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Positive periodic solutions of higher-dimensional functional difference equations with a parameter ☆

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Abstract

By using Krasnoselskii's fixed point theorem and upper and lower solutions method, we find some sets of positive values λ determining that there exist positive T-periodic solutions to the higher-dimensional functional difference equations of the form

$$x(n+1) = A(n)x(n) + \lambda h(n) f\big(x\big(n - \tau(n)\big)\big), \quad n \in Z,$$

where $A(n) = \operatorname{diag}[a_1(n), a_2(n), \ldots, a_m(n)]$, $h(n) = \operatorname{diag}[h_1(n), h_2(n), \ldots, h_m(n)]$, $a_j, h_j : Z \to R^+$, $\tau : Z \to Z$ are T -periodic, $j = 1, 2, \ldots, m$, $T \geqslant 1$, $\lambda > 0$, $x : Z \to R^m$, $f : R_+^m \to R_+^m$, where $R_+^m = \{(x_1, \ldots, x_m)^T \in R^m, x_j \geqslant 0, j = 1, 2, \ldots, m\}$, $R^+ = \{x \in R,\ x > 0\}$.

Keywords: Functional difference equation; Positive periodic solution; Fixed point theorem; Upper and lower solutions method

1. Introduction

The existence of positive periodic solutions of functional differential equations have been studied extensively in recent years [1–5]. However, relatively few papers have been published on the same problem for functional difference equations. In [6], Raffoul has

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studied the existence of positive periodic solutions for the following functional difference equation:

$$x(n+1) = a(n)x(n) + \lambda h(n) f\big(x\big(n - \tau(n)\big)\big), \tag{1.1}$$

where $a(n)$, $h(n)$ and $\tau(n)$ are T-periodic for T is an integer with $T \geqslant 1$, λ, $a(n)$, $f(x)$ and $h(n)$ are nonnegative with $0 < a(n) < 1$ for all $n \in \{0, 1, \ldots, T-1\}$.

Let $R_+^m = \{(x_1, \ldots, x_m)^T \in R^m,\ x_j \geqslant 0,\ j = 1, 2, \ldots, m\}$, $R^+ = \{x \in R,\ x > 0\}$, let Z be the set of all integers and N be the set of all nonnegative integers. Given $a < b \in Z$, let $[a, b] = \{a, a+1, \ldots, b\}$. Our aim in this paper is to study the existence and nonexistence of positive solutions of the following higher-dimensional functional difference equation:

$$x(n+1) = A(n)x(n) + \lambda h(n) f\big(x\big(n - \tau(n)\big)\big), \quad n \in Z, \tag{1.2}$$

where $A(n) = \mathrm{diag}[a_1(n), a_2(n), \ldots, a_m(n)]$, $h(n) = \mathrm{diag}[h_1(n), h_2(n), \ldots, h_m(n)]$, $a_j, h_j : Z \to R^+$, $\tau : Z \to Z$ are T -periodic, $j = 1, 2, \ldots, m$, $T \geqslant 1$, $\lambda > 0$, $f : R_+^m \to R_+^m$ is continuous. We denote BC the normed vector space of bounded function $\phi : Z \to R^m$ with the norm $\|\phi\| = \sum_{j=1}^m \sup_{n \in [0, T-1]} |\phi_j(n)|$, where $\phi = (\phi_1, \phi_2, \ldots, \phi_m)^T$ and $[0, T-1] := \{0, 1, 2, \ldots, T-1\}$. Particularly, for each $x = (x_1, x_2, \ldots, x_m)^T \in R^m$, we define the norm $|x|_0 = \sum_{j=1}^m |x_j|$.

In the sequel, we say that x is "positive" whenever $x \in R_+^m$, we denote $f = (f_1, f_2, \ldots, f_m)^T$, and denote the product of $x(n)$ from $n = a$ to $n = b$ by $\prod_{n=a}^{n=b} x(n)$ with the understanding that $\prod_{n=a}^{n=b} x(n) = 1$ for all $a > b$.

In Section 2, we make some preparation. In the first part of Section 3, by using Krasnoselskii's fixed point theorem, we obtain sufficient conditions for the existence of at least one positive T-periodic solution of (1.2); in the second part of Section 3, when f satisfies other conditions, we show that there exists $\lambda^* > 0$ such that (1.2) has at least one positive T-periodic solution for $\lambda \in (0, \lambda^*]$ and does not have any positive periodic solutions for $\lambda > \lambda^*$ by using the upper and lower solutions method [7].

