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Duality for Derived Categories and Cotilting Bimodules

Jun-ichi Miyachi

Department of Mathematics, Tokyo Gakugei University, Koganei-shi, Tokyo, 184, Japan

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INTRODUCTION

In commutative ring theory, the notion of Gorenstein rings has been studied widely. For rings having finite Krull dimension, Bass showed the characterization of Gorenstein rings as rings with finite self-injective dimension [B]. The notion of Cohen–Macaulay rings with dualizing modules was developed by Grothendieck and Hartshorne [Hr], and was studied by several authors. In developing these theories, they began by using the technique of local duality, and then used the technique of duality for derived categories [Hr]. In ring theory, the notion of Gorenstein rings was studied extensively (for example, [AR1], [AR2], [Ho], [I1], and [I2]). Miyashita introduced the notion of a tilting module of finite projective dimension [Ms]. Happel [Hp] and Cline, Parshall, and Scott [CPS] studied the relations between tilting modules and equivalences of derived categories. We studied the relation between tilting modules and localization of derived categories [Mc1]. Miyashita also introduced cotilting bimodules, and showed the existence of homological duality of modules [Ms]. The purpose of this paper is to study the relations between cotilting bimodules and duality for derived categories, and to study the non-commutative ring version of dualizing modules.

In Section 1, we define localization duality of triangulated categories by using the notion of localization of triangulated categories. In Section 2, we study a bimodule which induces a quotient duality and a localization duality of derived categories of modules (Theorem 2.8 and Corollary 2.5). Moreover, we consider the condition that a bimodule, in particular a cotilting bimodule, induces a duality for derived categories (Theorem 2.10 and Corollary 2.11). In Section 3, we approach the notion of approxima-

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tions, which was introduced by Auslander and Buchweitz [AB], by using localization duality for derived categories (Proposition 3.2 and Theorem 3.4). And we consider categories which are equivalent to the category of finitely presented modules having finite injective dimension (Propositions 3.6 and 3.7). In Section 4, we construct a finitely embedding cogenerator for the category of finitely presented modules by using a module which induces a quotient duality (Theorem 4.1, Proposition 4.8, and Corollary 4.9). Furthermore, we apply it to cotilting bimodules, and in particular, to rings with finite self-injective dimension (Propositions 4.4, 4.5; Corollaries 4.2, 4.3, 4.6, 4.7).

Throughout this paper, we assume that all rings have non-zero unity, and that all modules are unital.

1. PRELIMINARIES

Given a triangulated category $\mathscr{D}$ with a translation $\mathsf{T}_{\mathscr{D}}$, we define the opposite triangulated category $\mathscr{D}^{\mathrm{op}}$ by the following:

(a) $\mathsf{T}_{\mathscr{D}^{\mathrm{op}}}(X^{\mathrm{op}}) = \mathsf{T}_{\mathscr{D}}^{-1}(X)$;

(b) $X^{\mathrm{op}} \to Y^{\mathrm{op}} \to Z^{\mathrm{op}} \to \mathsf{T}_{\mathscr{D}^{\mathrm{op}}}(X^{\mathrm{op}})$ is a distinguished triangle if $\mathsf{T}_{\mathscr{D}}^{-1}(X) \to Z \to Y \to X$ is a distinguished triangle in $\mathscr{D}$.

Then there is the natural duality $\mathsf{D}_{\mathscr{D}}$: $\mathscr{D} \to \mathscr{D}^{\mathrm{op}}$, which induces the isomorphism between $\mathrm{Grot}(\mathscr{D})$ and $\mathrm{Grot}(\mathscr{D}^{\mathrm{op}})$, where $\mathrm{Grot}(\mathscr{D})$ is a Grothendieck group of $\mathscr{D}$ (see [G]).

We call a covariant ∂-functor Q: $\mathscr{U} \to \mathscr{V}$ between triangulated categories a quotient ∂-functor provided that there is an equivalent ∂-functor E: $\mathscr{U}/\mathrm{Ker}\,\mathsf{Q} \to \mathscr{V}$ such that $\mathsf{P} \circ \mathsf{E}$ is equal to Q, where P: $\mathscr{U} \to \mathscr{U}/\mathrm{Ker}\,\mathsf{Q}$ is the natural quotient (see [Mc1, Sect. 2; V, Chap. I, Sect. 2, No. 3]). We will call a contravariant ∂-functor G: $\mathscr{U} \to \mathscr{V}$ between triangulated categories a quotient duality if $\mathsf{D}_{\mathscr{V}} \circ \mathsf{G}$ is a quotient ∂-functor. Let G: $\mathscr{U} \to \mathscr{V}$ and F: $\mathscr{V} \to \mathscr{U}$ be contravariant ∂-functors. We call $\{\mathsf{G}, \mathsf{F}\}$ a right adjoint pair if $\mathsf{F} \circ \mathsf{D}_{\mathscr{V}}$ is the right adjoint of $\mathsf{D}_{\mathscr{V}} \circ \mathsf{G}$. In other words, there is a functorial isomorphism $\mathrm{Hom}_{\mathscr{U}}(X, \mathsf{F}Y) \cong \mathrm{Hom}_{\mathscr{V}}(Y, \mathsf{G}X)$ for all $\mathrm{X} \in \mathscr{U}$ and $Y \in \mathscr{V}$. We call $\{\mathscr{V}; \mathsf{G}, \mathsf{F}\}$ a localization duality of $\mathscr{U}$ provided that $\{\mathscr{V}^{\mathrm{op}}; \mathsf{D}_{\mathscr{V}} \circ \mathsf{G}, \mathsf{F} \circ \mathsf{D}_{\mathscr{V}}\}$ is a localization of $\mathscr{U}$. In other words, $\{\mathsf{G}, \mathsf{F}\}$ is a right adjoint pair, and the natural morphism $\mathrm{id}_{\mathscr{V}} \to \mathsf{G} \circ \mathsf{F}$ is an isomorphism. According to [Mc1, Proposition 2.3], if a quotient duality ∂-functor G: $\mathscr{U} \to \mathscr{V}$ has a contravariant ∂-functor F: $\mathscr{V} \to \mathscr{U}$ such that $\{\mathsf{G}, \mathsf{F}\}$ is a right adjoint pair, then $\{\mathscr{V}; \mathsf{G}, \mathsf{F}\}$ is a localization duality of $\mathscr{U}$. Also, by [Mc1, Theorem 2.5], ∂-functor G: $\mathscr{U} \to \mathscr{V}$ is a quotient duality if $\{\mathscr{V}; \mathsf{G}, \mathsf{F}\}$ is a localization duality of $\mathscr{U}$. By the above, if $\mathsf{G}\colon \mathscr{U} \to \mathscr{V}$ is a quotient duality, there is an epimorphism $\mathrm{Grot}(\mathscr{U}) \to \mathrm{Grot}(\mathscr{V})$.

Let $\mathscr{A}$ be an additive category, $K(\mathscr{A})$ a homotopy category of $\mathscr{A}$, and $K^+(\mathscr{A})$, $K^-(\mathscr{A})$, and $K^b(\mathscr{A})$ full subcategories of $K(\mathscr{A})$ generated by the bounded below complexes, the bounded above complexes, and the bounded complexes, respectively. For a full subcategory $\mathscr{B}$ of an abelian category $\mathscr{A}$, let $K^{*b}(\mathscr{B})$ be a full subcategory of $K^*(\mathscr{B})$ generated by complexes which have bounded homologies, and $K^*(\mathscr{B})_{\mathrm{Qis}}$ a quotient category of $K^*(\mathscr{B})$ by the multiplicative set of quasi isomorphisms, where $* = +$ or $-$. We denote $K^*(\mathscr{A})_{\mathrm{Qis}}$ by $D^*(\mathscr{A})$. For a thick abelian subcategory $\mathscr{C}$ of $\mathscr{A}$, we denote by $D^*_{\mathscr{C}}(\mathscr{A})$ a full subcategory of $D^*(\mathscr{A})$ generated by complexes of which all homologies belong to $\mathscr{C}$ (see [Hr] for details).

For a complex $X^{\cdot} := (X^i, d_i)$, we define the truncations

$$\sigma_{>n}(X^{\cdot})\colon \cdots \to 0 \to \mathrm{Im}\, d_n \to X^{n+1} \to X^{n+2} \to \cdots,$$
$$\sigma_{\le n}(X^{\cdot})\colon \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Ker}\, d_n \to 0 \to \cdots,$$
$$\sigma'_{\le n}(X^{\cdot})\colon \cdots \to X^{n-1} \to X^{n} \to \mathrm{Im}\, d_n \to 0 \to \cdots,$$
$$\tau_{>n}(X^{\cdot})\colon \cdots \to 0 \to X^{n+1} \to X^{n+2} \to \cdots,$$
$$\tau_{\le n}(X^{\cdot})\colon \cdots \to X^{n-1} \to X^{n} \to 0 \to \cdots.$$

For $m \le n$, we denote by $K^{[m,n]}(\mathscr{B})$ the full subcategory of $K(\mathscr{B})$ generated by complexes of the form $\cdots \to 0 \to X^m \to \cdots \to X^{n-1} \to X^n \to 0 \to \cdots$, and denote by $D^{[m,n]}(\mathscr{A})$ the full subcategory of $D(\mathscr{A})$ generated by complexes of which homology $H^i = 0$ ($i < m$ or $n < i$).

Let $\mathsf{F}\colon \mathscr{A} \to \mathscr{B}$ be a contravariant left exact additive functor between abelian categories. If $\mathscr{A}$ has enough projectives, and F has finite right homological dimension on $\mathscr{A}$, then $\mathbf{R}\mathsf{F}$, $\mathbf{R}^-\mathsf{F}$, and $\mathbf{R}^b\mathsf{F}$ exist, $\mathbf{R}\mathsf{F}|_{D^{\#}(\mathscr{A})} \cong \mathbf{R}^*\mathsf{F}$, and moreover, $\mathbf{R}^*\mathsf{F}$ has image in $D^{\#}(\mathscr{B})$, where $(*, \#) = (+, -)$, $(-, +)$, or (b, b) (see [Hr] for details).

