A class of semilinear hyperbolic Volterra integrodifferential equations in Hilbert spaces is considered. The main results in this paper are the global existence of solutions to the initial problem of the equation for large data and an asymptotic estimation of the solution. An application to a system relating to three-dimensional viscoelastic dynamics is presented.

1. INTRODUCTION

Many problems in viscoelasticity and in heat conduction with memory can be reduced to nonlinear Volterra integrodifferential equations. A typical example is the equation

\[ u_{tt} = \varphi(u_x)_x - \int_0^t a(t, s) g(u_x(x, s)) \, ds + f(t, x), \]

\[ 0 < x < 1, \quad t > 0, \quad (1.1) \]

\[ u(t, 0) = u(t, 1) = 0, \quad u(0, x) = u_0(x), \quad u_x(0, x) = u_1(x), \quad (1.2) \]

which describes the motion of a homogeneous one-dimensional viscoelastic body. In the past 10 years many authors have studied this problem. Most of their work concerned the global existence of classical solutions to the equation (1.1) for small data (see, for example, [2, 9, 11]), and the breakdown of classical solutions for large data (cf. [1, 10]). An interesting

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and important problem arises when the problem admits a global solution for large data. Engler [3] discussed the global existence of weak solutions to the equation, with singular kernel \( a(t) \), and its asymptotic stability. Hrusa [7] studied (1.1) when \( \varphi \) is linear. He proved that if \( |g'(\xi)| \) is bounded then the problem admits a global smooth solution, and if \( a(t) = \exp(-t), \ 0 < \sup g'(\xi) < \varphi'(0) \) then the solution to the homogeneous equation decays exponentially to zero as \( t \to \infty \).

In this paper, we study the abstract semilinear Volterra integrodifferential equation

\[
    u'' + Au - \int_0^t a(t,s)A^{1/2}g(u(s)) \, ds = f(t), \quad t > 0, \quad (1.3)
\]

with the initial condition

\[
    u(0) = u_0, \quad u'(0) = u_1. \quad (1.4)
\]

Here \( \mathcal{A} \) is a linear positive self-adjoint operator in a Hilbert space \( X \), and \( g \) is a nonlinear operator \( D(\mathcal{A}^{1/2}) \to X \). Travis and Webb have dealt with local existence of solutions to initial problems for an abstract integrodifferential equation (see [12]). We will show that if \( g \) is locally Lipschitz continuous and uniformly bounded as an operator from \( D(\mathcal{A}) \) into \( D(\mathcal{A}^{1/2}) \) then the problem (1.3), (1.4) admits a unique global solution. In studying the asymptotic behavior of solutions to the problem we restrict ourselves to the case where the equation has convolutional form, that is, \( a(t,s) = a(t-s) \), and \( g(u) = p(A^{1/2}u) \). If there exists a positive constant \( \delta_0 \) such that \( a(t)\exp(\delta_0 t) \) is completely monotone, and the Frechet derivative \( p'(v) \) satisfies some reasonable assumptions, then we can prove that the solution to the homogeneous equation decays exponentially to zero. It is easy to see that the kernel of the form

\[
    a(t) = \sum_{l=1}^N k_le^{-\lambda_l t}, \quad k_l, \lambda_l > 0
\]

satisfies the requirement. Such kernels are commonly used in rheology (see, for example, [4] and references therein).

This paper is organized as follows. In Section 2, we state the basic assumptions and main results of the work. Section 3 is devoted to the local and global existence of the solution to the problem. In Section 4, we show the asymptotic behavior of the solution. Finally, in Section 5, we present an application of our results to a system which relates to three-dimensional viscoelastic dynamics.
2. ASSUMPTIONS AND RESULTS

We first discuss the global existence of solutions to the problem

\[ u'' + Au - \int_0^t a(t, s) A^{1/2} g(u(s)) \, ds = f(t), \quad t > 0, \quad (2.1) \]
\[ u(0) = u_0, \quad u'(0) = u_1. \quad (2.2) \]

Let \( X \) and \( V \) be Hilbert spaces with scalar products \((\quad,\quad)_x, (\quad,\quad)_v\) and norms \(\|\quad\|_x, \|\quad\|_v\), respectively. Suppose that \(V \subset X\) is dense in \(X\) and the injection of \(V\) in \(X\) is compact.

Now we give the assumptions on the operator \(A\).

\((A_1)\) There is a bilinear continuous form \(a(u,v)\) on \(V\) which is coercive and symmetric; that is,

\[ |a(u,v)| \leq M \|u\|_V \|v\|_V, \]
\[ a(u,u) \geq \lambda \|u\|_V^2, \]
\[ a(u,v) = a(v,u), \quad \forall u,v \in V, \]

where \(M, \lambda\) are positive constants. Let \(A\) be the linear operator associated with the bilinear form; that is,

\[(Au,v) = a(u,v), \quad \forall u \in D(A), \quad v \in V.\]

Then we know that \(A\) is a strictly positive self-adjoint unbounded operator in \(X\); that is,

\[(Au,u) \geq \lambda \|u\|_V^2, \quad \forall u \in D(A).\]

