Characterizations of Certain Classes of Hankel Operators on the Bergman Spaces of the Unit Disk

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Let $f$ be an integrable function on the unit disk. The Hankel operator $H_f$ is densely defined on the Bergman space $A^p$ by $H_f g = fg - P(fg)$, where $g$ is a bounded analytic function and $P$ is the Bergman projection (orthogonal projection from $L^2$ to $A^2$) extended to $L^1$ via its integral formula. In this paper, the functions $f$ for which $H_f$ extends to a bounded operator from $A^p$ to $L^p$ are characterized, $1 < p < \infty$. Also characterized are the functions $f$ for which $H_f$ extends to a compact or Schatten class operator on $A^2$. The proofs can be extended to handle any smoothly bounded domain in $\mathbb{C}$ in place of the unit disk. © 1992 Academic Press, Inc.

1. INTRODUCTION

Let $D$ be the unit disk in the complex plane $\mathbb{C}$, endowed with two-dimensional area measure $dA$. Let $L^2$ denote $L^2(D, dA)$ and let $A^2$ denote the closed subspace of $L^2$ consisting of analytic functions. The orthogonal projection from $L^2$ to $A^2$ will be denoted $P$. Given a function $f \in L^2$ it is possible to define the Hankel operator $H_f$ on the bounded analytic functions $H^\infty$ by

$$H_f g = fg - P(fg).$$

In fact, since $P$ can be explicitly described in terms of integration against the Bergman kernel, it is possible to make sense of $H_f$ when $f$ is merely in $L^1$. The purpose of this paper is to prove necessary and sufficient condi-
tions on $f$ in order that $H_f$ be bounded in the $L^2$ norm, that is, in order that there exists a constant $C$ such that for all $g$ in $H^\infty$

$$\int |fg - P(fg)|^2 \, dA \leq C \int |g|^2 \, dA.$$ 

A standard side effect of the characterization is a similar characterization of the functions $f$ for which $H_f$ is (extends to be) a compact operator from $A^2$ to $L^2$. Since $H_f$ is just multiplication by $f$, followed by projection onto $L^2 \ominus A^2$ the orthogonal complement of $A^2$, we can obtain information on $H_f$ by investigating the form $\langle fg, h \rangle$, $g \in A^2$, $h \in L^2 \ominus A^2$. This will be our approach. It is only this so-called "big Hankel operator" that we will be considering. The "small" or "reduced" Hankel operator is defined by $h_f g = Q(fg)$ where $Q$ is the projection onto the conjugate analytic functions. See [12] and its references for information on that operator.

Previous work has obtained necessary and sufficient conditions for the boundedness or compactness of $H_f$, but always with some restrictions on the function $f$. The first may have been [4] in which S. Axler obtained among other things the result that if $f \in A^2$ then $H_f$ is bounded (respectively compact) if and only if $f$ belongs to the Bloch space $B$ (resp. the little Bloch space $B_0$).

In [6], Békollé, Berger, Coburn, and Zhu study the same problem in a more general setting (bounded symmetric domains in $\mathbb{C}^N$) but with the restriction that $H_f$ and $H_f$ be simultaneously bounded or, equivalently, that $f$ be real valued.

Another result, due to K. Stroethoff [18], characterizes the bounded functions $f$ for which $H_f$ is compact. The nature of his argument does not allow a limiting process to obtain a compactness criterion for unbounded $f$.

There have been several extensions and generalizations of these results. See, for example, [3, 5, 11, 19–23, 27]. The results in this paper are restricted (at present) to the disk $D$ or at least to smoothly bounded domains in the plane. However, the method used is general enough to obtain with essentially no extra effort the criteria for compactness, for $L^p$ boundedness when $p > 1$, and for membership in the Schatten classes $\mathcal{S}_p$ when $p \geq 1$. It is hoped that the method will eventually be extended to handle the case of the unit ball $B$ in $\mathbb{C}^N$ and even to strongly pseudoconvex domains. What is lacking at present are appropriate analogues of Lemmas 2 and 3 in Section 3. A similar problem occurs for weighted Bergman spaces. See Section 7 for a discussion of these problems.

The next section takes up some basic ideas needed for the main theorem. In Section 3 the main theorem is stated and proved, characterizing bounded Hankel operators $H_f$ in terms of $f$. In subsequent section I will
discuss extensions of the main result. Section 4 contains a proof of the corresponding compactness result, Section 5 a characterization of the boundedness in the $L^p$ norm, and Section 6 a characterization of the membership in the Schatten ideals $S_p$, $1 \leq p < \infty$. Finally, in Section 7, I will discuss the corresponding result in the unit ball in $\mathbb{C}^N$, $N > 1$, and for weighted $A^2$ spaces. While I am optimistic about soon finding a proof for the ball in $\mathbb{C}^N$, I do not think the same methods will be able to extend the result to the general setting of bounded symmetric domains dealt with in [6]. It does, however, seem likely that a similar result must be valid in that setting.

2. BACKGROUND AND PRELIMINARY RESULTS

The characterizations obtained here, in [4, 6, 18] all make use of the following ideas. Let $\rho$ denote the pseudohyperbolic metric,

$$\rho(z, w) = \frac{|z - w|}{|1 - \bar{z} w|}$$

and let $d$ denote the hyperbolic metric,

$$d(z, w) = \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$  

Both of these metrics are invariant under automorphisms of the disk. That is, if $\varphi: D \rightarrow D$ is analytic, one-to-one, and onto, then

$$\rho(\varphi(z), \varphi(w)) = \rho(z, w).$$

For each $a \in D$ let $\varphi_a$ be the Möbius transformation

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a} z}.$$  

Then $\varphi_a$ is an automorphism of $D$ which interchanges 0 and $a$. Two properties of $\varphi_a$ are required:

$$\varphi_a(\varphi_a(z)) = z$$

$$\varphi'_a(z) = \frac{1 - |a|^2}{(1 - \bar{a} z)^3}. \quad (2.1)$$

Given any automorphism $\varphi$ of $D$, let $U_\varphi$ be the operator defined on $L^2$ by

$$U_\varphi g = (g \circ \varphi) \varphi'.$$

Then, if $\|g\|$ denotes the $L^2$-norm of $g$, we have $\|U_\varphi g\| = \|g\|$. This is a consequence of the fact that $|\varphi'|^2$ is precisely the real Jacobian of the mapping.
Thus $U_\varphi$ is unitary with inverse $U_{\varphi^{-1}}$ and $U_\varphi$ preserves both $A^2$ and its orthogonal complement $L^2 \ominus A^2$. Thus $U_\varphi$ commutes with $P$. Note that (2.1) implies that $U_{\varphi_0} U_{\varphi_2} = I$. A good source for all these ideas is Kehe Zhu's book [26, Sect. 4.3].

Fix an arbitrary positive number $r$ and let $D(z) = D(z, r) = \{ w \in D : d(z, w) < r \}$. Define the 2-mean oscillation of a function $f$ on $D(z)$ by

$$
MO_r^2(f, z) = \left( \frac{1}{|D(z)|} \int_{D(z)} \frac{1}{|D(z)|} \int_{D(z)} f dA \right)^{1/2}.
$$

This is nothing more than the distance from $f$ to the constant functions in the space $L^2(D(z), (1/|D(z)|) dA)$. Define the space $BMO_r^2$ to be the set of functions $f$ in $L^2$ such that $MO_r^2(f, z)$ is bounded. Define $VMO_r^2$ to be the functions in $BMO_r^2$ for which $MO_r^2(f, z) \to 0$ as $|z| \to 1$. The space $BMO_r^2$ depends on the exponent 2. That is, if a space $BMO_p$ is defined analogously using $L^p$ norms, then $BMO_r^p \neq BMO_r^2$ unless $p = 2$. This differs from classical $BMO$ and the reason is that the disks $D(z)$ used to define the space are of fixed (hyperbolic) size. Nevertheless, the spaces $BMO_r^p$ are independent of $r$! Thus $r$ must be fixed, but it does not matter what it is fixed at. See K. Zhu [27]. Henceforth the notational dependence on $r$ is dropped.

The characterization in [6] is that for real valued functions $f$, $H_f$ is bounded (respectively compact) if and only if $f \in BMO_r^2$ (respectively $f \in VMO_r^2$). Zhu has shown that the same ideas can be extended to $L^p$ boundedness: For real $f$ the operator $H_f$ is bounded (compact) if and only if $f$ belongs to $BMO_r^p$ ($VMO_r^p$). It should be pointed out that all the results mentioned include additional necessary and sufficient conditions similar to the additional conditions in our main theorem.