2. Some preparation

In this paper, we always assume that

(P1) $0 < a_j(n) < 1$, $n \in [0, T-1]$, for $j = 1, 2, \ldots, m$.

For convenience, we introduce the definition of cone and the well-known Krasnosel'skii fixed point theorem.

Definition. Let X be a Banach space and K be a closed, nonempty subset of X. K is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geqslant 0$;
(ii) $u, -u \in K$ imply $u = 0$.

Theorem 2.1 (Krasnoselskii [8]). *Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are open bounded subsets of X with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$\varphi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|\varphi y\| \leqslant \|y\|$, $\forall y \in k \cap \partial\Omega_1$ *and* $\|\varphi y\| \geqslant \|y\|$, $\forall y \in k \cap \partial\Omega_2$; *or*
(ii) $\|\varphi y\| \geqslant \|y\|$, $\forall y \in k \cap \partial\Omega_1$ *and* $\|\varphi y\| \leqslant \|y\|$, $\forall y \in k \cap \partial\Omega_2$.

Then φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3. Main results

Let

$$X = \{x : Z \to R^m,\ x(n+T) = x(n)\};$$

then it is clear that $X \subset BC$, endowed with norm $\|x\| = \sum_{j=1}^m \|x_j\|_0$, where $\|x_j\|_0 = \sup_{n\in[0,T-1]} |x_j(n)|$.

Similar to the proof of Lemma 2.2 in [6], one can get the following lemma.

Lemma 3.1. *$x(n) \in X$ is a solution of Eq.* (1.2) *if and only if*

$$x(n) = \lambda \sum_{u=n}^{n+T-1} G(n,u)h(u)f\big(x\big(u-\tau(u)\big)\big), \tag{3.1}$$

where

$$G(n,u) = \operatorname{diag}\big[G_1(n,u), \ldots, G_m(n,u)\big] \tag{3.2}$$

and

$$G_j(n,u) = \frac{\prod_{s=u+1}^{n+T-1} a_j(s)}{1 - \prod_{s=n}^{n+T-1} a_j(s)}, \quad u \in [n, n+T-1],\ j = 1, 2, \ldots, m.$$

By (P1), we know that the denominator in $G_j(n,u)$ is not zero for $n \in [0, T-1]$. Note that due to (P1), we have

$$\bar{N}_j \equiv G_j(n,n) \leqslant G_j(n,u) \leqslant G_j(n, n+T-1) = G_j(0, T-1) \equiv M_j,$$
$$j = 1, 2, \ldots, m,$$

for $u \in [n, n+T-1]$, and

$$1 \geqslant \frac{G_j(n,u)}{G_j(n,n+T-1)} \geqslant \frac{G_j(n,n)}{G_j(n,n+T-1)} = \frac{\bar{N}_j}{M_j} > 0, \quad j = 1, 2, \ldots, m.$$

Let

$$\gamma = \min\left\{\frac{\bar{N}_j}{M_j},\ j = 1, 2, \ldots, m\right\}$$

and

$$\bar{N} := \min_{1 \leqslant j \leqslant m} \bar{N}_j, \qquad M := \max_{1 \leqslant j \leqslant m} M_j.$$

Then $\gamma \in (0,1)$.

Next, define a cone by

$$P = \{x \in X,\ x_j(n) \geqslant \gamma \|x_j\|_0,\ j = 1, 2, \dots, m\},$$

meanwhile, we define an operator $F : P \to P$ by

$$(Fx)(n) = \lambda \sum_{u=n}^{n+T-1} G(n,u) h(u) f\big(x\big(u - \tau(u)\big)\big)$$

for $x \in P$, $n \in Z$, where $G(n,u)$ is defined by (3.2). We denote

$$(Fx) = (F_1 x, F_2 x, \dots, F_m x)^T.$$

Lemma 3.2. *$F : P \to P$ is well defined.*

Proof. For each $x \in P$, since it is clear that $(Fx)(n+T) = (Fx)(n)$, $Fx \in X$. For any $x \in P$, we have

$$\begin{aligned}(F_j x)(n) &= \lambda \sum_{u=n}^{n+T-1} G_j(n,u) h_j(u) f_j\big(x\big(u - \tau(u)\big)\big) \\ &\leqslant \lambda \sum_{u=n}^{n+T-1} G_j(0, T-1) h_j(u) f_j\big(x\big(u - \tau(u)\big)\big).\end{aligned}$$

