Historia Mathematica 10 (1983) 63-89

NOTES

EUCLID'S INTENDED INTERPRETATION OF SUPERPOSITION

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There has been much debate over Euclid's method of superposition of geometric figures to deduce the well-known congruence theorems. The traditional interpretation is that Euclid intended to assume that a geometric figure could be physically displaced from an initial position and superpositioned on another figure so that the two figures would coincide, thus establishing a congruence. In this note, this view is challenged and a logical substitute is proposed. T. Heath is emphatic about Euclid's intent:

> *The phraseology of the propositions, e.g. I.4 and I.8 in which Euclid employs the method indicated, leaves no room for doubt that he regarded one figure as actually moved and placed upon the other.* [Heath 1926, 225]

Heath claims that either Euclid assumed the displacement of geometric figures to be permissible but failed to state the necessary axiom, or that such displacement was assumed to be an acceptable method of proof in geometry. Loosely speaking, the latter might be viewed as part of the underlying logic. Heath acknowledges that Euclid's use of the assumption was nominal:

> *... it is clear that Euclid disliked the method and avoided it whenever he could, e.g. in I.26, where he proves the equality of two triangles which have two angles respectively equal to two angles and one side of the one equal to the corresponding side of the other. It looks as though he found the method handed down by tradition ... and followed it, in the few cases where he does so, only because he had not been able to see his way to a satisfactory substitute.* [Heath 1926, 225]

These comments suggest that Euclid was aware of the logical incorrectness of failing to state the necessary axiom, but unable to frame a satisfactory alternative, followed a traditional method. This seems to us highly unlikely because we believe Euclid's purpose in writing the text was to provide a firm logical foundation for all his geometry. Proclus, the Greek

0315-0860/83 $3.00

commentator on the *Elements*, made it abundantly clear that Euclid attained a degree of logical perfection and organization.

> *Not long after these men came Euclid, who brought together the Elements, systematizing many of those of Theaetetus, and putting in irrefutable demonstrable form propositions that had been rather loosely established by his predecessors.* [Proclus 1970, 56]

Moreover, the tacit use of motion would make nonsense of construction problems. This has previously been noted by Klein [1939, 200]. In particular there would be no purpose in proving I.2--"To place at a given point (as an extremity) a straight line equal to a given straight line." Clearly, the proof given amounts to the reproduction of a line segment and is totally independent of physical displacement, despite the occurrence of "place." His use of such a word hints at the possibility that he viewed this reproduction as a substitute for the traditional method of displacement.

I.4 and I.8 are the only propositions where Heath claims displacement was used, and it is Euclid's use of "place" and "apply" that make this interpretation of motion possible. Both propositions and the pertinent parts of their proofs are given below.

PROPOSITION 4

> *If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.*
>
> *Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE and AC to DF and the angle BAC equal to the angle EDF. I say that the base BC is also equal to the base EF, the triangle ABC will be equal to the triangle DEF, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle ABC to the angle DEF, and the ACB to the angle DFE.*
>
> *For, if the triangle ABC be applied to the triangle DEF, and if the point A be placed on the point D and the straight line AB on DE, then the point B will also coincide with E, because AB is equal to DE.*

Again, AB coinciding with DE, the straight line AC will also coincide with DF, because the angle BAC is equal to the angle EDF; hence the point C will also coincide with the point F because AC is again equal to DF.

But B also coincided with E; hence the base BC will coincide with the base EF.

[For if, when B coincides with E and C with F, the base BC does not coincide with the base EF, two straight lines will enclose a space: which is impossible. Therefore the base BC will coincide with EF] and will be equal to it.... [Heath 1926, 247-248]

The remainder of the proof restates what has been done earlier.

PROPOSITION 8

If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.

Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE, and AC to DF; and let them have the base BC equal to the base EF;

I say that the angle BAC is also equal to the angle EDF.

For if the triangle ABC be applied to the triangle DEF, and if the point B be placed on the point E and the straight line BC on EF, the point C will also coincide with F, because BC is equal to EF. Then, BC coinciding with EF, BA, AC will also coincide with ED, DF; for, if the base BC coincides with the base EF, and the sides BA, AC do not coincide with ED, DF but fall beside them as EG, GF, then, given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there will have been constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely.each to that which has the same extremity with it. *each to that which has the same extremity with it.*

But they cannot be so constructed. Therefore, it is not possible that, if the base BC be applied to the base EF, the sides BA, AC should not coincide with ED, DF;.... [Heath 1926, 261-262]

The proof of I.8 depends on I.7, which states that for a given triangle *ABC* with base *AC*, it is impossible to construct a distinct triangle *ADC* on the base *AC* such that *D* and *B* are on the same side of *AC*, and the lengths of the sides *AD* and *DC* are equal to the lengths of the sides *AB* and *BC*, respectively.

There is no doubt that the language in the text and the method of proof suggests that the triangle *ABC* somehow has been superimposed on triangle *DEF*. The question is whether or not Euclid used physical displacement as a logical method for achieving this or, instead, intended that triangle *ABC* was to be reconstructed on triangle *DEF*.