In [18] (for the disk $D$) and [19] (for the ball in $\mathbb{C}^N$ and polydisks), Stroethoff has obtained compactness criteria that do not require $f$ to be real, but do require it to be bounded. The functions $U_{\varphi_a} 1 = \varphi_a$ are unit vectors in $A^2$ that tend to 0 weakly as $|a|$ tends to 1. If $H_f$ is compact, then $\| f \circ \varphi_a - P(f \circ \varphi_a) \| = \| H_f \varphi_a \| \to 0$ as $|a| \to 1$. This necessary condition turns out to be sufficient and Stroethoff showed this when $f$ is bounded. Stroethoff actually showed a little more, namely that $\| f \circ \varphi_a - P(f \circ \varphi_a) \|_{L^p} \to 0$ is also necessary and sufficient for any $p > 1$. This seems to be special to the case $f \in L^\infty$. A second necessary and sufficient condition obtained by Stroethoff is discussed below.

One characterization to be obtained here is that $H_f$ is bounded if and only if the following function is bounded:

$$
\inf_{h \in A^2} \frac{1}{|D(z)|} \int_{D(z)} | f - h |^2 dA.
$$
Thus our result replaces bounded distance to the constants (BMO) with bounded distance to analytic functions (BDA?). It will be shown later (Section 5) that for real valued functions and conjugates of analytic functions the two distances are comparable. Thus many of the previous results about boundedness of $H_f$, when specialized to the unit disk $D$, follow from ours. Stroethoff's second compactness criterion is that the above function tends to zero as $z$ tends to the boundary of $D$. Again, his proof is only for $f \in L^\infty$ and again he also gets an $L^p$, $p > 1$, version that seems to be special to this case. More on this in Sections 4 and 5.

There is a class of functions for which $H_f$ is almost trivially bounded. If $f$ satisfies

$$
\sup_{z \in D} \frac{1}{|D(z)|} \int_{D(z)} |f|^2 \, dA < \infty, \quad (2.2)
$$

then (see [17, 9, 16]) the measure $|f|^2 \, dA$ is a "Carleson measure" for the Bergman space and so the operator $M_f : A^2 \to L^2$ defined by $M_f g = fg$ is bounded. It is immediate that $H_f = (I - P) M_f$ is bounded. A second part of our characterization is that $H_f$ is bounded if and only if $f$ can be decomposed as $f = f_1 + f_2$ where $f_1$ satisfies (2.2) above and $(1 - |z|) \partial f_2(z)$ is bounded. Here $\partial = \partial / \partial z = (1/2)(\partial / \partial x + i(\partial / \partial y))$ denotes the usual "d-bar" derivative. This is similar to the decomposition in [6] which essentially replaces the $\partial$ with the full gradient. It should also be compared to the Bloch space result of Axler in [4] which says that for conjugate analytic symbol $f$, $H_f$ is bounded if and only if $(1 - |z|^2) \partial f$ is bounded. In case $f$ is real we have $|\partial f|^2 = (1/4)(|\partial f / \partial x|^2 + |\partial f / \partial y|^2) = (1/4) |\nabla f|^2$, while if $f$ is conjugate analytic then $|\partial f|^2 = (1/2) |\nabla f|^2$.

3. THE MAIN THEOREM

To make clear a point mentioned earlier, the projection $P$ from $L^2$ to $A^2$ is given explicitly by the formula

$$
(Pf)(z) = \frac{1}{\pi} \int_D \frac{f(w)}{(1 - z \bar{w})^2} \, dA(w). \quad (3.1)
$$

Thus $P(fg)$ is defined for any $f \in L^1$ and $g \in H^\infty$. If $f$ is an analytic function in $L^1$, then $P(fg) = fg$ and for such functions the Hankel operator $H_f$ is zero. If $f \in L^1$ and the Hankel operator is bounded, then $f - Pf = H_f 1 \in L^2$ and so in particular $Pf \in L^1$. Thus $H_f = H_f - Pf$ and we may replace $f$ with $f - Pf$ and suppose without any loss of generality that $f \in L^2$. The following is our main theorem.
THEOREM 1. Let \( f \in L^2 \) and let \( H_f \) be defined on \( H^\infty \) by
\( H_f g = fg - P(fg) \). Then the following are equivalent.

(a) \( H_f \) is bounded in the \( L^2 \) norm.

(b) \( \sup \{ \| f \circ \phi_a - P(f \circ \phi_a) \| : a \in D \} < \infty \).

(c) The function \( F(z) \) defined by
\[
F(z)^2 = \inf \left\{ \frac{1}{|D(z)|} \int_{D(z)} |f - h|^2 \, dA : h \in A^2 \right\}
\]
is bounded.

(d) \( f \) admits a decomposition \( f = f_1 + f_2 \) where \( f_2 \) is \( C^1 \) on \( D \) and satisfies
\[
(1 - |z|) \partial f_2(z)
\]
is bounded,
while \( f_1 \) satisfies (2.2), that is, the function
\[
G(z)^2 = \frac{1}{|D(z)|} \int_{D(z)} |f_1|^2 \, dA
\]
is bounded.

For each of the conditions (a) through (b) we have an associated quantity or norm: Define \( C_1 \) through \( C_4 \) as follows.

\[
C_1 = \| H_f \| = \sup \{ \| H_f g \| : \| g \| \leq 1 \}
\]
\[
C_2 = \sup_{a \in D} \| f \circ \phi_a - P(f \circ \phi_a) \|
\]
\[
C_3 = \| F \|_\infty = \sup_{z \in D} |F(z)|
\]
\[
C_4 = \inf \{ \| G \|_\infty + \| (1 - |z|) \partial f_2 \|_\infty : f = f_1 + f_2 \}.
\]

A consequence of the proof will be that there exist constants \( C(r) \), depending only on the hyperbolic radius of the disks \( D(z) \), such that
\[
C_j / C(r) \leq C_k \leq C(r) C_j \text{ for all } j, k \in \{1, 2, 3, 4\}.
\]

In the course of the proof we shall encounter many different constants. The symbol \( C \) without subscripts will denote an absolute constant which may be different from one occurrence to the next. The symbol \( C(r) \) will similarly denote different constants depending only on our initial choice of \( r \).

The plan of the proof will be to show that (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d) \( \Rightarrow \) (a). Only the last two implications are new. Note that a function in \( BMO^2 \) will
satisfy (c). Our implications (c) $\Rightarrow$ (d) $\Rightarrow$ (a) provide a new proof (at least in the unit disk) of the portion of [6] that says $f \in BMO^2$ implies $H_f$ is bounded.

**Proof.** Assume (a). Since $U_\varphi$ takes $H^\infty$ to itself and preserves the $L^2$ norm, we conclude that $U_\varphi H_f U^*_\varphi$ is bounded for any automorphism $\varphi$ and has the same norm as $H_f$. But simple calculations show that $U_\varphi H_f U^*_\varphi = H_{f \circ \varphi}$. Thus $\|f \circ \varphi - P(f \circ \varphi)\| = \|H_{f \circ \varphi} 1\| \leq \|H_{f \circ \varphi}\| = \|H_f\|$, and (b) is established with $C_2 \leq C_1$.

Now assume (b) and apply $U_\varphi^*$ to the function $f \circ \varphi_a - P(f \circ \varphi_a)$ to get a function with the same norm:

$$\|\varphi_a f - P(\varphi_a f)\| \leq C_2.$$

Writing this out as an integral gives (after a minor rearrangement)

$$\int_D \left| f - \frac{1}{\varphi_a} P(\varphi_a f) \right|^2 |\varphi_a|^2 \, dA \leq C^2_2.$$

Since $(1/|D(a)|) \chi_{D(a)} \leq C(r) |\varphi_a|^{-2}$ and $h = (1/\varphi_a) P(\varphi_a f) \in A^\infty$, we obtain (c) and the estimate $C_3^2 \leq C(r) C_2^2$.