Thus

$$\|F_j x\|_0 = \sup_{n \in [0, T-1]} \big|(F_j x)(n)\big| \leqslant \lambda \sum_{u=0}^{T-1} G_j(0, T-1) h_j(u) f_j\big(x\big(u - \tau(u)\big)\big)$$

and

$$\begin{aligned}(F_j x)(n) &= \lambda \sum_{u=n}^{n+T-1} G_j(n,u) h_j(u) f_j\big(x\big(u - \tau(u)\big)\big) \\ &\geqslant \lambda \bar{N}_j \sum_{u=0}^{T-1} h_j(u) f_j\big(x\big(u - \tau(u)\big)\big) \\ &= \lambda \bar{N}_j \sum_{u=0}^{T-1} \frac{G_j(0, T-1)}{M_j} h_j(u) f_j\big(x\big(u - \tau(u)\big)\big) \\ &\geqslant \gamma \|F_j x\|_0, \quad j = 1, 2, \dots, m.\end{aligned}$$

Therefore, $(Fx) \in P$. This completes the proof. □

Lemma 3.3. *$F : P \to P$ is completely continuous.*

Proof. We first show that F is continuous. By the continuity of f, for any given $y \in P$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in P$ with $\|x - y\| < \delta$, we have

$$\sup_{n\in[0,T-1]} \left| f_j\big(x\big(n-\tau(n)\big)\big) - f_j\big(y\big(n-\tau(n)\big)\big)\right| < \frac{\varepsilon}{\lambda M_j q_j T m}.$$

Hence

$$\begin{aligned}
&\left|(F_j x)(n) - (F_j y)(n)\right| \\
&\quad \leqslant \lambda \sum_{u=n}^{n+T-1} G_j(n,u) h_j(u) \left| f_j\big(x\big(u-\tau(u)\big)\big) - f_j\big(y\big(u-\tau(u)\big)\big)\right| \\
&\quad \leqslant \lambda M_j q_j T \sup_{n\in[0,T-1]} \left| f_j\big(x\big(n-\tau(n)\big)\big) - f_j\big(y\big(n-\tau(n)\big)\big)\right| < \frac{\varepsilon}{m}
\end{aligned}$$

for all $n \in [0, T-1]$, where $q_j = \max_{n\in[0,T-1]} h_j(n)$. This yields

$$\left\|(F_j x)(n) - (F_j y)(n)\right\|_0 < \frac{\varepsilon}{m}.$$

Thus,

$$\|Fx - Fy\| < \varepsilon.$$

Hence, F is continuous.

Next, we show that F maps bounded sets into relatively compact sets. Indeed, let d be a constant and $D = \{x \in X,\ \|x\| < d\}$ be a bounded set. We prove that $\overline{F(D)}$ is compact. To do this, we must show that any sequence in $F(D)$ contains a convergent subsequence. Thus, let $\{x^k\}_{k\in N}$ be a sequence in D. Let us show that $\{Fx^k\}_{k\in N}$ has a convergent subsequence. Since f is continuous on R^m, the sequence $\{f(x^k(0))\}_{k\in N}$ is bounded, then the sequence $\{f(x^k(0))\}_{k\in N}$ contains a convergent subsequence. So, let $\{x^{k,0}\}_{k\in N}$ be the subsequence of $\{x^k\}_{k\in N}$ such that $\{f(x^{k,0}(0))\}_{k\in N}$ is convergent.

Now, $\{f(x^{k,0}(1))\}_{k\in N}$ contains a convergent subsequence. So, let $\{x^{k,1}\}_{k\in N}$ be the subsequence of $\{x^{k,0}\}_{k\in N}$ such that $\{f(x^{k,1}(1))\}_{k\in N}$ is convergent. Observe that $\{x^{k,1}\}_{k\in N}$ is a subsequence of $\{x^k\}_{k\in N}$ and that $\{f(x^{k,1}(0))\}_{k\in N}$, $\{f(x^{k,1}(1))\}_{k\in N}$ are convergent.

