JOURNAL OF NUMBER THEORY 21, 256–259 (1985)

Some Further Remarks on Genus Field

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Communicated by H. Zassenhaus

Received January 1, 1982

Gurak's characterization of the relative (narrow) genus field of a normal extension is extended to any finite extension. Using this, a purely algebraic proof for a theorem of the author on the (narrow) genus field is given. © 1985 Academic Press, Inc.

1. INTRODUCTION

Let K/k be any finite extension with relative discriminant $\mathbf{D}$. The *narrow genus field* K^* of K/k (or the *genus field* as defined in [4]) is the maximal unramified (at finite primes) extension of K of the form $K\Omega$ with Ω/k abelian. For a prime $p \mid \mathbf{D}$, let $P_1, P_2, \ldots, P_s$ be the distinct prime divisors of p in K. Let $K_{(p)}$ denote the intersection of the local extensions of K at $P_1, P_2, \ldots, P_s$ and let $\tilde{K}_{(p)}$ denote the maximal abelian subextension of $K_{(p)}/k_p$ with local conductor p^{ρ_p} and local norm symbol $[\ \ , \tilde{K}_{(p)}/k_p]$. Let U_p and $U_{\mathbf{P}_i}$ $(i = 1, 2, \ldots, s)$ denote the unit groups of k_p and $K_{\mathbf{P}_i}$ respectively. Let $N_{A/B}$ denote the norm map from A to B. When there is no ambiguity, this is denoted by N. Let $[U_p : N_{K_{\mathbf{P}_i}/k_p} U_{\mathbf{P}_i}] = d_{\mathbf{P}_i}$. Let $e_p^* =$ G.C.F. of $d_{\mathbf{P}_i}$. Define a group G_p' of numerical characters of k with conductor p^{ρ_p} by forming the composites $\chi \circ [\ \ , \tilde{K}_{(p)}/k_p]^{-1}$ for each $\chi \in \mathbf{G}^*(\tilde{K}_{(p)}/k_p)$ (the character group of $\tilde{K}_{(p)}/k_p$). As the ramification index of $\tilde{K}_{(p)}/k_p$ is e_p^* (by Lemma 1 below), the order of $\mathbf{G}_p'$ is e_p^*. Set $\mathbf{G}' = \prod_{p \mid \mathbf{D}} \mathbf{G}_p'$ and let $\mathbf{G}_0'$ be the modified resolution of $\mathbf{G}'$ (as defined in Section 3 of [4]) with conductor m.

In Section 3, following the ideas of Gurak [4], we characterize K^* with the aid of $\mathbf{G}_0'$ and derive a formula for the narrow genus number, $[K^* : K]$, of K/k. We make use of this in Section 4 to give an algebraic proof for a theorem on the narrow genus field which was proved by means of some density theorems in [1] (corrections in [2]).

0022-314X/85 $3.00

2. Some Auxiliary Results

Lemma 1. *Let* $\tilde{G}_p$ *denote the Galois group of* $\tilde{K}_{(p)}/k_p$. *Then*

$$\prod_{\mathbf{P}|p} NK_{\mathbf{P}}^{\times} = NK_{(p)}^{\times} = N\tilde{K}_{(p)}^{\times},$$

$$\tilde{G}_p = k_p^{\times}/NK_{(p)}^{\times} = k_p^{\times}/N\tilde{K}_{(p)}^{\times}$$

and

$$e_p^* = \text{the ramification index of } \tilde{K}_{(p)}/k_p.$$

(Here F^* denotes the group of non-zero elements of a field F.)

This may be deduced from some well-known results in Classfield theory ([3, pp. 142–143, Propositions 3 and 4] and the fact that $\tilde{K}_{(p)} = \bigcap_{\mathbf{P}|p} \tilde{K}_{\mathbf{P}}$, where $\tilde{K}_{\mathbf{P}}$ is the maximal abelian subextension of $K_{\mathbf{P}}/k_p$). Theorem 2 of [1] is a special case of this.

Lemma 2. *Let* $\mathbf{P}_K^m$ *denote the group of principal ideals of* K *prime to* m, H_m *denote the narrow ray* mod$^{\times}$ m *and* K_0^* *denote the maximal abelian subextension of* K^*/k. *Then* K_0^* *is classfield to* $N_{K/k}\mathbf{P}_K^m \cdot H_m$, *where* m *is the conductor of* K_0^*/k.

This is Theorem 2 in [4].