Now let (c) be given. We are going to select a covering of $D$ by disks $D_j = D(z_j, r/2)$ and then use (c) to associate an analytic function to each such disk. First select a sequence $\{z_j\}$ which satisfies $d(z_j, z_k) \geq r/4$ for every $k \neq j$ and such that the collection $\{D(z_j, r/3): j = 1, 2, \ldots\}$ covers $D$. See [7, 14] for ways this can be done. Let $D_j = D(z_j, r/2)$. The two main properties of $\{D_j\}$ that we need are the following. Firstly, there is a positive integer $M(r)$ such that no point $z \in D$ lies in more than $M(r)$ of the disks $D_j$, that is,

$$\sum_j \chi_{D_j}(z) \leq M(r) \quad \text{for all } z \in D.$$

The same holds for the collection $\{D(z_j, cr)\}$ for any fixed finite $c$. Secondly, if $D_j$ and $D_k$ overlap and $z \in D_j \cap D_k$, then $D(z, r/2) \subset D(z_j, r) \cap D(z_k, r)$. The first of these two facts is well known and follows easily from the automorphism invariance of the hyperbolic metric $d$ and the observation that the disks $D(z_j, r/8)$ are disjoint. The second is a simple consequence of the triangle inequality for $d$.

By (c) associate to each $j$ an analytic function $h_j(z)$ such that

$$\frac{1}{|D(z_j)|} \int_{D(z_j)} |f - h_j|^2 \, dA < 4F(z_j)^2 \leq 4C^2_2.$$
(Recall that \(D(z_j) = D(z_j, r)\). We will show that \(|h_j - h_k|\) has an upper bound of \(C(r) C_3\) on \(D_j \cap D_k\). For \(z \in D_j \cap D_k\) then

\[
|h_j(z) - h_k(z)| \cdot |D(z, r/2)|^{1/2} \leq \left( C(r) \int_{D(z, r/2)} |h_j - h_k|^2 \, dA \right)^{1/2}
\]

\[
\leq \left( C(r) \int_{D(z_j)} |h_j|^2 \, dA \right)^{1/2}
\]

\[
+ \left( C(r) \int_{D(z_k)} |f - h_k|^2 \, dA \right)^{1/2}
\]

\[
\leq C(r)(F(z_j) |D(z_j)|^{1/2} + F(z_k) |D(z_k)|^{1/2}).
\]

Now, there exists a constant \(C(r)\) such that

\[
|D(z_j, r)| \leq C(r) |D(z, r/2)|
\]

whenever \(d(z, z_j) \leq r\). (See [26, p. 61].) This plus \(F(z) \leq C_3\) gives

\[
|h_j(z) - h_k(z)| \leq C(r) C_3.
\]

(3.2)

Now we are going to patch together the various \(h_j\) using a partition of unity \(\{\gamma_j\}\) subordinate to the covering \(\{D_j\}\) with one additional property:

\[
(1 - |z|) |\nabla \gamma_j(z)| \leq C(r).
\]

(3.3)

It is standard how to achieve this, but here are most of the details. Select any fixed \(C^\infty\) function \(\psi\) with support in \(D(0, r/2)\) and such that \(\psi = 1\) on \(D(0, r/3)\). Let \(\varphi_j = \varphi_{z_j}\) be the Möbius transformation taking \(z_j\) to 0 and let \(\psi_j = \psi \circ \varphi_j\). Finally let \(\gamma_j = \psi_j / \sum_k \psi_k\). Because the \(D(z_j, r/3)\) cover \(D\), the denominator stays bounded away from zero and because \(\{D_j\}\) is a locally finite covering, \(\gamma_j\) is \(C^\infty\). It is routine to bound \(\nabla \gamma_j\) by first order derivatives of those \(\varphi_j\) for which \(z \in D_j\) times constants depending only on properties of \(\psi\). This will give (3.3).

Now define \(f_2 = \sum h_j \gamma_j\) and \(f_1 = f - f_2\) and we will verify (d) for these functions. Note that \(f_2\) is actually \(C^\infty\), although we will only need \(C^1\) for (d) \(\Rightarrow\) (a).

\[
\left( \int_{D(z)} |f_1|^2 \, dA \right)^{1/2} = \left( \int_{D(z)} \left| \sum_j (f - h_j) \gamma_j \right|^2 \, dA \right)^{1/2}
\]

\[
\leq \sum_j \left( \int_{D(z)} |f - h_j|^2 |\gamma_j|^2 \, dA \right)^{1/2}
\]

\[
\leq \sum_j \left( \int_{D(z) \cap D_j} |f - h_j|^2 \, dA \right)^{1/2}
\]

\[
\leq M(3r) F(z_j) |D(z_j)|^{1/2}.
\]
The last inequality is because $D_j \subset D(z_j)$ but at most $M(3r)$ of the $D_j$ actually intersect $D(z)$. Thus

$$\frac{1}{|D(z)|} \int_{D(z)} |f_1|^2 dA \leq C(r) C_3^2,$$

where $C(r) \geq M(3r)^2 \sup \{||D(z_j)|/|D(z)||: D_j \cap D(z) \neq \emptyset\}$.

Turning now to $f_2$, fix a point $z$ and let $J$ be the set of integers $j$ such that $z \in D_j$. Then

$$f_2(z) = \sum_{j \in J} h_j \gamma_j.$$

Let us suppose for convenience that $1 \in J$ and write

$$f_2(z) = \sum_{j \in J} (h_1(z) + h_j(z) - h_1(z)) \gamma_j(z) = h_1(z) + \sum_{j \in J} (h_j(z) - h_1(z)) \gamma_j(z)$$

whence

$$|\bar{\partial} f_2(z)| = \left| \sum_{j \in J} (h_j(z) - h_1(z)) \bar{\partial} \gamma_j(z) \right| \leq M(r) C(r) C_3 (1 - |z|)^{-1}$$

because $J$ contains at most $M(r)$ integers and $|h_j(z) - h_1(z)| \leq C(r) C_3$ from (3.2), while $|\bar{\partial} \gamma_j| \leq |\nabla \gamma_j(z)| \leq C(r)(1 - |z|)^{-1}$ from (3.3).

Before we show (d) \Rightarrow (a) we need a few simple lemmas about the orthogonal complement $L^2 \ominus A^2$ of $A^2$. The following is mentioned in passing in [10]. A proof for $p = 2$ is contained in [2], and the proof for general $p$ is identical. Let $\partial = \partial/\partial z = (1/2)(\partial/\partial x - i(\partial/\partial y))$.

**Lemma 1.** Let $1 < p < \infty$, and let $p' = p/(p - 1)$. Define $(A^p)^\perp = \{k \in L^{p'}: \int g \bar{k} dA = 0 \text{ for all } g \in A^p\}$. Then $(A^p)^\perp$ is the closure in $L^{p'}$ of the set of all $\bar{\partial} \tau$ such that $\tau$ is $C^\infty$ with compact support in $D$. In particular,

$$L^2 \ominus A^2 = (A^2)^\perp = L^{2\text{-closure}}\{\bar{\partial} \tau: \tau \in C^\infty_0(D)\}.$$  

**Note.** We see $\bar{\partial} \tau$ instead of $\partial \tau$ in the description of $(A^p)^\perp$ because the pairing is conjugate linear in the second variable instead of linear.

The following is an easy consequence of the isometry of the Fourier transform on $L^2(R^2)$. 

LEMMA 2. If \( \tau \) is in \( C_0^\infty(D) \) then \( \bar{\partial}\tau \) and \( \partial\tau \) have the same norm in \( L^2 \).

Note. There is also an \( L^p \) analogue of this lemma which we will need briefly in Section 5: If \( 1 < p < \infty \) there is a constant \( C_p \) such that for all \( \tau \in C_0^\infty(D) \) we have \( \|\bar{\partial}\tau\|_p \leq C_p \|\partial\tau\|_p \). This follows from the Calderón-Zygmund theory of singular integrals: \( \bar{\partial}\tau \) is a singular integral transform of \( \partial\tau \) with kernel \( (\bar{z} - \bar{w})^{-2} \).

For a final lemma we need a sort of weighted Sobolev inequality. It is extremely simple and so probably not unknown, but I do not know a reference.