So we can define the powers \(A^s\) of \(A\) for \(s \in \mathbb{R}\). For every \(s > 0\), \(A^s\) is a self-adjoint operator in \(X\) with dense domain \(D(A^s) \subset X\). The space \(D(A^s)\) is endowed with the scalar product

\[(u,v)_{D(A^s)} = (A^s u, A^s v), \quad \forall s \in \mathbb{R},\]

which makes it a Hilbert space; we denote it by \(X_s\), with the scalar product \((\quad,\quad)_s\) and norm \(\|\quad\|_s\). It is clear that \(X_0 = X, X_1 = V\), and \(X_2 = D(A)\). And

\[D(A^{s_2}) \subset D(A^{s_1}), \quad \forall s_1 \leq s_2.\]

Each space is dense in the following one, the injection is continuous, and \(A^{s_2-s_1}\) is an isomorphism of \(X_{s_2}\) into \(X_{s_1}\).
For the operator $g$ we have the following assumptions:

\((G_1)\) $g$ maps $X_1$ into $X$ and is locally Lipschitzian; that is, there is a monotone increasing function $L_1(r)$ such that
\[
\|g(u_1) - g(u_2)\| \leq L_1(r)\|u_1 - u_2\|, \quad \forall u_1, u_2 \in X_1, \quad \|u_1\|, \|u_2\| \leq r.
\]

\((G_2)\) $g \in C(X_2, X_1)$ is uniformly bounded; that is, there exists a constant $L$ such that
\[
\|g(u)\| \leq L\|u\|_2, \quad \forall u \in X_2.
\]

For the existence theorem of local solutions to the problem, instead of $G$ we need only a weaker assumption.

\((G_2')\) $g \in C(X_2, X_1)$ and there is a monotone increasing function $L_2(r)$ such that
\[
\|g(u)\|_1 \leq L_2(r)\|u\|_2, \quad \forall u \in X_2, \quad \|u\|_1 \leq r.
\]

On the kernel $a(t, s)$, assume that

\((K_1)\) $a(t, s)$ and $a_t(t, s)$ are continuous on $[0, \infty) \times [0, \infty)$.

One of the main results of the paper is the following theorem.

**Theorem A.** Suppose that the hypotheses $(A_1), (G_1), (G_2)$, and $(K_1)$ hold. If $u_0 \in X_2, u_1 \in X_1, f$ and $f' \in L^1_{\text{loc}}([0, \infty); X)$, then the problem (2.1), (2.2) admits a unique global classical solution
\[
u \in C^k([0, \infty); X_{2-k}), \quad k = 0, 1, 2.
\]

Now we discuss the asymptotic stability of solutions to the following problem:

\[
u'' + Au - \int_0^t a(t, s)A^{1/2}p(A^{1/2}u(s)) \, ds = f(t), \quad (2.3)
\]
\[
u(0) = u_0, \quad \nu'(0) = u_1. \quad (2.4)
\]

We make the following assumptions on the above problem:

\((P_1)\) $p \in C^1(X, X), p(0) = 0$ and the Fréchet derivative $p'(v)$ satisfies the conditions
\[
\sup_{v \in X} \|p'(v)\|_{L(X, X)} < \hat{a}^{-1}(0),
\]
\[
(p'(v)u, u) \geq \beta\|u\|^2, \quad \forall u, v \in X,
\]
where \( \hat{a}(0) = \int_0^a \alpha(t) \, dt \) and \( \beta \) is a positive constant.

\( (P_1) \quad p \in C(X_1, X_2) \) is uniformly bounded and

\[
\sup_{u \in X_2} \frac{< A^{1/2} p(A^{1/2} u), Au >}{||u||_2^2} < \hat{a}^{-1}(0).
\]

\( (P_2) \) There exists a functional \( P(u) \in C^1(X; R) \) such that \( P(0) = 0 \) and

\[
p(v) = P'(v), \quad \forall v \in X.
\]

Finally, we assume that the kernel \( a(t, s) \) satisfies the following condition:

\( (K_2) \quad a(t, s) \equiv a(t - s) \neq 0 \) and \( a \in C^\infty([0, \infty)) \cap L^1(0, \infty); \) there exists a positive constant \( \delta_0 \) such that \( h(t) = e^{\delta_0 t} a(t) \) is completely monotone (see, for example, [5]), that is,

\[
(-1)^j b^{(j)}(t) \geq 0, \quad j = 0, 1, 2, \ldots .
\]

**Theorem B.** Suppose that the hypotheses \( (A_1), (P_1), (P_2), (P_3), \) and \( (K_2) \) hold. If \( u_0 \in X_2, u_1 \in X_1, f \) and \( f' \in L^2_{\text{loc}}([0, \infty); X) \), then the problem (2.3), (2.4) admits a unique global classical solution \( u \in C^4([0, \infty); X_{2-k}), k = 0, 1, 2, \) and there exist positive constants \( K \) and \( \sigma \) such that

\[
\sum_{k=0}^2 ||u^{(k)}(t)||_{2-k}^2 \\
\leq Ke^{-\sigma t} \left( \sum_{k=0}^2 ||u^{(k)}(0)||_{2-k}^2 + \int_0^t e^{\sigma s} \left( ||f(s)||^2 + ||f'(s)||^2 \right) ds \right). \tag{2.5}
\]