2. LOCALIZATION DUALITY OF DERIVED CATEGORIES

For a ring A, we denote by $\mathrm{Mod}\, A$ (resp., A-Mod) the category of right (resp., left) A-modules, and denote by $\mathrm{mod}\, A$ (resp., A-mod) the category of finitely presented right (resp., left) A-modules. We denote by $\mathrm{Inj}\, A$ (resp., A-Inj) the category of injective right (resp., left) A-modules, and denote by $\mathscr{P}_A$ (resp., ${}_A\mathscr{P}$) the category of finitely generated projective right (resp., left) modules. If A is a right coherent ring, then $\mathrm{mod}\, A$ is a thick abelian subcategory of $\mathrm{Mod}\, A$, and then $D^*(\mathrm{mod}\, A)$ is equivalent to

$K^{-,*}(\mathscr{P}_A)$. Moreover, $D^*(\text{mod } A)$ is equivalent to $D^*_{\text{mod } A}(\text{Mod } A)$, where $* = -$ or b (see [Hr]).

For a right A-module U_A over a ring A, we denote by $\text{add}\, U_A$ (resp., $\text{sum} U_A$) the category of right A-modules which are direct summands of finite direct sums of copies of U_A (resp., finite direct sums of copies of U_A), and denote by $\text{rac}(U_A)$ the full subcategory of mod A generated by modules M which satisfy $\text{Ext}^i_A(M, U) = 0$ for all $i > 0$. We denote injective dimension of U_A (resp., projective dimension of U_A) by $\text{idim}\, U_A$ (resp., $\text{pdim} U_A$).

Let A and B be rings, ${}_BU_A$ a $(B - A)$-bimodule. We will call ${}_BU_A$ a cotilting $(B - A)$-bimodule provided that it satisfies the following:

(C1) ${}_BU_A$ is finitely presented as both a right A-module and a left B-module;

(C2r) $\text{idim}\, U_A < \infty$; (C2$l$) $\text{idim}_B U < \infty$;

(C3r) $\text{Ext}^i_A(U, U) = 0$ for all $i > 0$; (C3l) $\text{Ext}^i_B(U, U) = 0$ for all $i > 0$;

(C4r) the natural ring morphism $B \to \text{Hom}_A(U, U)$ is an isomorphism;

(C4l) the natural ring morphism $A^{\text{op}} \to \text{Hom}_B(U, U)$ is an isomorphism.

In the case $B = A$, we will call a cotilting $(A - A)$-bimodule a dualizing A-bimodule.

LEMMA 2.1. *Let A and B be rings, ${}_BU_A$ a $(B - A)$-bimodule. Then* $\{\text{Hom}_A(-, {}_BU_A)\colon \text{Mod } A \to B\text{-Mod}, \text{Hom}_B(-, {}_BU_A)\colon B\text{-Mod} \to \text{Mod } A\}$ *is a right adjoint pair.*

LEMMA 2.2. *Let U_A be a right A-module over a right coherent ring A which satisfies the conditions* (C1), (C2r), (C3r). *For $X^{\cdot} \in K^+(\text{rac}(U_A))$ and $Y^{\cdot} \in K^+(\text{add}\, U_A)$, we have the functorial isomorphism* $\text{Hom}_{D(\text{mod } A)}(X^{\cdot}, Y^{\cdot}) \cong \text{Hom}_{K(\text{mod } A)}(X^{\cdot}, Y^{\cdot})$.

Proof. By [Hr, Chap. I, Proposition 3.3, Corollary 5.3], the natural functor $K^+(\text{rac}(U_A))_{\text{Qis}} \to D^+(\text{mod } A)$ is equivalent. Let $K^{+,\phi}(\text{rac}(U_A))$ be the full subcategory of $K^+(\text{rac}(U_A))$ consisting of complexes of which all homologies are null. Then $K^+(\text{rac}(U_A))_{\text{Qis}}$ is equal to $K^+(\text{rac}(U_A))/K^{+,\phi}(\text{rac}(U_A))$. According to [V, 5-3 Proposition] or [Mc1, Lemma 2.1], it suffices to show $\text{Hom}_{K^+(\text{rac}(U_A))}(K^{+,\phi}(\text{rac}(U_A)), K^+(\text{add}\, U_A)) = 0$. Let B be an endomorphism ring $\text{End}_A(U_A)$, and let $X^{\cdot}$ be an acyclic complex in

$K^+(\mathrm{rac}(U_A))$. Then $\mathrm{Hom}_A(X^{\cdot}, {}_BU_A)$ is an acyclic complex. For $Y^{\cdot} \in K^+(\mathrm{add}\, U_A)$, $\mathrm{Hom}_A(Y^{\cdot}, {}_BU_A)$ belongs to $K^-({}_B\mathscr{P})$. Hence we get

$$\begin{aligned}\mathrm{Hom}_{K(\mathrm{mod}\, A)}(X^{\cdot}, Y^{\cdot}) &\cong \mathrm{Hom}_{K(B\text{-}\mathrm{Mod})}(\mathrm{Hom}_A(Y^{\cdot}, {}_BU_A), \mathrm{Hom}_A(X^{\cdot}, {}_BU_A)) \\ &= 0.\end{aligned}$$

LEMMA 2.3. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C2r), (C3r). *Then* $\mathbf{R}^+\mathrm{Hom}_A(-, {}_BU_A)$: $D^+(\mathrm{mod}\, A) \to D^-(B\text{-}\mathrm{mod})$ *and* $\mathbf{R}^-\mathrm{Hom}_B(-, {}_BU_A)$: $D^-(B\text{-}\mathrm{mod}) \to D^+(\mathrm{mod}\, A)$ *exist, and* $\{\mathbf{R}^+\mathrm{Hom}_A(-, {}_BU_A), \mathbf{R}^-\mathrm{Hom}_B(-, {}_BU_A)\}$ *is a right adjoint pair.*

Proof. It is easy to see existence. For $X^{\cdot} \in D^+(\mathrm{mod}\, A)$ and $Y^{\cdot} \in D^-(B\text{-}\mathrm{mod})$, there exist $M^{\cdot} \in K^+(\mathrm{rac}(U_A))$ and $P^{\cdot} \in K^-({}_B\mathscr{P})$ such that $M^{\cdot} \cong X^{\cdot}$ in $D^+(\mathrm{mod}\, A)$ and $P^{\cdot} \simeq Y^{\cdot}$ in $D^-(B\text{-}\mathrm{mod})$. By Lemmas 2.1 and 2.2, we get the isomorphisms

$$\begin{aligned}&\mathrm{Hom}_{D(\mathrm{mod}\, A)}(X^{\cdot}, \mathbf{R}^-\mathrm{Hom}_B(Y^{\cdot}, {}_BU_A)) \\ &\quad\cong \mathrm{Hom}_{D(\mathrm{mod}\, A)}(M^{\cdot}, \mathrm{Hom}_B(P^{\cdot}, {}_BU_A)) \\ &\quad\cong \mathrm{Hom}_{K(\mathrm{mod}\, A)}(M^{\cdot}, \mathrm{Hom}_B(P^{\cdot}, {}_BU_A)) \\ &\quad\cong \mathrm{Hom}_{K(B\text{-}\mathrm{mod})}(P^{\cdot}, \mathrm{Hom}_A(M^{\cdot}, {}_BU_A)) \\ &\quad\cong \mathrm{Hom}_{D(B\text{-}\mathrm{mod})}(Y^{\cdot}, \mathbf{R}^+\mathrm{Hom}_A(X^{\cdot}, {}_BU_A)).\end{aligned}$$

PROPOSITION 2.4. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C2r), (C3r), *and* (C4r). *Then* $\{D^-(B\text{-}\mathrm{mod}); \mathbf{R}^+\mathrm{Hom}_A(-, {}_BU_A), \mathbf{R}^-\mathrm{Hom}_B(-, {}_BU_A)\}$ *is a localization duality of* $D^+(\mathrm{mod}\, A)$, *and the image of* $\mathbf{R}^+\mathrm{Hom}_A(-, {}_BU_A)|_{D^b(\mathrm{mod}\, A)}$ *is contained in* $D^b(B\text{-}\mathrm{mod})$.

Proof. By the condition (C2r), it is easy to see that the image of $\mathbf{R}^+\mathrm{Hom}_A(-, {}_BU_A)|_{D^b(\mathrm{mod}\, A)}$ is contained in $D^b(B\text{-}\mathrm{mod})$. According to Lemma 2.3, $\{\mathbf{R}^+\mathrm{Hom}_A(-, {}_BU_A), \mathbf{R}^-\mathrm{Hom}_B(-, {}_BU_A)\}$ is a right adjoint pair. For $X^{\cdot} \in D^-(B\text{-}\mathrm{mod})$, there is a complex $P^{\cdot} \in K^-({}_B\mathscr{P})$ such that $X^{\cdot}$ is isomorphic in $P^{\cdot}$ in $D^-(B\text{-}\mathrm{mod})$, and we have $\mathbf{R}^-\mathrm{Hom}_B(P^{\cdot}, {}_BU_A) \cong \mathrm{Hom}_B(P^{\cdot}, {}_BU_A)$ in $D^-(B\text{-}\mathrm{mod})$. Since $\mathrm{Hom}_B(P^{\cdot}, {}_BU_A) \in K^+(\mathrm{add}\, U_A)$, and $K^+(\mathrm{add}\, U_A)$ is contained in $K^+(\mathrm{rac}(U_A))$, we have $\mathbf{R}^+\mathrm{Hom}_A(\mathbf{R}^-\mathrm{Hom}_B(P^{\cdot}, {}_BU_A), {}_BU_A)$ is isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_B(P^{\cdot}, {}_BU_A), {}_BU_A)$ in $D^-(B\text{-}\mathrm{mod})$. The condition (C4$r$) implies that the canonical morphism $P^{\cdot} \to \mathrm{Hom}_A(\mathrm{Hom}_B(P^{\cdot}, {}_BU_A), {}_BU_A)$ is an isomorphism, and then the natural morphism $P^{\cdot} \to \mathbf{R}^{+}\operatorname{Hom}_A(\mathbf{R}^{-}\operatorname{Hom}_B(P^{\cdot}, {}_BU_A), {}_BU_A)$ is an isomorphism in $D^{-}(B\text{-mod})$. Hence the natural morphism

$$\mathrm{id}_{D^{-}(B\text{-mod})} \to \mathbf{R}^{+}\operatorname{Hom}_A\big(\mathbf{R}^{-}\operatorname{Hom}_B(-, {}_BU_A), {}_BU_A\big)$$

is an isomorphism.

