpha}}(\Gamma)$ is the analytic crossed product, and so $M^{\tilde{\alpha}}(\Gamma)$ is maximal. □

We shall say that a semigroup Σ of $\hat{G}$ is an archimedean ordered semigroup of $\hat{G}$ if, for each $\hat{h}, \hat{k} \in \Sigma$, there exists a positive integer n such that $n\hat{k} - \hat{h} \in \Sigma$. For example, in the case $\hat{G} = \mathbb{Z}$, the semigroups $n\mathbb{Z}_+ = n\{0, 1, 2, 3, \ldots\}$ or $\{n, n+1, n+2, \ldots\}$ $(\forall n \geqslant 0)$ are archimedean ordered semigroups.

Lemma 4.2. *If Γ is an archimedean ordered semigroup of $\hat{G}$ which satisfies $\Gamma \cup (-\Gamma) = \hat{G}$, then Γ is a maximal semigroup.*

Proof. Suppose that Σ is a semigroup of $\hat{G}$ such that $\Gamma \subsetneq \Sigma$. Then there exists a non-zero element $\hat{h}$ in Σ such that $\hat{h} \notin \Gamma$. By the assumption, we note that $-\hat{h} \in \Gamma$. Hence, for each $\hat{k} \in \Gamma$, there is a positive integer $n > 0$ such that $n(-\hat{h}) - \hat{k} \in \Gamma$ because Γ has an archimedean order. Thus we have

$$-\hat{k} = n\hat{h} + \{n(-\hat{h}) - \hat{k}\} \in \Sigma,$$

and so $-\Gamma \subseteq \Sigma$. This implies that $\hat{G} = \Gamma \cup (-\Gamma) \subseteq \Sigma \subseteq \hat{G}$. This completes the proof. □

Theorem 4.1 suggests that the condition (4.1) plays an important role in the maximality of $M^{\tilde{\alpha}}(\Gamma)$. If Σ is an archimedean ordered semigroup in $\hat{G}$ such that $\Sigma \cup (-\Sigma) = \hat{G}$ and $\Sigma \cap (-\Sigma) \neq \{\hat{0}\}$, then $M^{\tilde{\alpha}}(\Sigma)$ does not need to be a maximal σ-weakly closed subalgebra of M. Indeed, we shall gave an example which satisfies such a situation as follows:

Example 4.3. Let $G = \mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ be the two dimensional torus (the bitorus) and $L^2(\mathbb{T}^2)$ be the usual Lebesgue space with respect to the Haar measure on $\mathbb{T}^2$. Let M be the von Neumann algebra acting on $L^2(\mathbb{T})$ generated by L_1 and L_2, where L_1 (respectively L_2) is the multiplication operator with the projection map on the first coordinate (respectively second coordinate). We note that M is usually known as the algebra $L^\infty(\mathbb{T}^2)$, that is M is spatially isomorphic to the crossed product $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{Z}^2$ acting on the Hilbert space $\ell^2(\mathbb{Z}^2) = \ell^2(\mathbb{Z}^2, \mathbb{C})$. Thus we may identify M and $L^2(\mathbb{T}^2)$ with $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{Z}^2$ and $\ell^2(\mathbb{Z}^2)$, respectively. For each $s, t \in \mathbb{R}$, we define

$$(W_{s,t}\xi)(m,n) = e^{-2\pi i m s} e^{-2\pi i n t} \xi(m,n) \quad \big(\forall \xi \in \mathcal{H},\ \forall (m,n) \in \mathbb{Z}^2\big).$$

The automorphism group $\tilde{\alpha}$ of $\mathbb{R}^2$ on M is implemented by the unitary operator $W_{s,t}$, that is,

$$\tilde{\alpha}_{s,t}(x) = W_{s,t} x W_{s,t}^* \quad \big(\forall x \in M,\ \forall (s,t) \in \mathbb{R}^2\big).$$

Let Γ be a semigroup of $\mathbb{Z}^2$ which satisfies $\Gamma \cap (-\Gamma) = \{\hat{0}\}$ and $\Gamma \cup (-\Gamma) = \hat{G}$ as follows:

$$\Gamma = \big\{(k,l) \in \mathbb{Z}^2 \bigm| k = 0 \text{ and } l \geqslant 0, \text{ or } k > 0\big\}.$$

Then Γ induces a lexicographic order in $\mathbb{Z}^2$. For this semigroup, we can define the analytic crossed product $M^{\tilde{\alpha}}(\Gamma)$ of M with the diagonal which is a factor as follows:

$$M^{\tilde{\alpha}}(\Gamma) = \overline{\mathrm{alg}\big\{L_1^m L_2^n \bigm| (m,n) \in \Gamma\big\}}^{\sigma-w}.$$

By Theorem 1.1, we see that the analytic crossed product $M^{\tilde{\alpha}}(\Gamma)$ is not maximal. Indeed, if we put $\Sigma = \{(k,l) \in \mathbb{Z}^2 \mid k \geqslant 0\}$, then Σ is the archimedean ordered semigroup of $\hat{G}$ which satisfies

$\Sigma \cup (-\Sigma) = \hat{G}$, $\Sigma \cap (-\Sigma) \neq \{\hat{0}\}$ and $\Gamma \subsetneq \Sigma \subsetneq \hat{G}$. By Lemma 4.2, we note that Σ is a maximal semigroup in $\hat{G}$. By Lemmas 3.1 and 3.4, $M^{\tilde{\alpha}}(\Sigma)$ is a σ-weakly closed subalgebra of M which contains $M^{\tilde{\alpha}}(\Gamma)$ properly. Moreover, by [5, Section 3, Example (1)], there is not a σ-weakly closed subalgebra of M which contains $M^{\tilde{\alpha}}(\Gamma)$ and is maximal among the proper σ-weakly closed subalgebras of M. That is, $M^{\tilde{\alpha}}(\Sigma)$ is not a maximal σ-weakly closed subalgebra of M, in spite of the facts that Σ is maximal and the diagonal of $M^{\tilde{\alpha}}(\Sigma)$ is a factor.