Now we apply the results to study the motion of a three-dimensional viscoelastic body. The configuration of the body is a bounded domain \( \Omega \subset R^3 \) with smooth boundary \( \partial \Omega \) and the displacement is a three-dimensional vector field \( u \). A typical problem is to determine \( u(t, x) \) such that

\[
u_{tt} = \text{div} \sigma + f(t, x) \quad \text{in} \ (0, \infty) \times \Omega, \tag{2.6}
\]

\[
u(t, x) = 0 \quad \text{for} \quad t \in (0, \infty), \quad x \in \partial \Omega, \tag{2.7}
\]

\[
u(0, x) = u^0(x), \quad u_i(0, x) = u_i^1(x), \tag{2.8}
\]
where $\sigma = (\sigma_{ij})$, 
\[
\sigma_{ij} = \sum_{k,l=1}^{2} c_{ijkl} \frac{\partial u_k}{\partial x_l} - \int_0^t a(t,s) p_{ij} (\nabla u(s,x)) \, ds,
\]
and $a$ is a scalar function.

On $c_{ijkl}$ and $p_{ij}$, we assume that
\begin{itemize}
  \item [(H_1)] the tensor $(c_{ijkl})$ is symmetric, 
  \[c_{ijkl} = c_{klij},\]
  \end{itemize}
and satisfies the uniform stability condition; that is, there exists a constant $\alpha > 0$ such that
\[
\sum_{i,j,k,l=1}^{3} \int_{\Omega} c_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \, dx \geq \alpha \| \nabla u \|_X^2, \quad \forall u \in (H^1_0(\Omega))^3, \tag{2.10}
\]
where $X = (L^2(\Omega))^3$;
\begin{itemize}
  \item [(H_2)] $p_{ij}(\xi) \in C^1$, $p_{ij}(0) = 0$, the tensor 
  \[b_{ijkl}(\xi) = \frac{\partial p_{ij}(\xi)}{\partial \xi_{kl}}\]
  is symmetric, 
  \[b_{ijkl}(\xi) = b_{klij}(\xi),\] \tag{2.11}
\end{itemize}
and there exists a constant $\beta > 0$ such that
\[
\sum_{i,j,k,l=1}^{3} \int_{\Omega} b_{ijkl}(\nabla v) \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \, dx \geq \beta \| \nabla u \|_X^2, \quad u, v \in (H^1_0(\Omega))^3. \tag{2.12}
\]
\begin{itemize}
  \item [(H_3)] There exists a constant $\rho_1 > 0$ such that 
  \[\sum_{i,j,k,l=1}^{3} \int_{\Omega} ((1-\rho_1)c_{ijkl} - \hat{a}(0)b_{ijkl}(\nabla u)) \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \, dx \geq 0, \quad \forall u, v \in (H^1_0(\Omega))^3.\]
\end{itemize}
Assumptions $(H_1)$–$(H_3)$ are motivated by mechanics. The following condition is similar to the one proposed by Dafermos and Nohel [2].
There exists a constant $r_0 > 0$ such that

$$3 \sum_{i, \ldots, r=1}^{\infty} \int_{\Omega} \left( (1 - r_0) c_{ijkl} - \hat{a}(0) b_{ijkl}(\nabla u) \right) c_{pqrs} \frac{\partial^2 u_k}{\partial x_i} \frac{\partial^2 u_q}{\partial x_r} \, dx \geq 0,$$

$$\forall u, v \in (H^2(\Omega) \cap H_0^1(\Omega))^3.$$

For the problem (2.6)–(2.8), we have

**Theorem C.** Suppose that $(H_1)$ and $(K_1)$ hold, $p_{ij} \in C^1$, and $b_{ijkl}(\xi)$ is uniformly bounded, i.e.,

$$|b_{ijkl}(\xi)| \leq C$$

for a suitable constant $C$. If $u^0 \in (H^2(\Omega) \cap H_0^1(\Omega))^3$, $u^1 \in (H_0^1(\Omega))^3$, and $f_i \in L^2_{loc}([0, \infty); X)$, then the problems (2.6) and (2.7) admit a unique solution;

$$u \in C^k([0, \infty); (H^{2-k}(\Omega))^3).$$

Moreover, if assumptions $(H_2)$–$(H_4)$ and $(K_2)$ hold, then the solution $u$ satisfies the estimate

$$\sum_{k=0}^{2} \left\| \left( \frac{\partial}{\partial t} \right)^k u(t, \cdot) \right\|_{(H^{2-k}(\Omega))^3}^2 \leq Ke^{-\sigma t} \left( \sum_{k=0}^{2} \left\| \left( \frac{\partial}{\partial t} \right)^k u(0, \cdot) \right\|_{(H^{2-k}(\Omega))^3}^2 + \int_0^t e^{\sigma s} \left( \| f(s, \cdot) \|_X^2 + \| f_i(s, \cdot) \|_X^2 \right) \, ds \right)$$

for suitable positive constants $K$ and $\sigma$.