LEMMA 3. Let \( 1 < p < \infty \). There is a constant \( C_p \) such that if \( \tau \in C_0^\infty(D) \), then

\[
\int_D \frac{|\tau(z)|^p}{(1-|z|)^p} \, dA(z) \leq C_p \int_D |\nabla \tau|^p \, dA.
\]

Proof. First put \( z = re^{i\theta} \), then write

\[
|\tau(re^{i\theta})|^p = \int_r^\infty \frac{\partial}{\partial s} |\tau(se^{i\theta})|^p \, ds \leq \int_r^1 p |\tau(se^{i\theta})|^{p-1} |\nabla \tau(se^{i\theta})| \, ds.
\]

Multiply both sides of this inequality by \( r(1-r^2)^{-p} \) and integrate with respect to \( r \), using Fubini's Theorem on the right. This yields

\[
\int_0^1 \frac{|\tau(re^{i\theta})|^p}{(1-r^2)^p} \, r \, dr \leq \int_0^1 p |\tau(se^{i\theta})|^{p-1} |\nabla \tau(se^{i\theta})| \int_0^s \frac{r}{(1-r^2)^p} \, dr \, ds.
\]

Now

\[
\int_0^s \frac{r}{(1-r^2)^p} \, dr = \frac{1}{p-1} \left( \frac{1}{(1-s^2)^{p-1}} - 1 \right) \leq C_p \frac{s}{(1-s^2)^{p-1}}.
\]

If we put this in the above inequality, integrate with respect to \( \theta \), and use Hölder’s inequality, we get

\[
\int_D \frac{|\tau(z)|^p}{(1-|z|^2)^p} \, dA(z) \leq C_p \left( \int_D \frac{|\tau(z)|^p}{(1-|z|^2)^p} \, dA(z) \right)^{1-1/p} \left( \int_D |\nabla \tau(z)|^p \, dA(z) \right)^{1/p}.
\]

Dividing by the first factor on the right proves the lemma (after observing that \( 1 - |z|^2 \leq 2(1 - |z|) \)).
We are now ready to prove (d) ⇒ (a). Let $f = f_1 + f_2$ as in (d) and let $g \in H^\infty$. Then, as mentioned in the Introduction, we have

$$\|H_{f_1}g\|^2 \leq \|f_1g\|^2 = \int_D |g|^2 |f_1|^2 dA \leq C(r) \|g\|^2,$$  \hspace{1cm} (3.4)$$

where $C(r)$ as usual depends only on $r$ and

$$B = \sup_z \frac{1}{|D(z)|} \int_{D(z)} |f_1|^2 dA = \sup_z G(z)^2 \leq C_4^2.$$

This explicit form for the right hand side of (3.4) is easiest to see using the method of [13]: Dominate $|g(z)|^2$ by $\int_{D(z)} |g|^2 dA/|D(z)|$ and interchange the order of integration in (3.4). Thus $\|H_{f_1}\| \leq C(r) C_4$. Finally we show that $\|H_{f_2}\| \leq C C_4$. Let $\tau \in C^\infty_\circ(D)$, and consider

$$|\langle H_{f_2}, g, \partial \tau \rangle| = |\langle f_2 g, \partial \tau \rangle|$$

$$= \left| \int f_2 g \overline{\partial \tau} dA \right|$$

$$= \left| \int \bar{\partial} f_2 g \bar{\tau} dA \right|$$

$$\leq \|(1 - |z|) \bar{\partial} f_2 g\| \frac{\tau}{1 - |z|}$$

$$\leq CC_4 \|g\| \|\nabla \tau\|$$

$$\leq CC_4 \|g\| \|\partial \tau\|.$$  \hspace{1cm} (3.5)$$

The third equality is just integration by parts. The three inequalities are (in order) Cauchy–Schwarz, Lemma 3, and Lemma 2. Finally, by Lemma 1 we can get the norm of $H_{f_2}g$ by taking the supremum over all $\tau$ with $\|\partial \tau\| \leq 1$. This shows $\|H_{f_2}\| \leq C C_4$. When added to the estimate for $f_1$, we get $\|H_f\| \leq C(r) C_4$. This completes the proof of (d) ⇒ (a) and of Theorem 1.

4. COMPACT HANKEL OPERATORS

Given the large number of analogous results, it is not at all surprising that the characterization of compact Hankel operators is just the "little oh" version of the boundedness condition. The proof is a matter of tracing through the main theorem and showing that where one of the constants $C_j$ occurs one can actually put a quantity tending to zero as $|z|$ tends to one.
THEOREM 2. Let \( f \in L^2 \). The following are equivalent.

(a) \( H_f \) is (extends to) a compact operator from \( A^2 \) to \( L^2 \).

(b) \( \| f \circ \varphi_a - P(f \circ \varphi_a) \| \to 0 \) as \( |a| \to 1 \), \( a \in \mathbb{D} \).

(c) The function \( F(z) \) defined (as before) by

\[
F(z)^2 = \inf \left\{ \frac{1}{|D(z)|} \int_{D(z)} |f - h|^2 \, dA : h \in A^2 \right\}
\]

tends to zero as \( |z| \) tends to 1.

(d) \( f \) admits a decomposition \( f = f_1 + f_2 \) so that \( (1 - |z|) \hat{f}_2(z) \to 0 \) as \( |z| \to 1 \), and \( G(z) \to 0 \) as \( |z| \to 1 \) where \( f_2 \in C^1(\mathbb{D}) \) and (as before)

\[
G(z)^2 = \frac{1}{|D(z)|} \int_{D(z)} |f_1|^2 \, dA.
\]

Proof. First observe that the functions \( \| f \circ \varphi_a - P(f \circ \varphi_a) \| \)

\( (= \| f \circ \varphi_a - P(f \circ \varphi_a) \|) \), \( F(z) \), \( G(z) \), and \( (1 - |z|) \hat{f}_2(z) \) are all continuous on \( \mathbb{D} \)

and so the conditions (a) through (d) all imply the boundedness of the respective functions and of \( H_f \).

Assuming (a) we get

\[
\| H_f(\varphi_a') \| = \| U_{\varphi_a}(f \circ \varphi_a - P(f \circ \varphi_a)) \|
\]

\[
= \| f \circ \varphi_a - P(f \circ \varphi_a) \|.
\]

Since \( H_f \) is compact and \( \varphi_a \to 0 \) weakly as \( |a| \to 1 \), it follows that

\( \| H_f(\varphi_a') \| \to 0 \). This and the above prove (b).

Given (b) we observe that the proof of Theorem 1 actually showed that

\[
F(a)^2 \leq C(r) \| f \circ \varphi_a - P(f \circ \varphi_a) \|^2
\]

and (c) is an immediate consequence.

A careful reading of the proof of the implication (c) \( \Rightarrow \) (d) in Theorem 1

reveals that the function \( f_1 \) produced there actually satisfies

\[
\frac{1}{|D(z)|} \int_{D(z)} |f_1|^2 \, dA \leq C(r) \sup \{ F(w)^2 : w \in D(z, 2r) \}
\]

and the right hand side goes to zero at the boundary of \( \mathbb{D} \) whenever \( F(z) \)

does. Similarly

\[
|h_j(z) - h_k(z)| \leq C(r) \sup \{ F(w)^2 : w \in D(z, 2r) \}
\]

whenever \( D_j \cap D_k \neq \emptyset \). This implies \( (1 - |z|) \hat{f}_2(z) \to 0 \) as \( |z| \to 1 \) in the
same way that the boundedness of $h_j - h_k$ led to the boundedness in Theorem 1(d).

Finally, assume (d). It follows easily (see [16]) that $M_{f_1}$ is compact and therefore $H_{f_1} = (I - P) M_{f_1}$ is also compact. It remains to be shown that $H_{f_2}$ is compact. Let $g_n \in A^2$ be a sequence tending weakly to zero. Then $\|g_n\|$ is a bounded sequence, while $g_n(z) \to 0$ uniformly on compact sets in $D$. Let $\epsilon > 0$. Choose a compact set $K$ so that $(1 - |z|) |\tilde{\delta} f_2(z)| < \epsilon$ when $z \in D \setminus K$ and then choose an integer $n_\epsilon$ such that $|g_n(z)| < \epsilon$ when $z \in K$ and $n \geq n_\epsilon$. Then

$$|\langle H_{f_2} g_n, \partial \tau \rangle| = \left| \int_K + \int_{D \setminus K} \tilde{\delta} f_2 g_n \partial \tau \ dA \right|$$

$$\leq (\epsilon \|(1 - |z|) \tilde{\delta} f_2\|_{\infty} + \epsilon \|g_n\|) \frac{1}{1 - |z|}$$

$$\leq C(r) \epsilon \|\partial \tau\|, \quad n \geq n_\epsilon.$$  

Thus, $\|H_{f_2} g_n\| \leq C(r) \epsilon$, $n \geq n_\epsilon$, and so $\|H_{f_2} g_n\| \to 0$ as $n \to \infty$. This shows that $H_{f_2}$ is compact and therefore also $H_f$. This completes the proof.