Let me limit the discussion temporarily to I.4. Within this proof it is quite clear that either "place" and "apply" both mean displacement, or they both mean construction. That is, it would be pointless first to displace the triangle *ABC* and then to reproduce the line segment *BC* (from its original position) superimposed on *DE*. In this sense there is a semantic interdependence of these two words within this proof. The same semantic interdependence exists between "place" and "apply" within the proof of I.8. But here the similarity of their meaning is even more evident. Near the beginning of the proof (lines 13-15 in the Heath translation) Euclid placed the straight line *BC* on *EF*. Further on (lines 31-33) he referred to this placement as application. So "place" and "apply" are used synonymously in the proof of I.8.

Since the phraseology of both propositions is so similar it is reasonable to believe that the meaning of "place" and "apply" in I.4 is the same as the meaning in I.8.

The connection between the meaning of "apply" and construction is established in the proof of I.8. In the *reductio ad absurdum* (line 25) Euclid referred to what had been done earlier in the proof, namely, the application of triangle *ABC* to *DEF* (line 13) as construction.

Therefore there is no evidence in the text itself to suggest that motion was intended. Euclid's purpose, as we have earlier stated it, makes it unlikely that Euclid was bound by a Greek tradition of displacement. Euclid hinted at the possibility that the reproduction of geometric figures was to be viewed as a substitute for displacement by using "place" in I.2 even though motion was not involved. In I.8 Euclid referred to the application of a triangle as construction and within each proof established a semantic interdependence between "place" and "apply."

Any logical substitute that is proposed can at best be plausible; for whatever Euclid intended, he failed to convey this in an explicit way.

Contrary to modern Euclidean geometry, such as the version found in [Hilbert 1902], Euclid attempted to make mathematical sense of equal geometric figures occupying different positions.

Modern treatments do not distinguish points in the plane from their positions. Both are defined as an ordered pair of numbers. More accurately, position is not addressed in the formal language of mathematics, where, instead, the primitive relation of congruence is employed. By defining figures in the plane to be collections of points, the equality of figures means the equality of sets. Obviously this is not what Euclid intended.

In what sense then can a figure such as a triangle occupy more than one position? Euclid did not describe what he meant by equal geometric figures which do not coincide. On the other hand, his language in I.4, I.8 and other parts of the text clearly indicates that he had an idea of equality in mind and yet kept it tacit. Even if motion is permitted, equality of figures would only make sense if he spoke of a triangle existing in different positions at different times. But, although Euclid did not introduce the concept of time, he was very explicit about the ontological status of figures: they were constructed. Heath [1926, 145-146] points out that the existence of figures was not assumed by Euclid prior to their construction. Hence it is reasonable to believe that whenever Euclid discussed geometric figures, he assumed that they had been constructed and that the possibility exists that two constructions could yield equal results even though the results did not coincide.

The Euclidean constructions have an algorithmic nature on which Proclus commented. In a discussion of the distinction between problems (propositions that predicate the possibility of construction) and theorems (prepositions whose proofs are not purely construction), Proclus cited the following view which had been expressed by Carpus (an engineer):

> *And the enunciation of a problem, he says, is simple, requiring no additional technical knowledge at all; it only demands that something clearly possible be done, such as constructing an isosceles triangle or, given two straight lines, cutting off from the greater a length equal to the less. What is unclear or difficult about these?* [Proclus 1970, 188]

Propositions I.1, I.2 and I.3 assert that certain constructions can be carried out. They are examples of problems. There proofs consist solely of algorithms to obtain the geometric results. In contrast, Propositions I.4 and I.8 are examples of theorems.

Adopting the above view, the following interrelationship of problems and theorems may be proposed: Proposition I.1 provides one algorithm for producing a specific triangle. It is possible, however, that other algorithms may be given to produce the same triangle. The point of Proposition I.4 is that regardless of how triangles *ABC* and *DEF* are constructed (i.e., suppose that different algorithms are used for each triangle), if two pairs of corresponding sides and their included angles are "equal," the two triangles are "equal." In other words the effect is the same as if identical algorithms had been used for both constructions.

However, unless it is assumed that an algorithm is well-defined, there is no guarantee that repeated applications of the same algorithm will produce equal geometric results. Indeed, an algorithm must be sufficiently explicit to avoid ambiguities--i.e., to ensure that the construction predicated in the proposition can actually be carried out. This requires that a specific (unique) figure be produced. To put it another way, once Euclid had provided the proof, his belief in the correctness of the proof includes the assumption that the algorithm is well-defined.

Thus, it is almost certain that Euclid assumed his algorithms to be well-defined and that the geometric result of applying an algorithm was unique. This assumption, together with Propositions I.2 and I.3, allows for the duplication of a given triangle. Specifically, if a triangle *ABC* is given, it is possible to reconstruct a "copy" *DEF* which is "equal" to triangle *ABC*.