Motivated by this fact, we shall characterize $M^{\alpha}(\Sigma)$ with some maximality in general case.

Proposition 4.4. *Let N be a factor and $M = N \rtimes_{\alpha} \hat{G}$. Let Σ be an archimedean ordered semigroup of $\hat{G}$ which satisfies $\Sigma \cup (-\Sigma) = \hat{G}$. If $\mathfrak{A}$ is an $\tilde{\alpha}$-invariant σ-weakly closed subalgebra of M which contains $M^{\tilde{\alpha}}(\Sigma)$, then $\mathfrak{A} = M^{\tilde{\alpha}}(\Sigma)$ or $\mathfrak{A} = M$. That is, $M^{\tilde{\alpha}}(\Sigma)$ is maximal among $\tilde{\alpha}$-invariant σ-weakly closed subalgebras of M.*

Proof. We may assume that $M^{\tilde{\alpha}}(\Sigma) \subsetneq \mathfrak{A}$. Since $\mathfrak{A}$ is $\tilde{\alpha}$-invariant, there is an element $x \in \mathfrak{A}$ and $\hat{h} \notin \Sigma$ such that $0 \neq \varepsilon_{\hat{h}}(x) \in \mathfrak{A}$. Since $\mathrm{Sp}_{\tilde{\alpha}}(\varepsilon_{\hat{h}}(x)) = \{\hat{h}\}$, there exists $y \in \pi_{\alpha}(N)$ such that $\varepsilon_{\hat{h}}(x) = y\lambda(\hat{h})$. Thus we have

$$\pi_{\alpha}(N)y\pi_{\alpha}(N)\lambda(\hat{h}) = \pi_{\alpha}(N)\varepsilon_{\hat{h}}(x)\pi_{\alpha}(N) \subseteq \mathfrak{A}.$$

Since N is a factor and $\pi_{\alpha}(N)y\pi_{\alpha}(N)$ is a two-sided ideal of $\pi_{\alpha}(N)$, the σ-weakly closure of $\pi_{\alpha}(N)y\pi_{\alpha}(N)$ coincides with $\pi_{\alpha}(N)$, and so $\lambda(\hat{h})$ lies in $\mathfrak{A}$. Let S be the semigroup generated by Σ and $\hat{h}$. Since $\mathfrak{A}$ is the algebra which contains $M^{\tilde{\alpha}}(\Sigma)$ and $\lambda(\hat{h})$, we have $M^{\tilde{\alpha}}(S) \subseteq \mathfrak{A}$. However, by Lemma 4.2, Σ is maximal and hence $S = \hat{G}$. Therefore, we have

$$M = M^{\tilde{\alpha}}(\hat{G}) = M^{\tilde{\alpha}}(S) \subseteq \mathfrak{A} \subseteq M.$$

This completes the proof. □

Corollary 4.5. *Keep the notation as in Example* 4.3. *Then every σ-weakly closed subalgebra of $L^{\infty}(\mathbb{T}^2)$ which contains $M^{\tilde{\alpha}}(\Sigma)$ properly is not $\tilde{\alpha}$-invariant.*

Remark 4.6. The invariance of subalgebras for an automorphism group is an interesting problem because this property is convenient to study the structure of subalgebras. For example, let $G = \mathbb{T}$. Then $\hat{G} = \mathbb{Z}$ and so $\hat{G}_+ = \mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$. Let M be an arbitrary finite von Neumann algebra with a faithful, normal, tracial state τ and $\alpha = \{\alpha_t\}_{t\in\mathbb{T}}$ be an automorphism group of $\mathbb{T}$ on M such that $\tau \circ \alpha_t = \tau$ ($\forall t \in \mathbb{T}$). Then Solel showed in [8, Corollary 4.4] that every σ-weakly closed subalgebra of M which contains $M^{\alpha}(\mathbb{Z}_+)$ is α-invariant. He also proved the same result when $G = \hat{G} = \mathbb{R}$ and $\hat{G}_+ = \mathbb{R}_+$. That is, let M be a σ-finite von Neumann algebra and $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ be an automorphism group of $\mathbb{R}$ on M. Then every σ-weakly closed subalgebra of M that contains $M^{\alpha}(\mathbb{R}_+)$ is α-invariant [9, Proposition 2.1].

Remark 4.7. Lemma 3.1 raises the question; is every σ-weakly closed subalgebra of M able to describe as a subalgebra $M^{\tilde{\alpha}}(\Gamma)$ for some semigroup Γ in $\hat{G}$? Example 4.3 gives a counterexample for this question. In general, let $\hat{G}_+$ be a positive semigroup of $\hat{G}$ which induces an archimedean order in $\hat{G}$ and let N be not a factor. Then every σ-weakly closed subalgebra of $M = N \rtimes_{\alpha} \hat{G}$ which contains $M^{\tilde{\alpha}}(\hat{G}_+)$ never has the form $M^{\tilde{\alpha}}(\Lambda)$ for some semigroup Λ of $\hat{G}$. However we are interested in another case. More precisely, when is a σ-weakly closed subalgebra of $M^{\tilde{\alpha}}(\hat{G}_+)$ of the form $M^{\tilde{\alpha}}(\Lambda)$ for some semigroup Λ of $\hat{G}$?

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