COROLLARY 2.5. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C2r), (C2l), (C3r), *and* (C4r). *Then* $\{D^b(B\text{-mod}); \mathbf{R}^b\operatorname{Hom}_A(-, {}_BU_A), \mathbf{R}^b\operatorname{Hom}_B(-, {}_BU_A)\}$ *is a localization duality of* $D^b(\operatorname{mod} A)$.

Proof. By the condition (C2l), it is easy to see that the image of $\mathbf{R}^b\operatorname{Hom}_B(-, {}_BU_A)$ is contained in $D^b(\operatorname{mod} A)$. We are done by Proposition 2.4.

We will call an A-module M an endo−artinian module provided that M is Artinian as an $\operatorname{End}_A(M)$-module. For a finitely generated A-module M, let $n(M)$ be the number of non-isomorphic indecomposable modules which are direct summands of M.

COROLLARY 2.6. *Let A be a right Artinian ring, and U_A a finitely generated endo−artinian right A-module which satisfies the conditions* (C2r) *and* (C3r). *Assume that injective dimension of ${}_BU$ is finite, where* $B := \operatorname{End}_A(U)$. *Then we have $n(U_A) \le n(A_A)$.*

Proof. By Proposition 2.5, $\{D^b(B\text{-mod}); \mathbf{R}^b\operatorname{Hom}_A(-, {}_BU_A), \mathbf{R}^b\operatorname{Hom}_B(-, {}_BU_A)\}$ is a localization duality of $D^b(\operatorname{mod} A)$. According to Section 1, we have a surjection $\operatorname{Grot}(D^b(\operatorname{mod} A)) \to \operatorname{Grot}(D^b(B\text{-mod}))$. Since U_A is a finitely generated endo−artinian right A-module, B is an Artinian ring. Then we have $\operatorname{Grot}(D^b(\operatorname{mod} A)) \cong \mathbf{Z}^{n(A)}$ and $\operatorname{Grot}(D^b(B\text{-mod})) \cong \mathbf{Z}^{n(B)}$, and hence, we have $n(U_A) \le n(A_A)$.

As in [Mc1], we have the following lemma, testing whether $\mathsf{G}\colon \mathscr{U} \to \mathscr{V}$ is a quotient duality.

LEMMA 2.7. *Let $\mathsf{G}\colon \mathscr{U} \to \mathscr{V}$ be a contravariant ∂-functor between triangulated categories. Assume there exists a family $\mathscr{T}$ of objects in $\mathscr{U}$ satisfying the following conditions*:

(a) *For every $N \in \mathscr{V}$, there exists an object $M \in \mathscr{U}$ such that $N \cong \mathsf{G}M$;*

(b) *Given $X, Y \in \mathscr{U}$, for all $f \in \operatorname{Hom}_{\mathscr{V}}(\mathsf{G}X, \mathsf{G}Y)$, there exist distinguished triangles*

$$X \xrightarrow{s_X} T_X \to Z \to \qquad \textit{and} \qquad Y \xrightarrow{s_Y} T_Y \to Z' \to ,$$

where $T_X, T_Y \in \mathscr{T}$ *and* $Z, Z' \in \operatorname{Ker} \mathsf{G}$, *and* $f' \in \operatorname{Hom}_{\mathscr{U}}(T_Y, T_X)$ *such that* $f \circ \mathsf{G}s_X = \mathsf{G}s_Y \circ \mathsf{G}f'$. *Then* $\mathsf{G}\colon \mathscr{U} \to \mathscr{V}$ *is a quotient duality.*

THEOREM 2.8. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C3r). *If injective dimension of U_A is at most one, then* $\mathbf{R}^b \operatorname{Hom}_A(-, {}_BU_A)\colon D^b(\operatorname{mod} A) \to D^b(B\text{-mod})$ *is a quotient duality.*

Proof. Let $\mathsf{F} \coloneqq \operatorname{Hom}_A(-, {}_BU_A)$. Let $\mathscr{T}$ be the family of complexes $X^{\cdot}$: $\cdots \to 0 \to X^m \to \cdots \to X^{n-1} \to X^n \to 0 \to \cdots$ (for all $m \le n$) $\in K^b(\operatorname{mod} A)$, where $X^n \in \operatorname{rac}(U_A)$ and $X^i \in \operatorname{add} U_A$ $(m \le i < n)$. It suffices to show that $\mathscr{T}$ satisfies the conditions of Lemma 2.7. By assumption, if M is a finitely presented A-submodule of some module which belongs to $\operatorname{rac}(U_A)$, then M belongs to $\operatorname{rac}(U_A)$. Since $D^b(B\text{-mod})$ is equivalent to $K^{-,b}({}_B\mathscr{P})$, given $N^{\cdot} \in D^{[m,n]}(B\text{-mod})$, there exists a complex $U^{\cdot} \in K^+(\operatorname{add} U_A)$ such that $\mathsf{F}U^{\cdot}$ is isomorphic to $N^{\cdot}$ in $D^b(B\text{-mod})$. Furthermore, we have $\mathsf{F}(\sigma'_{\le -t} U^{\cdot}) \cong \sigma_{>t}\mathsf{F}U^{\cdot} \cong \mathsf{F}U^{\cdot}$ in $D^b(B\text{-mod})$ $(t < m-1)$. Since $\operatorname{Im} d_{1-m}$ is finitely cogenerated by U_A, $\operatorname{Im} d_{1-m} \in \operatorname{rac}(U_A)$, and then, $\sigma'_{\le -t} U^{\cdot}$ belongs to $\mathscr{T}$. Hence the condition (a) of Lemma 2.7 is satisfied. Given $X^{\cdot}$ and $Y^{\cdot} \in D^{[m,n]}(\operatorname{mod} A)$, there exist $P^{\cdot}_X$ and $P^{\cdot}_Y \in K^{[m-1,n]}(\operatorname{rac}(U_A))$ such that $P^{\cdot}_X \cong X^{\cdot}$ and $P^{\cdot}_Y \cong Y^{\cdot}$ in $D^b(\operatorname{mod} A)$. For $\mathsf{F}P^{\cdot}_X$ and $\mathsf{F}P^{\cdot}_Y$, there exist $U^{\cdot}_X$ and $U^{\cdot}_Y \in K^+(\operatorname{add} U_A)$, which have $t_X\colon P^{\cdot}_X \to U^{\cdot}_X$ and $t_Y\colon P^{\cdot}_Y \to U^{\cdot}_Y$ in $K^+(\operatorname{mod} A)$, such that $\mathsf{F}t_X$ and $\mathsf{F}t_Y$ are isomorphisms in $D^-(B\text{-mod})$. As above, for all $t \ge n+1$, we can take $s_X\colon P^{\cdot}_X \to \sigma'_{\le t} U^{\cdot}_X$ and $s_Y\colon P^{\cdot}_Y \to \sigma'_{\le t} U^{\cdot}_Y$ such that $t_X = (\sigma'_{\le t} U^{\cdot} \to U^{\cdot}_X) \circ s_X$ and $t_Y = (\sigma'_{\le t} U^{\cdot}_Y \to U^{\cdot}_Y) \circ s_Y$, and that $\mathsf{F}s_X$ and $\mathsf{F}s_Y$ are isomorphisms in $D^b(B\text{-mod})$. Then mapping cones of s_X and s_Y belong to $\operatorname{Ker} \mathbf{R}^b\mathsf{F}$. Since $\mathsf{F}U^{\cdot}_Y$ belongs to $D^{[-n,-m+1]}(B\text{-mod})$, $\sigma_{\le t}\mathsf{F}U^{\cdot}_Y$ is acyclic for all $t \ge n+1$. Then we have the isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{D(B\text{-mod})}(\mathbf{R}^b\mathsf{F}X^{\cdot}, \mathbf{R}^b\mathsf{F}Y^{\cdot}) &\cong \operatorname{Hom}_{D(B\text{-mod})}(\mathsf{F}U^{\cdot}_X, \mathsf{F}U^{\cdot}_Y) \\
&\cong \operatorname{Hom}_{K(B\text{-mod})}(\mathsf{F}U^{\cdot}_X, \mathsf{F}U^{\cdot}_Y) \\
&\cong \operatorname{Hom}_{K(B\text{-mod})}(\mathsf{F}U^{\cdot}_X, \sigma_{>-t}\mathsf{F}U^{\cdot}_Y) \\
&\cong \operatorname{Hom}_{K(B\text{-mod})}(\sigma_{>-t-1}\mathsf{F}U^{\cdot}_X, \sigma_{>-t}\mathsf{F}U^{\cdot}_Y) \\
&\cong \operatorname{Hom}_{K(B\text{-mod})}(\mathsf{F}\sigma'_{\le t+1}U^{\cdot}_X, \mathsf{F}\sigma'_{\le t}U^{\cdot}_Y) \\
&\cong \operatorname{Hom}_{K(\operatorname{mod} A)}(\sigma'_{\le t}U^{\cdot}_Y, U^{\cdot}_Y, \sigma'_{\le t+1}U^{\cdot}_X).
\end{aligned}$$

Hence the condition (b) of Lemma 2.7 is satisfied.