Remark. While (c) does imply (d) in the present theorem, one does not need the full strength of (d) to obtain (a). An examination of the proof shows that all that is needed is the following ostensibly weaker condition:

$$(d_2) \quad \text{For each } \epsilon > 0 \text{ it is possible to decompose } f \text{ into } f_1 + f_2 \text{ in such a way that } \lim \sup_{|z| \to 1} G(z) \leq \epsilon \text{ and } \lim \sup_{|z| \to 1} (1 - |z|) |\tilde{\delta} f_2(z)| \leq \epsilon. $$

Stroethoff's result in [18] is the equivalence of (a), (b), and (c) for bounded $f$. If $f$ is bounded, one can replace the $L^2$ norms in (b) and (c) with $L^p$ norms. To see this, consider part (c). It is clear that

$$\frac{1}{|D(z)|} \int_{D(z)} |f - h|^p \ dA \leq \left( \frac{1}{|D(z)|} \int_{D(z)} |f - h|^2 \ dA \right)^{p/2}$$

when $p \leq 2$. If $p > 2$ then fix $z$ and select an $h$ minimizing $(1/|D(z)|) \int_{D(z)} |f - h|^2 \ dA$. Then $|h(w)|^2 < C(1/|D(z)|) \int_{D(z)} |f|^2 \ dA \leq C \|f\|_{L^2}^2$ for $w \in D(z, r/2)$. When $p > 2$ this leads to $(1/|D(z)|) \int_{D(z)} |f - h|^p \ dA < C \|f\|_{L^\infty}^{p-2} (1/|D(z)|) \int_{D(z)} |f - h|^2 \ dA$ provided the radius $r$ is replaced by $r/2$. But since any $r$ may be chosen to begin with, the $L^p$ version obtained in [18] is a consequence of the $L^2$ version. We will see in the next section that this is not the case when $f$ is not bounded.
5. BOUNDEDNESS IN $L^p$, $p > 1$

One can also ask about the boundedness of $H_f$ in the $L^p$ norm. That is, ask for which $f$ one has the following estimate: There exists a constant $C$ such that

$$\|fg - P(fg)\|_p \leq C \|g\|_p, \quad g \in H^\infty.$$ 

As in the case $p = 2$ there is no loss of generality to suppose at the outset that $f \in L^p$. The proof for $L^p$ boundedness is identical (nearly) to the proof of Theorem 1. There is only a slight complication in condition (b) of the theorem.

**Theorem 3.** Let $1 < p < \infty$ and let $f \in L^p$. Then the following are equivalent.

(a) $H_f$ is bounded in the $L^p$ norm.

(b) $\sup_{a \in D} \text{dist}_{L^p}(f \circ \varphi_a, A^p) < \infty$.

(c) $\sup_{z \in D} \inf \{(1/|D(z)|) \int_{D(z)} |f - h|^p dA : h \in A^p\} < \infty$.

(d) $f = f_1 + f_2$ where $(1 - |z|) \delta f_2(z)$ is bounded and

$$\sup_{z \in D} \frac{1}{|D(z)|} \int_{D(z)} |f_1|^p dA < \infty. \quad (5.1)$$

**Proof.** In place of $U_\varphi$ we have the following isometries on $L^p$:

$$V_\varphi = (\varphi')^{2/p} f \circ \varphi.$$ 

Then the $V_\varphi$ are invertible for any automorphism $\varphi$ of $D$ and $V_\varphi^{-1} = V_{\varphi^{-1}}$. Now, with $\varphi = \varphi_a$,

$$V_{\varphi_a} H_f V_{\varphi_a}^{-1}(1) = V_{\varphi_a} f V_{\varphi_a}(1) - V_{\varphi_a} P f V_{\varphi_a}(1)$$

$$= f \circ \varphi_a - h_a, \quad (5.2)$$

where $h_a = V_{\varphi_a} P f V_{\varphi_a}(1) \in A^p$. All we need here is the fact that $P$ is bounded in the $L^p$ norm for all $1 < p < \infty$. Since the $V_{\varphi_a}$ are isometries, (5.2) implies that $\text{dist}_{L^p}(f \circ \varphi_a, A^p) \leq \|H_f(1)\|_p \leq \|H_f\|$. So (b) follows from (a).

Now assume (b) so that there is a constant $C$ and analytic functions $h_a$ with

$$\int |f \circ \varphi_a - h_a|^p dA \leq C, \quad a \in D.$$
Apply $V_{\varphi_a}$ to get

$$\int |f - h_a \cdot \varphi_a|^p |\varphi_a|^2 \, dA \leq C, \quad a \in D.$$  

This gives (c) in the same way as in Theorem 1, via $|\varphi_a|^2 \geq c(1/|D(a)|) \chi_{D(a)}$.

If we assume (c) we obtain $f = f_1 + f_2$ in exactly the same way as in Theorem 1 except that $L^p$ integrals appear everywhere in place of $L^2$ integrals. Thus (d) follows from (c).

To get from (d) to (a) it suffices to have the $L^p$ versions of Lemmas 1, 2, and 3 and the $L^p$ version of the boundedness for $M_f$. We estimate the $L^p$ norm of $H_f g$ by the $L^p$ norm of $f_1 g$ as in the $p = 2$ case, but using the fact that (5.1) is just the requirement for $|f_1|^p \, dA$ to be a Carleson measure for $A^p$ [17, 9, 13]. We estimate the $L^p$ norm of $H_f g$ just as in (3.5), except that the duality between $L^p$ and $L^{p'}$ is used and Hölder's inequality replaces Cauchy–Schwarz. We need to make use of the facts that $H_f g \in (A^{p'})'$ and that $L^{p'} = A^p \oplus (A^p)_{1}$, which follow easily from the boundedness of $P$ in $L^p$ norm.

As promised in Section 2, I will show how the $BMO-VMO$ conditions are implied by the $BDA$ conditions (c) in Theorems 1, 2, and 3.

**Proposition 1.** For each $p > 1$ there is a constant $C_p$ such that if $f$ is real or the conjugate of an analytic function, then

$$\inf_{c \in C} \frac{1}{|D(z)|} \int_{D(z)} |f - c|^p \, dA \leq \inf_{h \in A^p} \frac{1}{|D(z)|} \int_{D(z)} |f - h|^p \, dA. \quad (5.3)$$

**Proof.** After a translation and dilation it is enough to prove the proposition with the unit disk and ordinary Lebesgue measure in place of $D(z)$ and $(1/|D(z)|) \, dA$. In this proof, norms will be the $L^p$ norm. It is well known (see [26, p. 75, Exercise 9] but correct it by changing $f$ to $f - f(0)$) that if $g \in A^p$ then $\|g - g(0)\| \leq C_p \|\Im g\|$. Now if $f$ is real and its distance from $A^p$ is 1, let $g \in A^p$ be chosen with $\|f - g\| \leq 2$. Then $\|\Im g\| = \|\Im(f - g)\| \leq 2$ as well. Thus $\|g - g(0)\| \leq 2C_p$. Putting these two together gives $\|f - g(0)\| \leq \|f - g\| + \|g - g(0)\| \leq 2 + 2C_p$. This gives (5.3) (after a translation and dilation). A similar argument may be used when $f$ is conjugate analytic.

It is easy to prove a compactness criterion for $H_f$ on $A^p$ and it is just the "little oh" analogue of Theorem 3 or the $L^p$ analogue of Theorem 2. The results in [18] imply that for $f \in L^\infty$ the operator $H_f$ is compact as an operator on $A^p$ if and only if it is compact as an operator on $A^2$. We show now that this is not true for unbounded $f$: Taking $f$ to be real, we may use...
the BMO versions of conditions (c) of the various theorems. If $p > 2$ we need to construct a function such that

$$\inf_{c \in C} \left( \frac{1}{|D(z)|} \right) \int_{D(z)} |f - c|^2 \, dA \to 0$$

while

$$\inf_{c \in C} \left( \frac{1}{|D(z)|} \right) \int_{D(z)} |f - c|^p \, dA \to 0$$

as $|z| \to 1, z \in D$. Simply define $f$ separately on a sequence of disjoint disks $\{D(z_k)\}$. Require that $\int_{D(z_k)} f \, dA = 0$ and $\int_{D(z_k)} |f|^2 \, dA = o(|D(z_k)|)$ but $\int_{D(z_k)} |f|^p \, dA \neq o(|D(z_k)|)$. This is possible because the inclusion map is not continuous from $L^2$ to $L^p$.