Again, $\{f(x^{k,1}(-1))\}_{k\in N}$ contains a convergent subsequence. So, let $\{x^{-1,k,1}\}_{k\in N}$ be the subsequence of $\{x^{k,1}\}_{k\in N}$ such that $\{f(x^{-1,k,1}(-1))\}_{k\in N}$ is convergent. Observe that $\{x^{-1,k,1}\}_{k\in N}$ is a subsequence of $\{x^k\}_{k\in N}$ and that $\{f(x^{-1,k,1}(-1))\}_{k\in N}$, $\{f(x^{-1,k,1}(0))\}_{k\in N}$, $\{f(x^{-1,k,1}(1))\}_{k\in N}$ are convergent.

Continuing in this fashion we find, for each $l \in Z$, a subsequence $\{x^{-(l+1),k,(l+1)}\}_{k\in N}$ of $\{x^{-l,k,l}\}_{k\in N}$ such that $\{f(x^{-(l+1),k,(l+1)}(l+1))\}_{k\in N}$, $\{f(x^{-(l+1),k,(l+1)}(-(l+1)))\}_{k\in N}$ are convergent. Observe that also the sequences $\{f(x^{-(l+1),k,(l+1)}(-l))\}_{k\in N}, \ldots,$ $\{f(x^{-(l+1),k,(l+1)}(l))\}_{k\in N}$ are convergent.

Consider now the sequence $\{x^{-s,s,s}\}_{s\in N}$. Observe that it is a subsequence of $\{x^k\}_{k\in N}$, and also that $\{f(x^{-s,s,s}(n))\}_{s\in N}$ is convergent for all $n \in Z$. Let us show that $\{Fx^{-s,s,s}\}_{s\in N}$ is a Cauchy sequence.

Therefore, for any $\varepsilon > 0$, there exists $\bar{k} \in N$ such that $e, g \in N$ with $e, g \geqslant \bar{k}$,

$$\sup_{n\in[0,T-1]} \left| f_j\big(x^{-e,e,e}\big(n-\tau(n)\big)\big) - f_j\big(x^{-g,g,g}\big(n-\tau(n)\big)\big)\right| < \frac{\varepsilon}{\lambda M_j q_j T m},$$
$$j = 1, 2, \ldots, m.$$

Consequently, if $e, g \in N$ with $e, g \geqslant \bar{k}$, for all $n \in [0, T-1]$, we have

$$\begin{aligned}
&\left|(F_j x^{-e,e,e})(n) - (F_j x^{-g,g,g})(n)\right| \\
&\quad \leqslant \lambda \sum_{u=n}^{n+T-1} G_j(n,u) h_j(u) \left| f_j\big(x^{-e,e,e}\big(u-\tau(u)\big)\big) - f_j\big(x^{-g,g,g}\big(u-\tau(u)\big)\big)\right| \\
&\quad \leqslant \lambda M_j q_j T \sup_{n\in[0,T-1]} \left| f_j\big(x^{-e,e,e}\big(n-\tau(n)\big)\big) - f_j\big(x^{-g,g,g}\big(n-\tau(n)\big)\big)\right| < \frac{\varepsilon}{m}
\end{aligned}$$

and

$$\begin{aligned}
\left\|(F x^{-e,e,e})(n) - (F x^{-g,g,g})(n)\right\| &= \sum_{j=1}^{m} \left\|(F_j x^{-e,e,e})(n) - (F_j x^{-g,g,g})(n)\right\|_0 \\
&< m\frac{\varepsilon}{m} < \varepsilon.
\end{aligned}$$

This proves that $\{F x^{-s,s,s}\}_{s\in N}$ is a Cauchy sequence, and with this the proof is complete. □

From now on, by using different methods, we will obtain different results. So we will continue in two parts.

(I) In this part, when f satisfies certain conditions, we obtain some conditions determined by λ under which there exists at least one positive periodic solution of (1.2) by using fixed point theorem.