COROLLARY 2.9. *Let A be a right Artinian ring, and U_A a finitely generated endo–artinian right A-module which satisfies the condition* (C3r). *If the injective dimension of U_A is at most one, then we have* $n(U_A) \le n(A_A)$.

Proof. The same as Corollary 2.6.

For a right A-module U_A, we denote by $\operatorname{coresol}(U_A)$ the full subcategory of $\operatorname{mod} A$ generated by modules M which have an exact sequence $0 \to M \to U^0 \to U^1 \to \cdots \to U^i \to \cdots$, where $U^i \in \operatorname{add} U_A$ $(i \geq 0)$.

THEOREM 2.10. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C2r), (C3r), *and* (C4r). *Then the following are equivalent*:

(a) $\mathbf{R}^+\operatorname{Hom}_A(-, {}_BU_A)$: $D^+(\operatorname{mod} A) \to D^-(B\text{-mod})$ *is a duality*;

(b) $\operatorname{rac}(U_A)$ *coincides with* $\operatorname{coresol}(U_A)$;

(c) *for every $X \in \operatorname{rac}(U_A)$, there is an exact sequence* $0 \to X \to V \to W \to 0$ *in* $\operatorname{mod} A$, *with $V \in \operatorname{add} U_A$ and $W \in \operatorname{rac}(U_A)$.*

In this case, ${}_BU_A$ satisfies the conditions (C3l) *and* (C4l).

Proof. (a) $\Rightarrow$ (b) Let M be a right A-module which belongs to $\operatorname{rac}(U_A)$. Then $\mathbf{R}^+\operatorname{Hom}_A(M, {}_BU_A)$ is isomorphic to a left B-module $\operatorname{Hom}_A(M, {}_BU_A)$ in $D^-(B\text{-mod})$. We take a projective resolution $P^{\cdot}$ of $\operatorname{Hom}_A(M, {}_BU_A)$. Then we have the exact sequence

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to \operatorname{Hom}_A(M, {}_BU_A) \to 0,$$

where $P^i \in {}_B\mathscr{P}$ $(i \leq 0)$. According to Lemma 2.3, a quasi inverse of $\mathbf{R}^+\operatorname{Hom}_A(-, {}_BU_A)$ is $\mathbf{R}^-\operatorname{Hom}_B(-, {}_BU_A)$. Then $\operatorname{Hom}_B(P^{\cdot}, {}_BU_A)$ is isomorphic to M in $D^+(\operatorname{mod} A)$. Therefore, we have the exact sequence

$$\begin{aligned} 0 \to M \to \operatorname{Hom}_B\left(P^0, {}_BU_A\right) &\to \operatorname{Hom}_B\left(P^{-1}, {}_BU_A\right) \\ &\to \operatorname{Hom}_B\left(P^{-2}, {}_BU_A\right) \to \cdots . \end{aligned}$$

It is easy to see that $\operatorname{Hom}_B(P^i, {}_BU_A) \in \operatorname{add} U_A$.

(b) $\Rightarrow$ (a) Let η: $\operatorname{id}_{D^+(\operatorname{mod} A)} \to \mathbf{R}^-\operatorname{Hom}_B(\mathbf{R}^+\operatorname{Hom}_A(-, {}_BU_A), {}_BU_A)$ be the natural morphism of functors. Since $\operatorname{idim} U_A := n < \infty$, a derived functor $\mathbf{R}^-\operatorname{Hom}_B(\mathbf{R}^+\operatorname{Hom}_A(-, {}_BU_A), {}_BU_A)$ is a way-out right ∂-functor. According to [Hr, Chap. I, Proposition 7.1], it suffices to show that $\eta(M)$ is an isomorphism for all $M \in \operatorname{mod} A$. Let $\cdots \to P^{-2} \xrightarrow{f_{-2}} P^{-1} \xrightarrow{f_{-1}} P^0 \to M \to 0$ be a finitely generated projective resolution of M, and $M_i := \operatorname{Im} f_i$. Then M_{-n} belongs to $\operatorname{rac}(U_A)$, and the complex $M_{-n} \to P^{-n+1} \to \cdots \to P^{-2} \xrightarrow{f_{-2}} P^{-1} \xrightarrow{f_{-1}} P^0$ is isomorphic to M in $D^+(\operatorname{mod} A)$. Therefore it suffices to show that $\eta(X^{\cdot})$ is an isomorphism for all $X^{\cdot} \in K^b(\operatorname{rac}(U_A))$. Let $X^{\cdot}$ be a complex $X^0 \to X^1 \to \cdots \to X^k \in K^b(\operatorname{rac}(U_A))$;

then we have a distinguished triangle $X^k[-k] \to X^{\cdot} \to \tau_{\leq k-1}X^{\cdot} \to$ in $D^+(\text{mod } A)$. By assumption, we have the exact sequence

$$0 \to X^k \to U^0 \to U^1 \to \cdots \to U^j \to \cdots,$$

where $U^j \in \text{add}\, U_A$ $(j \geq 0)$. Let $U^{\cdot}$ be a complex $U^0 \to U^1 \to \cdots \to U^j \to \cdots$; then $U^{\cdot} \in K^+(\text{add}\, U_A)$ such that X^k is isomorphic to $U^{\cdot}$ in $D^+(\text{mod } A)$. According to the condition (C4r), $U^{\cdot} \to \text{Hom}_B(\text{Hom}_A(U^{\cdot}, {}_BU_A), {}_BU_A)$ is an isomorphism in $D^+(\text{mod } A)$, and hence $\eta(X^k[-k])$ is an isomorphism in $D^+(\text{mod } A)$. We have the following morphism between distinguished triangles,

$$\begin{array}{ccccccc} X^k[-k] & \longrightarrow & X^{\cdot} & \longrightarrow & \tau_{\leq k-1}X^{\cdot} & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ \mathbf{R}^-\mathsf{G}\mathbf{R}^+\mathsf{F}(X^k[-k]) & \longrightarrow & \mathbf{R}^-\mathsf{G}\mathbf{R}^+F(X^{\cdot}) & \longrightarrow & \mathbf{R}^-\mathsf{G}\mathbf{R}^+\mathsf{F}(\tau_{\leq k-1}X^{\cdot}) & \longrightarrow \end{array}$$

where $F := \text{Hom}_A(-, {}_BU_A)$ and $\mathsf{G} := \text{Hom}_B(-, {}_BU_A)$. By induction on k, $\eta(\tau_{\leq k-1}X^{\cdot})$ is an isomorphism, and hence $\eta(X^{\cdot})$ is an isomorphism. We are done by Proposition 2.4.

(b) $\Leftrightarrow$ (c) By idim $U_A < \infty$, it is easy.

By the duality, it is easy to see that ${}_BU_A$ satisfies the condition (C3l), and we have ring-isomorphisms $A^{\text{op}} \cong \text{End}_A(A_A)^{\text{op}} \cong \text{End}_B(\text{Hom}_A(A, {}_BU_A)) \cong \text{End}_B({}_BU)$.

Remark. The condition (c) of Theorem 2.10 is an important condition of abelian categories with suitable subcategories of "maximal Cohen–Macaulay objects" in [AB]. Theorem 2.10 says that the condition (c) is indispensable for duality theory.

COROLLARY 2.11. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a cotilting $(B-A)$-bimodule. Then* $\mathbf{R}^*\text{Hom}_A(-, {}_BU_A)\colon D^*(\text{mod } A) \to D^{\#}(B\text{-mod})$ *is a duality, where* $(^*, ^{\#}) = (\textit{nothing}, \textit{nothing})$, $(+, -)$, $(-, +)$, *or* (b, b).

Proof. By symmetry, according to Corollary 2.5, $\mathbf{R}^b\text{Hom}_A(-, {}_BU_A)\colon D^b(\text{mod } A) \to D^b(B\text{-mod})$ is a duality with a quasi inverse $\mathbf{R}^b\text{Hom}_B(-, {}_BU_A)$. Let M be a finitely presented right A-module which belongs to $\text{rac}(U_A)$. Then we have $\mathbf{R}^b\text{Hom}_A(M, {}_BU_A) \cong \text{Hom}_A(M, {}_BU_A)$ in $D^b(B\text{-mod})$. We take a projective resolution $P^{\cdot}$ of $\text{Hom}_A(M, {}_BU_A)$ in B-mod. By the duality, $\text{Hom}_B(P^{\cdot}, {}_BU_A)$ is a coresolution of M. The condition (C4r) implies that M belongs to $\text{coresol}(U_A)$. Therefore, $\text{rac}(U_A)$ is contained in $\text{coresol}(U_A)$. Conversely, the condition (C2r) implies that $\text{coresol}(U_A)$ is contained in $\text{rac}(U_A)$. Hence the condition (b) of Theorem 2.10 is satisfied. By symmetry, we have $\mathbf{R}^*\text{Hom}_A(-, {}_BU_A)\colon D^*(\text{mod } A) \to D^{\#}(B\text{-mod})$ is a duality, where $(^*, ^{\#}) = (+, -)$ or $(-, +)$. For every $X^{\cdot} \in D(\text{mod } A)$, there exists a distinguished triangle $\sigma_{\le 0}X^{\cdot} \to X^{\cdot} \to \sigma_{>0}X^{\cdot} \to$. Then we have the following morphism between distinguished triangles in $D(\mathrm{mod}\, A)$,

$$\begin{array}{ccccccc} \sigma_{\le 0}X^{\cdot} & \to & X^{\cdot} & \to & \sigma_{>0}X^{\cdot} & \to \\ \downarrow & & \downarrow & & \downarrow & \\ \mathbf{R}\mathsf{G}\mathbf{R}\mathsf{F}(\sigma_{\le 0}X^{\cdot}) & \to & \mathbf{R}\mathsf{G}\mathbf{R}\mathsf{F}(X^{\cdot}) & \to & \mathbf{R}\mathsf{G}\mathbf{R}\mathsf{F}(\sigma_{>0}X^{\cdot}) & \to \end{array}$$

where $\mathsf{F} := \mathrm{Hom}_A(-, {}_BU_A)$ and $\mathsf{G} := \mathrm{Hom}_B(-, {}_BU_A)$. Since $\sigma_{\le 0}X^{\cdot} \in D^-(\mathrm{mod}\, A)$ and $\sigma_{>0}X^{\cdot} \in D^+(\mathrm{mod}\, A)$, the natural morphism $X^{\cdot} \to \mathbf{R}\,\mathrm{Hom}_B(\mathbf{R}\,\mathrm{Hom}_A(X^{\cdot}, {}_BU_A), {}_BU_A)$ is an isomorphism. By the symmetry, we complete the proof.