Let triangle *ABC* and a point *D* be given. Using I.2, "place" the line segment *AC* at the point *D* so that *A* and *D* coincide. Similarly, place *AC* at the point *A* superimposed on *AC*. (Actually line segments *DF* and *A'C'* are constructed which are "equal" to *AC*.) Draw a circle with radius *AB* and center located at *D* and also a circle with radius *AB* and center located at *A*. Propositions I.2 and I.3 allow for this transfer of distances. Similarly, draw circles with centers at *C* and *F* and radii *BC*. Let *E* denote the point of intersection of the circles with centers at *D* and *F*. Denote by *B'* the point of intersection of the circles centered at *A* and *C* and lying on the same side of *AC* as *B*. Since these circles have radii *AB* and *CB*, they must cut off the segments *AB* and *CB* at *B*. Hence *B'* coincides with *B*. Since triangles *A'B'C'* and *ABC* coincide, they are equal. The triangles *A'B'C'* and *DEF* are "equal," since they were constructed via the same algorithm (which is assumed to be well-defined). Hence, triangles *ABC* and *DEF* are "equal."

This reproduction of a given triangle scarcely varies from the algorithm provided by Euclid in his proof of I.1. Consequently, it shares the same logical deficiency which has been noted by Heath [1926, 242], namely, that two circles will meet at a unique point on one side of a segment. The altered proofs of I.4 and I.8 are outlined below.

In I.4, interpret the application of triangles *ABC* to the triangle *DEF* to mean that a "copy" of triangle *ABC*, say, *A'B'C'*, is constructed so that *A'* coincides with *D* and *A'B'* with *DE*.

(Triangles *ABC* and *A'B'C'* are "equal.") Similarly in I.8, a copy of triangle *ABC*, say, *A'B'C'*, is constructed so that *B'* and the line segment *B'C'* coincide with *E* and *EF*, respectively. The remainder of both proofs follow as given by Euclid with *A'*, *B'* and *C'* substituted in the text for *A*, *B* and *C*. Both arguments demonstrate that triangles *A'B'C'* and *DEF* coincide and are therefore equal. Since triangles *ABC* and *A'B'C'* are "equal" and triangles *A'B'C'* and *DEF* are equal, it follows that triangles *ABC* and *DEF* are "equal."

The assumption that algorithms are well-defined does not weaken the *Elements* in the sense that certain postulates or proofs are without any purpose. In particular, the proofs of I.4 and I.8 are necessary in the following sense: if two triangles are given (constructed) such that either if their two sides and the included angle are congruent or if their corresponding sides are congruent, it is not assumed that the same algorithm was used in their construction. Similarly, although Euclid has provided us with one algorithm to construct right angles, it is not redundant to postulate the equality of all right angles. If we assume the displacement of geometric figures, the proofs of I.2, I.3 and I.23 are pointless. Granted, the underlying assumption permits an alternate proof of I.23 by reconstructing the given angle; but it is still eloquent to employ I.8 as Euclid does. As for I.26, either of the two underlying assumptions would shorten the proof of the first part of the proposition. However, neither assumption is of any obvious advantage in proving the second part.

The proofs of I.4 and I.8 are straightforward once the triangle reproduction theorem is available. Thus the question remains: how could this theorem have been in Euclid's mind and why did he choose to omit the proof? Perhaps Euclid believed that enough information had been provided in the proofs of the first three propositions and that to prove the reproduction theorem would have been redundant. Suppose that his goal was to prove I.4 by substituting a reconstruction method for the traditional method of displacing geometric figures. It would have been natural to first attempt the reconstruction of a given line segment. Once Euclid realized (as he undoubtedly did) that this could be accomplished by first constructing an equilateral triangle on the given line segment, such a construction would become the starting point. For a given line segment, the triangle would be completely determined once the third point had been located. Eulcid determined this point by drawing two circles having equal radii, with their centers at the end points of the given line segment. Having completed the proof he would have had to assume that his demonstration was adequate to complete the construction of a specific triangle. The fact that his algorithm was not well-defined is beside the point. He undoubtedly assumed it was. Thus, once having proved I.1 and I.2, it would have become clear to Euclid that a given equilateral triangle (and hence any given triangle) could be reconstructed by first "placing" the base at a given point and then following the (well-defined) algorithm from I.1 to construct simultaneously one triangle superimposed on the given triangle and another on the "placed" line segment. However, if the idea of a well-defined algorithm remained suppressed, as I suspect it was, to have provided the reader with the details would have involved little more than a repetition of the construction already given in I.1. That this idea was to remain unexpressed is not surprising; indeed, as late as the 17th century the most explicit definition of a function characterized this concept as a succession of operations [Kline 1972, 339]. No mention of uniqueness was made until Euler distinguished between single-valued and multivalued functions. The modern definition of a function as a set of ordered pairs and the emphasis on such properties as uniqueness and well-definedness are relatively recent formulations. Undoubtedly these properties were often presupposed despite the lack of any formal justification for doing so.

ACKNOWLEDGMENTS

This article is an outgrowth of the research done to complete a doctoral dissertation (SUNYAB) under the guidance of Gerald Rising, John Corcoran and Milton Baker. I am also grateful to David Rice (D'Youville College) for verifying the accuracy of the Heath translation of Euclid's propositions which are fundamental to this paper.

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