Theorem 3.1. *Suppose that* (P1), (P2) *and the conditions*

(H1) $\displaystyle \lim_{x_j\to 0+0} \frac{f_j(x)}{x_j} = l_j, \quad 0 < l_j < \infty,\ j = 1, 2, \ldots, m,$

(H2) $\displaystyle \lim_{x_j\to +\infty} \frac{f_j(x)}{x_j} = L_j, \quad 0 < L_j < \infty,\ j = 1, 2, \ldots, m,$

hold. Then, for each λ satisfying

$$\frac{1}{\gamma \bar{N} \bar{q} T L''} < \lambda < \frac{1}{M q T l'} \tag{3.3}$$

or

$$\frac{1}{\gamma \bar{N} \bar{q} T l''} < \lambda < \frac{1}{M q T L'}, \tag{3.4}$$

where $q = \max_{1\leqslant j\leqslant m} q_j$, $\bar{q}_j = \min_{n\in[0,T-1]} h_j(n)$, $\bar{q} = \min_{1\leqslant j\leqslant m} \bar{q}_j$, $l' = \max_{1\leqslant j\leqslant m} l_j$, $L' = \max_{1\leqslant j\leqslant m} L_j$, $l'' = \min_{1\leqslant j\leqslant m} l_j$, $L'' = \min_{1\leqslant j\leqslant m} L_j$, (1.2) *has at least one positive T-periodic solution.*

Proof. Suppose (3.3) holds. Let $\varepsilon > 0$ be given such that

$$\frac{1}{\gamma \bar{N} \bar{q} T (L'' - \varepsilon)} \leqslant \lambda \leqslant \frac{1}{M q T (l' + \varepsilon)}.$$

By (H1), there exists $\bar{E}_1 > 0$ such that $f_j(x) \leqslant (l_j + \varepsilon)x_j \leqslant (l' + \varepsilon)x_j$ for $0 < x_j \leqslant \bar{E}_1$. Define

$$\Omega_1 = \bigl\{x \in P,\ \|x_j\|_0 < \bar{E}_1,\ j = 1, 2, \ldots, m\bigr\},$$

then, if $x \in P \cap \partial\Omega_1$,

$$\begin{aligned}(F_j x)(n) &= \lambda \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u) f_j\bigl(x\bigl(u - \tau(u)\bigr)\bigr)\\ &\leqslant \lambda M q (l' + \varepsilon) \sum_{u=n}^{n+T-1} x_j\bigl(u - \tau(u)\bigr) \leqslant \lambda M q T (l' + \varepsilon)\|x_j\|_0 \leqslant \|x_j\|_0.\end{aligned}$$

Thus,

$$\|F_j x\|_0 \leqslant \|x_j\|_0$$

and

$$\|Fx\| = \sum_{j=1}^{m} \|F_j x\|_0 \leqslant \sum_{j=1}^{m} \|x_j\|_0 = \|x\| \quad \text{for all } x \in P \cap \partial\Omega_1.$$

Next, we construct Ω_2. By (H2), there exists $\bar{E}_2$ such that $f_j(x) \geqslant (L_j - \varepsilon)x_j \geqslant (L'' - \varepsilon)x_j$ for all $x_j \geqslant \bar{E}_2$. Let $E_2' = \max\{2\bar{E}_1, \bar{E}_2/\gamma\}$. Define

$$\Omega_2 = \bigl\{x \in P,\ \|x_j\|_0 < E_2'\bigr\}.$$

Since $x \in P$, if $x \in P \cap \partial\Omega_2$, then $\min_{n \in [0, T-1]} x_j(n) \geqslant \gamma\|x_j\|_0$ and

$$\begin{aligned}(F_j x)(n) &= \lambda \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u) f_j\bigl(x\bigl(u - \tau(u)\bigr)\bigr)\\ &\geqslant \lambda \bar{N}\bar{q}(L'' - \varepsilon) \sum_{u=n}^{n+T-1} x_j\bigl(u - \tau(u)\bigr) \geqslant \lambda \bar{N}\bar{q} T (L'' - \varepsilon)\gamma\|x_j\|_0 \geqslant \|x_j\|_0.\end{aligned}$$

Thus,

$$\|F_j x\|_0 \geqslant \|x_j\|_0$$

and

$$\|Fx\| = \sum_{j=1}^{m} \|F_j x\|_0 \geqslant \sum_{j=1}^{m} \|x_j\|_0 = \|x\| \quad \text{for all } x \in P \cap \partial\Omega_2.$$

By Theorem 2.1, F has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$, that is, (1.2) has at least one positive periodic solution. Similarly, we know that the conclusion still holds when λ satisfies (3.4). The proof is completed. □

(II) In this part, we show that there exists $\lambda^* > 0$ such that (1.2) has at least one positive T-periodic solution for $\lambda \in (0, \lambda^*]$ and does not have any positive periodic solutions for $\lambda > \lambda^*$ when f satisfies other conditions.