EXAMPLES. (1) For a non-commutative ring A, in the case of even dualizing modules, there exist many dualizing A-bimodules. Let A be a connected finite dimensional hereditary k-algebra over an algebraically closed field k of infinite representation type. For $M \in \mathrm{mod}\, A$, we define $\mathrm{Tr}\, M := \mathrm{Cok}(\mathrm{Hom}_A(f, A))$, where $P_1 \to^f P_0 \to M \to 0$ is a minimal projective presentation of M, and $D := \mathrm{Hom}_k(-, k)$. Then $(\mathrm{Tr}\, D)^n(A)$ is a dualizing A-bimodule for all $n \ge 0$.

(2) More generally, let A and B be finite dimensional k-algebras over a field k, ${}_AT_B$ a tilting $(A-B)$-bimodule of finite projective dimension. Then $\mathsf{D}({}_AT_B)$ is a cotilting $(B-A)$-bimodule.

(3) Let A be a ring $\left(\begin{smallmatrix} F & V \\ 0 & G \end{smallmatrix}\right)$, where F, G are division rings, and V is an $(F-G)$-bimodule such that $\dim_F V = \dim V_G = \infty$. Then A is a coherent ring and also a dualizing A-bimodule.

Let R be a commutative Cohen−Macaulay ring with a dualizing R-module ω. A finitely generated R-module M is called a maximal Cohen−Macaulay R-module if $\mathrm{depth}_{\mathfrak{p}R_{\mathfrak{p}}} M_{\mathfrak{p}}$ is equal to the Krull dimension of $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$, or equivalently if $\mathrm{Ext}^i_R(M, \omega) = 0$ for all $i > 0$ (see [AB]).

PROPOSITION 2.12. *Let R be a commutative Cohen−Macaulay ring with a dualizing R-module ω. If A is an R-algebra which is finitely generated maximal Cohen−Macaulay as an R-module, then* $\mathrm{Hom}_R(A, \omega)$ *is a dualizing A-bimodule.*

Proof. It is clear that $\mathrm{Hom}_R(A, \omega)$ is finitely generated as both a right and a left A-module. We take an injective coresolution $E^{\cdot}$ of ω. Since A is a maximal Cohen−Macaulay R-module, we have an injective coresolution $\mathrm{Hom}_R(A, E^{\cdot})$ of $\mathrm{Hom}_R(A, \omega)$ as both a right and a left A-module. Then $\mathrm{idim}_A \mathrm{Hom}_R(A, \omega)$ and $\mathrm{idim}\, \mathrm{Hom}_R(A, \omega)_A$ are finite. In order that $\mathrm{Hom}_R(A, \omega)$ satisfy the conditions (C3r), (C4r), it suffices to show that the natural morphism $A_A \to \mathbf{R}^b \mathrm{Hom}_A(\mathbf{R}^b \mathrm{Hom}_A(A_A, \mathrm{Hom}_R(A, \omega)),$

$\mathrm{Hom}_R(A, \omega))$ is an isomorphism in $D^b(\mathrm{mod}\, A)$. We have the isomorphisms

$$\begin{aligned}
&\mathbf{R}^b \mathrm{Hom}_A\big(\mathbf{R}^b \mathrm{Hom}_A(A_A, \mathrm{Hom}_R(A, \omega)), \mathrm{Hom}_R(A, \omega)\big) \\
&\quad \cong \mathbf{R}^b \mathrm{Hom}_A(\mathrm{Hom}_R(A, \omega), \mathrm{Hom}_R(A, \omega)) \\
&\quad \cong \mathrm{Hom}_A(\mathrm{Hom}_R(A, \omega), \mathrm{Hom}_R(A, E^{\cdot})) \\
&\quad \cong \mathrm{Hom}_R(\mathrm{Hom}_R(A_A, \omega), E^{\cdot}) \\
&\quad \cong \mathbf{R}^b \mathrm{Hom}_R\big(\mathbf{R}^b \mathrm{Hom}_R(A, \omega), \omega\big) \\
&\quad \cong A_A .
\end{aligned}$$

Similarly, $\mathrm{Hom}_R(A, \omega)$ satisfies the conditions (C3l), (C4l).

Remark. More generally, for dualizing bimodule complexes, we can drop the condition of maximal Cohen−Macaulay in Proposition 2.12 (see [Mc2] for details).

3. APPLICATIONS TO APPROXIMATIONS

Auslander and Buchweitz introduced the notion of Cohen−Macaulay approximations in connection with Cohen−Macaulay rings with dualizing modules [AB]. In this section, from the point of view of derived categories, we approach this theory.

For a left B-module ${}_BU$, we denote by $\mathrm{fresol}({}_BU)$ the full subcategory of B-mod generated by modules M which have an exact sequence $0 \to U^{-n} \to \cdots \to U^{-1} \to U^0 \to M \to 0$ for some n, where $U^i \in \mathrm{add}_B U$ $(-n \le i \le 0)$.

LEMMA 3.1. *Let A be a right coherent ring, B a left cohernet ring, and ${}_BU_A$ a $(B-A)$-bimodule which satisfies the condition* (C1). *Assume that* $\mathbf{R}^b\mathrm{Hom}_A(-, {}_BU_A)$: $D^b(\mathrm{mod}\, A) \to D^b(B\text{-mod})$ *is a quotient duality. Then for every finitely presented left B-module C, there exist exact sequences* $0 \to Y_C \to X_C \to C \to 0$ *and* $0 \to C \to Y^C \to X^C \to 0$ *with* $X_C, X^C \in \mathrm{coresol}({}_BU)$, *and* $Y_C, Y^C \in \mathrm{fresol}({}_BU)$.

Proof. Since $D^b(\mathrm{mod}\, A)$ is equivalent to $K^{-,b}(\mathscr{P}_A)$, for every finitely presented left B-module C, there exists a complex $P^{\cdot}$ in $K^{-,b}(\mathscr{P}_A)$ such that $\mathrm{Hom}_A(P^{\cdot}, {}_BU_A)$ is isomorphic to C in $D^b(B\text{-mod})$. Then we have $\mathrm{H}^i\mathrm{Hom}_A(P^{\cdot}, {}_BU_A) = 0$ $(i \ne 0)$ and $\mathrm{H}^0\mathrm{Hom}_A(P^{\cdot}, {}_BU_A) \cong C$ in B-mod.

Clearly, $\mathrm{Hom}_A(P^{\cdot}, {}_BU_A)$ belongs to $K^+(\mathrm{add}_B U)$. Let $\mathrm{Hom}_A(P^{\cdot}, {}_BU_A)$ be the complex

$$0 \to U^{-s} \to \cdots \to U^{-1} \xrightarrow{d_{-1}} U^0 \xrightarrow{d_0} U^1 \to U^2 \to \cdots \to U^i \to \cdots,$$

where $U^i \in \mathrm{add}_B U$ $(-s \le i)$. Then we have the exact sequences

$$0 \to \mathrm{Im}\, d_{-1} \to \mathrm{Ker}\, d_0 \to C \to 0,$$
$$0 \to C \to \mathrm{Cok}\, d_{-1} \to \mathrm{Im}\, d_0 \to 0.$$

Furthermore, $\mathrm{Ker}\, d_0$ and $\mathrm{Im} d_0$ belong to $\mathrm{coresol}({}_BU)$, and $\mathrm{Cok}\, d_{-1}$ and $\mathrm{Im}\, d_{-1}$ belong to $\mathrm{fresol}({}_BU)$.

Remark. Applying the technique in Lemma 3.1 to the situation of Proposition 2.4, we also get a generalization of a result of Auslander and Buchweitz [AB, Theorem 1.8].

The situations of Corollary 2.5, Theorem 2.8, and Corollary 2.11 satisfy the conditions of Lemma 3.1. In particular, under the conditions of Corollary 2.11, mod A and B-mod are abelian categories with suitable subcategories of "maximal Cohen−Macaulay objects" (see [AB] for details). For a ring B, the stable category $B\text{-mod}/\mathrm{add}_B U$ has the same objects as B-mod, its homomorphisms are $\mathrm{Hom}_{B\text{-mod}/\mathrm{add}_B U}(X, Y) = \mathrm{Hom}_B(X, Y)/\{f\colon X \to Y \mid f$ factors through an object in $\mathrm{add}_B U\}$ for $X, Y \in B\text{-mod}/\mathrm{add}_B U$. Let $\Pi\colon B\text{-mod} \to B\text{-mod}/\mathrm{add}_B U$ be the natural functor. We get results similar to those of Auslander and Buchweitz [AB, Theorems A, B].