From Lemmas 1 and 2 we get:

Lemma 3. *The ramification index of a* K_0^**-prime divisor of* p *is* e_p^*.

Lemma 4. *The Galois group of a normal extension is generated by its inertia subgroups.*

This is Lemma 6.5 in [5, p. 265].

3. Characterization of Narrow Genus Field

We first characterize the narrow genus field of a finite extension K/k in terms of numerical characters. Let F_0 denote the maximal abelian subextension of F/k. Let $\bar{k}$ denote the narrow classfield of k, h^+ the narrow class number $[\bar{k}: k]$ of k and g^+ the narrow genus number $[K^*: K]$ of K. For an abelian extension F/k, let $\mathbf{G}(F/k)$ denote the group of numerical Artin characters of F/k. (For more details about this group, see Section 3 of [4].) Further, let $U_{K/k}$ be the subgroup of U_k^+ (group of totally positive units of k) which are local norms at all k-primes.

THEOREM 1. *K_0^* is the unique abelian extension containing k with $\mathbf{G}(K_0^*/k) = \mathbf{G}_0'$. Further, the narrow genus number*

$$g^+ = \frac{h_k^+ |\mathbf{G}_0'|}{[K_0 : k]} = \frac{h_k^+ \prod_{p|m} e_p^*}{[K_0 : k][U_k^+ : U_{K/k}]}.$$

This is proved in [4] (Theorem 7) for the case K/k normal.

Proof. Let $(K_0^*)^*$ be the narrow genus field of K_0^*/k. Taking K and K_0^* for K in Lemma 2 and applying Lemmas 1 and 3, we find that $K_0^* = (K_0^*)^*$. Also taking K and K_0^* for K in Section 1 to form the associated numerical character groups, we find that both groups are the same in view of Lemma 1 and Lemma 3. Then, by Theorem 7 in [4], $\mathbf{G}((K_0^*)^*/k) = \mathbf{G}_0'$. The proof for the narrow genus number formula could be given using an argument similar to the one given in the proof of Theorem 4 in [4].

4. Narrow Genus Field for Finite Extension of the Rationals

Using Theorem 1, we give a purely algebraic proof for the following theorem which determines the narrow genus field of a finite extension of the rationals. Earlier, the author used some density theorems to prove this in [1].

THEOREM 2. *Let K/Q be a finite algebraic extension and let $\Omega^{(p)}$ denote the unique abelian extension of the rationals of degree e_p^* and conductor p^{ρ_p}. Then*

$$K^* = K \prod_p \Omega^{(p)}$$

where p run through all the rational primes with $e_p^* > 1$.

(Here e_p^* and p^{ρ_p} have the same meaning as in Section 1 except that k is taken as $\mathbb{Q}$.)

Proof. Let $p_1, p_2, ..., p_r$ be the primes for which $e_p^* > 1$. Now by Lemma 4, K_0^* can be viewed as the inertia field of a subfield $K^{(p_1)}$ such that p_1 is fully ramified in $K_0^*/K^{(p_1)}$. Since $K^{(p_1)}/\mathbb{Q}$ is normal, we can repeat the argument and get $K^{(p_1)}$ as the inertia field of $K^{(p_2)}$ such that p_2 is fully ramified in $K^{(p_1)}/K^{(p_2)}$. Continuing this process, we get $K^{(p_t)} = \mathbb{Q}$ for some $t \leq r$. Also we see that the degree of $K_0^* = e_{p_1}^* e_{p_2}^* \cdots e_{p_t}^* \neq \prod_{i=1}^r e_{p_i}^*$ unless $t = r$. But by Theorem 1, the degree of $K_0^* = e_{p_1}^* e_{p_2}^* \cdots e_{p_r}^*$. Therefore $t = r$. This shows that $K^{(p_{r-1})}$ is an abelian extension of $\mathbb{Q}$ where p_r is fully ramified. In the above argument, taking p_i ($i = 1, 2, \ldots, r$) instead of p_{r-1} and vice versa, we see that for every p_i there is an abelian extension of $\mathbb{Q}$ in K_0^*, where p_i is fully ramified. Thus, K_0^* contains abelian subfields of degrees $e_{p_i}^*$ ($i = 1, 2, \ldots, r$). These abelian subfields must have conductors $P^{\rho_{p_i}}$. Also the composite of these abelian subfields has degree equal to that of K_0^*. So $K_0^* = \prod_{p_i} \Omega(p_i)$ and the theorem follows.

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