Lemma 3.4. *Suppose that*

$$\lim_{x_j \to +\infty} \frac{f_j(x)}{x_j} = +\infty, \quad j = 1, 2, \ldots, m. \tag{3.5}$$

Let I be a compact subset of $(0, +\infty)$. Then there exists a constant $C_I > 0$ such that $\|x\| < C_I$ for all $\lambda \in I$ and all possible T-periodic positive solutions x of (1.2) associated with λ.

Proof. Suppose to the contrary that there is a sequence $\{x^i\}_{i \in N}$ of T-periodic positive solutions of (1.2) associated with $\{\lambda_i\}$ such that $\lambda_i \in I$ for all i and $\|x^i\| \to +\infty$ as $i \to +\infty$. Since $x^i \in P$,

$$\min_{n \in [0, T-1]} x_j^i(n) \geqslant \gamma \|x_j^i\|_0, \quad j = 1, 2, \ldots, m.$$

By (3.5), there exists $H_f > 0$ such that $f_j(x) \geqslant \sigma x_j$ for all $x_j \geqslant H_f$, and there exist i_0, j_0 such that $\gamma \|x_{j_0}^{i_0}\|_0 \geqslant H_f$, where σ satisfies

$$\lambda_{i_0} \sigma \gamma \bar{N} \bar{q} T > 1.$$

Thus, we have

$$\begin{aligned}\|x_{j_0}^{i_0}\|_0 \geqslant x_{j_0}^{i_0}(n) &= \lambda_{i_0} \sum_{u=n}^{n+T-1} G_{j_0}(n, u) h_{j_0}(u) f_{j_0}\bigl(x^{i_0}\bigl(u - \tau(u)\bigr)\bigr) \\ &\geqslant \lambda_{i_0} \sigma \bar{N} \bar{q} \sum_{u=n}^{n+T-1} x_{j_0}^{i_0}\bigl(u - \tau(u)\bigr) \geqslant \lambda_{i_0} \gamma \sigma \bar{N} \bar{q} T \|x_{j_0}^{i_0}\|_0 > \|x_{j_0}^{i_0}\|_0.\end{aligned}$$

This is a contradiction. The proof is completed. □

Lemma 3.5. *Suppose that*

$$\textit{if } \|x\| \geqslant \|y\| \quad \textit{then } f_j(x) \geqslant f_j(y) \quad \textit{and} \quad f_j(0) > 0, \quad j = 1, 2, \ldots, m. \tag{3.6}$$

Let (1.2) has a T-periodic positive solution $x(n)$ associated with $\bar{\lambda} > 0$. Then (1.2) also has a positive T-periodic solution associated with $\lambda \in (0, \bar{\lambda})$.

Proof. In view of (3.1) and (3.6), we have

$$\begin{aligned}x_j(n) &= \bar{\lambda} \sum_{u=n}^{n+T-1} G_j(n, u) h_j(u) f_j\bigl(x\bigl(u - \tau(u)\bigr)\bigr) \\ &> \lambda \sum_{u=n}^{n+T-1} G_j(n, u) h_j(u) f_j\bigl(x\bigl(u - \tau(u)\bigr)\bigr)\end{aligned}$$

and

$$\lambda \sum_{u=n}^{n+T-1} G_j(n, u) h_j(u) f_j(0) > 0.$$

Let

$$\bar{x}_j^0(n) = x_j(n),$$
$$\bar{x}_j^{k+1}(n) = \lambda \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u)f_j\big(\bar{x}^k\big(u-\tau(u)\big)\big), \quad k = 0,1,2,\ldots,$$
$$\underline{x}_j^0(n) = 0 \quad \text{and} \quad \underline{x}_j^{k+1}(n) = \lambda \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u)f_j\big(\underline{x}^k\big(u-\tau(u)\big)\big),$$
$$k = 0,1,2,\ldots.$$