PROPOSITION 3.2. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a cotilting $(B-A)$-bimodule. For a finitely presented left B-module C, the following hold.*

(a) *there exist exact sequence* $0 \to Y_C \to X_C \to C \to 0$ *and* $0 \to C \to Y^C \to X^C \to 0$ *with* $X_C, X^C \in \mathrm{rac}({}_BU)$ *and* $Y_C, Y^C \in \mathrm{fresol}({}_BU)$.

(b) *For other exact sequences* $0 \to Y'_C \to X'_C \to C \to 0$ *and* $0 \to C \to Y^{C\prime} \to X^{C\prime} \to 0$ *which satisfy the condition* (a), *there exist morphisms between exact sequences,*

$$\begin{array}{ccccccccc} 0 \to & Y_C & \to & X_C & \to & C & \to 0 \\ & \downarrow\beta & & \downarrow\alpha & & \| & \\ 0 \to & Y'_C & \to & X'_C & \to & C & \to 0 \end{array} \qquad \begin{array}{ccccccccc} 0 \to & C & \to & Y^C & \to & X^C & \to 0 \\ & \| & & \downarrow\delta & & \downarrow\gamma & \\ 0 \to & C & \to & Y^{C\prime} & \to & X^{C\prime} & \to 0 \end{array}$$

such that $\Pi(\alpha)$, $\Pi(\beta)$, $\Pi(\gamma)$, *and* $\Pi(\delta)$ *are isomorphisms in* $B\text{-mod}/\mathrm{add}_B U$.

Proof. (a) By Theorem 2.10, Corollary 2.11, and Lemma 3.1, it is clear.

(b) For $0 \to Y_C \to X_C \to C \to 0$, we get the following resolution and coresolution,

$$0 \to U^{-n} \to \cdots \to U^{-2} \to U^{-1} \to Y_C \to 0,$$

$$0 \to X_C \to U^0 \to U^1 \to \cdots \to U^i \to \cdots,$$

where $U^i \in \operatorname{add} U_A$ for all i. Then we get the complex $U^{\cdot}$: $\cdots \to 0 \to U^{-n} \to \cdots \to U^{-2} \to U^{-1} \to U^0 \to U^1 \to \cdots \to U^i \to \cdots \in K^+(\operatorname{add}_B U)$. Similarly, for $0 \to Y'_C \to X'_C \to C \to 0$, we get a complex $V^{\cdot}$: $\cdots \to 0 \to V^{-n} \to \cdots \to V^{-2} \to V^{-1} \to V^0 \to V^1 \to \cdots \to V^i \to \cdots \in K^+(\operatorname{add}_B U)$. Since Y'_C is contained in $\operatorname{fresol}({}_BU)$, $\operatorname{Ext}^1_B(X_C, Y'_C) = 0$. Then we have an exact sequence $0 \to \operatorname{Hom}_B(X_C, Y'_C) \to \operatorname{Hom}_B(X_C, X'_C) \to \operatorname{Hom}_B(X_C, C) \to 0$. Hence we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_C & \longrightarrow & X_C & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow\beta & & \downarrow\alpha & & \| & & \\ 0 & \longrightarrow & Y'_C & \longrightarrow & X'_C & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Since $Y_C, Y'_C \in \operatorname{fresol}({}_BU)$, by Lemma 2.1, β can be extended to a morphism $\tau_{\leq -1}U^{\cdot} \to \tau_{\leq -1}V^{\cdot}$. Also, by $X_C, X'_C \in \operatorname{rac}({}_BU) = \operatorname{coresol}({}_BU)$, α can be extended to a morphism $\tau_{\geq 0}U^{\cdot} \to \tau_{\geq 0}V^{\cdot}$. Then α and β can be extended to a quasi isomorphism $f: U^{\cdot} \to V^{\cdot}$. According to Lemma 2.2, there exists a morphism $g: V^{\cdot} \to U^{\cdot}$ such that $f \circ g$ and $g \circ f$ are isomorphic in $K^+(\operatorname{add}_B U)$, that is, homotopic to $\operatorname{id}_{V^{\cdot}}$ and $\operatorname{id}_{U^{\cdot}}$, respectively. Hence we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y'_C & \longrightarrow & X'_C & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow\beta' & & \downarrow\alpha' & & \| & & \\ 0 & \longrightarrow & Y_C & \longrightarrow & X_C & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

such that $\operatorname{id}_{X_C} - \alpha' \circ \alpha$ and $\operatorname{id}_{Y_C} - \beta' \circ \beta$ factor through U^{-1}, and that $\operatorname{id}_{X_{C'}} - \alpha \circ \alpha'$ and $\operatorname{id}_{Y_{C'}} - \beta \circ \beta'$ factor through V'^{-1}. For $0 \to C \to Y^C \to X^C \to 0$ and $0 \to C \to Y^{C\prime} \to X^{C\prime} \to 0$, we are done by same technique.

Auslander and Buchweitz called a sequence of (a) in Proposition 3.2 a $\operatorname{rac}({}_BU)$-approximation, and called a sequence of (b) in Proposition 3.2 a $\operatorname{fresol}({}_BU)$-hull [AB]. Auslander and Reiten showed the minimality of such an approximation and a hull, in the case with B a commutative complete Cohen−Macaulay ring or an artin algebra (see [AR1]). A map $g: X \to C$ is called right minimal provided that f is an isomorphism for all $f: X \to X$

which satisfy $g \circ f = g$. A left minimal map is defined dually. A right minimal $\mathrm{rac}({}_BU)$-approximation is called a minimal $\mathrm{rac}({}_BU)$-approximation. A left minimal $\mathrm{fresol}({}_BU)$-hull is called a minimal $\mathrm{fresol}({}_BU)$-hull (see [AR1] for details). We also get a result similar to that of Auslander and Reiten [AR1, Proposition 1.1 or Proposition 1.5] (cf. [Ms, Theorem 6.1]).

PROPOSITION 3.3. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a cotilting $(B-A)$-bimodule. Then* $\mathrm{Hom}_A(-, {}_BU_A)\colon \mathrm{rac}(U_A) \to \mathrm{rac}({}_BU)$ *is a duality with a quasi inverse* $\mathrm{Hom}_B(-, {}_BU_A)$.

Proof. By the proof of Corollary 2.11.

For an additive category $\mathscr{A}$, we will call $\mathscr{A}$ semiperfect if $\mathrm{End}_{\mathscr{A}}(X)$ is a semiperfect ring for every object $X \in \mathscr{A}$.

THEOREM 3.4. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a cotilting $(B-A)$-bimodule.*

(a) *If* $\mathrm{rac}({}_BU)$ *is semiperfect, then there exists a unique minimal* $\mathrm{rac}({}_BU)$*-approximation in B*-mod.

(b) *If* $\mathrm{fresol}({}_BU)$ *is semiperfect, then there exists a unique minimal* $\mathrm{fresol}({}_BU)$*-hull in B*-mod.

Proof. (a) According to Proposition 3.2, it suffices to show minimality. Let an exact sequence $0 \to Y \to X \to C \to 0$ be a $\mathrm{rac}({}_BU)$-approximation of C. By the proof of Proposition 3.2, we have an exact sequence $0 \to \mathrm{Hom}_B(X, Y) \to \mathrm{Hom}_B(X, X) \to \mathrm{Hom}_B(X, C) \to 0$. Since $\mathrm{End}_B(X)$ is a semiperfect ring and $\mathrm{Hom}_B(X, C)$ is a finitely generated $\mathrm{End}_B(X)$-module, $\mathrm{Hom}_B(X, C)$ has a projective cover $\mathrm{Hom}_B(X, X') \to \mathrm{Hom}_B(X, C)$. By $X' \in \mathrm{add}\, X$, we have $\mathrm{Hom}_{\mathrm{End}_B(X)}(\mathrm{Hom}_B(X, X'), \mathrm{Hom}_B(X, C)) \cong \mathrm{Hom}_B(X', C)$, and then there exists a morphism $X' \to C$ such that $\mathrm{Hom}_B(X', X') \to \mathrm{Hom}_B(X', C)$ is a projective cover. Then we get the following morphism between exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_B(X, Y) & \longrightarrow & \mathrm{Hom}_B(X, X) & \longrightarrow & \mathrm{Hom}_B(X, C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Hom}_B(X, Y') & \longrightarrow & \mathrm{Hom}_B(X, X') & \longrightarrow & \mathrm{Hom}_B(X, C) & \longrightarrow & 0
\end{array}$$

where $Y' = \mathrm{Ker}(X' \to C)$ and the vertical arrows are epimorphisms. Since $\mathrm{Hom}_B(X, X')$ is a projective $\mathrm{End}_B(X)$-module, and

$$\mathrm{Hom}_{\mathrm{End}_B(X)}(\mathrm{Hom}_B(X, X), \mathrm{Hom}_B(X, X')) \cong \mathrm{Hom}_B(X, X'),$$

we get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & X'' & = & X'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where all rows and columns are exact and the vertical arrows are split maps. It is easy to see that $X' \to C$ is a right minimal map. By $Y \in \operatorname{fresol}({}_BU)$, we get $\operatorname{Ext}^i_B(U, Y) = 0$ for all $i > 0$, and therefore, $\operatorname{Ext}^i_B(U, X'') = 0$ for all $i > 0$. Since $X'' \in \operatorname{rac}({}_BU) = \operatorname{coresol}({}_BU)$ and $\operatorname{idim}_B U < \infty$, we have $X'' \in \operatorname{add}_B U$. We take a resolution of Y: $0 \to U^{-n} \to \cdots \to U^{-2} \to U^{-1} \to Y \to 0$, where $U^i \in \operatorname{add} U_A$ for all i. Since $\operatorname{Ext}^i_B(X'', \operatorname{fresol}({}_BU)) = 0$ for all $i > 0$, we have the commutative diagram

$$\begin{array}{ccccccccccccc}
 & & & & & & & & X'' & = & X'' & & \\
 & & & & & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U^{-n} & \longrightarrow & \cdots & \longrightarrow & U^{-2} & \longrightarrow & U^{-1} & \longrightarrow & Y & \longrightarrow & 0
\end{array}$$

where the vertical arrows are split monomorphisms. Then Y' belongs to $\operatorname{coresol}({}_BU)$, and hence we get a minimal $\operatorname{rac}({}_BU)$-approximation of C: $0 \to Y' \to X' \to C \to 0$.