6. SCHATTEN CLASSES

The Schatten ideal $\mathcal{S}_p$ consists of all the operators $T$ on the Hilbert space for which the singular numbers $s_n(T)$ form a sequence belonging to $l^p$. The singular numbers of the operator $T$ are defined by

$$s_n = s_n(T) = \inf \{ \| T - K \| : \text{rank } K \leq n \}. \quad (6.1)$$

An equivalent definition, better for some purposes, is

$$s_n = \inf \{ \| T | W \| : \text{codim } W = n \}. \quad (6.2)$$

The $\mathcal{S}_p$ norm $\| T \|_p$ of an operator $T$ is the $l^p$ norm of its singular numbers.

Let $H_f$ be a bounded Hankel operator and let $f = f_1 + f_2$ as in part (d) of Theorem 1. The argument in the proof of (d) $\Rightarrow$ (a) actually shows that for any function $g \in A^2$

$$\| H_{f_1} g \| \leq \| M_{f_1} g \|$$

and

$$\| H_{f_2} g \| \leq C \| M_{(1 - |z|) \partial f_2} g \|.$$}

Thus, from (6.2), the singular numbers for $H_{f_1}$ and $H_{f_2}$ are dominated by those for $M_{f_1} |A^2$ and $M_{(1 - |z|) \partial f_2} |A^2$. Fortunately the Schatten ideal characterization of these multiplication operators from $A^2$ to $L^2$ is available in [15].

Previous results on the Schatten class membership of Hankel operators can be found in [3, 24, 25].

Let $d\mu(z) = (1 - |z|^2)^{-1} \, dA(z)$ be the automorphism invariant measure on $D$. Remember that $D(z) = D(z, r)$ where $r$ is some fixed but arbitrary positive number.

**Theorem 4.** Let $1 \leq p < \infty$ and let $f \in L^2$. Assume that $H_f$ is bounded in the $L^2$ norm. Then the following are equivalent.
(a) \( H_f \) belongs to \( \mathcal{S}_p \).

(b) If \( p \geq 2 \)

\[
\int \| f \circ \varphi_\zeta - P(f \circ \varphi_\zeta) \| ^p d\lambda(\zeta) < \infty, \tag{6.3}
\]

while for all \( p \geq 1 \)

\[
\left( \int \left( \int_{D(0)} | f \circ \varphi_\zeta - P(f \circ \varphi_\zeta) |^2 \, dA \right)^{p/2} \right)^{1/2} d\lambda(\zeta) < \infty. \tag{6.4}
\]

(c) \( \int F(z)^p \, d\lambda(z) < \infty \) where \( F(z) \) is defined as in Theorem 1(c).

(d) \( f = f_1 + f_2 \) where \( f_1 \) satisfies

\[
G(z) = \left( \frac{1}{|D(z)|} \int_{D(\zeta)} |f_1|^2 \, dA \right)^{1/2} \in L^p(D, d\lambda),
\]

and \( (1 - |z|) \, \bar{\partial} f_2 \) satisfies the same condition as \( f_1 \).

For each of the conditions (b), (c), and (d), there is an equivalent condition in which sums over separated sequences replace integration with respect to \( d\lambda \). This gives us the equivalent theorem that follows. We will defer to the end of the section the proof that the two forms are equivalent. What we will actually prove is the following theorem.

**Theorem 4'.** Let \( 1 \leq p < \infty \) and let \( f \in L^2 \). Assume that \( H_f \) is bounded in the \( L^2 \) norm. Then the following are equivalent.

(a') \( H_f \) belongs to \( \mathcal{S}_p \).

(b') For every separated sequence \( \{\xi_k\} \) in \( D \) (that is, \( \inf_{j \neq k} d(\xi_j, \xi_k) > 0 \)) we have if \( p \geq 2 \)

\[
\sum \| f \circ \varphi_{\xi_k} - P(f \circ \varphi_{\xi_k}) \|^p < \infty, \tag{6.5}
\]

while for all \( p \geq 1 \) we have

\[
\sum_k \left( \int_{D(0)} | f \circ \varphi_{\xi_k} - P(f \circ \varphi_{\xi_k}) |^2 \, dA \right)^{p/2} < \infty. \tag{6.6}
\]

(c') For every separated sequence as in (b'), \( \sum_k F(\xi_k)^p < \infty \).

(d') \( f = f_1 + f_2 \) such that for every separated sequence as in (b')

\[
\sum_k \left( \frac{1}{|D(\xi_k)|} \int_{D(\xi_k)} |f_1|^2 \, dA \right)^{p/2} < \infty
\]

and the same holds with \( (1 - |z|) \, \bar{\partial} f_2 \) in place of \( f_1 \).
Proof. Let \( H_f \in \mathcal{S}_p \). For any bounded operator \( A \) from a Hilbert space into \( A^2, H_fA \) is in \( \mathcal{S}_p \) with the Schatten norm \( |H_fA|_p \leq |H_f|_p \|A\| \). Then, if \( p \geq 2 \), we have \([8, p. 95]\) for every orthonormal sequence \( \{e_k\} \)

\[
\sum_k \|H_fAe_k\|^p \leq |H_fA|_p \leq \|A\| |H_f|_p.
\]

It is well known \([1, 7]\) that if \( \{\zeta_k\} \) is a separated sequence then the operator \( A \) taking \( e_k \) to \( \varphi_{\zeta_k} \) is bounded. Thus

\[
\sum_k \|H_f\varphi_k\|^p < \infty,
\]

where we write \( \varphi_k \) for \( \varphi_{\zeta_k} \). Thus

\[
\sum \|f \circ \varphi_k - P(f \circ \varphi_k)\|^p = \sum \|H_{f \circ \varphi_k}1\|^p = \sum \|H_f\varphi_k\|^p < \infty.
\]

This is just (6.5).

In case \( 1 \leq p \leq 2 \) we use the following fact \([8, p. 94]\): For an operator \( T \) and any orthonormal sequence \( \{e_k\} \)

\[
\sum |\langle Te_k, e_k\rangle|^p \leq |T|_p^p.
\]

We apply this to \( B^*H_fA \) where \( A \) is as before but \( B \) is defined by \( Be_k = c_k \chi_{D(\zeta_k)} H_f(\varphi'_k) \) where \( c_k \) is chosen to make \( Be_k \) a unit vector. That is, \( c_k = (\int_{D(\zeta_k)} |H_f\varphi'_k|^2 \, dA)^{-1/2} \). Then \( B \) is clearly bounded. (If the collection \( \{D(\zeta_k)\} \) is disjoint this is immediate. In general it is a finite union of disjoint sequences.) Thus

\[
\sum_k \left( \int_{D(\zeta_k)} |H_{f \circ \varphi_k}1|^2 \, dA \right)^{p/2} = \sum_k \left( \int_{D(\zeta_k)} |\varphi'_k(H_f \circ \varphi'_k) \circ \varphi_k|^2 \, dA \right)^{p/2} = \sum_k \left( \int_{D(\zeta_k)} |H_f\varphi'_k|^2 \, dA \right)^{p/2} = \sum_k (c_k \langle H_f \varphi'_k, \chi_{D(\zeta_k)} H_f \varphi'_k \rangle)^p = \sum_k |\langle B^*H_fAe_k, e_k\rangle|^p < \infty.
\]

The second last equality above combines the definition of \( c_k \) and that of the inner product. The second is a change of variables in each integral \( z \rightarrow \varphi_k(z) \). Condition (6.6) is just the finiteness of the first sum. This same sum is finite for \( p > 2 \) because of (6.5). Thus (a) implies (b').
As in the proof of Theorem 1 we easily obtain

$$F(\zeta)^2 \leq C \int_{D(\zeta)} |H_f \varphi_\zeta|^2 \, dA = C \int_{D(0)} |H_{f \circ \varphi_\zeta}|^2 \, dA$$

and so (c') follows from (b').

We have already seen in Section 4 that $G(z) \leq C(r) \overline{F}(z)$, where $\overline{F}(z) = \sup\{F(w) : w \in D(z, 2r)\}$. It is easy to verify from its definition that $\overline{F} \leq C(r) F_3$, where $F_3$ is the same as $F$ except it uses $D(z, 3r)$ instead of $D(z, r)$. Put another way, $G_{1/3}(z) \leq C(r) F(z)$, where $G_{1/3}$ is defined the same as $G$ except it uses $D(z, r/3)$ instead of $D(z, r)$. Thus (d') is immediate with this change in $r$. However, once we have shown that (d') implies (a) for any $r$, we will have (d') equivalent to the rest for all $r$.

Finally, let us assume (d'). This is (nearly) the condition obtained in [15] which is equivalent to a Toeplitz operator belonging to $S_{p/2}$. In that paper the disk $D$ was broken down into convenient pieces which are roughly the size of $D(\zeta_k)$ but have the advantage of being disjoint and covering $D$. Nevertheless the result obtained there is easily shown to be equivalent to the following.