Clearly, we have

$$\bar{x}_j^0(n) > \bar{x}_j^1(n) \geqslant \cdots \geqslant \bar{x}_j^k(n) \geqslant \underline{x}_j^k(n) \geqslant \cdots \geqslant \underline{x}_j^1(n) > \underline{x}_j^0(n).$$

If we let $x_j(n) = \lim_{k\to\infty} \bar{x}_j^k(n)$, then $x_j(n)$ satisfies (3.1), and we have

$$x_j(n) > \underline{x}_j^1(n) > \underline{x}_j^0(n) = 0.$$

This completes our proof. □

Lemma 3.6. *Suppose that* (3.5) *and* (3.6) *hold. Then there exists* $\lambda_* > 0$ *such that* (1.2) *has a* T*-periodic positive solution.*

Proof. Let

$$\beta_j(n) = \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u), \quad M_{f_j} = \max_{n\in[0,T-1]} f_j\big(\beta\big(n-\tau(n)\big)\big),$$
$$j = 1,2,\ldots,m,$$

and

$$\lambda_* = \frac{1}{\max_{1\leqslant j\leqslant m} M_{f_j}}.$$

Then

$$\beta_j(n) = \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u) \geqslant \lambda_* \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u)f_j\big(\beta\big(u-\tau(u)\big)\big)$$

and

$$\lambda_* \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u)f_j(0) > 0.$$

Let

$$\bar{x}_j^0(n) = \beta_j(n),$$
$$\bar{x}_j^{k+1}(n) = \lambda_* \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u)f_j\big(\bar{x}^k\big(u-\tau(u)\big)\big), \quad k = 0,1,2,\ldots,$$

$$\underline{x}_j^0(n) = 0 \quad \text{and} \quad \underline{x}_j^{k+1}(n) = \lambda_* \sum_{u=n}^{n+T-1} G_j(n,u) h_j(u) f_j\big(\underline{x}^k\big(u-\tau(u)\big)\big),$$
$$k = 0, 1, 2, \ldots.$$

Clearly, we have

$$\bar{x}_j^0(n) \geqslant \bar{x}_j^1(n) \geqslant \cdots \geqslant \bar{x}_j^k(n) \geqslant \underline{x}_j^k(n) \geqslant \cdots \geqslant \underline{x}_j^1(n) > \underline{x}_j^0(n).$$

If we now let $x_j(n) = \lim_{k\to\infty} \bar{x}_j^k(n)$, then $x_j(n)$ satisfies (3.1), and we have

$$x_j(n) \geqslant \underline{x}_j^1(n) > \underline{x}_j^0(n) = 0.$$

This completes our proof. □

Theorem 3.2. *Suppose that* (3.5) *and* (3.6) *hold. then there exists* $\lambda^* > 0$ *such that* (1.2) *has at least one positive* T*-periodic solution for* $\lambda \in (0, \lambda^*]$ *and does not have any* T*-periodic positive solution for* $\lambda > \lambda^*$.

Proof. Suppose to the contrary that there is a sequence $\{x^k\}_{k\in N}$ of T-periodic positive solutions of (1.2) associated with $\{\lambda_k\}$ such that $\lim_{k\to\infty} \lambda_k = \infty$. Then either we have $\|x^{k_i}\| \to +\infty$ as $i \to \infty$ or there is $Q > 0$ such that $\|x^k\| \leqslant Q$. Assume the former case holds. Note that $x^k \in P$ and thus

$$\min_{n\in[0,T-1]} x_j^k(n) \geqslant \gamma \big\|x_j^k\big\|_0, \quad j = 1, 2, \ldots, m.$$

By (3.5), we may choose $H_f > 0$ and $\sigma > 0$ such that $f_j(x) \geqslant \sigma x_j$ when $x_j \geqslant H_f$, and there exist i_0, j_0 such that $\gamma\|x_{j_0}^{k_{i_0}}\|_0 \geqslant H_f$. On the other hand, there exists $\{n_{j_0}^{k_{i_0}}\} \in [0, T-1]$ such that $x_{j_0}^{k_{i_0}}(n_{j_0}^{k_{i_0}}) = \|x_{j_0}^{k_{i_0}}\|_0$. By (1.2), we have