(b) Similarly.

EXAMPLE. Let R be a commutative complete local Noetherian ring, A an R-algebra which is finitely generated as an R-module. Then A-mod is semiperfect. Moreover, if A satisfies the conditions of Proposition 2.12, then A has a dualizing A-bimodule.

A $(B-A)$-cotilting bimodule U is called a strong cotilting bimodule if $\operatorname{fresol}({}_BU) = \{Y \in B\text{-mod} \mid \operatorname{idim} Y < \infty\}$ and $\operatorname{fresol}(U_A) = \{X \in \operatorname{mod} A \mid \operatorname{idim} X < \infty\}$ [AR2]. Since U is not necessarily a strong cotilting bimodule, in general, $\operatorname{fresol}({}_BU)$ does not coincide with $\{Y \in B\text{-mod} \mid \operatorname{idim} Y < \infty\}$, and there is no induced equivalence between $\{Y \in B\text{-mod} \mid \operatorname{idim} Y < \infty\}$ and $\{X \in \operatorname{mod} A \mid \operatorname{pdim} X < \infty\}$. But, in the case of Artinian rings, we have the correspondences between categories of complexes. Let $\mathscr{T}({}_BU)$ be the triangulated subcategory of $D^b(B\text{-mod})$ generated by ${}_BU$. Let $D^b(B$-

mod$)_{\mathrm{fid}}$ be the triangulated subcategory of $D^b(B\text{-mod})$ generated by complexes which are isomorphic to complexes in $K^b(B\text{-Inj})$.

LEMMA 3.5. *Let A be a right Artinian ring, B a left Artinian ring.*

(a) *For a complex $X^{\cdot} \in D^b(B\text{-mod})$, the following are equivalent*:

(1) $X^{\cdot} \in D^b(B\text{-mod})_{\mathrm{fid}}$;

(2) *for every $Y^{\cdot} \in D^b(B\text{-mod})$, there exists some integer n such that* $\mathrm{Hom}_{D^b(B\text{-mod})}(Y^{\cdot}, X^{\cdot}[i]) = 0$ *for all $i > n$.*

(b) *For a complex $X^{\cdot} \in D^b(\mathrm{mod}\ A)$, the following are equivalent*:

(1) $X^{\cdot} \in K^b(\mathscr{P}_A)$;

(2) *for every $Y^{\cdot} \in D^b(\mathrm{mod}\ A)$, there exists some integer n such that* $\mathrm{Hom}_{D^b(\mathrm{mod}\ A)}(X^{\cdot}, Y^{\cdot}[i]) = 0$ *for all $i > n$.*

Proof. (a) (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (1) According to [A], it is easy to see that if there exists some integer n such that $\mathrm{Hom}_{D^b(B\text{-mod})}(B/\mathrm{rad}\ B, X^{\cdot}[i]) = 0$ for all $i > n$, then $X^{\cdot} \in D^b(B\text{-mod})_{\mathrm{fid}}$.

(b) Similarly.

The following correspondence is the complex version of Sharp or Auslander–Buchweitz and Reiten for Artinian rings (see [S], [AB], [AR2]).

PROPOSITION 3.6. *Let A be a right Artinian ring, B a left Artinian ring, and ${}_BU_A$ a cotilting $(B-A)$-bimodule. Then $\mathscr{T}({}_BU)$ coincides with $D^b(B$-mod$)_{\mathrm{fid}}$, and ${}_BU_A \otimes -$ induces an equivalence $K^b({}_A\mathscr{P}) \to D^b(B\text{-mod})_{\mathrm{fid}}$.*

Proof. By the duality, it is easy to see that $\mathbf{R}^b\mathrm{Hom}_B(-, {}_BU_A)$ induces a duality functor $\mathrm{Hom}_A(-, {}_BU_A)$: $K^b(\mathscr{P}_A) \to \mathscr{T}({}_BU)$. In general, there exists a duality $\mathrm{Hom}_A(-, A)$: $K^b({}_A\mathscr{P}) \to K^b(\mathscr{P}_A)$. For a complex $P^{\cdot} \in K^b({}_A\mathscr{P})$, we have $\mathrm{Hom}_A(\mathrm{Hom}_A(P^{\cdot}, A), {}_BU_A) \cong {}_BU_A \otimes P^{\cdot}$ in $D^b(B\text{-mod})$. Then ${}_BU_A \otimes -$ induces an equivalence $K^b({}_A\mathscr{P}) \to \mathscr{T}({}_BU)$. According to Lemma 3.5, it is easy to see that $\mathbf{R}^b\mathrm{Hom}_B(-, {}_BU_A)$ also induces a duality between $K^b(\mathscr{P}_A)$ and $D^b(B\text{-mod})_{\mathrm{fid}}$. Hence $\mathscr{T}({}_BU)$ coincides with $D^b(B\text{-mod})_{\mathrm{fid}}$.

The following result is a generalization of a result of Levin and Vasconcelos [LV] (cf. [I2, Theorem 2]).

PROPOSITION 3.7. *Let A be a coherent ring with* $\mathrm{idim}_A A$, $\mathrm{idim}\ A_A < \infty$. *For every finitely presented left A-module, its injective dimension is finite if and only if its projective dimension is finite.*

Proof. Since $\mathrm{idim}_A A$, $\mathrm{idim}\ A_A < \infty$, A is a dualizing A-bimodule. Let C be a finitely presented left A-module. It is clear if $\mathrm{pdim}\, C < \infty$, then

$\operatorname{idim} C < \infty$. Suppose $\operatorname{idim} C < \infty$. We take a $\operatorname{coresol}({}_A A)$-approximation: $0 \to Y \to X \to C \to 0$. Since $Y \in \operatorname{fresol}({}_A A)$, $\operatorname{idim} Y < \infty$. Then $\operatorname{idim} X$ is finite. For X, we have a coresolution: $0 \to X \to U^0 \to U^1 \to \cdots$, where $U^i \in \operatorname{add}_A A$. Let $X^i := \operatorname{Im}(U^{i-1} \to U^i)$; then $\operatorname{Ext}^1_A(X^1, X) \cong \operatorname{Ext}^l_A(X^i, X)$ for all $i \geq 1$. Therefore, we have $\operatorname{Ext}^1_A(X^1, X) = 0$, and $X \to U^0$ is a split monomorphism. Hence X is projective. Since $\operatorname{pdim} Y$ and $\operatorname{pdim} X$ are finite, we have $\operatorname{pdim} C < \infty$.

4. FINITELY EMBEDDING COGENERATORS

According to [Hr], an injective resolution of a dualizing module of a Cohen–Macaulay ring includes every injective indecomposable module in some term of it. To prove this, they used localization of commutative rings (see [B], [Hr]). We approach this problem by using categorical localization.

Let $\mathscr{A}$ be an abelian category, $\mathscr{B}$ a full subcategory of $\mathscr{A}$. We call an object $X \in \mathscr{A}$ a finitely embedding cogenerator for $\mathscr{B}$ provided that every object in $\mathscr{B}$ admits an injection to some finite direct sum of copies of X in $\mathscr{A}$.

THEOREM 4.1. *Let A be a right coherent ring, B a left coherent ring, ${}_B U_A$ a $(B-A)$-bimodule which satisfies the condition* (C1), *and let* $0 \to {}_B U \to E^0 \to E^1 \to \cdots$ *be an injective coresolution of ${}_B U$ in B*-Mod. *If the image of* $\mathbf{R}^- \operatorname{Hom}_A(-, {}_B U_A)$: $D^-(\operatorname{mod} A) \to D^+(B\text{-mod})$ *contains B*-mod, *then* $\bigoplus_{K \geq 0} E^k$ *is a finitely embedding cogenerator for B*-mod, *and* $\prod_{k \geq 0} E^k$ *is a finitely embedding injective cogenerator for B*-mod.