### 3. Existence of the Global Solution

The problem discussed in this section is

$$u'' + Au - \int_0^t a(t, s) A^{1/2} g(u(s)) \, ds = f(t), \quad (3.1)$$

$$u(0) = u_0, \quad u'(0) = u_1. \quad (3.2)$$

The goal of the section is to prove Theorem A.
Theorem 3.1. Suppose that assumptions \((A_1), (G_1), (G_2),\) and \((K_1)\) hold. If \(u_0 \in X_2, u_1 \in X_1, f \in C([0, \infty); X),\) and \(f' \in L_{10}^{1, \infty}(0, \infty); X)\), then there exists \(T > 0\) such that the problem (3.1), (3.2) admits a unique classical solution

\[ u \in C^k([0, T]; X_{2-k}), \quad k = 0, 1, 2. \]

Moreover, if \([0, T_{\text{max}}]\) is the maximal interval of the solution, then

\[ \lim_{t \to T_{\text{max}}} \left( \|u(t)\|_2^2 + \|u'(t)\|_2^2 + \|u''(t)\|_2^2 \right) = \infty, \quad (3.3) \]

provided \(T_{\text{max}} < \infty\).

By using an iteration method, we can prove this theorem on the local existence of the classical solution (see, for example, [7]). We omit the details.

By Theorem 3.1, the proof of Theorem A follows immediately from the following lemma.

Lemma 3.1. Suppose that the assumptions of Theorem A are fulfilled and \(u(t) \in C^k([0, T); X_{2-k}), \ k = 0, 1, 2\) is the solution of the problems (3.1) and (3.2). Then

\[ \limsup_{t \to T} \left( \|u(t)\|_2^2 + \|u'(t)\|_2^2 + \|u''(t)\|_2^2 \right) < \infty. \quad (3.4) \]

Proof. Formally differentiating (3.1) with respect to \(t\) and taking the scalar product of it with \(u''\), we can obtain by the assumption \((G_2)\) that

\[
\frac{d}{dt}\left( \|u''(t)\|_2^2 + \|u'(t)\|_2^2 \right) \leq C(T)\left( \|u''(t)\|_2^2 + \|u(t)\|_2^2 \right) + \|f'(t)\|_2^2
\]

\[+ C(T) \int_0^t \|u(s)\|_2^2 \, ds, \quad 0 < t < T, \quad (3.5)\]

where \(C(T)\) is a positive constant depending on \(T\). By (3.5), (3.1), and \((G_2)\), it is not difficult to obtain

\[
\|u''(t)\|_2^2 + \|u'(t)\|_2^2 + \|u(t)\|_2^2
\]

\[\leq C\left( \|u_0\|_2^2 + \|u_1\|_2^2 + f_T \right)
\]

\[+ C(T) \int_0^t \left( \|u''(s)\|_2^2 + \|u'(s)\|_2^2 + \|u(s)\|_2^2 \right) \, ds, \quad 0 < t < T, \quad (3.6)\]

\[\箱\]
where \( C \) is a constant and \( \tilde{f}_T = \sup_{[0,T]} \| f(t) \|^2 + \int_0^T \| f'(s) \|^2 \, ds \). Then (3.4) follows from Gronwall's inequality and (3.6). This completes the proof of Lemma 3.1.

4. ASYMPTOTIC BEHAVIOR OF THE SOLUTION

This section is devoted to the discussion of the asymptotic behavior of solutions to the problem

\[
\begin{align*}
\dot{u}''(t) + Au(t) - \int_0^t a(t-s)A^{1/2}p\left(A^{1/2}u(s)\right) \, ds &= f(t), \\
\dot{u}(0) &= u_0, \quad \dot{u}'(0) = u_1.
\end{align*}
\]

First, we note that under the assumptions of Theorem B, the problem admits a global solution by Theorem A. Therefore, it suffices to show the estimate (2.5).

For the first order norm of the solution, we have

\[
\|u'(t)\|^2 + \|u(t)\|^2 \leq K_1 e^{-\sigma_1 t} \left( \|u_0\|^2 + \|u_1\|^2 + \int_0^t e^{\sigma_1 s} \|f(s)\|^2 \, ds \right),
\]

**Theorem 4.1.** Assume that the hypotheses of Theorem B hold. Let \( u \) be the solution to the problem. Then there exist constants \( K_1 \) and \( \sigma_1 \) such that

\[
\|u'(t)\|^2 + \|u(t)\|^2 \leq K_1 e^{-\sigma_1 t} \left( \|u_0\|^2 + \|u_1\|^2 + \int_0^t e^{\sigma_1 s} \|f(s)\|^2 \, ds \right).
\]

**Proof.** Set

\[
\begin{align*}
E_1(t) &= \frac{1}{2} \left( \|u'(t)\|^2 + \|u(t)\|^2 \right) - \tilde{a}(t)P\left(\dot{v}(t)\right), \\
H(t,s) &= P\left(\dot{v}(t)\right) - P\left(\dot{v}(s)\right) + (p(v(s)),v(s) - v(t)), \\
E_2(t) &= \int_0^t a(t-s)H(t,s) \, ds,
\end{align*}
\]

where \( \tilde{a}(t) = \int_0^t a(s) \, ds \) and \( v(t) = A^{1/2}u(t) \).