$$\begin{aligned}
a_{j_0}\big(n_{j_0}^{k_{i_0}}\big)\big\|x_{j_0}^{k_{i_0}}\big\|_0 &= a_{j_0}\big(n_{j_0}^{k_{i_0}}\big)x_{j_0}^{k_{i_0}}\big(n_{j_0}^{k_{i_0}}\big)\\
&= x_{j_0}^{k_{i_0}}\big(n_{j_0}^{k_{i_0}}+1\big) - \lambda_{k_{i_0}} h_{j_0}\big(n_{j_0}^{k_{i_0}}\big) f_{j_0}\big(x^{k_{i_0}}\big(n_{j_0}^{k_{i_0}} - \tau\big(n_{j_0}^{k_{i_0}}\big)\big)\big)\\
&\leqslant \big\|x_{j_0}^{k_{i_0}}\big\|_0 - \lambda_{k_{i_0}} h_{j_0}\big(n_{j_0}^{k_{i_0}}\big)\sigma x_{j_0}^{k_{i_0}}\big(n_{j_0}^{k_{i_0}} - \tau\big(n_{j_0}^{k_{i_0}}\big)\big)\\
&\leqslant \big\|x_{j_0}^{k_{i_0}}\big\|_0 - \lambda_{k_{i_0}} h_{j_0}\big(n_{j_0}^{k_{i_0}}\big)\sigma\gamma\big\|x_{j_0}^{k_{i_0}}\big\|_0\\
&= \big\|x_{j_0}^{k_{i_0}}\big\|_0\big[1 - \lambda_{k_{i_0}} h_{j_0}\big(n_{j_0}^{k_{i_0}}\big)\sigma\gamma\big],
\end{aligned}$$

we have

$$\lambda_{k_{i_0}} \leqslant \frac{1 - a_{j_0}(n_{j_0}^{k_{i_0}})}{h_{j_0}(n_{j_0}^{k_{i_0}})\sigma\gamma}.$$

Note that $(1 - a_j(n))/h_j(n)$ is bounded. Thus, we obtain a contradiction.

Next, suppose that the latter case holds. In view of (3.6), there exists $\sigma_1 > 0$ such that $f_j(0) \geqslant \sigma_1 Q$. Then, as above, we will obtain

$$\begin{aligned}a_j(n_j^k)\|x_j^k\|_0 &= a_j(n_j^k)x_j^k(n_j^k) = x_j^k(n_j^k+1) - \lambda_k h_j(n_j^k) f_j(x^k(n_j^k - \tau(n_j^k))) \\ &\leqslant \|x_j^k\|_0 - \lambda_k h_j(n_j^k)\sigma_1 Q \leqslant \|x_j^k\|_0 - \lambda_k h_j(n_j^k)\sigma_1\|x_j^k\|_0 \\ &= \|x_j^k\|_0\big[1 - \lambda_k h_j(n_j^k)\sigma_1\big]\end{aligned}$$

for all k. A contradiction will again be reached. Thus, there exists λ^* such that (1.2) has at least one positive T-periodic solution for $\lambda \in (0, \lambda^*)$ and no T-periodic positive solutions for $\lambda > \lambda^*$.

Finally, we assert that (1.2) has at least one T-periodic positive solution for $\lambda = \lambda^*$. Indeed, let $\{\lambda_i\}$ satisfy $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda^*$ and $\lim_{k\to\infty} \lambda_k = \lambda^*$. Since x^i is T-periodic positive solution of (1.2) associated with λ_i and Lemma 3.4 implies that the set $\{x^i\}$ of solutions is uniformly bounded in P, moreover, x^i is the fixed point of the operator equation $Fx = x$ and F is completely continuous, the sequence $\{x^i\}$ has a subsequence $\{x^{i_g}\}$ converging to $x \in P$. Thus, we have

$$\begin{aligned}x_j(n) &= \lim_{g\to\infty} x_j^{i_g}(n) = \lim_{g\to\infty} \lambda_{i_g} \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u) f_j\big(x^{i_g}\big(u - \tau(u)\big)\big) \\ &= \lambda^* \sum_{u=n}^{n+T-1} G_j(n,u)h_j(u) f_j\big(x\big(u - \tau(u)\big)\big), \quad j = 1, 2, \ldots, m.\end{aligned}$$

Hence x is a T-periodic positive solution of (1.2) associated with $\lambda = \lambda^*$. The proof is complete. □

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