Proof. By assumption, for every $X \in B$-mod, there exists a complex $P^\cdot$ in $K^-(\operatorname{sum} A_A)$ such that $\operatorname{Hom}_A(P^\cdot, {}_B U_A)$ is isomorphic to X in $D^b(B\text{-mod})$. We may assume $\operatorname{Hom}_A(P^\cdot, {}_B U_A)$ is the following complex $U^\cdot$,

$$\cdots \to 0 \to U^{-s} \to \cdots \to U^{-1} \to U^0 \to U^1 \to \cdots,$$

where $U^i \in \operatorname{sum}_B U$ for all i. Then we have $H^i(U^\cdot) = 0$ $(i \neq 0)$ and $H^0(U^\cdot) \cong X$ in B-mod. Let $E^\cdot$ be the complex $E^0 \to E^1 \to \cdots \to E^j \to \cdots$, then U^i has an injective coresolution $E^\cdot_i$ which is some finite direct sum of copies of $E^\cdot$ for all i. Then $E^\cdot_i$ is isomorphic to U^i in $D^+(B\text{-Mod})$ for all i. By induction on k, we construct a complex $V^\cdot_k \in K^+(B\text{-Inj})$ which has a quasi isomorphism $\tau_{\leq k} U^\cdot \to V^\cdot_k$ in $K^+(B\text{-Mod})$ as follows. First, we take $V^\cdot_{-s} := E^\cdot_{-s}$. Assume we have a complex $V^\cdot_{k-1}$ which satisfies the above condition. Since $\tau_{\leq k-1} U^\cdot \to V^\cdot_{k-1}$ and $U^k \to E^\cdot_k$ are quasi isomorphisms in $K^+(B\text{-Mod})$, for a distinguished triangle $\tau_{\leq k} U^\cdot \to \tau_{\leq k-1} U^\cdot \to$

$U^k[-k+1] \to$ in $K^+(B\text{-Mod})$, we have the following commutative diagram in $D^+(B\text{-Mod})$:

$$\begin{array}{ccc} \tau_{\le k-1}U^{\cdot} & \longrightarrow & U^k[-k+1] \\ \downarrow & & \downarrow \\ V^{\cdot}_{k-1} & \longrightarrow & E^{\cdot}_k[-k+1] \end{array}$$

Since $V^{\cdot}_{k-1}$ and $E^{\cdot}_k$ belong to $K^+(B\text{-Inj})$, we can consider that the above diagram is commutative in $K^+(B\text{-Mod})$. Then we can choose $V^{\cdot}_{k-1} \to E^{\cdot}_k[-k+1]$ as a map between complexes. By taking a mapping cone of $V^{\cdot}_{k-1} \to E^{\cdot}_k[-k+1]$, we have the following morphism between distinguished triangles in $K^+(B\text{-Mod})$:

$$\begin{array}{ccccccc} \tau_{\le k}U^{\cdot} & \longrightarrow & \tau_{\le k-1}U^{\cdot} & \longrightarrow & U^k[-k+1] & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ V^{\cdot}_k & \longrightarrow & V^{\cdot}_{k-1} & \longrightarrow & E^{\cdot}_k[-k+1] & \longrightarrow \end{array}$$

Since $V^{\cdot}_k$ is a mapping cone of $V^{\cdot}_{k-1} \to E^{\cdot}_k[-k+1]$, it is clear that $V^{\cdot}_k$ belongs to $K^+(B\text{-Inj})$. And, since $\tau_{\le k-1}U^{\cdot} \to V^{\cdot}_{k-1}$ and $U^k \to E^{\cdot}_k$ are quasi isomorphisms in $K^+(\mathrm{rac}(U_A))$, $\tau_{\le k}U^{\cdot} \to V^{\cdot}_k$ is a quasi isomorphism in $K^+(B\text{-Mod})$. By construction, $V^{\cdot}_1$ is the complex

$$\cdots \to 0 \to I^{-s} \to \cdots \to I^{-1} \xrightarrow{d_{-1}} I^0 \xrightarrow{d_0} I \to \cdots,$$

where $I^i \in \mathrm{add}(\bigoplus_{k \ge 0} E^k)$ for all i. Also, we have $H^i(V^{\cdot}_1) = 0$ $(i < 0)$ and $\mathrm{H}^0(V^{\cdot}_1) \cong X$ in B-mod. Then we have the exact sequences

$$0 \to I^{-s} \to \cdots \to I^{-1} \to \mathrm{Im}\, d_{-1} \to 0 \tag{1}$$

$$0 \to \mathrm{Im}\, d_{-1} \to \mathrm{Ker}\, d_0 \to X \to 0. \tag{2}$$

Since I^i is injective $(-s \le i \le -1)$, $\mathrm{Im}\, d_{-1}$ is injective. Therefore, the exact sequence (2) splits, and hence X admits an injection to I^0. By $I^0 \in \mathrm{add}(\bigoplus_{k \ge 0} E^k)$, X is embedded in some finite direct sum of copies of $\bigoplus_{k \ge 0} E^k$.

COROLLARY 4.2. *Let A be a right coherent ring, B a left coherent ring, ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C2l), (C3l), *and* (C4l), *and let $0 \to {}_BU \to E^0 \to E^1 \to \cdots \to E^n \to 0$ be an injective coresolution of ${}_BU$ in B-Mod. If* $\mathrm{rac}({}_BU)$ *coincides with* $\mathrm{coresol}({}_BU)$, *then $\bigoplus_{k=0}^n E^k$ is a finitely embedding injective cogenerator for B*-mod.

Proof. It is easy to see that the situation of the left version of Theorem 2.10 satisfies the condition of Theorem 4.1.

COROLLARY 4.3. *Let A be a right coherent ring, B a left coherent ring, ${}_BU_A$ a $(B-A)$-bimodule which satisfies the condition* (C1), *and let* $0 \to {}_BU \to E^0 \to E^1 \to \cdots$ *be an injective coresolution of ${}_BU$ in B*-Mod. *If* $\mathbf{R}^b\mathrm{Hom}_A(-, {}_BU_A)$: $D^b(\mathrm{mod}\ A) \to D^b(B\text{-mod})$ *is a quotient duality, then* $\bigoplus_{k \geq 0} E^k$ *is a finitely embedding cogenerator for B*-mod, *and* ${}_k\Pi_0 E^k$ *is a finitely embedding injective cogenerator for B*-mod.

Remark. In Corollary 4.3, we do not need finiteness of injective dimension of U_A. Indeed, let A be a commutative Noetherian regular ring of infinite Krull dimension. According to [Hr, Chap. V, Sect. 8], injective dimension of A is infinite, but A induces a duality $\mathbf{R}^b\mathrm{Hom}_A(-, A)$: $D^b(\mathrm{mod}\ A) \to D^b(\mathrm{mod}\ A)$.

We have a result better than a result of Miyashita [Ms, Corollary in Sect. 6].

PROPOSITION 4.4. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C2r), (C2l), (C3r), *and* (C4r). *Let* $0 \to {}_BU \to E^0 \to E^1 \to \cdots \to E^n \to 0$ *be an injective coresolution of ${}_BU$ in B*-Mod. *Then* $\bigoplus_{k=0}^{n} E^k$ *is a finitely embedding injective cogenerator for B*-mod.

Proof. By Corollaries 2.5 and 4.3.

PROPOSITION 4.5. *Let A be a right coherent ring, B a left Noetherian ring, and ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C2r), (C2l), (C3r), *and* (C4r). *Let* $0 \to {}_BU \to E^0 \to E^1 \to \cdots$ *be an injective coresolution of ${}_BU$ in B*-Mod. *Then every injective indecomposable left B-module is isomorphic to a direct summand of some E^k.*

Proof. According to Proposition 4.4, $\bigoplus_{k \geq 0} E^i$ is a finitely embedding cogenerator for B-mod. By [Ma], every injective indecomposable left B-module I is an injective hull of some cyclic left B-module M. Since B is a left Noetherian ring, there is an injection from M to a finite direct sum of copies $\bigoplus_{k \geq 0} E^k$. By injectivity of $\bigoplus_{k \geq 0} E^k$ there is an injection from I to a finite direct sum of copies $\bigoplus_{k \geq 0} E^k$. According to [Ma] and the Krull−Schmidt−Azumaya theorem, we complete the proof.

COROLLARY 4.6. *Let A be a coherent ring, and* $0 \to {}_AA \to E^0 \to E^1 \to \cdots \to E^n \to 0$ *an injective coresolution of ${}_AA$ in A*-Mod. *If* $\mathrm{idim}_A A$, *and* $\mathrm{idim}\ A_A$ *are finite, then* $\bigoplus_{k=0}^{n} E^k$ *is a finitely embedding injective cogenerator for A*-mod.

Proof. By assumption, A is a dualizing A-bimodule. We are done by Proposition 4.4.

The following result is a generalization of a result of Hoshino [Ho, Theorem II], and is better than a result of Iwanaga [I1, Theorem 2].

COROLLARY 4.7. *Let A be a right coherent and left Noetherian ring, and $0 \to {}_AA \to E^0 \to E^1 \to \cdots$ an injective coresolution of ${}_AA$ in A-*Mod. *If* $\dim {}_AA$ *and* $\operatorname{idim} A_A$ *are finite, then every injective indecomposable left A-module is isomorphic to a direct summand of some E^k.*

Proof. By assumption, A is a cotilting A-bimodule. We are done by Proposition 4.5.

PROPOSITION 4.8. *Let A be a right coherent ring, B a left coherent ring, ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C3r), *and let $0 \to {}_BU \to E^0 \to E^1 \to \cdots$ be an injective coresolution of ${}_BU$ in B-*Mod. *If the injective dimension of U_A is at most one, then $\bigoplus_{k \geq 0} E^k$ is a finitely embedding cogenerator for B-*mod, *and* $\prod_{k \geq 0} E^k$ *is a finitely embedding injective cogenerator for B-*mod.

Proof. By Theorem 2.8 and Corollary 4.3.

COROLLARY 4.9. *Let A be a right coherent ring, B a left Noetherian ring, ${}_BU_A$ a $(B-A)$-bimodule which satisfies the conditions* (C1), (C3r), *and let $0 \to {}_BU \to E^0 \to E^1 \to \cdots$ be an injective coresolution of ${}_BU$ in B-*Mod. *If the injective dimension of U_A is at most one, then every injective indecomposable left B-module is isomorphic to a direct summand of some E^k.*

Proof. The same as that for Corollary 4.6.

EXAMPLE. Let A be a ring $\left(\begin{smallmatrix} F & V \\ 0 & G \end{smallmatrix}\right)$, where F, G are division rings, and V is an $(F-G)$-bimodule such that $\dim_F V < \infty$ and $\dim V_G = \infty$. Then A is a right coherent and left Artinian ring, and A satisfies the conditions of Corollary 4.9.

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