**Theorem.** Let $\mu$ be a positive measure on $D$. Define an operator $T_\mu$ on $A^2$ as follows: $\left< T_\mu g, h \right> = \int_D g \overline{h} \, d\mu$, $g, h \in A^2$. Then $T_\mu \in S_p$ if and only if for every separated sequence $\{\zeta_k\}$ we have

$$\sum_k \left( \frac{\mu(D(\zeta_k))}{|D(\zeta_k)|} \right)^p < \infty.$$ 

In our case we want the Schatten class membership of $M_\psi : A^2 \to L^2$ for $\psi = f_1$ or $(1 - |z|) \delta_{f_2}$. To get this from the above theorem observe that

$$\left< M_\psi g, M_\psi h \right> = \int g \overline{h} |\psi|^2 \, dA = \left< T_{|\psi|^2} g, h \right>$$

so $M_\psi^* M_\psi = T_{|\psi|^2}$. Thus [8, Chapt. II, Sect. 2], $M_\psi \in S_p$ if and only if $T_{|\psi|^2} \in S_{p/2}$. Finally, by the problem above on Toeplitz operators, the condition in (d') is exactly what is needed to have both $T_{|f_1|^2}$ and $T_{|(1 - |z|) \delta_{f_2}|^2}$ belong to $S_{p/2}$. Thus the corresponding multiplication operators $M_{f_1}$ and $M_{(1 - |z|) \delta_{f_2}}$ belong to $S_p$ and so, by the remarks preceding the theorem, $H_f$ belongs to $S_p$. This finishes the proof.

Let us now see how the conditions (b), (c), and (d) are equivalent to their primed counterparts.

To see that (6.3) follows from (6.5), fix a sequence $\{z_k\}$ such that
\( \{D(z_k)\} \) is disjoint and select a \( \zeta_k \) in \( D(z_k, r/2) \) which maximizes \( \|f \circ \varphi_{\zeta} - P(f \circ \varphi_{\zeta})\| \). Then this sequence is separated and

\[
\int_{\bigcup_{k} D(z_k, r/2)} \|f \circ \varphi_{\zeta} - P(f \circ \varphi_{\zeta})\|^p \, d\lambda(\zeta)
\leq \sum_k \|f \circ \varphi_{\zeta_k} - P(f \circ \varphi_{\zeta_k})\|^p \, \lambda(D(z_k, r/2))
\leq C(r) \sum_k \|f \circ \varphi_k - P(f \circ \varphi_k)\|^p < \infty
\]

by (6.5). If we repeat this a finite number of times with appropriately chosen \( \{z_k\} \), we get (6.3).

Now suppose that we have (6.3) and let \( \{\zeta_k\} \) be a separated sequence. In the usual way we rewrite (6.3) as

\[
\int \|H_f \varphi'_{\zeta}\|^p \, d\lambda(\zeta) < \infty.
\]

Recalling that \( \varphi'_{\zeta}(z) = -(1 - |\zeta|^2)(1 - \zeta z)^{-2} \) it is easy to see that \( (1 - |\zeta|^2)^{-p} \|H_f \varphi'_{\zeta}\|^p = \|H_f (1 - \zeta (\cdot))^{-2}\|^p \) is subharmonic in \( \zeta \). From this it follows that

\[
\|H_f \varphi'_{\zeta_k}\|^p \leq C(r) \frac{(1 - |\zeta_k|^2)^p}{|D(\zeta_k)|} \int_{D(\zeta_k)} (1 - |\zeta|^2)^{-p} \|H_f \varphi'_{\zeta}\|^p \, dA(\zeta)
\leq C(r) \int_{D(\zeta_k)} \|H_f \varphi'_{\zeta}\|^p \, d\lambda(\zeta).
\]

Summing over all \( k \) gives (6.5).

Now I will prove the equivalence of (c) and (c'). For (d) the proof goes much the same way, and it will be omitted. So assume (c') and let \( \{z_k\} \) be some separated sequence with \( \{D(z_k)\} \) covering \( D \). Choose a point \( \zeta_k \) in each disk \( D(z_k) \) so that

\[
F'(\zeta_k)^2 = \inf_{h \in \mathcal{A}} \frac{1}{|D(\zeta_k)|} \int_{D(\zeta_k)} |f - h|^2 \, dA
\]

is maximized. Then

\[
\int (F(\zeta))^p \, d\lambda(\zeta) \leq \sum_{D(z_k)} F(\zeta_k)^p \, d\lambda(\zeta) \leq C(r) \sum F(\zeta_k)^p \lambda(D(z_k)).
\]

Since \( \{\zeta_k\} \) is a finite union of separated sequences and \( \lambda(D(z)) \) is independent of \( z \), this shows that (c) follows from (c').
Finally, assume (c) and let \( \{ \zeta_k \} \) be any separated sequence. If \( z \) is any point in \( D(\xi_k, r/2) \), then \( D(\xi_k, r/2) \subset D(z) \). Then for any \( h \in A^2 \)

\[
\frac{1}{|D(\xi_k, r/2)|} \int_{D(\xi_k, r/2)} |f - h|^2 \, dA \leq C(r) \int_{D(z)} |f - h|^2 \, dA. \tag{6.7}
\]

Select an \( h \) which nearly minimizes the right hand side, and then replace the left hand side by its infimum to obtain \( F_{1/2}(\xi_k) \leq C(r) F(z) \) for any \( z \in D(\xi_k, r/2) \), where \( F_{1/2} \) is defined like \( F \) except with \( \tau \) replaced by \( r/2 \). This easily implies that \( F_{1/2}(\xi_k)^p \leq C(r) \int_{D(\xi_k)} F^p \, d\lambda \) and summing this gives \((c')\) except that \( \tau \) is replaced with \( r/2 \). But the equivalence of \((c')\) with \((a)\), which is independent of \( r \), shows that \((c')\) follows for any \( r \).

To close this section I would like to point out that for \( p = 1 \), \((6.3)\) and \((6.5)\) are not equivalent to the rest of the statements. For \( 1 < p < 2 \) they probably still are but I do not have a proof at present. To see that \((6.3)\) does not follow from \((d)\) when \( p = 1 \), select \( f(z) = \tilde{z}(1 - |z|^2)^x \), with \( x > 1 \). Then if we let \( f = f_1 \) we get \([[(1/|D(z)|) \int_{D(z)} |f_1|^2 \, dA]^{1/2} \leq C(r)(1 - |z|^2)^x \). Since the right hand side of this integrable with respect to \( d\lambda \), \((d)\) is satisfied with \( p = 1 \). Nevertheless, \((6.3)\) is not. For if we estimate \( \|f \circ \varphi \circ z - P(f \circ \varphi \circ z)\| = \|f \circ \varphi \circ z - P(f \circ \varphi \circ z)\| \) by \( \sup \{|\langle f \circ \varphi \circ z, k \rangle| : k \in L^2(\mathbb{D}) \} \) we see that is is clearly greater than a constant multiple of \( |\langle \zeta(1 - |z|^2)^x \varphi \circ z, \zeta \rangle| = |\zeta|^2 (1 - |z|^2)^x (1 - |\zeta|^2)(1 - \zeta z)^{-2} \, dA| \). Because of the rotational symmetry, this last is a nonzero multiple of \( |\varphi \circ z(0)| = C_\alpha(1 - |\zeta|^2) \). But this is not integrable with respect to \( d\lambda(\zeta) \) and so \((6.3)\) fails for \( p = 1 \).

7. Remarks

I will discuss here without proof some additional extensions (and hoped for future extensions) of Theorems 1 through 4.