Taking the scalar product of (4.1) with \( u' + \epsilon_1 u + \epsilon_2 \int_0^t a(t-s)(u(t) - u(s)) \, ds \), where \( \epsilon_1 \) and \( \epsilon_2 \) are small positive constants determined later, we
have
\[
\frac{d}{dt} E(t; \epsilon_1, \epsilon_2)
= -a(t) P(v(t)) + \int_0^t a'(t-s) H(t, s) \, ds
+ \epsilon_1 \|u'(t)\|^2 - \epsilon_1 (Au(t), u(t)) + \epsilon_1 \tilde{a}(t) \langle p(v(t)), v(t) \rangle
- \epsilon_1 \int_0^t a(t-s)(p(v(t)) - p(v(s)), v(t)) \, ds + \epsilon_1 (f(t), u(t))
- \epsilon_2 \left( v(t) - \tilde{a}(t) p(v(t)), \int_0^t a(t-s)(v(t) - v(s)) \, ds \right)
- \epsilon_2 \left( \int_0^t a(t-s)(p(v(t)) - p(v(s))) \, ds, \right.
\left. \int_0^t a(t-s)(v(t) - v(s)) \, ds \right)
+ \epsilon_2 \left( u'(t), \int_0^t a'(t-s)(u(t) - u(s)) \, ds \right) - \epsilon_2 \tilde{a}(t) \|u'(t)\|^2
+ \epsilon_2 \left( f(t), \int_0^t a(t-s)(u(t) - u(s)) \, ds \right)
\]
where
\[
E(t; \epsilon_1, \epsilon_2) = E_1(t) + E_2(t) + \epsilon_1 (u'(t), u(t))
- \epsilon_2 \left( u'(t), \int_0^t a(t-s)(u(t) - u(s)) \, ds \right).
\]
By the assumptions \((P_1)\) and \((P_2)\), we can obtain that
\[
\|p(v(t))\| \leq (1 - \rho) \hat{\alpha}^{-1}(0) \|u(t)\|_1, \quad (4.5)
\]
\[
|\tilde{a}(t)(p(v(t)), v(t))| \leq (1 - \rho) \|u(t)\|_2^2, \quad (4.6)
\]
\[
H(t, s) \geq \frac{\beta}{2} \|u(t) - u(s)\|_2^2, \quad (4.7)
\]
\[
\left| \int_0^t a(t-s)(p(v(t)) - p(v(s)), v(t)) \, ds \right|
\leq \frac{\rho}{2} \|u(t)\|_2^2 + (\beta \rho \hat{\alpha}(0))^{-1} E_2(t) \quad (4.8)
\]
for some $0 < \rho < 1$. Using the above estimations, we get from (4.4) that

$$\frac{d}{dt} E(t; \epsilon_1, \epsilon_2) \leq -\left( \frac{1}{2} \tilde{a}(t) \epsilon_2 - \epsilon_1 \right) \|u'(t)\|^2 - \left( \frac{1}{4} \rho \epsilon_1 - \gamma \epsilon_2 \right) \|u(t)\|_2^2$$

$$+ \left( (\beta \rho \tilde{a}(0))^{-1} \epsilon_1 + C_1^2 (1 + \gamma^{-1} C_2^2) \epsilon_2 \right) E_2(t)$$

$$+ \left( 1 - C_1^2 \tilde{a}^{-1}(t) \epsilon_2 \right) \int_0^t \epsilon'(t-s) H(t, s) \, ds$$

$$+ \left( f(t), u'(t) \right) + (\rho^{-1} b_1 \epsilon_1 + \gamma^{-1} C_1^2 \epsilon_2) \|f(t)\|^2 \quad (4.9)$$

for some positive constants $C_3, C_2, b_1$ and any $\gamma > 0$.

Taking $\epsilon_1 = \frac{1}{2} \tilde{a}(1) \epsilon_2, \epsilon_2$ small enough and $\gamma = (\rho/32) \tilde{a}(1)$, and noting that

$$a'(t) + \delta_0 a(t) \leq 0, \quad (4.10)$$

$$\frac{1}{2} (E_1(t) + E_2(t)) \leq E(t; \epsilon_1, \epsilon_2) \leq 2 (E_1(t) + E_2(t)), \quad (4.11)$$

we can get from (4.9) that

$$\frac{d}{dt} E(t; \epsilon_1, \epsilon_2) \leq -C_3 (E_1(t) + E_2(t)) + (f(t), u'(t))$$

$$+ C_4 \|f(t)\|^2, \quad \forall t \geq 1, \quad (4.12)$$

for suitable positive constants $C_3$ and $C_4$.

By using $(P_2), (P_2)$, and Taylor’s theorem, we can show that

$$E_1(t) \geq \frac{\rho}{2} (\|u'(t)\|^2 + \|u(t)\|^2). \quad (4.13)$$

Then it follows from (4.12) that

$$\frac{d}{dt} E(t; \epsilon_1, \epsilon_2) \leq -\sigma_1 E(t; \epsilon_1, \epsilon_2) + C_5 \|f(t)\|^2, \quad \forall t \geq 1 \quad (4.14)$$

for suitable positive constants $\sigma_1$ and $C_5$. The estimation (4.3) follows from (4.14) easily. This completes the proof of Theorem 4.1.

Now we estimate the second-order norm of the solution. Multiplying (4.1) by $e^{\delta t}, \delta \in (0, \delta_0]$, and formally differentiating the resulting equation,
we have
\[
\frac{d}{dt} \left( e^{\delta t}(u^*(t) + Au(t) - f(t)) \right)
\]
\[
= a_{\delta}(0) e^{\delta t} A_{1/2} p(v(t)) + \int_0^t a_{\delta}'(t-s) e^{\delta s} A_{1/2} \rho v(s) ds,
\]
(4.15)
where \(a_{\delta}(t) = e^{\delta t} a(t)\). Let \(r_{\delta}(t)\) be the resolvent kernel associated with \(a_{\delta}'(t)\), that is, the unique solution of the resolvent equation
\[
a_{\delta}(0) r_{\delta}(t) + \frac{a_{\delta}'(t)}{a_{\delta}(0)} + \int_0^t a_{\delta}'(t-s) r_{\delta}(s) ds = 0.
\]
(4.16)
Applying \(r_{\delta}(t)\), (4.15) can be written (see, for example, [8])
\[
e^{\delta t}(u^{**} + a(0) r_{\delta}(0) u^* + Au + a(0) Q(u))
\]
\[
+ a(0) \int_0^t r_{\delta}'(t-s) e^{\delta s} A u(s) ds + a(0) \int_0^t r_{\delta}'(t-s) e^{\delta s} u''(s) ds
\]
\[
+ \delta e^{\delta t} (u^* + Au) = f_1(t),
\]
(4.17)
where
\[
Q(u) = r_{\delta}(0) A u - A_{1/2} p( A_{1/2} u),
\]
and
\[
f_1(t) = (a(0) r_{\delta}(0) + \delta) e^{\delta t} f(t)
\]
\[
+ e^{\delta t} f'(t) + a(0) \int_0^t r_{\delta}'(t-s) e^{\delta s} f(s) ds.
\]
The smoothness of \(r_{\delta}(t)\) is ensured by Lemma 4.1 below.

Proceeding from the equation (4.17), we will show the following result.

**Theorem 4.2.** Under the assumptions of Theorem B, there exist positive constants \(K_2\) and \(\sigma_2\) such that the solution \(u\) to the problem satisfies the estimation
\[
\|u^{**}(t)\|^2 + \|u'(t)\|^2 + \|u(t)\|^2
\]
\[
\leq K_2 e^{-\sigma t} \left( \|u_0\|^2 + \|u_1\|^2 + \|u''(0)\|^2 + \|u'(t)\|^2 + \|u(t)\|^2 \right)
\]
\[
+ B_2(\sigma) \int_0^t e^{-\sigma(t-s)} \left( \|u'(s)\|^2 + \|u(s)\|^2 + \|f(s)\|^2 + \|f'(s)\|^2 \right) ds
\]
(4.18)
for \(0 \leq \sigma \leq \sigma_2\), where \(B_2(\sigma)\) is a constant depending on \(\sigma\).
Before proving Theorem 4.2, we give some lemmas.

**Lemma 4.1.** Assume that $K$ holds. Then for $\delta \in (0, \delta_0]$ we have (i) $r_\delta(t) \in C^\infty [0, \infty)$; (ii) $\int_0^\infty |r_\delta(t)| \, dt, \int_0^\infty |r_\delta'(t)| \, dt$ are convergent and bounded independently of $\delta \in (0, \delta_0]$.

**Proof.** For the conclusion (i), see [9].

We prove (ii) by using Laplace transforms. Transforming (4.16), we get

$$\hat{r}_\delta(z) = \frac{1}{z \hat{a}_\delta(z)} - \frac{1}{a(0)},$$

where we denote the Laplace transform of $r_\delta(t)$ by $\hat{r}_\delta(z)$. By using (4.19) and (4.16), we have

$$\hat{r}_\delta(z) = \frac{1}{\hat{a}_\delta(z)} - \frac{z}{a(0)} + \frac{a'(0)}{a(0)^2}.$$  \hspace{1cm} (4.20)

As in [9], we can verify that $\hat{r}_\delta(z)$ is analytic in $\Re(z) > -(\delta_0 - \delta)$ and is infinitely differentiable in $\Re(z) \geq -(\delta_0 - \delta)$. For large $z$, after integrating by parts, we find

$$\hat{a}_\delta(z) = a(0)(z - \delta)^{-1} + a'(0)(z - \delta)^{-2}$$

$$+ a''(0)(z - \delta)^{-3} + O((z - \delta)^{-4}).$$

Thus

$$\frac{1}{\hat{a}_\delta(z)} = \frac{1}{a(0)}(z - \delta) - \frac{a'(0)}{a(0)^2}$$

$$- \left( \frac{a''(0)}{a(0)^3} - \frac{a'(0)^2}{a(0)^4} \right) (z - \delta)^{-1} + O((z - \delta)^{-2}).$$  \hspace{1cm} (4.21)

Noting that $a'_\delta(0) = \delta a(0) + a'(0)$, we obtain from (4.20) and (4.21) that

$$\hat{r}_\delta(z) = \frac{a'(0)^2 - a(0)a''(0)}{a(0)^3} (z - \delta)^{-1} + O((z - \delta)^{-2}).$$  \hspace{1cm} (4.22)

And it is not difficult to verify from (4.22) that

$$r_\delta'(t) = \frac{1}{2\pi i^2} \int_{-\infty}^\infty e^{i\eta t} \frac{d^2}{dz^2} \hat{r}_\delta(i\eta) \, d\eta.$$  \hspace{1cm} (4.23)
From (4.22) and (4.23), we can conclude that \( \int_0^t |r_s'(t)| \, dt \) is bounded by a constant which is independent of \( \delta \in (0, \delta_0] \).