Weighted Bergman Spaces. The method employed in Section 6 works almost without change to produce the corresponding theorems for certain weighted Bergman spaces \( A^{p, \beta} \), with \(-1 < \beta < 1/(p - 1)\). Note the limitations on the exponent \( \beta \). These are defined to be the Banach spaces of analytic functions belonging to \( L^p(dA_\beta) \) where \( dA_\beta(z) = (1 - |z|^2)^\beta \, dA(z) \). The definition of the Hankel operator \( H^\beta \) is formally the same except that the projection \( P \) should be interpreted as (or replaced by) the projection \( P_\beta \) of \( L^{2, \beta} = L^2(D, dA_\beta) \) onto \( A^{2, \beta} \). All four theorems are true as stated for these weighted spaces if the norms \( \| \cdot \| \) of functions and operators are interpreted in this weighted sense. The only non-routine part of the verification of this statement is the appropriate handling of \((d) \Rightarrow (a)\): The implication \((a) \Rightarrow (b)\) is obtained the same way (in all four theorems) using the
appropriate replacements for the $U_\phi$ and $V_\phi$. The implication (b) $\Rightarrow$ (c) uses the same ideas and produces (c) exactly as stated and without reinter-
pretation. Thus, (d) follows immediately. To get (d) $\Rightarrow$ (a) it is necessary to have a replacement for the three lemmas of Section 3. It is for this reason that $\beta$ is required to be less than $1/(p-1)$: I am unable to obtain the proper analogue of Lemma 2 for larger values of $\beta$. The subspace $L^{2,\beta}\ominus A^{2,\beta}$ is obtained as follows: An $L^{2,\beta}$ function $f$ is analytic if and only if $\bar{\partial}f$ is zero in the sense of distributions, i.e., if and only if

$$\int_D f \bar{\partial}\tau \, dA = \int_D f \frac{\bar{\partial}\tau}{(1-|z|^2)^{\beta}} dA_\beta = 0$$

for all $\tau \in C_0^\infty(D)$. Thus

$$L^{2,\beta}\ominus A^{2,\beta} = L^{2,\beta}\cdot \text{closure} \left\{ \frac{\bar{\partial}\tau}{(1-|z|^2)^{\beta}} : \tau \in C_0^\infty(D) \right\}.$$}

This is the proper analogue of Lemma 1 for $p = 2$. The general case is similar. The appropriate analogue of Lemma 3 is also valid. What one needs is the following inequality:

$$\int_D \frac{|\tau(z)|^p}{(1-|z|^2)^{p\beta + p - \beta}} dA(z) \leq C \int_D \frac{|\nabla \tau|^p}{(1-|z|^2)^{p\beta - \beta}} dA(z).$$

This follows exactly as in Lemma 3 except that after reaching inequality (3.4) one multiplies by $r(1-r^2)^{-p\beta - p + \beta}$ instead of merely $r(1-r^2)^{-p}$.

There is some difficulty with Lemma 2. At present I cannot obtain the appropriate extension required for weighted spaces. What would be needed is

$$\int_D \frac{|\bar{\partial}\tau|^p}{(1-|z|^2)^{p\beta - \beta}} dA(z) \leq C \int_D \frac{|\bar{\partial}\tau|^p}{(1-|z|^2)^{p\beta - \beta}} dA(z),$$

for all $\tau \in C_0^\infty(D)$. I do not know if this is valid for all $\beta > 1$ but it is valid as long as the weight $(1-|z|^2)^{-p\beta + \beta}$ has an extension to all of $\mathbb{C}$ that belongs to Muckenhoupt's class $A_\beta$. Muckenhoupt's class is the class of weights for which Calderon–Zygmund singular integrals are bounded in $L^p$ of the weight. Since $\bar{\partial}\tau$ is a singular integral of $\bar{\partial}^\tau$ we will have the above inequality when $(1-|z|^2)^{-p\beta + \beta}$ is in $A_\beta$. This is the case only for the range $-1 < \beta < 1/(p-1)$. Thus, for this range of $\beta$, we obtain (d) $\Rightarrow$ (a) for the weighted spaces $A_\beta$.

**Entire Functions.** Another situation where the same difficulty shows up and for which there is not even this partial result is the so-called Fock space. In this case the domain is the entire plane $\mathbb{C}$ instead of the disk and
there is of necessity a weight, namely \( \exp(-|z|^2/2) \). The appropriate analogues of (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d) still hold. What is needed to obtain a version of Theorems 1–4 here is a proof that if \( f_2 \) is \( C^\infty \) with \( \partial f_2 \) bounded, then \( H_f \) is bounded (or the appropriate compactness or Schatten class statement). Now analogues of Lemmas 1 and 3 are available (with essentially the same proofs), but Lemma 2 would take the form (when \( p = 2 \)) \[
\int_{\mathbb{C}} |\partial \tau(z)|^2 \exp(|z|^2/2) \, dA(z) \leq C \int_{\mathbb{C}} |\partial \tau(z)|^2 \exp(|z|^2/2) \, dA(z).
\]
The weight \( \exp(|z|^2/2) \) does not belong to \( A_2 \) so we cannot even get a partial result here. It seems possible that this inequality might still be valid and so the results of Theorems 1 through 4 might still hold. It might even be that the theorems hold without this inequality being valid. I do not know what the true situation might be as of this writing.

Several Complex Variables. If the unit disk is replaced with the unit ball \( B_N \) in \( \mathbb{C}^N \) and \( dA \) is replaced with the \( 2N \)-dimensional volume measure \( dV \), then the forward implications of Theorem 1 still go through in appropriate form from (a) to (d). It might be added that (d) is even a little stronger than might be expected: \( \partial f_2 \) is a vector and its component in the radial direction at \( z \) is no larger than \( C(1 - |z|)^{-1} \) while its components in the complex directions orthogonal to the radius through \( z \) are no larger than \( C(1 - |z|)^{-1/2} \). Now the orthogonal complement to \( A^2 \) in the ball is the closure of the set of all \( h \) such that \( h \, dV \) has the form \( \partial \tau \) where \( \tau \) is a \( C^\infty \) form of type \( (N-1, N) \) with compact support in \( B_N \). In place of Lemmas 2 and 3 we would need an inequality like the following: Let \( R\tau(z) \) denote the radial component of \( \tau(z) \). Then

\[
\inf_{\partial \tau = \partial \sigma} \int_{B_N} \frac{|R\tau(z)|^2}{(1 - |z|^2)^2} + \frac{|\tau(z) - R\tau(z)|^2}{(1 - |z|^2)} \, dV(z) \leq C \int_{B_N} |\partial \sigma(z)|^2 \, dV(z).
\]

I do not have such an inequality at this writing but I am hopeful that it might be true. A more ambitious inequality in which the complicated integrand on the left above is replaced by \( |\tau|^2 (1 - |z|^2)^{-2} \) seems less likely to be true, though I do not have a counterexample. I am able to prove the above inequality for the case \( \tau = R\tau \).

Other Domains in \( \mathbb{C} \). The theorems can be adapted to obtain valid results in any bounded domain in \( \mathbb{C} \) bounded by \( C^1 \) curves. This is almost trivial if the domain is simply connected. If the domain is multiply connected, then condition (b) of the theorems will have to be abandoned and one goes directly from (a) to (c). This is easily done in the disk: simply apply the operator \( H_f \) to the appropriate function (usually \( \phi_\alpha \)) and estimate as in the last part of the (b) \( \Rightarrow \) (c) proof. For a domain \( \mathcal{W} \) with circles for boundaries, assume the unit circle is the outer boundary. Then use the same functions \( \phi_\alpha \) with the points \( a \) that are close to this boundary.
(and hence far from the other boundaries). View the disks $D(a)$ not as hyperbolic disks in $D$ but merely as a collection of disks whose radii are proportional to their distances from the boundary. In this way, $(a) \Rightarrow (c)$ is easily obtained and with the same point of view, so is $(c) \Rightarrow (d) \Rightarrow (a)$. If $W$ does not have circles for boundary curves, it can be transformed to one that does via a conformal map with bounded distortion. The result is the following.

**Theorem 5.** Let $W$ be a bounded domain in the complex plane with $C^1$ boundary. Let $P$ denote the projection from $L^2(W, dA)$ to $A^2(W) = \mathcal{H} \cap L^2(W, dA)$ and define $Hfg = fg - P(fg)$ for any $f$ in $L^2$. Then the following are equivalent.

(a) $H_f$ is bounded in the $L^2$ norm.

(c) Let $0 < \eta < 1$. If $D(z) = \{ w \in W : |z - w| < \eta \text{ dist}(z, C \setminus W) \}$, then

$$\sup_{z \in W} \inf_{h \in A^2} \frac{1}{|D(z)|} \int_{D(z)} |f - h|^2 dA < \infty.$$ 

(d) $f = f_1 + f_2$ where

$$\sup_{z \in W} \frac{1}{|D(z)|} \int_{D(z)} |f_1|^2 dA < \infty$$

and $\text{dist}(z, C \setminus W) \partial f_2$ is bounded.

All that one needs to show that $(d) \Rightarrow (a)$ are the appropriate analogues of Lemmas 1 through 3 with the distance to the boundary of $W$ in place of $1 - |z|$. These are easily proved in the same way as the originals.

**References**