A similar method can be used to prove the conclusion on \( \int_0^t |r_s''(t)| \, dt \).

**Lemma 4.2.** Let \( a(t) \) be completely monotone on \([0, \infty)\) and let \( a \in L^1(0, \infty) \). Then
\[
\frac{\text{Re}(\hat{a}(i\xi))}{|\hat{a}(i\xi)|^2} \geq \hat{a}^{-1}(0).
\]

**Proof.** By Bernstein's theorem (see [5]), it follows that there exists a positive measure \( \mu \) such that
\[
a(t) = \int_0^\infty e^{-\sigma t} \mu(\sigma) \, d\sigma, \quad \int_0^\infty \sigma^{-\frac{1}{2}} \mu(\sigma) < \infty. \tag{4.24}
\]
Then it is not difficult to show the Lemma by (4.24).

**Lemma 4.3.** Let \( a(t) \) satisfy the assumption \((K_2)\). Then for any \( T > 0 \) and any \( w \in C([0, T]; X) \), it holds that
\[
\int_0^T r_s(0)\|w(t)\|^2 + \left( \int_0^t r_s'(t-s) w(t-s) \, ds, w(t) \right) \, dt
\geq \hat{a}_s^{-1}(0) \int_0^T \|w(t)\|^2 \, dt, \tag{4.25}
\]
where \( r_s \) is the resolvent kernel associated with \( a_s \).

**Proof.** By applying Parseval's formula, Lemma 4.2 and (4.20), we can get the estimate (4.25). We omit the details.

Now we proceed to prove Theorem 4.2.

**Proof of Theorem 4.2.** Taking the scalar product of (4.17) with \( e^{s(t)}u'' \) and integrating the resulting equation with respect to \( t \) from 0 to \( T \), we get the estimate
\[
e^{2\delta T} \left( \frac{1}{2} \|u''(T)\|^2 + \frac{1}{2} \|u'(T)\|^2 \right)
+ a(0)r_s(0) \int_0^T \|e^{s(t)}u''(t)\|^2 \, dt - a(0) \int_0^T e^{2\delta t} \left( \frac{d}{dt} Q(u(t)), u'(t) \right) \, dt
\leq C \left( U_2 + \delta \int_0^T e^{2\delta t} \left( \|u(t)\|^2_1 + \|u'(t)\|^2_1 + \|u''(t)\|^2 \right) \, dt \right)
+ B(\delta) \left( e^{2\delta T} \left( \|u'(T)\|^2 + \|u(T)\|^2_1 \right) \right.
\left. + \int_0^T e^{2\delta t} \left( \|u(t)\|^2_1 + \|u'(t)\|^2 \right) \, dt + F(T) \right) \tag{4.26}
\]
where

\[ U_2 = \|u_0\|_2^2 + \|u_1\|_1^2 + \|u''(0)\|^2, \]

\[ F(T) = \int_0^T e^{2sI} \left( \|f(t)\|^2 + \|f'(t)\|^2 \right) dt. \]

Hereafter \( C \) and \( B(\delta) \) denote various constants which are independent of and dependent on \( \delta \in (0, \delta_0] \), respectively.

Taking the scalar product of (4.17) with \( e^{\delta t}u' \) and integrating the resulting equation with respect to \( t \) from 0 to \( T \), we can obtain

\[
e^{2\delta T} \Biggl( (u''(T), u'(T)) + \frac{1}{2} u(0)r_0'(0)\|u'(T)\|^2 \Biggr)
+ \int_0^T e^{2\delta t} \left( \|u'(t)\|_1^2 - \|u''(t)\|^2 \right) dt
\leq C \left( U_2 + \delta \int_0^T e^{2\delta t} \left( \|u(t)\|_2^2 + \|u'(t)\|_1^2 + \|u''(t)\|^2 \right) dt \right)
+ B(\delta) \left( \int_0^T e^{2\delta t} \left( \|u(t)\|_2^2 + \|u'(t)\|_1^2 \right) dt + F(T) \right). \tag{4.27} \]

Multiplying (4.27) by \((a(0)r_0(0) - \epsilon)\), adding the resulting expression to (4.26), and taking \( \epsilon \) small enough, we can obtain an estimate as follows:

\[
e^{2\delta T} \left( \|u''(T)\|^2 + \|u'(T)\|_1^2 \right) + \int_0^T e^{2\delta t} \left( \|u''(t)\|^2 + \|u'(t)\|_1^2 \right) dt
\leq C \left( U_2 + \delta \int_0^T e^{2\delta t} \left( \|u(t)\|_2^2 + \|u'(t)\|_1^2 + \|u''(t)\|^2 \right) dt \right)
+ B(\delta) \left( e^{2\delta T} \left( \|u(T)\|_2^2 + \|u'(T)\|_1^2 \right) \right.
+ \left. \int_0^T e^{2\delta t} \left( \|u(t)\|_1^2 + \|u'(t)\|_2^2 \right) dt + F(T) \right). \tag{4.28} \]

By the assumptions, it is easy to verify that

\[
\int_0^T e^{2\delta t} \left( A^{1/2} p(A^{1/2} u(t)), Au(t) \right) dt \leq (1 - \rho) \hat{\omega}^{-1}(0) \int_0^T e^{2\delta t} \|u(t)\|_2^2 dt \tag{4.29} \]
for some $0 < \rho < 1$ and that

$$
\left| a(0) \int_0^T \left[ \int_0^t r_\delta(t-s)e^{\delta t}u^*(s) \, ds, e^{\delta t}Au(t) \right] dt \right|
\leq \frac{1}{4} \rho a(0) \hat{a}^{-1}(0) \int_0^T e^{2\delta t}\|u(t)\|^2 \, dt
+ \rho^{-1} a(0) \hat{a}(0) \left( \int_0^T |r_\delta(t)| \, dt \right)^2 \int_0^T e^{2\delta t}\|u^*(t)\|^2 \, dt. \quad (4.30)
$$

Taking the scalar product of (4.17) with $e^{\delta t}Au$, integrating the resulting expression with respect to $t$ from 0 to $T$, and applying Lemma 4.3, (4.29), and (4.30), we get

$$
e^{2\delta T}\|u(T)\|^2 + \int_0^T e^{2\delta t}\|u(t)\|^2 \, dt
\leq C \left( e^{2\delta T} (\|u^*(T)\|^2 + \|u'(T)\|^2) + \int_0^T e^{2\delta t} (\|u^*(t)\|^2 + \|u'(t)\|^2) \, dt \right)
+ CU_2 + C\delta \int_0^T e^{2\delta t}\|u(t)\|^2 \, dt + CF(T). \quad (4.31)
$$

Combining (4.28) with (4.31) and taking $\delta$ small enough, we can complete the proof of Theorem 4.2.

**Proof of Theorem B.** In fact, Theorem B is an immediate consequence of Theorems 4.1 and 4.2. The only thing we should do to prove the theorem is to estimate $\int_0^T (e^{-\alpha(t-s)}(\|u'(s)\|^2 + \|u(s)\|^2) \, ds$. This is not difficult. We omit it. The proof is completed.

### 5. APPLICATIONS

In this section we study the following problem:

$$
\frac{\partial^2 u_i}{\partial t^2} - \sum_{j,k,l=1}^3 \epsilon_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} + \int_0^t a(t,s) \sum_{j=1}^3 \frac{\partial}{\partial x_j} p_{ij} \left( \nabla u(s,x) \right) ds \quad (5.1)
$$

$$
= f(t,x) \quad \text{in} \ (0,\infty) \times \Omega,
$$

$$
u_i(t,x) = 0, \quad (t,x) \in (0,\infty) \times \partial \Omega, \quad (5.2)
$$

$$
u_i(0,x) = u_i^0(x), \quad \frac{\partial u_i}{\partial t}(0,x) = u_i^1(x), \quad x \in \Omega, \quad i = 1, 2, 3. \quad (5.3)
$$
Now we prove Theorem C for the problem (5.1)–(5.3). Let $X = (L^2(\Omega))^3$, $V = (H^1_0(\Omega))^3$, and

$$a(u, v) = \sum_{i,j,k,l=1}^{3} c_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial v_k}{\partial x_l} dx. \quad (5.4)$$

It is easy to verify that $a(u, v)$ satisfies the assumption $(A_1)$ and

$$Au = - \left( \sum_{i,j,k,l=1}^{3} c_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} \right), \quad \forall u \in (H^2(\Omega) \cap H^1_0(\Omega))^3. \quad (5.5)$$

In order to define the nonlinear operator $p$ in (2.3), as in [6] we need the following Lemma:

**Lemma 5.1.** Let $A$ be the operator defined by $a(u, v)$. Then there exist a linear bounded operator $B_j$ in $X$ and a positive constant $C$ such that

$$B_j A^{1/2} A^{1/2} = - A^{1/2} B_j^*, \quad (6.6)$$

$$C \|u\|^2 \leq \sum_{j=1}^{3} \|B_j u\|^2 \leq \alpha^{-1} \|u\|^2, \quad \forall u \in X, \quad (5.7)$$

where $B_j^*$ is the adjoint operator of $B_j$ and $\alpha$ is the constant in $(H_2)$.

**Proof.** Let $B_j = (\partial / \partial x_j) A^{-1/2}$. Then we can show the lemma easily.

Using Lemma 5.1, we have

$$\frac{\partial}{\partial x_j} \left( p_j \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right) \right) = - A^{1/2} B_j^* \left( p_j \left( B_1 A^{1/2} u, B_2 A^{1/2} u, B_3 A^{1/2} u \right) \right).$$

Let

$$p(v) = \sum_{j=1}^{3} B_j^* \left( p_j \left( B_1 v, B_2 v, B_3 v \right) \right), \quad \forall v \in X. \quad (5.8)$$

Then by applying the assumptions $(H_2)$, $(H_3)$, and $(H_4)$, we can show that the operator $p(v)$ defined by (5.8) satisfies the assumptions $(P_1)$, $(P_2)$, and $(P_3)$.

Now we complete the proof of Theorem C by Theorems A and B.
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REFERENCES